International Journal of Control

Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713393989

On controllability and control laws for discrete linear repetitive processes

L. Hladowski [a]; K. Galkowski [a]; E. Rogers [b]; D. H. Owens [c]

[a] Institute of Control and Computation Engineering, University of Zielona Gora, Zielona Gora, Poland
[b] School of Electronics and Computer Science, University of Southampton, Southampton, SO17 1BJ, UK
[c] Department of Automatic Control and Systems Engineering, University of Sheffield, Sheffield, S1 3JD, UK

First published on: 18 September 2009

To cite this Article Hladowski, L., Galkowski, K., Rogers, E. and Owens, D. H.(2010) 'On controllability and control laws for discrete linear repetitive processes', International Journal of Control, 83: 1, 66 — 73, First published on: 18 September 2009 (iFirst)

To link to this Article: DOI: 10.1080/00207170903100206

URL: http://dx.doi.org/10.1080/00207170903100206

International Journal of Control
Vol. 83, No. 1, January 2010, 66–73

Taylor & Francis
Taylor & Francis Group

On controllability and control laws for discrete linear repetitive processes

L. Hladowski[a*], K. Galkowski[a], E. Rogers[b] and D.H. Owens[c]

[a]*Institute of Control and Computation Engineering, University of Zielona Gora, ul. Podgorna 50, 65-246, Zielona Gora, Poland;* [b]*School of Electronics and Computer Science, University of Southampton, Southampton, SO17 1BJ, UK;* [c]*Department of Automatic Control and Systems Engineering, University of Sheffield, Sheffield, S1 3JD, UK*

(*Received 27 April 2008; final version received 8 June 2009*)

Repetitive processes are a distinct class of 2D systems (i.e. information propagation in two independent directions) of both systems theoretic and applications interest. They cannot be controlled by the direct extension of existing techniques from either standard (termed 1D here) or 2D systems theory. This article develops significant new results on the relationships between one physically motivated concept of controllability for the so-called discrete linear repetitive processes and the structure and design of control laws, including the case when disturbances are present.

Keywords: repetitive processes; 2D dynamics; controllability; systems theory

1. Introduction

The unique characteristic of a repetitive, or multipass, process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

To introduce a formal definition, let $\alpha < +\infty$ denote the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(p)$, $0 \le p \le \alpha - 1$, generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(p)$, $0 \le p \le \alpha - 1$, $k \ge 0$.

Physical examples of these processes include long-wall coal cutting and metal rolling operations, see, for example, Edwards (1974) and Rogers, Galkowski and Owens (2007). Also in recent years, applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications include classes of iterative learning control schemes (Moore, Chen and Bahl 2005) and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle (Roberts 2000). Recently, it has been shown by Hladowski, Cai, Galkowski, Rogers, Freeman and Lewin (2008), with experimental verification on a gantry robot system, how the repetitive process setting can be used to design control laws that consider both trial-to-trial error convergence and transient response along the trials where these can conflict.

Attempts to control these processes using standard (or 1D) systems theory/algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2D systems structure, i.e. information propagation occurs from pass-to-pass and along a given pass and also the initial conditions are reset before the start of each new pass. In seeking a rigorous foundation on which to develop a control theory for these processes, it is natural to attempt to exploit structural links which exist between, in particular, the class the of so-called discrete linear repetitive processes and 2D linear systems described by the extensively studied Roesser or Fornasini–Marchesini state-space models (see the original references cited in, for example, Rogers et al. (2007)). Discrete linear repetitive processes are distinct from such 2D linear systems in the sense that information propagation in one of the two separate directions (along the pass) only occurs over a finite duration. Moreover, many key elements of 2D discrete linear systems theory provide, at very best, only partial answers to key systems theoretic questions for repetitive processes.

*Corresponding author. Email: L.Hladowski@issi.uz.zgora.pl

ISSN 0020–7179 print/ISSN 1366–5820 online

DOI: 10.1080/00207170903100206
http://www.informaworld.com

This is the case for the so-called pass profile controllability of discrete linear repetitive processes which has no 2D Roesser or Fornasini–Marchesini model interpretation. Also, a fundamental question for applications is when does there exist implementable control action which can force the process to produce a pre-specified pass profile on a given pass? Previous work has shown that the existence condition for such action is precisely this pass profile controllability property, which is completely characterised by a matrix rank test defined in terms of matrices from the process state-space model. This article develops significant new results for important sub-cases that could arise in applications due to key structural properties of the defining state-space model and/or the presence of disturbances.

Throughout this article $\lceil a \rceil$ is used to denote the smallest integer which is at least equal to a and $\lfloor a \rfloor$ is used to denote the largest integer which is at most equal to a, e.g. $\lceil 2 \rceil = \lfloor 2 \rfloor = 2$, $\lceil 2.1 \rceil = 3$, $\lfloor 2.1 \rfloor = 2$, $\lfloor 2.9 \rfloor = 2$.

2. Background

The state-space model of the discrete linear repetitive processes initially considered here has the following form (see, for example, Rogers et al. 2007) over $0 \leq p \leq \alpha - 1$, $k \geq 0$, where on pass k $x_k(p)$ is the $n \times 1$ state vector, $y_k(p)$ is the $m \times 1$ pass profile vector, and $u_k(p)$ is the $r \times 1$ vector of control inputs

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + B\,u_{k+1}(p) + B_0 y_k(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + D\,u_{k+1}(p) + D_0 y_k(p). \end{aligned} \tag{1}$$

To complete the process description, it is necessary to specify the boundary conditions, i.e. the state initial vector and the initial pass profile, respectively. Here it suffices to consider the simplest possible form for these, i.e.

