Generic Infinite Traces and Path-Based Coalgebraic Temporal Logics

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Abstract

This paper gives a general coalgebraic account of the notions of *possibly infinite trace* and *possibly infinite execution* in state-based, dynamical systems, by extending the generic theory of finite traces and executions developed by Hasuo and coauthors [8]. The systems we consider are modelled as coalgebras of endofunctors obtained as the composition of a computational type (e.g. nondeterministic or stochastic) with a general transition type. This generalises existing work by Jacobs [10] that only accounts for a nondeterministic computational type. We subsequently introduce path-based temporal (including fixpoint) logics for coalgebras of such endofunctors, whose semantics is based upon the notion of possibly infinite execution. Our approach instantiates to both nondeterministic and stochastic computations, yielding, in particular, path-based fixpoint logics in the style of CTL* for nondeterministic systems, as well as generalisations of the logic PCTL for probabilistic systems.

Keywords: coalgebra, trace semantics, temporal logic, nondeterminism, probability

1 Introduction

Path-based temporal logics are commonly used as specification logics, particularly in the context of automatic verification. Instances of such logics include the logic CTL* with its fragments CTL and LTL for transition systems [3], and the logic PCTL for probabilistic transition systems [7]. In spite of the similarities shared by these logics, no general, unified account of path-based temporal logics exists.

Coalgebras are by now recognised as a truly general model of dynamical systems, instances of which subsume transition systems, their probabilistic counterparts, and many other interesting state-based models [14]. Moreover,

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the modal logics associated with coalgebraic models [13,1,2] are natural logics for specifying system behaviour, that also instantiate to familiar logics in particular cases. These logics can be classified into one-step modal logics, wherein the semantics of modal operators depends solely on the one-step behaviour of system states (as considered e.g. in [13,1]), and extensions of such logics with (e.g. fixpoint) operators whose interpretation depends on the long-term, possibly infinite behaviour of system states [2]. While (some of) the logics in the second category are able to express application-relevant temporal properties of states, their syntax does not directly refer to the computation paths from particular states, as is the case for logics such as CTL* and PCTL. Indeed, there is still no general, coalgebraic account of the notion of (infinite) computation path, as used in the semantics of CTL* and PCTL. Worse still, in the case of probabilistic transition systems, adding standard fixpoints to the corresponding one-step modal language (as considered in [12,1]) is not very useful, as it does not appear to allow the specification of properties such as: "the likelihood of some state property p holding eventually is greater than some $q \in [0,1]$".

In what follows, we give a general account of the notion of computation path, and of path-based temporal logics such as CTL* and PCTL. Following [10,8], we model systems as coalgebras of a signature functor obtained as the composition of a computational type T (called *branching type* in [8]) with a transition type F, and require that T distributes over F in a suitable way. As examples, we consider nondeterministic and probabilistic systems, with the non-empty powerset functor $\mathcal{P}^+ : \mathsf{Set} \to \mathsf{Set}$ on the category of sets and respectively the probability measure functor $\mathcal{G}_1 : \mathsf{Meas} \to \mathsf{Meas}$ on the category of measurable spaces describing the computational types needed to recover the usual notions of computation path for such systems. While the transition type describes the type of individual transitions (typically linear) and determines the notion of computation path, the computational type describes how the transitions from particular states are structured (e.g. using sets, or probability distributions). The distributivity of T over F then allows computation paths from individual states to be similarly structured.

Our approach to defining infinite computation paths builds on earlier work by Jacobs [10] where *infinite trace maps* were defined for coalgebras of type $\mathcal{P} \circ F$, with $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ the powerset functor and $F : \mathsf{Set} \to \mathsf{Set}$ a polynomial functor. We generalise this to arbitrary computational types T (subject to some additional constraints), thereby obtaining notions of *possibly infinite trace* and *possibly infinite execution* of a state in a $T \circ F$-coalgebra, that are parametric in T and F. We subsequently introduce path-based temporal (including fixpoint) logics for coalgebras of endofunctors of type $T \circ F$, whose semantics is defined in terms of the possibly infinite executions from a particular state. By instantiating our approach, we recover known temporal logics and obtain new variants of known logics. Specifically, taking T to be the non-empty powerset monad $\mathcal{P}^+$ and $F = \mathsf{Id}$ sheds new light on the logic CTL* [3], which we recover as a fragment of a path-based fixpoint logic for $\mathcal{P}^+ \circ \mathsf{Id}$. Varying F to $A \times \mathsf{Id}$ with A a set of labels yields an interesting variant of CTL* interpreted over labelled transition systems. On the other hand, taking $T = \mathcal{G}_1$ and $F = \mathsf{Id}$ allows us to recover the logic PCTL [7] as an instance of a generic temporal logic with Until operators.

