Multidim Syst Sign Process (2011) 22:83–96
DOI 10.1007/s11045-010-0108-1

An approach to iterative learning control for spatio-temporal dynamics using *n*D discrete linear systems models

Błażej Cichy · Krzysztof Gałkowski · Eric Rogers · Anton Kummert

Received: 9 December 2009 / Accepted: 6 April 2010 / Published online: 15 May 2010

Abstract Iterative Learning Control (ILC) is now well established in terms of both the underlying theory and experimental application. This approach is specifically targeted at cases where the same operation is repeated over a finite duration with resetting between successive executions. Each execution is known as a trial and the key idea is to use information from previous trials to update the control input used on the current one with the aim of improving performance from trial-to-trial. In this paper, the subject area is the application of ILC to spatio-temporal systems described by a linear partial differential equation (PDE) using a discrete approximation of the dynamics, where there are a number of construction methods that could be applied. Here explicit discretization is used, resulting in a multidimensional, or nD, discrete linear system on which to base control law design, where n denotes the number of directions of information propagation and is equal to the total number of indeterminates in the PDE. The resulting control laws can be computed using Linear Matrix Inequalities (LMIs) and a numerical example is given. Finally, a natural extension to robust control is noted and areas for further research briefly discussed.

This work has been partially supported by the Ministry of Science and Higher Education in Poland under the project N N514 293235.

B. Cichy (✉) · K. Gałkowski
Institute of Control and Computation Engineering, University of Zielona Góra, ul. Podgórna 50, 65–246 Zielona Góra, Poland
e-mail: B.Cichy@issi.uz.zgora.pl

K. Gałkowski
e-mail: K.Galkowski@issi.uz.zgora.pl

E. Rogers
School of Electronics and Computer Science, University of Southampton, Southampton SO17 1BJ, UK
e-mail: etar@ecs.soton.ac.uk

A. Kummert
Faculty of Electrical, Information and Media Engineering, University of Wuppertal, Rainer-Gruenter-Strasse 21, 42119 Wuppertal, Germany
e-mail: kummert@uni-wuppertal.de

Keywords Iterative learning control · nD linear systems · PDEs · Explicit discretization

1 Introduction

Iterative learning control (ILC) is a technique for controlling systems operating in a repetitive, or trial-to-trial, mode with the requirement that a reference trajectory $y_{ref}(t)$ defined over a finite interval in t known as the trial length is followed to a high precision. Examples of such systems include robotic manipulators that are required to repeat a given task, chemical batch processes or, more generally, the class of tracking systems. Since the original work (Arimoto et al. 1984), the general area of ILC has been the subject of intense research effort. Initial sources for the, by now very large, literature here are the survey papers (Bristow et al. 2006; Hyo-Sung et al. 2007). The first of these places emphasis on applications and the second gives a categorization of the main approaches up to 2004.

A major objective of ILC is to achieve convergence of the trial-to-trial error and often this is the only objective. It is possible that enforcing fast convergence could lead to unsatisfactory performance along the trial. One way of addressing this problem is to use a 2D systems setting where one direction of information propagation is from trial-to-trial and the other along the trial. The fact that the trial length is finite strongly suggests the use of a repetitive process setting for analysis.

The unique characteristic of a repetitive, or multipass (Rogers et al. 2007), process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. In particular, a pass is completed and then the process is reset before the start of the next one. On each pass, an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem where the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction. Industrial examples of these processes are detailed in, for example, Rogers et al. (2007).

There has been work on the use of a 2D discrete linear systems setting to design linear ILC control schemes based on the well known Roesser (1975) and Fornasini-Marchesini (1978) state-space models, see, for example, Kurek and Zaremba (1993), but the focus of these results is entirely on trial-to-trial error convergence. More recently, ILC algorithms designed in the repetitive process setting have been experimentally tested with results that clearly show how the trade-off between error convergence and along the trial performance can be treated in this setting (Hładowski et al. 2010). Also, for general information on nD, $n \geq 2$, systems theory and applications see Bose (1982) and recent overview paper (Bose 2007).

Currently, the vast majority of the work reported on ILC considers finite-dimensional systems but there has been some work reported on its application to distributed parameter systems governed by partial differential equations (PDEs), for example, Choi et al. (2001), Qu (2002), Moore and Chen (2006), Xu et al. (2009), Zhao (2005). In Xu et al. (2009) the case of systems described by parabolic PDEs is used to make the point that the key feature, aside from applying a control signal, which must be present for successful application of ILC, that is, repetition, is often encountered in systems modeled by PDEs. Application areas for ILC of systems described by PDEs discussed in Xu et al. (2009) include the control of tokamak plasmas. In this application area the use of transformer action to produce the tokamak plasma current means that existing tokamaks operate in a pulsed mode, also known as a discharge. One approach to their control is based on creating desired profiles during early stage operation with the aim of maintaining these during subsequent phases. Here the role for ILC would be to regulate the control actuation after successive discharges in order to minimize the matching error, that is, the difference between actual and desired signals.

In terms of developing ILC for PDEs, an obvious approach is to work directly with the defining equations, where, for example, Xu et al. (2009) considers the design of proportional, or P-Type, and derivative, or D-Type, control laws for parabolic PDEs, such as the controlled heat equation, using semigroup theory. See also Zhao (2005) where a number of other possible application areas are considered, such as velocity and tension control for axially moving materials and electrostatic microbridge actuators. Note also that distributed sensors/actuators have a long history in numerous areas and more recent developments in supporting technologies have led to renewed activity into their effective application, see, for example, Maxwell and Asokanthan (2004) and Zhao and Rahn (2007).