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \geq 0 \\ y_0(p) &= y(p), \quad 0 \leq p \leq \alpha - 1 \end{aligned} \tag{2}$$

where d_{k+1} is an $n \times 1$ vector with constant entries and $y(p)$ is an $m \times 1$ vector whose entries are the known functions of p.

The structure of the boundary conditions and, in particular, the state initial vector sequence $\{x_{k+1}(0)\}_{k \geq 0}$ is critical to the stability properties of the example considered since, unlike other classes of linear systems, these alone can cause instability. For example, if we let $x_{k+1}(0)$ be a function of points along the previous pass, such as

$$x_{k+1}(0) = d_{k+1} + K_1 y_k(\alpha - 1) \tag{3}$$

where K_1 is an $n \times m$ matrix, then Rogers et al. (2007) gives an example which is stable with $K_1 = 0$, but could be unstable when $K_1 \neq 0$. In applications there is therefore a critical need to adequately model this sequence. Note also that if the state initial vector sequence is of the form (2) then clear structural links exist with repetitive control systems, which in repetitive process analysis is the same as using a single variable $V = k\alpha + p$ to convert the dynamics into those of an equivalent infinite length single pass process. Likewise, the previous pass profile sequence (3) can be used to establish a link with delay differential systems. The details of both these cases are again given in Rogers et al. (2007) and it is also clear that these links cannot be used to solve all systems theory and control law design problems for even processes described by (1) and (2).

The terms $B_0 y_k(p)$ and $D_0 y_k(p)$ describe the contributions of the previous pass dynamics to the current pass state and pass profile vectors, respectively. In the longwall coal cutting, the pass profile at any point along a pass is the height of the stone/coal interface above some datum line. As the cutting machine rests on the previous pass profile during the production of the current one, and the objective of each pass is to remove the maximum amount of coal without penetrating the stone/coal interface, it is clear how the previous pass profile influences the current one. Note also that the model for the contributions of the previous pass profile here is the simplest possible and clearly adequate modelling of this feature is critical if relevant control systems theory is to be developed.

In this article, we will make extensive use of the 1D equivalent model of processes described by the state-space model and boundary conditions given above. Note that 1D equivalent models also arise in the analysis of other classes of 2D linear systems, such as in Porter and Aravena (1984). For discrete linear repetitive processes of the form considered here, the 1D equivalent model has the distinct advantage of being defined in terms of vectors with dimensions which are fixed at the outset and the matrices involved have constant entries. It is also important to note that not all systems theoretic questions for discrete linear repetitive processes can be solved by using this approach.

To summarise the construction of this model (with full details in, for example, Galkowski, Rogers and Owens (1998)), first introduce the change of variables

$$\upsilon_k(p) = y_{k-1}(p),\ k \geq 0, 0 \leq p \leq \alpha - 1, l = k + 1$$

and define the so-called global pass profile, state and input super-vectors as

$$Y(l) := \begin{bmatrix} \upsilon_l(0) \\ \upsilon_l(1) \\ \vdots \\ \upsilon_l(\alpha-1) \end{bmatrix}, \; X(l) := \begin{bmatrix} x_l(1) \\ x_l(2) \\ \vdots \\ x_l(\alpha) \end{bmatrix},$$

$$U(l) := \begin{bmatrix} u_l(0) \\ u_l(1) \\ \vdots \\ u_l(\alpha-1) \end{bmatrix}$$

respectively. Then the 1D equivalent model is given by

$$\begin{aligned} Y(l+1) &= \Phi Y(l) + \Delta U(l) + \Theta d_l \\ X(l) &= \Gamma Y(l) + \Sigma U(l) + \Psi d_l \end{aligned} \tag{4}$$

where

$$\Phi = \begin{bmatrix} D_0 & 0 & \cdots & 0 \\ CB_0 & D_0 & \cdots & 0 \\ CAB_0 & CB_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{\alpha-2}B_0 & CA^{\alpha-3}B_0 & \dots & D_0 \end{bmatrix},$$

$$\Delta = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{\alpha-2}B & CA^{\alpha-3}B & \dots & D \end{bmatrix}$$

$$\Theta^T = \begin{bmatrix} C^T & (CA)^T & (CA^2)^T & \cdots & (CA^{\alpha-1})^T \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} B_0 & 0 & \dots & 0 \\ AB_0 & B_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{\alpha-1}B_0 & A^{\alpha-2}B_0 & \dots & B_0 \end{bmatrix},$$

$$\Sigma = \begin{bmatrix} B & 0 & \dots & 0 \\ AB & B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A^{\alpha-1}B & A^{\alpha-2}B & \dots & B \end{bmatrix}$$

$$\Psi^T = \begin{bmatrix} A^T & (A^2)^T & \dots (A^\alpha)^T \end{bmatrix}$$

Pass profile controllability is defined as follows (see, for example, Hladowski, Galkowski, Rogers, and Owens 2006).