The paper is structured as follows. The remainder of this section gives a brief overview of the logics CTL* and PCTL, our main examples. Section 2 recalls some basic definitions and results required later and some details of the generic theory of finite traces [8]. Section 3 defines infinite traces and executions and studies their properties. Section 4 uses infinite executions to define general path-based coalgebraic logics, including fixpoint logics and temporal logics with Until operators. A summary of the results and an outline of future work are given in Section 5.

Transition systems and the logic CTL*

The semantics of CTL* [5] is based on the notion of computation path. Given a transition system with set of states S and accessibility relation $R \subseteq S \times S$, a *computation path* from a state s_0 is an infinite sequence of states $s_0 s_1 \ldots$ such that $s_i R s_{i+1}$ for $i \in \omega$. The syntax of CTL* consists of *path formulas*, formalising properties of computation paths and employing operators such as **X** (in the *next* state along the path), **F** (at some *future* state along the path), **G** (*globally* along the path) and **U** (*until* operator), and *state formulas*, formalising properties of states and employing operators (**A** and **E**) that quantify (universally, respectively existentially) over the computation paths from a particular state. Every state formula is also a path formula, with the latter requiring that the first state of a path satisfies the given state formula. For example, the property "along every path, the system will eventually reach a success state" is formalised as $\mathbf{A}\,\mathbf{F}\,\mathit{success}$, or equivalently as $\mathbf{A}(\mathsf{tt}\,\mathbf{U}\,\mathit{success})$, where tt denotes the true proposition and *success* denotes an atomic proposition. The assumption one typically makes of the transition system of interest is that each state s has at least one outgoing transition. (For states where this is not the case, self-loops are added to the original transition system.) This allows one to focus only on the *infinite* computation paths.

Probabilistic transition systems and the logic PCTL

In the probabilistic transition system model, the state transitions are governed by a probability distribution on the target states – this assigns a probability value to each outgoing transition from a particular state, with the values for transitions from the same state summing up to 1. The logic PCTL [7] for probabilistic transition systems is similar in spirit to CTL*: its syntax consists of path and state formulas, with similar operators ($\mathbf{X}$ and $\mathbf{U}$) for the path formulas, and its semantics is based on the same notion of computation path; the main difference is that, instead of stating that a path formula holds in all/some of the paths from a particular state, the basic state formulas of PCTL, of the form $[\varphi]_{\sim p}$ with φ a path formula, $\sim \in \{<, \leq, >, \geq\}$ and $p \in [0, 1]$, refer to the likelihood of φ holding along the paths from a particular state. For example, $[\mathsf{tt}\, \mathbf{U}\, \mathit{success}]_{\geq 1}$ states that the likelihood of eventually reaching a success state is 1. To interpret state formulas, one computes probability measures over the computation paths from each state of a given model.

The previous examples suggest that a general account of computation paths (to be referred to as *infinite executions* in what follows) should first define the shape of a *potential* infinite execution (in the above cases, any infinite sequence of states), and then provide a suitable structure on the *actual* infinite executions from each state of a particular model (e.g. a *set* of computation paths, or a *probability measure* over computation paths). The former should be sufficient to allow an interpretation of path formulas (of a generic path-based logic still to be defined), whereas the latter should support an interpretation of state formulas (of the same logic).