This paper investigates, at the basic algorithm development level, the design of ILC control laws for systems described by linear PDEs based on first discretizing the defining equations and then using the resulting model for design. In particular, the use of explicit discretization is investigated leading to an nD discrete linear systems state-space model where if ζ denotes the number of spatial indeterminates $n = \zeta + 1$. The resulting control law design algorithms can be computed using Linear Matrix Inequalities (LMIs) and a numerical example is given.

The models obtained by the approach proposed in this work are of the local type and hence the state-space dimension is low and it is obviously necessary to ensure that they adequately capture the dynamics of the defining PDEs. In particular numerical instability must be prevented by imposing limits on the time and space discretization periods, which can be calculated by means of standard numerical analysis methods, such as those in Strikwerda (1989), or standard software tools. In the conclusions section we give a comparative discussion the results reported in this paper and some areas for further research are briefly discussed.

We begin in the next section with a summary of the necessary background on repetitive processes. Throughout this paper $M \succ 0$ (respectively $\prec 0$) denotes a real symmetric positive (respectively negative) definite matrix.

2 Linear repetitive processes

The unique characteristic of a repetitive, or multipass (Rogers et al. 2007), process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. In particular, a pass is completed and then the process is reset before the start of the next one. On each pass, an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem where the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

In order to explain how such a process arises in an industrial application, consider the longwall coal cutting process, see the relevant references cited in Rogers et al. (2007), where coal is extracted by hauling the cutting machine along the coal face riding on a semi-flexible conveyor. At the end of each pass, the machine is hauled back in reverse to the starting position and then the machine and conveyor are pushed forward to rest on the newly cut pass profile, that is, the height of the stone/coal interface about some fixed datum line. The control objective is to steer the cutting head such that the maximum amount of coal is extracted without penetrating the stone/coal interface at either the top or bottom of the coal seam, and the basic geometry shows that the previous pass profile function critically influences the next one and hence it is a repetitive process. The influence of the previous pass profile on the current one can lead to undulations in the floor profiles that build up from pass-to-pass and when excessive productive work must stop to enable them to be removed.

Consider the case of discrete dynamics along the pass and let $\alpha < \infty$ denote the pass length and $k \geq 0$ the pass number or index. Then such processes evolve over the subset of the positive quadrant in the 2D plane defined by $\{(p, k) : 0 \leq p \leq \alpha - 1, k \geq 0\}$, and the most basic discrete linear repetitive process state-space model (Rogers et al. 2007) has the following form

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p) \end{aligned} \tag{1}$$

Here on pass k, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the pass profile vector, and $u_k(p) \in \mathbb{R}^r$ is the vector of control inputs.

In order to complete the process description it is necessary to specify the boundary conditions, that is, the pass state initial vector sequence and the initial pass profile. The simplest form of these is

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \geq 0 \\ y_0(p) &= f(p), \quad 0 \leq p \leq \alpha - 1 \end{aligned} \tag{2}$$

where the $n \times 1$ vector d_{k+1} has known constant entries and $f(p)$ is an $m \times 1$ vector whose entries are known functions of p.

The stability theory (Rogers et al. 2007) for linear repetitive processes is based on an abstract model in a Banach space setting which includes a large number of such processes as special cases. In this setting, a bounded linear operator mapping a Banach space into itself describes the contribution of the previous pass dynamics to the current one and the stability conditions are described in terms of properties of this operator. Noting again the unique feature of these processes, that is, oscillations that increase in amplitude from pass-to-pass, this theory is based on ensuring that such a response cannot occur by demanding that the output sequence of pass profiles generated $\{y_k\}$ has a bounded input bounded output stability property defined in terms of the norm on the underlying Banach space.

Two distinct forms of stability can be defined in this setting which are termed asymptotic stability and stability along the pass respectively. The former requires this property with respect to the, finite and fixed, pass length and the latter uniformly, that is, independent of the pass length. Asymptotic stability guarantees the existence of the limit profile defined as the strong limit as $k \to \infty$ of the sequence $\{y_k\}_k$ and for processes described by (1) and (2) this is described by a standard, or 1D, discrete linear systems state-space model with state matrix $A_{lp} := A + B_0(I - D_0)^{-1}C$. Hence it is possible for asymptotic stability to result in a limit profile which is unstable as a 1D discrete linear system, for example, $A = -0.5$, $B = 0$, $B_0 = 0.5 + \beta$, $C = 1$, $D = 0$, $D_0 = 0$, where β is a real scalar satisfying $|\beta| \geq 1$. Stability along the pass prevents this from happening by demanding that the stability property be independent of the pass length, which can be analyzed mathematically by letting $\alpha \to \infty$.