Definition 2.1: Discrete linear repetitive processes described by (1) and (2) are said to be pass profile controllable if $\exists$ a pass number K^* and control input vectors $u_{k+1}(p)$, defined over $0 \le p \le \alpha-1$, $0 \le k \le K^*$, which will drive the process to an arbitrarily specified pass profile on pass K^*.

Other work (Hladowski et al. 2006) has shown that pass profile controllability holds if, and only if,

$$\text{rank}\begin{bmatrix} D & D_0 D & \cdots D_0^{m-1}D \end{bmatrix} = m. \tag{5}$$

In terms of applications, this is clearly a highly relevant problem, i.e. it is required to force the process to produce a pre-specified pass profile on a particular pass. It is known (Rogers et al. 2007) that this property is required for the existence of a control law to guarantee the most basic form of stability for these processes (and is a necessary condition for stronger versions). In iterative learning control, for example, this imposes the requirement that a particular trajectory is learnt after K^* trials have elapsed and it is also clear that there is no Roesser/Fornasini–Marchesini model 2D systems model version of this problem.

In many practical cases, pass profile controllability cannot hold since $D = 0$ (direct feedthrough term from the current pass input vector to current pass profile vector) and for these we can make no progress by this route. This article shows that a weaker form of this property can still be present in such cases and also that it is possible to use state feedback to minimise its effects on onward analysis.

The cases when pass profile controllability does not hold follow immediately from (5), i.e. when $D_0 = 0$ and D is not of full rank. Given the role of pass profile controllability in the control of discrete linear repetitive processes, it is clearly of interest to ask if a weaker form of this property holds in such cases. For further analysis, we require the following definition.

Definition 2.2: Define the natural number δ as follows:

(i) $\delta = 1$ when $D_0 = 0$ and $CB_0 \neq 0$, or
(ii) $\delta = 2, 3, \dots$ when

$$\begin{aligned} &\forall t = 0, 1, \dots, \delta-2, \quad CA^tB_0 = 0 \\ &\forall t > \delta - 2 \qquad\qquad\; CA^tB_0 \neq 0 \end{aligned} \tag{6}$$

and $D_0 = 0$.

Suppose now that we partition $Y(l)$ as

$$Y(l) := \left[(Y^1(l))^T \middle| (Y^2(l))^T \middle| (Y^3(l))^T\right]^T$$

where the vectors $Y^1(l)$ and $Y^3(l)$ are of dimension $\delta m \times 1$ and the vector $Y^2(l)$ is of dimension $m(\alpha - 2\delta) \times 1$. Then (4) can be rewritten as

$$Y^i(l+1) = \Phi_{i1} Y^1(l) + \Phi_{i2} Y^2(l) + \Phi_{i3} Y^3(l) + \Delta_i U(l) + \Theta_i d_l \quad i = 1,2,3$$

where (see also Hladowski et al. 2006)

$$Y(l) := \begin{bmatrix} Y^1(l) \\ Y^2(l) \\ Y^3(l) \end{bmatrix} \tag{7}$$

and

$$Y^1(l) := \begin{bmatrix} \upsilon_l(0) \\ \upsilon_l(1) \\ \vdots \\ \upsilon_l(\delta-1) \end{bmatrix}, \; Y^2(l) := \begin{bmatrix} \upsilon_l(\delta) \\ \upsilon_l(\delta+1) \\ \vdots \\ \upsilon_l(\alpha-\delta-1) \end{bmatrix},$$

$$Y^3(l) := \begin{bmatrix} \upsilon_l(\alpha-\delta) \\ \upsilon_l(\alpha-\delta+1) \\ \vdots \\ \upsilon_l(\alpha-1) \end{bmatrix}$$

$$\Phi_{11} = \Phi_{12} = \Phi_{13} = \Phi_{33} = \Phi_{23} = 0$$

$$\Phi_{21} = \begin{bmatrix} CA^{\delta-1}B_0 & & 0 \\ \vdots & \ddots & \\ CA^{2\delta-2}B_0 & \cdots & CA^{\delta-1}B_0 \\ \vdots & \vdots & \\ CA^{\alpha-\delta-2}B_0 & \cdots & CA^{\alpha-2\delta-1}B_0 \end{bmatrix}$$

$$\Phi_{22} = \left[\begin{array}{ccc|c} & 0 & & 0 \\ \hline CA^{\delta-1}B_0 & & 0 & \\ \vdots & \ddots & & 0 \\ CA^{\alpha-2\delta-2}B_0 & \cdots & CA^{\delta-1}B_0 & \end{array}\right]$$

$$\Phi_{31} = \begin{bmatrix} CA^{\alpha-\delta-1}B_0 & \cdots & CA^{\alpha-2\delta}B_0 \\ \vdots & \ddots & \vdots \\ CA^{\alpha-2}B_0 & \cdots & CA^{\alpha-\delta-1}B_0 \end{bmatrix} \tag{8}$$

In this case, we have that $Y^1(l)$, $l > 1$, is never pass profile controllable and $Y^3(l)$ does not influence any future pass profile. Hence, such a process is not pass profile controllable and instead we can consider the following property.