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2 Preliminaries

We recall that a *measurable space* is given by a pair (X, Σ_X) with X a set and Σ_X a σ-algebra of (*measurable*) subsets of X, whereas a *measurable map* between (X, Σ_X) and (Y, Σ_Y) is given by a function $f : X \to Y$ with the property that $f^{-1}(V) \in \Sigma_X$ for each $V \in \Sigma_Y$. We write Meas for the category of measurable spaces and measurable maps. A measurable space (X, Σ_X) is called *discrete* if $\Sigma_X = \mathcal{P}X$. A *subprobability measure* on a measurable space (X, Σ_X) is then a function $\mu : \Sigma_X \to [0, 1]$ such that $\mu(\emptyset) = 0$ and $\mu(\bigcup_{i\in\omega} X_i) = \sum_i \mu(X_i)$ for countable families $(X_i)_{i\in\omega}$ of pairwise disjoint measurable subsets of X. Thus, $\mu(X) \leq 1$ for any subprobability measure μ on (X, Σ_X). If $\mu(X) = 1$, then μ is called a *probability measure*. Given a measurable space (X, Σ_X) and $x \in X$, the *Dirac probability measure* δ_x is defined by $\delta_x(U) = 1$ iff $x \in U$ and $\delta_x(U) = 0$ otherwise.

We write $\mathcal{G} : \mathsf{Meas} \to \mathsf{Meas}$ for the *subprobability measure functor* [6], sending a measurable space (X, Σ_X) to the set $\mathcal{M}(X, \Sigma_X)$ of subprobability measures on (X, Σ_X), equipped with the σ-algebra generated by the sets $\{\mu \mid \mu(U) \geq q\}$ with $U \in \Sigma_X$ and $q \in [0, 1]$. A related functor, considered in [8], is the *subprobability distribution functor* $\mathcal{S} : \mathsf{Set} \to \mathsf{Set}$, sending a set X to the set of *subprobability distributions over* X, i.e. functions $\mu : X \to [0, 1]$ with $\sum_{x \in X} \mu(x) \leq 1$ [2].

Given a functor $F : \mathsf{C} \to \mathsf{C}$, an *$F$-coalgebra* is given by a pair (X, γ) with X a C-object and $\gamma : X \to FX$ a C-arrow. As previously mentioned, we work in the setting of coalgebras of endofunctors obtained as the composition of a computational type with a transition type. The computational type is specified by a monad T on a category C, whereas the transition type is captured by an endofunctor F on C. As in [8], a crucial assumption is the existence of a distributive law $\lambda : F \circ T \Rightarrow T \circ F$ of T over F. Such a distributive law must be compatible with the monad structure, i.e. $\lambda \circ F\eta = \eta_F$ and $\lambda \circ F\mu = \mu_F \circ T\lambda \circ \lambda_T$, where $\eta : \mathsf{Id} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ denote the unit and multiplication of the monad T.

As examples of computational types, we consider (variants of):

- the powerset monad $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$, modelling nondeterministic computations, with unit given by singletons and multiplication given by unions,
- the subprobability measure monad $\mathcal{G} : \mathsf{Meas} \to \mathsf{Meas}$, modelling probabilistic computations, with unit given by the Dirac measures and multiplication given by integration (see [6] for details).

Both of the above monads are *strong* and *commutative*, i.e. they come equipped with a *strength map* $st_{X,Y} : X \times TY \to T(X \times Y)$ as well as a *double strength map* $dst_{X,Y} : TX \times TY \to T(X \times Y)$, for each choice of C-objects X, Y [3]:

- the powerset monad has strength given by $st_{X,Y}(x, V) = \{x\} \times V$ and double strength given by $dst_{X,Y}(U, V) = U \times V$, for $x \in X$, $U \in \mathcal{P}X$ and $V \in \mathcal{P}Y$,
- the subprobability measure monad has strength given by $st_{(X,\Sigma_X),(Y,\Sigma_Y)}(x, \nu)(U, V) = \nu(V)$ iff $x \in U$ and $st_{(X,\Sigma_X),(Y,\Sigma_Y)}(x, \nu)(U, V) = 0$ otherwise, and double strength given by $dst_{(X,\Sigma_X),(Y,\Sigma_Y)}(\mu, \nu)(U, V) = \mu(U) \cdot \nu(V)$, for $x \in X$, $\mu \in \mathcal{M}(X, \Sigma_X)$, $\nu \in \mathcal{M}(Y, \Sigma_Y)$, $U \in \Sigma_X$, $V \in \Sigma_Y$.