The contributions to the current pass state and pass profile vectors at any instance, say $y_{k+1}(i), 0 \leq i \leq \alpha - 1$, in the along the pass direction in (1) are only from $y_k(i)$, that is, the same instant on the previous pass. This is the simplest possible case since in some applications, such as longwall coal cutting (Rogers et al. 2007), it is necessary to assume that there are contributions from $y_k(j)$, $i \neq j$, in order to adequately model the process dynamics. In Gałkowski et al. (2006) the following model was introduced as one possible representation in such cases

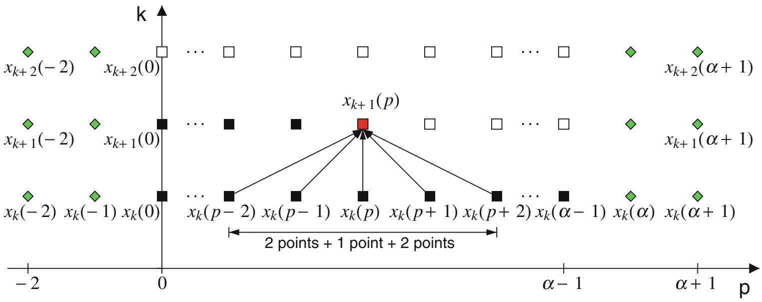

Fig. 1 Illustrating the updating structure of (3) for $\gamma = 2$

$$
\begin{aligned}
x_{k+1}(p) &= \sum_{i=-\gamma}^{\gamma} A_{(i+\gamma+1)} x_k(p+i) + Bu_k(p) \\
y_k(p) &= Cx_k(p)
\end{aligned}
\tag{3}
$$

where on pass k, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the pass profile vector, and $u_k(p) \in \mathbb{R}^r$ is the vector of control inputs. The associated boundary conditions are

$$
\begin{aligned}
x_0(p) &= g(p), \quad 0 \le p \le \alpha - 1 \\
x_k(i) &= 0, \quad -\gamma \le i < 0, \ k > 0 \\
x_k(\alpha + i) &= g_k, \quad 0 \le i < \gamma, \ k > 0
\end{aligned}
\tag{4}
$$

where $g(p)$ is an $n \times 1$ vector whose entries are known functions of p, g_k is an $n \times 1$ vector with known constant entries, and the sequence $\{g_k\}$ is bounded. The updating structure of this model can be visualized as in Fig. 1 for $\gamma = 2$.

Commonly used 2D discrete linear systems state-space models, such as those due to Roesser (1975) and Fornasini and Marchesini (1978) respectively, and discrete linear repetitive processes described by (1) are causal in the right-upper quadrant, that is, any the value of any variable at (p, k) cannot depend on values at (p', k') where $p' > p$, or $k' > k$, or both these relations hold. The model of (3) is clearly non-causal in this sense but physical motivation does not contradict this as the along the pass dynamics are often in the form of spatial, as opposed to temporal, updating and hence right-upper quadrant causality is not necessarily required as the system remains recursive.

This absence of upper-right quadrant causality of processes described by (3) means they cannot be analyzed by applying control related analysis from either Roesser or Fornasini Marchesini state-space model based 2D discrete linear systems theory. The model of (3) and (4) is termed a wave repetitive process since the pass-to-pass information propagation is in the form of a wave of points. Moreover, it is this model that can arise in the discretization of spatio-temporal systems described by PDEs as we illustrate next for a parabolic PDE. Note also that more general forms of spatio-temporal causality occur in signal processing, see Rabenstein and Trautmann (2003), and are related to semi-causal and minimum neighbor systems (Levy et al. 1990).

Consider the parabolic PDE

$$
\frac{\partial x(t, w)}{\partial t} = \widehat{\alpha}^2 \frac{\partial^2 x(t, w)}{\partial w^2} + \delta u(t, w) \tag{5}
$$

where $x(t, w)$ is the variable of interest, $u(t, w)$ is the input variable, and t and w are the time and space variables respectively. One case where such an equation arises is heat conduction in a metal bar and to construct a discrete approximation we use

$$\begin{aligned} \frac{\partial x(t,w)}{\partial t} &\approx \frac{x(t+T,w)-x(t,w)}{T} \\ \frac{\partial^2 x(t,w)}{\partial w^2} &\approx \frac{1}{12h^2}\left[-x(t,w-2h)+16x(t,w-h)-30x(t,w)\right. \\ &\qquad \left. +16x(t,w+h)-x(t,w+2h)\right] \end{aligned} \tag{6}$$

where T and h are the time and space discretization periods respectively. Also introduce $x_k(p) \widehat{=} x(kT, ph)$ and $u_k(p) \widehat{=} u(kT, ph)$ to obtain the following discrete approximation of (5)

$$\begin{aligned} x_{k+1}(p) = x_k(p) + \frac{\widehat{\alpha}^2 T}{12h^2}&\left[-x_k(p-2)+16x_k(p-1)-30x_k(p)\right. \\ &\left. +16x_k(p+1)-x_k(p+2)\right] + Bu_k(p) \end{aligned} \tag{7}$$

where $B \widehat{=} \delta$. Equivalently, we can write

$$\begin{aligned} x_{k+1}(p) &= A_1 x_k(p-2) + A_2 x_k(p-1) + A_3 x_k(p) + A_4 x_k(p+1) \\ &\quad + A_5 x_k(p+2) + Bu_k(p) \\ y_k(p) &= Cx_k(p) \end{aligned} \tag{8}$$

with $C = I$ and

$$A_1 = A_5 = -\lambda, \quad A_2 = A_4 = 16\lambda, \quad A_3 = -30\lambda + 1 \tag{9}$$

where

$$\lambda \widehat{=} \widehat{\alpha}^2 \frac{T}{12h^2} \tag{10}$$

which is a special case of (3) with $\gamma = 2$. In contrast to their interpretations in (3) the discrete integer variables k and p in (8) are temporal and spatial respectively. In all cases that can be approximated by (3) and (4), γ is equal to the order of the PDE in the spatial variables. Also if the time indeterminate is higher order then we obtain a higher order in k version of (8).