Definition 2.3: Discrete linear repetitive processes described by (1) and (2) for which $\exists \delta \geq 1$, satisfying (i) or (ii) as appropriate in Definition 2.2, are said to be relaxed pass profile controllable if $\exists$ a pass number K^* and control input vectors $u_k+1(p)$, defined over $0 \leq p \leq \alpha - 1$, $0 \leq k \leq K^*$, such that the process dynamics are transferred from an initial pass profile segment of the form

$$\widehat{Y}(1) = \left[(Y^1(1))^T \middle| (Y^2(1))^T \right]^T$$

to a prescribed pass profile segment on pass K^* of the form

$$\widetilde{Y}^* := \widetilde{Y}(K^*) = \left[(Y^2(K^*))^T \middle| (Y^3(K^*))^T \right]^T. \tag{9}$$

The key points here are that on each pass the first δ points are not pass profile controllable and the last δ points do not contribute to the dynamics of any subsequent pass profile. This is illustrated in Figure 1. It is also obvious that the maximum allowed value for δ is

$$\delta_{\max} = \left\lfloor \frac{\alpha - 1}{2} \right\rfloor \tag{10}$$

since for $\delta > \delta_{\max}$ $Y^2(l)$ does not exist.

Let $\tau = \min(\lfloor \frac{\alpha}{\delta} \rfloor - 3, K^* - 4)$ and also introduce the so-called relaxed pass profile controllability matrix defined in terms of the 1D equivalent model as

$$\Upsilon(K^*) = \left[\begin{array}{c|c|c|c|c|c|c} \Delta_2 & \Phi_{22}\Delta_2 + \Phi_{21}\Delta_1 & \Phi_{22}^2\Delta_2 + \Phi_{22}\Phi_{21}\Delta_1 & \cdots & \Phi_{22}^{\tau}\Delta_2 + \Phi_{22}^{\tau-1}\Phi_{21}\Delta_1 & \Phi_{22}^{\tau}\Phi_{21}\Delta_1 & 0 \\ \Delta_3 & \Phi_{32}\Delta_2 + \Phi_{31}\Delta_1 & \Phi_{32}(\Phi_{22}\Delta_2 + \Phi_{21}\Delta_1) & \cdots & \Phi_{32}(\Phi_{22}^{\tau-1}\Delta_2 + \Phi_{22}^{\tau-2}\Phi_{21}\Delta_1) & \Phi_{32}(\Phi_{22}^{\tau}\Delta_2 + \Phi_{22}^{\tau-1}\Phi_{21}\Delta_1) & \Phi_{32}\Phi_{22}^{\tau}\Phi_{21}\Delta_1 \end{array} \right].$$

Then we have the following result.

Theorem 2.4 (Hladowski et al. 2006)**:** *Discrete linear repetitive processes described by* (1) *and* (2) *for which* $D_0 = 0$ *and* (6) *holds are relaxed pass profile controllable if, and only if,* $\exists$ K^* *such that*

$$\operatorname{rank} \tilde{\Upsilon}(\tilde{K}^*) = m(\alpha - \delta).$$

The maximum value of $\tilde{K}^*$ *is equal to* $\tilde{K}^*_{\max} = (\alpha - \delta) + 1$.

Next, we investigate the particular case when the pass profile segment represented by $Y_3(l)$ (i.e. the last part of the pass) remains free as well, i.e. not the subject of the control action, which can happen in many practical applications, for example metal rolling.

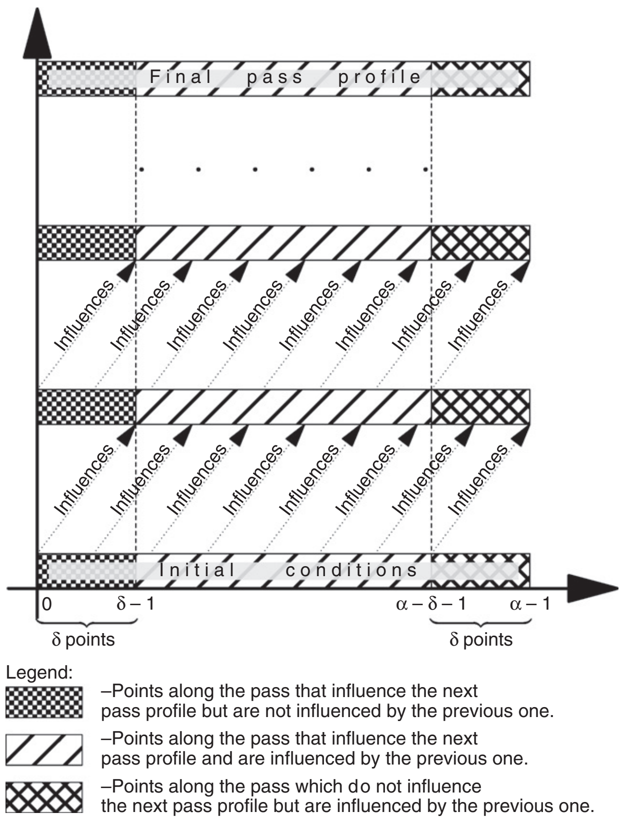

Figure 1. Illustrating relaxed pass profile controllability.

2.1 *Two sided relaxed pass profile controllability*

Definition 2.5: Discrete linear repetitive processes described by (1) and (2) for which $\exists\ \delta \geq 1$, satisfying (i) or (ii) as appropriate in Definition 2.2, are said to be two sided relaxed pass profile controllable if $\exists$ a pass number K^* and control input vectors $u_{k+1}(p)$, defined over $0 \leq p \leq \alpha - 1,\ 0 \leq k \leq K^*$, such that the process dynamics are transferred from an initial pass profile segment of the form

$$\widehat{Y}(1) := \left[\,(Y^1(1))^T (Y^2(1))^T\,\right]^T$$

to a prescribed pass profile segment on pass K^* of the form

$$\widetilde{Y}^* := \widetilde{Y}(K^*) := Y^2(K^*).$$

Note here the critical difference between relaxed and two sided relaxed pass profile controllability, the former requires us to achieve $Y^2(K^*)$ and $Y^3(K^*)$ whereas the second only requires us to achieve $Y^2(K^*)$.