It is shown in [8] that any commutative monad on Set has a canonical distributive law over any *shapely polynomial functor* (i.e. a functor built from identity and constant functors using finite products and arbitrary coproducts). This provides examples of distributive laws of the powerset monad over shapely polynomial functors. Moreover, the construction of the canonical dis-

[2] Thus, a subprobability distribution can take non-zero values on at most countably-many elements of X.

[3] Moreover, these are natural in X and Y.

tributive law (by induction on the structure of the shapely functor) generalises straightforwardly to any category with products and coproducts, thereby also providing examples of distributive laws of the subprobability measure monad over shapely polynomial functors on Meas.

As in [8], the *Kleisli category* of a monad (T, η, μ) on a category C will play an important rôle when defining the notions of infinite trace and infinite execution for systems whose computational type is given by T. This category, denoted $\mathsf{Kl}(T)$, has the same objects as C, and C-arrows $f : X \to TY$ as arrows from X to Y. The composition of two $\mathsf{Kl}(T)$-arrows $f : X \to Y$ and $g : Y \to Z$ is given by the C-arrow $\mu_Z \circ Tg \circ f$. We let $K : \mathsf{Kl}(T) \to \mathsf{C}$ denote the functor defined by:

- $K(X) = TX$,
- $K(f) = \mu_Y \circ Tf$ for $f : X \to Y$ in $\mathsf{Kl}(T)$,

and write $J : \mathsf{C} \to \mathsf{Kl}(T)$ for its left adjoint, defined by:

- $J(X) = X$,
- $J(f) = Tf \circ \eta_X = \eta_Y \circ f$ for $f : X \to Y$ in C.

Later we will make use of the following property of the functor J:

Lemma 2.1 *If the functor $T : \mathsf{C} \to \mathsf{C}$ (weakly) preserves the limit $(Z, (\pi_i)_{i\in\omega})$ of an ω^{op}-chain $(f_i)_{i\in\omega}$, then so does $J : \mathsf{C} \to \mathsf{Kl}(T)$.*

Proof. Assume first that T weakly preserves the limit $(Z, (\pi_i : Z \to Z_i)_{i\in\omega})$ of $(f_i : Z_{i+1} \to Z_i)_{i\in\omega}$. To show that $(JZ, (J\pi_i : JZ \to JZ_i)_{i\in\omega})$ is a weakly limiting cone for $(Jf_i : JZ_{i+1} \to JZ_i)_{i\in\omega}$ in $\mathsf{Kl}(T)$, let $(X, (\delta_i : X \to JZ_i)_{i\in\omega})$ denote an arbitrary cone for $(Jf_i)_{i\in\omega}$ in $\mathsf{Kl}(T)$. Hence, in C, $\mu_{Z_i} \circ T\eta_{Z_i} \circ Tf_i \circ \delta_{i+1} = \delta_i$, that is, $Tf_i \circ \delta_{i+1} = \delta_i$ for all $i \in \omega$. This makes $(\delta_i)_{i\in\omega}$ a cone over $(Tf_i)_{i\in\omega}$ in C, and the weak limiting property of $(TZ, (T\pi_i)_{i\in\omega})$ in C now yields a mediating map $m : X \to TZ$ such that $T\pi_i \circ m = \delta_i$ in C for all $i \in \omega$. This is equivalent to $\mu_{Z_i} \circ T\eta_{Z_i} \circ T\pi_i \circ m = \delta_i$ in C for $i \in \omega$, that is, $J\pi_i \circ m = \delta_i$ in $\mathsf{Kl}(T)$ for $i \in \omega$. The proof of the stronger statement, in the case when T preserves the limit of $(f_i)_{i\in\omega}$, is similar.

As mentioned above, we assume the existence of a distributive law λ of the monad T over the endofunctor F. It is known (see e.g. [8]) that such distributive laws $\lambda : F \circ T \Rightarrow T \circ F$ are in one-to-one correspondence with liftings of the functor $F : \mathsf{C} \to \mathsf{C}$ to $\mathsf{Kl}(T)$. In particular, the lifting $\overline{F} : \mathsf{Kl}(T) \to \mathsf{Kl}(T)$ induced by a distributive law $\lambda : F \circ T \Rightarrow T \circ F$ is defined by:

- $\overline{F}A = FA$,
- $\overline{F}f = \lambda_B \circ Ff$ for $f : A \to B$ in $\mathsf{Kl}(T)$.