Before proceeding to any analysis based on a discrete approximation, it is necessary to verify its numerical accuracy. For the particular example considered here, one approach is to use the Raleigh-quotients approach see, for example, Strikwerda (1989). In particular, introduce

$$C(\Omega) = 1 - 2\lambda\cos 2\Omega + 32\lambda\cos\Omega - 30\lambda \tag{11}$$

Then using the Raleigh-quotients numerical stability holds when

$$C_{min} = 1 - 64\lambda \geq -1 \tag{12}$$

and hence the time and space discretization periods used must satisfy

$$0 \leq T \leq \frac{3}{8}\frac{h^2}{\widehat{\alpha}^2} \tag{13}$$

The choice of discretization method will be discussed again in the conclusions section of this paper. Next, we move on to formulate the ILC design problem, develop a control law design algorithm, and then give a numerical example.

3 ILC for wave repetitive processes

Suppose that the objective is to achieve pre-specified $y_k^*(p)$, over the fixed domain $R = \{(k, p) : k = 0, 1, \ldots, N;\ p = 0, 1, \ldots, \alpha - 1\}$, which is saptio-temporal. Then one possible route is to extend ILC from the temporal to spatio-temporal domain by applying the following sequential procedure: (i) apply a computed input $u_k(p)$ over R, (ii) compute the error between resulting and actual values over R when (i) is complete, (iii) use this error to update the control, (iv) apply this new control, (v) repeat (ii)–(iv) until the error has been reduced to an acceptable level. Clearly we need an extra index to denote the trials here and for this we use the integer l as a superscript on variables and the following is the process model over R

$$\begin{aligned} x_{k+1}^l(p) &= \sum_{i=-\gamma}^{\gamma} A_{(i+\gamma+1)} x_k^l(p+i) + Bu_k^l(p) \\ y_k^l(p) &= Cx_k^l(p) \end{aligned} \tag{14}$$

The tracking error $e_k^l(p)$ over R is

$$e_k^l(p) \widehat{=} y_k^*(p) - y_k^l(p) \tag{15}$$

and it is easy to see that

$$e_{k+1}^{l+1}(p) = e_{k+1}^l(p) - (y_{k+1}^{l+1}(p) - y_{k+1}^l(p)) \tag{16}$$

Hence, on substituting from (14),

$$e_{k+1}^{l+1}(p) = \sum_{i=-\gamma}^{\gamma} \mathsf{C}_{(i+\gamma+1)} \eta_k^{l+1}(p+i) + \mathsf{D}\Delta u_k^{l+1}(p) + e_{k+1}^l(p) \tag{17}$$

where

$$\begin{aligned} \mathsf{C}_{(i+\gamma+1)} &= -CA_{(i+\gamma+1)}, \quad i = -\gamma, \ldots, 0, \ldots \gamma \\ \mathsf{D} &= -CB \end{aligned} \tag{18}$$

and

$$\begin{aligned} \eta_k^{l+1}(p) &\ \widehat{=}\ x_k^{l+1}(p) - x_k^l(p) \\ \Delta u_k^{l+1}(p) &\ \widehat{=}\ u_k^{l+1}(p) - u_k^l(p) \end{aligned} \tag{19}$$

which describes the updating in l. Similarly, the updating in p can be written as

$$\eta_{k+1}^{l+1}(p) = \sum_{i=-\gamma}^{\gamma} A_{(i+\gamma+1)} \eta_k^{l+1}(p+i) + B\Delta u_k^{l+1}(p) \tag{20}$$

Consider now the control law

$$\Delta u_k^{l+1}(p) = \sum_{i=-\gamma}^{\gamma} K_{(i+\gamma+1)} \eta_k^{l+1}(p+i) + K_{(2\gamma+2)} e_{k+1}^l(p) \tag{21}$$

and apply it to (17) and (20) to obtain the following model for the controlled dynamics

$$\begin{aligned} \eta_{k+1}^{l+1}(p) &= \sum_{i=-\gamma}^{\gamma} \Phi_{(i+\gamma+1)}\eta_k^{l+1}(p+i) + \Psi_1 e_{k+1}^{l}(p) \\ e_{k+1}^{l+1}(p) &= \sum_{i=-\gamma}^{\gamma} \Upsilon_{(i+\gamma+1)}\eta_k^{l+1}(p+i) + \Psi_2 e_{k+1}^{l}(p) \end{aligned} \tag{22}$$

where

$$\begin{aligned} \Psi_1 = BK_{(2\gamma+2)}, \quad \Psi_2 = I - CBK_{(2\gamma+2)} \\ \Phi_{(i+\gamma+1)} = A_{(i+\gamma+1)} + BK_{(i+\gamma+1)}, \quad \Upsilon_{(i+\gamma+1)} = -C\Phi_{(i+\gamma+1)} \end{aligned} \tag{23}$$

for all $i = -\gamma, \ldots, 0, \ldots \gamma$.