Let $\tau = \min(\lceil \frac{\alpha}{\delta} \rceil - 3, K^* - 4)$ and introduce the so-called two sided relaxed pass profile controllability matrix formed in terms of the 1D equivalent model as

$$\begin{aligned}\tilde{\Upsilon}(K^*) = \Big[\Delta_2 \Big| \Phi_{22}\Delta_2 + \Phi_{21}\Delta_1 \Big| \Phi_{22}^2\Delta_2 + \Phi_{22}\Phi_{21}\Delta_1 \Big| \\ \Phi_{22}^3\Delta_2 + \Phi_{22}^2\Phi_{21}\Delta_1 \Big| \cdots \Big| \Phi_{22}^{\tau}\Delta_2 \\ + \Phi_{22}^{\tau-1}\Phi_{21}\Delta_1 \Big| \Phi_{22}^{\tau}\Phi_{21}\Delta_1\Big]. \qquad (11)\end{aligned}$$

Then we have the following result.

Theorem 2.6: *Discrete linear repetitive processes described by* (1) *and* (2) *for which* $\exists\,\delta \geq 1$ *as defined in Definition* 2.2, *is two sided relaxed pass profile controllable if, and only if,* $\exists \tilde{K}^*$ *such that*

$$\text{rank}\,\tilde{\Upsilon}(\tilde{K}^*) = m(\alpha - 2\delta). \qquad (12)$$

Also the maximum value of $\tilde{K}^*$ *is equal to*

$$\tilde{K}^*_{\max} = (\alpha - 2\delta) + 1.$$

Proof: Two sided relaxed pass profile controllability is a special case of relaxed pass profile controllability where only $Y^2 = Y^2_{\text{ref}}$ is required. Consider now the general response formula

$$\begin{aligned}Y^2(K) = \Phi_{22}^{K-1}Y^2(1) + \Phi_{22}^{K-2}\Phi_{21}Y^1(1) \\ + \sum_{j=1}^{K-2}\Big[\Phi_{22}^{j}\Delta_2 + \Phi_{22}^{j-1}\Phi_{21}\Delta_1\Big] \\ \times U(K-j-1) + \Delta_2 U(K-1).\end{aligned}$$

The matrix-vector form of this last expression is defined in terms of (11) and then the condition of Definition 2.5 requires that (12) holds. Moreover, the formula for $\tilde{K}^*_{\max}$ is immediate. □

3. Control of relaxed pass profile controllable processes

In some applications it will clearly be required to deal explicitly with disturbances. Hence, in this section we consider an extension of (1) in the form

$$\begin{aligned}x_{k+1}(p+1) = Ax_{k+1}(p) + Bu_{k+1}(p) + B_0y_k(p) + Ew_{k+1}(p) \\ y_{k+1}(p) = Cx_{k+1}(p) + Du_{k+1}(p) + D_0y_k(p) + Fw_{k+1}(p)\end{aligned} \qquad (13)$$

over $0 \leq p \leq \alpha - 1,\ k \geq 0$, where $w_k(p)$ is a $q \times 1$ disturbance vector affecting the state and pass profile updating equations which can vary both along the pass and from pass-to-pass.

The control goal now is to drive the system to the required reference pass profile, denoted in the 1D equivalent model by Y_{ref}. We assume that (13) is relaxed pass profile controllable (see Definition 2.3) and hence the first δ points of the pass profile vector cannot be influenced by the control term. Therefore only a part of the output super-vector can be specified, as Y^1 is unobtainable. Hence, the required limit pass profile (Y_{ref}) consists only of the second (Y^2) and third (Y^3) sub-vectors of the Y and is written as

$$Y_{\text{ref}} = \begin{bmatrix} \widetilde{X} \\ Y^2_{\text{ref}} \\ Y^3_{\text{ref}} \end{bmatrix}, \begin{bmatrix} Y^2_{\text{ref}} \\ Y^3_{\text{ref}} \end{bmatrix} = \begin{bmatrix} y_{\text{ref}}(\delta) \\ y_{\text{ref}}(\delta+1) \\ \vdots \\ y_{\text{ref}}(\alpha-1) \end{bmatrix}$$

where $\widetilde{X}$ denotes a vector of the required dimensions whose elements cannot be influenced by control action.