The following property of this lifting will be used later:

Lemma 2.2 *The lifting $\overline{F}$ satisfies $\overline{F} \circ J = J \circ F$.*

Proof. For $f : X \to Y$ in C, the C-arrows that define the Kleisli maps $\overline{F}Jf$ and JFf are $\lambda_Y \circ F\eta_Y \circ Ff$ and respectively $\eta_{FY} \circ Ff$. By the compatibility of the distributive law λ with the monad structure, these coincide.

In what follows we also assume that $\mathsf{Kl}(T)$ is *DCpo-enriched*, that is, each homset $\mathsf{Kl}(T)(X, Y)$ is a partial order, with directed collections of maps $(f_i : X \to Y)_{i\in I}$ admitting a join $\bigsqcup_{i\in I} f_i : X \to Y$, and with composition of arrows preserving directed joins: $g \circ (\bigsqcup_{i\in I} f_i) = \bigsqcup_{i\in I}(g \circ f_i)$ and $(\bigsqcup_{i\in I} f_i) \circ h = \bigsqcup_{i\in I}(f_i \circ h)$. We note that the Kleisli categories of the monads $\mathcal{P}$ and $\mathcal{G}$ are *DCpo*-enriched, with the order on $\mathsf{Kl}(\mathcal{P})(X, Y)$ being defined pointwise via the inclusion order on $\mathcal{P}(Y)$, and the order on $\mathsf{Kl}(\mathcal{G})((X, \Sigma_X), (Y, \Sigma_Y))$ being defined pointwise from the dcpo $\leq_Y$ on $\mathcal{G}(Y, \Sigma_Y)$ given by $\mu \leq_Y \nu$ iff $\mu(U) \leq \nu(U)$ for all $U \in \Sigma_Y$.

Finite traces and executions

In [8], the authors consider coalgebras (X, γ) of endofunctors of the form $T \circ F$ with the monad $T : \mathsf{Set} \to \mathsf{Set}$ and the endofunctor $F : \mathsf{Set} \to \mathsf{Set}$ being related by a distributive law $\lambda : F \circ T \Rightarrow T \circ F$, and with the Kleisli category of T being *$DCpo_\perp$-enriched*; that is, in addition to *DCpo*-enrichedness, the orders on $\mathsf{Kl}(T)(X, Y)$ are required to have a bottom element. In this setting, the elements of the carrier I_F of the initial F-algebra provide the potential *finite traces* of states of $T \circ F$-coalgebras, and a *finite trace map* $ftr_\gamma : X \to T(I_F)$ is defined via finality in $\mathsf{Kl}(T)$. The crucial observation is that the initial F-algebra in Set lifts to a final $\overline{F}$-coalgebra in $\mathsf{Kl}(T)$ (where, as before, $\overline{F} : \mathsf{Kl}(T) \to \mathsf{Kl}(T)$ is the lifting of F to $\mathsf{Kl}(T)$ induced by λ). Thus, the finite trace map arises as the unique coalgebra morphism from the $\overline{F}$-coalgebra in $\mathsf{Kl}(T)$ induced by a $T \circ F$-coalgebra in Set to the final $\overline{F}$-coalgebra. The resulting notion of trace of a state of a $T \circ F$-coalgebra is referred to as *fat trace* in [11], as it retains the structure specified by the transition type F and therefore may involve branching.

A *finite execution map* for a $T \circ F$-coalgebra (X, γ) is also defined in [11], as the finite trace map obtained by regarding (X, γ) as a $T \circ F \circ (X \times \mathsf{Id})$-coalgebra. Here we propose a variant of this notion obtained by replacing the functor $F \circ (X \times \mathsf{Id})$ with the functor $X \times F$. The reason for this variation is that we expect finite executions starting in a state of a coalgebra to incorporate the state itself.