This is a 3D linear system with two non-temporal directions of information propagation p and l (space and the number of trials respectively), and one temporal (k). Equivalently it can be viewed as a discrete linear repetitive process where it is rectangles of information that is updated from trial-to-trial (in the l direction).

4 ILC design

It has been shown in previous work (Hładowski et al. 2010) that stability along the pass, or trial in this application area, theory for discrete linear repetitive processes described by (1) can be used to design ILC algorithms for finite-dimensional discrete linear systems and experimental verification on a gantry robot produced very good agreement between predicted and measured performance. This work also demonstrates that a repetitive process setting for analysis and design enables control law design for trial-to-trial error convergence and along the trial response.

The design algorithms in Hładowski et al. (2010) used a Lyapunov function characterization of stability along trial, where the function used has the form

$$V(k, p) = x_{k+1}^T(p)Qx_{k+1}(p) + y_k(p)Wy_k(p) \tag{24}$$

with $Q \succ 0$ and $W \succ 0$, that is, the sum of quadratic terms in the current pass state and previous pass profiles respectively for given k and p.

In the case of (22) a candidate Lyapunov function for given k, l, and p is

$$V(k, l, p) = V_1(k, l, p) + V_2(k, l, p) \tag{25}$$

where

$$\begin{aligned} V_1(k, l, p) &\widehat{=} (e_{k+1}^l)^T(p)P_{(2\gamma+2)}e_{k+1}^l(p) \\ V_2(k, l, p) &\widehat{=} \sum_{i=-\gamma}^{\gamma} (\eta_k^{l+1})^T(p+i)P_{(i+\gamma+1)}\eta_k^{l+1}(p+i) \end{aligned} \tag{26}$$

and $P_i \succ 0, i = 1, \ldots, 2\gamma+2$. Here V_1 represents the local error vector energy at $(k+1, p, l)$ and V_2 represents the energy of the signal η for given k, l summed over the window of points $-\gamma + p, \ldots, p, \ldots, \gamma + p$. Summing over $p = 0, 1, \ldots, \alpha - 1$, gives another candidate

Lyapunov function as

$$V(k,l) = \sum_{p=0}^{\alpha-1} V(k,l,p) \,\widehat{=}\, V_1(k,l) + V_2(k,l) \tag{27}$$

which has the same structure as (24).

Now define the increment for (25) as

$$\Delta V(k,l,p) \,\widehat{=}\, (e_{k+1}^{l+1})^T(p) P_{(2\gamma+2)} e_{k+1}^{l+1}(p) + (\eta_{k+1}^{l+1})^T(p) \left(\sum_{i=-\gamma}^{\gamma} P_{(i+\gamma+1)} \right) \eta_{k+1}^{l+1}(p)$$
$$- (e_{k+1}^{l})^T(p) P_{(2\gamma+2)} e_{k+1}^{l}(p) - \sum_{i=-\gamma}^{\gamma} \eta_k^{l+1T}(i) P_{(i+\gamma+1)} \eta_k^{l+1}(i) \tag{28}$$

and hence (by summation over $p = 0, 1, \ldots, \alpha - 1$, and taking into account the initial conditions)

$$\Delta V(k,l) = V_1(k,l+1) - V_1(k,l) + V_2(k+1,l) - V_2(k,l) \tag{29}$$

The increment (29) has the same structure as that for (24). Moreover, it has been shown elsewhere (Rogers et al. 2007) that stability along the pass of processes described by (1) holds when the increment of the Lyapunov function (24) is negative definite for all possible values of α and k. It is also straightforward to argue that this stability theory extends to processes for which (25) (and also (27)) is a candidate Lyapunov function. Hence the proof of the following result is omitted here.

Theorem 1 *An ILC scheme described by (22) is stable over* $R = \{(k,p) : k = 0, 1, \ldots, N;\ p = 0, 1, \ldots, \alpha - 1\}$ *for any choice of the positive integers* N *and* $\alpha > 1$ *if*

$$\Delta V(k,l) < 0$$
$$\forall e_k^l(p), \eta_k^{l+1}(i), \quad p = 0, 1, \ldots, \alpha - 1, \quad 0 \le k \le N, \quad l = 0, 1, \ldots.$$

Introduce the following notation

$$M = \begin{bmatrix} \Phi_1 & \ldots & \Phi_{(2\gamma+1)} & \Psi_1 \\ \vdots & \ddots & \vdots & \vdots \\ \Phi_1 & \ldots & \Phi_{(2\gamma+1)} & \Psi_1 \\ \Upsilon_1 & \ldots & \Upsilon_{(2\gamma+1)} & \Psi_2 \end{bmatrix} \tag{30}$$

$$P = \left(\bigoplus_{i=-\gamma}^{\gamma} P_{(\gamma+i+1)} \right) \bigoplus P_{(2\gamma+2)} \tag{31}$$

where $\oplus$ denotes the direct sum of square matrices.