The 1D equivalent model for this case can (by an obvious extension to the construction in the disturbance free case) be written as

$$\begin{aligned} Y^1(l+1) &= \Delta_1 U(l) + \Theta_1 d_l + \Omega_1^Y W(l) \\ Y^2(l+1) &= \Phi_{21} Y^1(l) + \Phi_{22} Y^2(l) \\ &\quad + \Delta_2 U(l) + \Theta_2 d_l + \Omega_2^Y W(l) \\ Y^3(l+1) &= \Phi_{31} Y^1(l) + \Phi_{32} Y^2(l) \\ &\quad + \Delta_3 U(l) + \Theta_3 d_l + \Omega_3^Y W(l) \end{aligned} \tag{14}$$

for $l \geq 1$, where

$$\Omega_1^Y = \left[\begin{array}{cccc|c} F & 0 & \cdots & 0 & \\ CE & F & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & 0 \\ CA^{\delta-1}E & CA^{\delta-2}E & \dots & F & \end{array}\right]$$

$$\Omega_2^Y = \left[\begin{array}{cccccc|c} CA^{\delta}E & \cdots & F & 0 & \cdots & 0 & \\ CA^{\delta+1}E & \cdots & CE & F & \cdots & 0 & \\ \vdots & & \vdots & \vdots & \ddots & \vdots & 0 \\ CA^{\alpha-\delta-1}E & \cdots & CA^{\alpha-2\delta-1}E & CA^{\alpha-2\delta-2}E & \cdots & F & \end{array}\right]$$

$$\Omega_3^Y = \begin{bmatrix} CA^{\alpha-\delta}E & \cdots & F & 0 & \cdots & 0 \\ CA^{\alpha-\delta+1}E & \cdots & CE & F & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ CA^{\alpha-1}E & \cdots & CA^{\delta-1}E & CA^{\delta-2}E & \dots & F \end{bmatrix}$$

and finally

$$W(l) = \left[(w_l(0))^T (w_l(1))^T \dots (w_l(\alpha-1))^T \right]^T$$

denotes the disturbance super-vector. The structure of Φ_{22} is also critical to what follows, i.e.

$$\Phi_{22} = \left[\begin{array}{ccc|c} & 0 & & 0 \\ \hline CA^{\delta-1}B_0 & & 0 & \\ \vdots & \ddots & & 0 \\ CA^{\alpha-2\delta-2}B_0 & \cdots & CA^{\delta-1}B_0 & \end{array}\right]. \tag{15}$$

The matrix Φ_{22} is nilpotent and we have the following result (Hladowski et al. 2006).

Theorem 3.1: *Suppose that Φ_{22} has the structure of (15). Then in the general case $\Phi_{22}^{l^*} = 0$, where*

$$l^* = \left\lceil \frac{\alpha}{\delta} - 2 \right\rceil. \tag{16}$$

Note here the updating of $Y^1(l+1)$ is static in the sense that it does not depend on any of the sub-vectors of Y on the previous pass, i.e. $Y^1(l)$, $Y^2(l)$, $Y^3(l)$.

Introduce now the so-called incremental vectors as

$$\begin{aligned} \hat{Y}^2(l+1) &= Y^2(l+1) - Y^2_{\text{ref}} \\ \hat{Y}^3(l+1) &= Y^3(l+1) - Y^3_{\text{ref}}. \end{aligned} \tag{17}$$

Now solve (17) for $Y_2(l+1)$ and $Y_3(l+1)$ and then substitute in (14) to obtain

$$\begin{aligned} Y^1(2) &= \Delta_1 U(1) + \Theta_1 d_1 + \Omega_1^Y W(1), \\ \hat{Y}^2(2) &= \Phi_{21} Y^1(1) + \Phi_{22} Y^2(1) + (\Phi_{22} - I) Y^2_{\text{ref}} \\ &\quad + \Delta_2 U(1) + \Theta_2 d_1 + \Omega_2^Y W(1), \\ \hat{Y}^3(2) &= \Phi_{31} Y^1(1) + \Phi_{32} Y^2(1) \\ &\quad + \Phi_{32} Y^2_{\text{ref}} - Y^3_{\text{ref}} \\ &\quad + \Delta_3 U(1) + \Theta_3 d_1 + \Omega_3^Y W(1) \end{aligned}$$

for $l = 1$ and

$$\begin{aligned} \hat{Y}^2(l+1) &= \Phi_{22} \hat{Y}^2(l) + (\Phi_{22} - I) Y^2_{\text{ref}} \\ &\quad + \Delta_2 U(l) + \Phi_{21} \Delta_1 U(l-1) \\ &\quad + \Omega_2^Y W(l) + \Phi_{21} \Omega_1^Y W(l-1) \\ &\quad + \Theta_2 d_l + \Phi_{21} \Theta_1 d_{l-1}, \\ \hat{Y}^3(l+1) &= \Phi_{32} \hat{Y}^2(l) + \Phi_{32} Y^2_{\text{ref}} - Y^3_{\text{ref}} \\ &\quad + \Delta_3 U(l) + \Phi_{31} \Delta_1 U(l-1) \\ &\quad + \Theta_3 d_l + \Phi_{31} \Theta_1 d_{l-1} \\ &\quad + \Omega_3^Y W(l) + \Phi_{31} \Omega_1^Y W(l-1) \end{aligned} \tag{18}$$

for $l > 1$. The goal now is to find the set of appropriate input super-vectors $\{U(l)\}_{l=1,2,\dots,l^\ddagger}$, such that $\hat{Y}^2(l^\ddagger) = 0$ and $\hat{Y}^3(l^\ddagger) = 0$ because by (17) this implies that $Y^2(l^\ddagger) = Y^2_{\text{ref}}$ and $Y^3(l^\ddagger) = Y^3_{\text{ref}}$. To achieve the goal, the following control algorithm (based on decoupling of the disturbance ($W(l)$) and initial condition (d_l) vectors and exploiting the nilpotency property of Φ_{22}) is proposed:

Control algorithm A

(a) Set $U(1) = 0$, $l = 2$
(b) Find $U(l)$ such that

$$\begin{aligned} \Delta_2 U(l) &= (I - \Phi_{22}) Y^2_{\text{ref}} - \Phi_{21} \Delta_1 U(l-1) - \Omega_2^Y W(l) \\ &\quad - \Phi_{21} \Omega_1^Y W(l-1) - \Theta_2 d_l - \Phi_{21} \Theta_1 d_{l-1} \\ \Delta_3 U(l) &= -\Phi_{32} Y^2_{\text{ref}} - Y^3_{\text{ref}} - \Phi_{31} \Delta_1 U(l-1) - \Theta_3 d_l \\ &\quad - \Phi_{31} \Theta_1 d_{l-1} - \Omega_3^Y W(l) - \Phi_{31} \Omega_1^Y W(l-1) \end{aligned} \tag{19}$$

If no such $U(l)$ exists, stop as the algorithm cannot be applied.

(c) If $\hat{Y}^2(l)=0$ and $\hat{Y}^3(l)=0$, stop. The required solution is found, $l^{\ddagger}=l$.

(d) Increase l by one and go to (a)

Theorem 3.2: *Control algorithm A achieves* $Y^2(l^{\ddagger})=Y^2_{\text{ref}}$ *and* $Y^3(l^{\ddagger})=Y^3_{\text{ref}}$ *if* (19) *has solutions for* $l=2,3,\ldots,l^{\ddagger}$.

Proof: Substituting (19) into (18) yields

$$\hat{Y}^2(l+1)=\Phi_{22}\hat{Y}^2(l)\quad \hat{Y}^3(l+1)=\Phi_{32}\hat{Y}^2(l)$$

or, equivalently,

$$\hat{Y}^2(l+1)=\Phi_{22}^{l}\hat{Y}^2(1)\quad \hat{Y}^3(l+1)=\Phi_{32}\hat{Y}^2(l)$$

for $l>1$. By Theorem 3.1 we can write

$$\hat{Y}^2(l^*+1)=\Phi_{22}^{l^*}\hat{Y}^2(1)=0\quad \hat{Y}^3(l^*+1)=\Phi_{32}\hat{Y}^2(l^*).$$

Also

$$\hat{Y}^2(l^*+2)=0\qquad \hat{Y}^3(l^*+2)=\Phi_{32}\hat{Y}^2(l^*+1)=0 \tag{20}$$

and by (17) the required result is obtained and the proof is complete. □

Theorem 3.3: *Control algorithm A requires at most* $l^*_{max}=\lceil\frac{\alpha}{\delta}\rceil$ *steps to yield the required result.*

Proof: As shown in Theorem 3.1, $\Phi_{22}^{l^*}=0$. Moreover by (20), $\hat{Y}^3(l^*+2)=0$, hence the algorithm needs at most l^*+2 steps. Introducing (16) yields the required result. □

3.1 *Control of two sided relaxed pass profile controllable processes with disturbances*

The essential difference between two sided relaxed and relaxed pass profile controllability is the lack of requirements on Y^3, and hence in the former case the equations defining Y^3 must be deleted from the analysis since Y^3 is not subject to control action. Substituting the first incremental vector of (17) into the middle equation of (14) yields

$$\begin{aligned}Y^1(2)&=\Delta_1U(1)+\Theta_1d_1+\Omega_1^YW(1)\\ \hat{Y}^2(2)&=\Phi_{21}Y^1(1)+\Phi_{22}Y^2(1)+(\Phi_{22}-I)Y^2_{\text{ref}}\\ &\quad+\Delta_2U(1)+\Theta_2d_1+\Omega_2^YW(1)\end{aligned} \tag{21}$$

for $l=1$ and

$$\begin{aligned}\hat{Y}^2(l+1)&=\Phi_{22}\hat{Y}^2(l)+(\Phi_{22}-I)Y^2_{\text{ref}}+\Delta_2U(l)\\ &\quad+\Phi_{21}\Delta_1U(l-1)+\Omega_2^YW(l)\\ &\quad+\Phi_{21}\Omega_1^YW(l-1)+\Theta_2d_l+\Phi_{21}\Theta_1d_{l-1}\end{aligned} \tag{22}$$

for $l>1$. The task now is to enforce $\hat{Y}^2(l)=0$ and hence attain the required reference pass profile vector $Y^2(l)=Y^2_{\text{ref}}$. To achieve this, we modify the previous algorithm (Control algorithm A) to the following:

Control algorithm B

(a) Set $U(1)=0$, $l=2$

(b) Find $U(l)$ such that the following equation holds:

$$\begin{aligned}\Delta_2U(l)&=(I-\Phi_{22})Y^2_{\text{ref}}-\Phi_{21}\Delta_1U(l-1)\\ &\quad-\Omega_2^YW(l)+\Phi_{21}\Omega_1^YW(l-1)\\ &\quad-\Theta_2d_l+\Phi_{21}\Theta_1d_{l-1}\end{aligned} \tag{23}$$

If no such $U(l)$ exist, stop as the algorithm cannot be applied.

(c) If $\hat{Y}^2(l)=0$, stop. The required solution is found, $l^{\ddagger}=l$

(d) Increase l by one and go to (a)

Now we have the following results whose proof is very similar to that for Theorem 3.2 and hence is omitted here.