Theorem 2 *An ILC scheme described by (22) is stable over* $R = \{(k,p) : k = 0, 1, \ldots, N;\ p = 0, 1, \ldots, \alpha - 1\}$ *for any choice of the positive integers* N *and* $\alpha > 1$ *if*

$$M^T P M - P \prec 0 \tag{32}$$

Proof Follows immediately from

$$\Delta V(k,l,p) = X(k,l,p)^T \left(M^T P M - P\right) X(k,l,p)$$

where

$$X(k,l,p) \,\widehat{=}\, \left[(\eta_k^{l+1})^T(p-\gamma)\cdots(\eta_k^{l+1})^T(p)\cdots(\eta_k^{l+1})^T(p+\gamma)(e_{k+1}^l)^T(p)\right]^T$$

and summing over $p = 0, 1, \ldots, \alpha - 1$.

The following result gives an LMI based condition for stability together with a control law design algorithm.

Theorem 3 *An ILC scheme described by (22) is stable over $R = \{(k,p) : k = 0, 1, \ldots, N;\ p = 0, 1, \ldots, \alpha - 1\}$ for any choice of the positive integers N and $\alpha > 1$ if there exist matrices N_i, and matrices $P_i \succ 0, i = 1, \ldots, 2\gamma + 2$, such that*

$$\begin{bmatrix} -P & Y^T \\ Y & -P \end{bmatrix} \prec 0 \tag{33}$$

where

$$Y = \begin{bmatrix} A_1 P_1 + BN_1 & \ldots & A_\omega P_\omega + BN_\omega & BN_{\omega+1} \\ \vdots & \ddots & \vdots & \vdots \\ A_1 P_1 + BN_1 & \ldots & A_\omega P_\omega + BN_\omega & BN_{\omega+1} \\ -CA_1 P_1 - CBN_1 & \ldots & -CA_\omega P_\omega - CBN_\omega & P_{\omega+1} - CBN_{\omega+1} \end{bmatrix} \tag{34}$$

$$\omega = 2\gamma + 1 \tag{35}$$

and P is defined in (31). If (33) holds, then stabilizing control law matrices in (21) are given by

$$K_i = N_i P_i^{-1}, \quad i = 1, \ldots, 2\gamma + 2 \tag{36}$$

Proof Apply the Schur's complement formula to (32), then pre- and post-multiply the result by $I \oplus P$, where I is the identity matrix with compatible dimensions, to obtain

$$\begin{bmatrix} -P & PM^T \\ MP & -P \end{bmatrix} \prec 0 \tag{37}$$

Introducing

$$Y = MP \tag{38}$$

and expanding the product of (38) gives

$$Y = MP = \begin{bmatrix} A_1 P_1 + BK_1 P_1 & \ldots & A_\omega P_\omega + BK_\omega P_\omega & BK_{\omega+1} P_{\omega+1} \\ \vdots & \ddots & \vdots & \vdots \\ A_1 P_1 + BK_1 P_1 & \ldots & A_\omega P_\omega + BK_\omega P_\omega & BK_{\omega+1} P_{\omega+1} \\ -CA_1 P_1 - CBK_1 P_1 & \ldots & -CA_\omega P_\omega - CBK_\omega P_\omega & P_{\omega+1} - CBK_{\omega+1} P_{\omega+1} \end{bmatrix} \tag{39}$$

where M is defined in (30) and ω is defined in (35). The proof is completed by setting

$$K_i P_i = N_i, \quad i = 1, \ldots, 2\gamma + 2 \tag{40}$$

in (34). □

5 Numerical example

Consider the PDE of (5) with, in normalized units, $\widehat{\alpha} = 1$, $B = \delta = 0.2$, $h = 1$ and $T = 0.0833$, where from (13) the maximum possible value of this parameter is 0.375, $N = 10$ and $\alpha = 61$. Hence from (8)

$$A_1 = A_5 = -6.9444 \times 10^{-3}, \quad A_2 = A_4 = 0.1111, \quad A_3 = 0.7917, C = 1$$

Suppose also that the desired set of values for the output over the domain R is that of Fig. 2, see also Xu et al. (2009), with boundary conditions

$$\begin{aligned}
&x_k^0(p) = u_k^0(p) = 0, \quad 0 \le p \le \alpha - 1, \ 0 \le k \le N \\
&x_0^l(p) = y_0^*(p), \quad 0 \le p \le \alpha - 1, \ l \ge 0 \\
&x_k^l(p) = y_k^l(p) = y_k^R(p), \quad p \in \{-2, -1\} \cup \{\alpha, \alpha + 1\}, \ 0 \le k \le N, \ l \ge 0 \\
&u_k^l(p) = e_k^l(p) = 0, \quad p \in \{-2, -1\} \cup \{\alpha, \alpha + 1\}, \ 0 \le k \le N, \ l \ge 0
\end{aligned}$$

where $y_k^R(p)$ are extrapolated values of the signal reference shown on Fig. 2 at the boundary points i.e. $p \in \{-2, -1\} \cup \{\alpha, \alpha + 1\}$.

The LMIs of Theorem 3 are feasible in this case and yield the following control law matrices K_i, $i = 1, \ldots, 6$,

$$K_1 = K_5 = 0.1505, \quad K_2 = K_4 = -2.4074, \quad K_3 = -0.4861, \quad K_6 = 1.1762$$

As representatives of the effects of the application of ILC in this case, Fig. 3 shows the error dynamics for $l = 10$ and $l = 30$ respectively. These confirm that the error is converging as l increases with no evidence of unacceptable transient dynamics.