Theorem 3.4: *Control algorithm B achieves* $Y^2(l^{\ddagger})=Y^2_{\text{ref}}$ *if* (23) *has solutions for* $l=2,3,\ldots,l^{\ddagger}$.

Theorem 3.5: *Control algorithm B requires at most* $l^*_{max}=\lceil\frac{\alpha}{\delta}-1\rceil$ *steps to yield the required result.*

Proof: Note that we only need to force $\hat{Y}^2(l)$ to zero. Using Theorem 3.1 we obtain $\hat{Y}^2(l^*+1)=\Phi_{22}^{l^*}Y_2(1)=0$. Introducing (16) now gives the required result. □

4. Modifying the value of δ

It is clear from the analysis so far in this paper that if the value of δ is large, then we cannot control a large part of the pass profile vector. In this section, we show how auxiliary control action can be used to beneficially change the value of δ.

Consider the application of the control law to (1)

$$u_{k+1}(p)=K_1x_{k+1}(p)+K_2y_k(p)+v_k(p) \tag{24}$$

resulting in the controlled process state-space model

$$\begin{aligned}x_{k+1}(p+1)&=A^*x_{k+1}(p)+Bv_k(p)+B_0^*y_k(p)\\ y_{k+1}(p)&=C^*x_{k+1}(p)+Dv_k(p)+D_0^*y_k(p)\end{aligned} \tag{25}$$

where

$$A^*=A+BK_1,\ B_0^*=B_0+BK_2$$
$$C^*=C+DK_1,\ D_0^*=D_0+DK_2$$

and the new external vector v could subsequently be used to introduce further control action. Also it is often beneficial to reduce the value of δ and hence minimise the uncontrollable part of the process pass profile sequence. This can be achieved by using (24) if we can find appropriate values for K_1 and K_2. It is obvious that any process described by (1) must satisfy one of the following:

- $D_0 = 0$ and the process is both relaxed pass profile and pass profile uncontrollable. In this case, the only option is to use (24) with $DK_2 = 0$ to increase $\delta < \delta_{\max}$, and then test for pass profile controllability of the new state-space model.
- $D_0 \neq 0$ and the process is pass profile uncontrollable. In this case, there is no δ in the sense of Definition 2.2 and hence the process is relaxed pass profile uncontrollable. In this case it is therefore first necessary to force D_0 to zero. To achieve this, we have to find K_2 such that

$$D_0 = -DK_2. \tag{26}$$

  If (26) has solutions, it is necessary to determine the value of δ and then check the requirements of Theorem 2.4. If the controlled process is relaxed pass profile controllable, further reduction of δ can be achieved by modifying K_1, as in the next case.
- Relaxed pass profile controllability holds but a change of δ is required. In this case it is obvious that applying (24) with $DK_2 \neq 0$ immediately means extended pass profile controllability cannot hold. To reduce δ, apply (24) with $DK_2 = 0$ and then attempt to find K_1 such that δ for the modified process is less than that for the uncontrolled process and moreover relaxed pass profile controllability holds for this δ. As shown in (10), it is only necessary to test for $\delta \leq \delta_{\max}$.

If the relevant case here is achieved, then the control law designs given earlier in this article can be applied to the modified process.

5. Conclusions

This article has produced new results on pass profile controllability of discrete linear repetitive processes where this property is well defined physically. The particular focus has been on what can be achieved in the presence of structural properties of the defining state-space model that will often arise in applications.

Acknowledgement

This work has been partially supported by the Ministry of Science and Higher Education in Poland under the project No. N514 293235.

References

Edwards, J.B. (1974), 'Stability Problems in the Control of Multipass Processes', *Proceedings of the Institution of Electrical Engineers*, 121(11), 1425–1431.

Galkowski, K., Rogers, E., and Owens, D.H. (1998), 'Matrix Rank Based Conditions for Reachability/controllability of Discrete Linear Repetitive Processes', *Linear Algebra and its Applications*, 275–276, 201–224.

Hladowski, L., Cai, Z., Galkowski, K., Rogers, E., Freeman, C., and Lewin, P.L. (2008), 'Using 2D Systems Theory to Design Output Signal Based Iterative Learning Control Laws with Experimental Verification', in *47th IEEE Conference on Decision and Control*, pp. 3026–3031.

Hladowski, L., Galkowski, K., Rogers, E., and Owens, D. (2006), 'Relaxed Pass Profile Controllability of Discrete Linear Repetitive Processes', *International Journal of Control*, 79(8), 938–958.

Moore, K.L., Chen, Y., and Bahl, V. (2005), 'Monotonically Convergent Iterative Learning Control for Linear Discrete-time Systems', *Automatica*, 41(9), 1529–1537.

Porter, W.A., and Aravena, J.L. (1984), '1-D Models for M-D Processes', *IEEE Transactions on Circuits and Systems*, 31(8), 742–744.

Roberts, P.D. (2000), 'Numerical Investigations of a Stability Theorem Arising From 2-Dimensional Analysis of an Iterative Optimal Control Algorithm', *Multidimensional Systems and Signal Processing*, 11(1/2), 109–124.

Rogers, E., Galkowski, K., and Owens, D.H. (2007), *Control Systems Theory and Applications for Linear Repetitive Processes, Vol. 349 of Lecture Notes in Control and Information Sciences*, Berlin: Springer Verlag.