There is clearly a need to check that the control effort used is not excessive. To obtain $u_k^l(p)$ for (14) first replace k by $k + 1$ in (15) and l by $l - 1$ in (19). Then we obtain

$$\begin{aligned}
u_k^l(p) &= u_k^{l-1}(p) + K_1 \left(x_k^l(p - \gamma) - x_k^{l-1}(p - \gamma)\right) \\
&\quad + \cdots + K_{(2\gamma+1)} \left(x_k^l(p + \gamma) - x_k^{l-1}(p + \gamma)\right) + K_{(2\gamma+2)} \left(x_{k+1}^*(p) - x_{k+1}^{l-1}(p)\right)
\end{aligned} \tag{41}$$

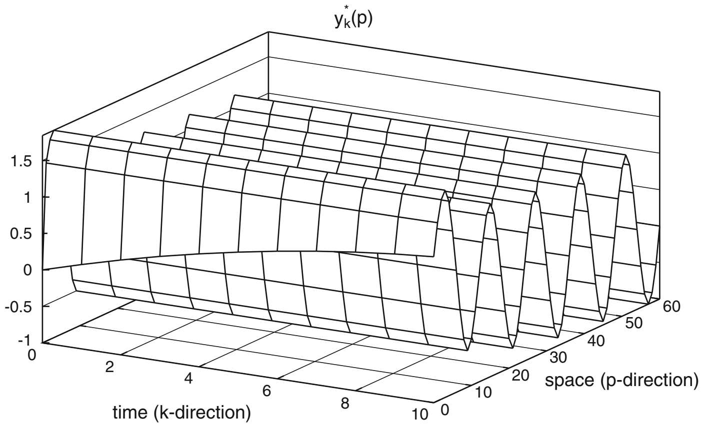

Fig. 2 Reference for the ILC design

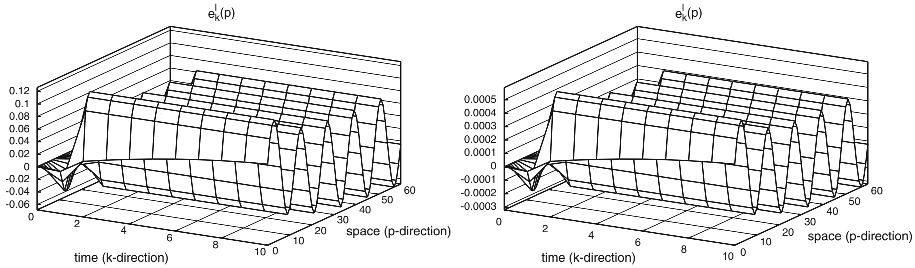

Fig. 3 Error dynamics on trials $l = 10$ (*left*) and $l = 30$ (*right*) respectively

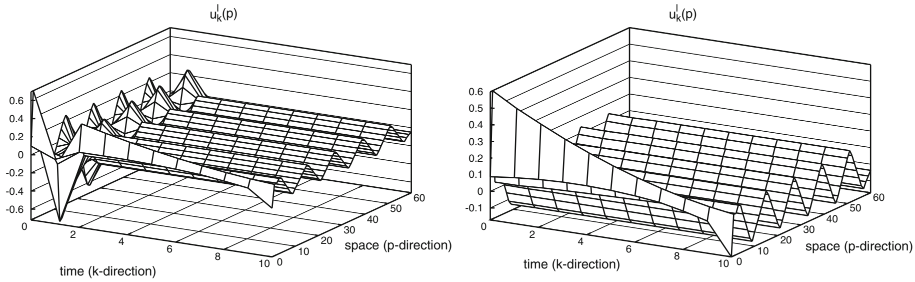

Fig. 4 Control input on trials $l = 10$ (*left*) and $l = 30$ (*right*) respectively

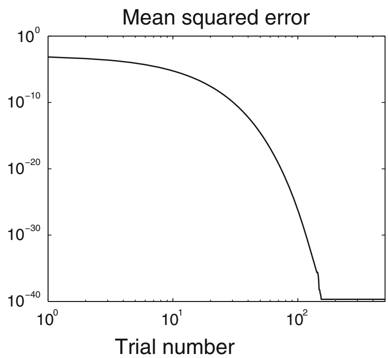

Fig. 5 Mean squared error for $e_k^l(p)$

Figure 4 shows the control inputs corresponding to $l = 10$ and $l = 30$ which are also acceptable. Finally, Fig. 5 shows the mean square error plotted against trial number.

6 Conclusions

This paper has addressed the design of ILC for systems described by PDEs where there are a number of possible starting points. One of these is to do analysis and design without any form of discretization and in some cases analytical formulas for control law design and the response of the resulting controlled system can be found. An alternative is to employ discretization for either control law design or design and implementation. This latter approach is of particular interest when discrete sensors and actuators are to be employed and in the case of the former actual application of the control law will involve also discrete approximation.

ILC for PDEs is still in the early stages of development and the advantages and disadvantages of competing methods are not yet fully understood. Here we have used direct discretization on the defining PDEs to obtain a model on which to base control law design. This results in a model of the dynamics in the form of a discrete linear repetitive process and when the trial index is added control law design proceeds from a 3D linear system model with two non-temporal directions of information propagation and one temporal. The resulting ILC design algorithms can be computed using LMIs. Moreover, the resulting control law has a well defined structure which has attractions in terms of implementation architectures. If the k and p axes are finite then an alternative is to use a lifting approach to formulate the design problem in terms of standard discrete linear systems theory. This approach has also been applied (Rogers et al. 2007) to discrete linear repetitive processes, such as those described by (1) and (2), and leads to computations with much larger dimensioned matrices. Moreover, stability in such a case cannot prevent unbounded temporal dynamics whereas the stability theory used here does not allow this situation to arise.

Clearly there is much further research to be done on this approach to ensure that an adequate discrete model for design is produced in the most efficient way and also for verifying its numerical stability. The parabolic PDE has been used here to illustrate the design algorithms as opposed to an actual solution to a given problem. Alternative discretization methods need to be considered together with algorithms for pre-specified actuator/sensor configurations such as boundary control. One other critical area is to develop a robust control theory and supporting design algorithms. In which context, the structure of the model for design here allows direct extension to robust control based, for example, on polytopic and norm bounded approaches that have already been investigated in the repetitive process literature (Rogers et al. 2007). If, however, lifting is employed as in Moore and Chen (2006) then this is a much more difficult task.

References

Arimoto, S., Kawamura, S., & Miyazaki, F. (1984). Bettering operation of robots by learning. *Journal of Robotic Systems, 1*(2), 123–140.

Bose, N. K. (1982). *Applied multidimensional systems theory*. New York: Van Nostrand Reinhold.

Bose, N. K. (2007). Multidimensional systems and signal processing—good theory for good practice. *IEEE Circuits and Systems Magazine, 7*(4), 20–41.

Bristow, D. A., Tharayil, M., & Alleyne, A. (2006). A survey of iterative learning control. *IEEE Control Systems Magazine, 26*(3), 96–114.

Choi, J., Seo, B. J., & Lee, K. S. (2001). Constrained digital regulation of hyperbolic PDE systems: A learning control approach. *Korean Journal of Chemical Engineering, 18*(5), 606–611.

Fornasini, E., & Marchesini, G. (1978). Doubly indexed dynamical systems: State-space models and structural properities. *Theory of Computing Systems, 12*(1), 59–72.

Gałkowski, K., Cichy, B., Rogers, E., & Lam, J. (2006). Stabilization of a class of uncertain "wave" discrete linear repetitive processes. In: *Proceedings of the 45th IEEE conference on decision and control*, pp. 1435–1440.

Hładowski, Ł., Gałkowski, K., Cai, Z., Rogers, E., Freeman, C. T., & Lewin, P. L. (2010). Experimentally supported 2D systems based iterative learning control law design for error convergence and performance. *Control Engineering Practice, 18*(4), 339–348.

Hyo-Sung, A., YangQuan, C., & Moore, K. L. (2007). Iterative learning control: Brief survey and categorization. *IEEE Transactions on Systems Man and Cybernetics Part C: Applications and Reviews, 37*(6), 1099–1121.

Kurek, J. E., & Zaremba, M. B. (1993). Iterative learning control synthesis based on 2-D system theory. *IEEE Transactions on Automatic Control, 38*(1), 121–125.

Levy, B. C., Adams, M. B., & Wilsky, A. S. (1990). Solution and linear estimation of 2-D nearest neighbor models. *Proceedings of the IEEE, 78*(4), 627–641.
Maxwell, N. D., & Asokanthan, S. F. (2004). Modal characteristics of a flexible beam with multiple distributed actuators. *Journal of Sound and Vibration, 269*(1–2), 19–31.
Moore, K. L., Chen, Y. Q. (2006). Iterative learning approach to a diffusion control problem in an irrigation application. In: *Proceedings of the 2006 IEEE international conference on mechatronics and automation*, pp. 1329–1334.
Qu, Z. (2002). An iterative learning algorithm for boundary control of a stretched moving string. *Automatica, 38*(5), 821–827.
Rabenstein, R., & Trautmann, L. (2003). Towards a framework for continuous and discrete multidimensional systems. *International Journal of Applied Mathematics and Computer Science, 13*(1), 73–86.
Roesser, R. P. (1975). A discrete state-space model for linear image processing. *IEEE Transactions on Automatic Control, 20*(1), 1–10.
Rogers, E., Gałkowski, K., & Owens, D. H. (2007). *Control systems theory and applications for linear repetitive processes, Lecture notes in control and information sciences, vol. 349*. New York: Springer.
Strikwerda, J. C. (1989). *Finite difference schemes and partial differential equations*. USA: Wadsworth and Brooks.
Xu, C., Arastoo, R., Schuster, E. (2009). On iterative learning control of parabolic distributed parameter systems. In: *Proceedings of the 17th mediterranean conference on control and automation*, pp. 510–515.
Zhao, H. (2005). *Passive, iterative, and repetitive control for flexible distributed parameter systems*. Ph.D. thesis, Department of Mechanical and Nuclear Engineering, The Pennsylvania State University.
Zhao, H., & Rahn, C. D. (2007). Stability of damped membranes and plates with distributed inputs. *Journal of Sound and Vibration, 302*(3), 564–576.

Author Biography

Błażej Cichy received the M.Sc. degree from the University of Zielona Gora, Poland in 2003. Currently, he is completing his Ph.D research at the same university under the supervision of Prof. Krzysztof Gałkowski. His research interests are on multidimensional (nD) systems, linear repetitive processes, iterative learning control, and the application of numerical methods to control problems. For his activity, he has been awarded the President of University of Zielona Gora Teamwork Second Class Award.