Definition 2.3 Let $T : \mathsf{C} \to \mathsf{C}$ be a strong monad, $F : \mathsf{C} \to \mathsf{C}$ be an endofunctor, and $\lambda : F \circ T \Rightarrow T \circ F$ be a distributive law of T over F. Also, for a $T \circ F$-coalgebra (X, γ), let (I_X, ι_X) denote an initial $(X \times F)$-algebra, and let $\lambda_X : (X \times F) \circ T \Rightarrow T \circ (X \times F)$ denote the natural transformation given by $(\lambda_X)_Y = st_{X,FY} \circ (id_X \times \lambda_Y)$. The *finite execution map*

$\mathsf{fexec}_\gamma : X \to TI_X$ is the C-map underlying the unique $\overline{X \times F}$-coalgebra morphism from $(X, st_{X,FX} \circ \langle id_X, \gamma\rangle)$ to the final $\overline{X \times F}$-coalgebra.

Modal logics for coalgebras

Our path-based coalgebraic temporal logics will be based on the notion of predicate lifting, as introduced by Pattinson [13]. However, the semantics of these logics will differ somewhat from the standard semantics of coalgebraic modal logics induced by predicate liftings, as defined e.g. in loc. cit. Also, the notion of predicate lifting used here is more general than the original one of [13], and applies to endofunctors on both Set and Meas.

We begin by fixing a category C with forgetful functor $U : \mathsf{C} \to \mathsf{Set}$, and a contravariant functor $P : \mathsf{C} \to \mathsf{Set}^{\mathrm{op}}$ such that P is a subfunctor of $\hat{\mathcal{P}} \circ U$, with $\hat{\mathcal{P}} : \mathsf{Set} \to \mathsf{Set}^{\mathrm{op}}$ the contravariant powerset functor. Thus, for each state space X, PX specifies a set of admissible predicates. As instances of P we will consider the contravariant powerset functor $\hat{\mathcal{P}} : \mathsf{Set} \to \mathsf{Set}^{\mathrm{op}}$ (in the case of coalgebras of endofunctors on Set), and the functor taking a measurable space to the carrier of its underlying σ-algebra (in the case of coalgebras of endofunctors on Meas).

Now given an endofunctor $F : \mathsf{C} \to \mathsf{C}$ and $n \in \omega$, an *n-ary predicate lifting for F* is a natural transformation $\lambda : P^n \Rightarrow P \circ F$. For simplicity of presentation, we assume all predicate liftings to be unary, however, our results generalise straightforwardly to predicate liftings with arbitrary finite arities. We briefly recall the syntax and standard coalgebraic semantics of coalgebraic modal logics induced by predicate liftings. Given a set Λ of predicate liftings for F, the modal language $\mathcal{L}_\Lambda$ has formulas given by:

$$\mathcal{L}_\Lambda \ni \varphi ::= \mathsf{tt} \mid \neg\varphi \mid \varphi \wedge \varphi \mid [\lambda]\varphi \qquad (\lambda \in \Lambda)$$

A coalgebraic semantics for this language is obtained by defining $[\![\varphi]\!]_\gamma \subseteq PC$ for each F-coalgebra (C, γ), by structural induction on $\varphi \in \mathcal{L}_\Lambda$. The interesting case is $[\![[\lambda]\varphi]\!]_\gamma = (P\gamma)(\lambda_C([\![\varphi]\!]_\gamma))$ for $\varphi \in \mathcal{L}_\Lambda$ and $\lambda \in \Lambda$. In Section 4, we will see a novel use of modalities arising from predicate liftings, namely to interpret state formulas in path-based temporal logics. There, we will typically require our predicate liftings to be *monotone*, in that $A \subseteq B$ implies $\lambda_X(A) \subseteq \lambda_X(B)$ for all X and all $A, B \in PX$.

3 Possibly Infinite Traces and Executions

Our aim is to define a notion of possibly infinite execution of a state in a coalgebra, to be used in the semantics of path-based coalgebraic temporal logics. Some initial steps in this direction were made in [10], where a notion of infinite trace was defined for coalgebras of type $\mathcal{P} \circ F$, with $F : \mathsf{Set} \to \mathsf{Set}$

a polynomial functor equipped with a distributive law $\lambda : F \circ \mathcal{P} \Rightarrow \mathcal{P} \circ F$. Specifically, it was observed in loc. cit. that the final F-coalgebra in Set (whose elements represent potential infinite traces) gives rise to a weakly final $\overline{F}$-coalgebra in $\mathsf{Kl}(\mathcal{P})$. Then, for a $\mathcal{P} \circ F$-coalgebra, an infinite trace map was obtained using weak finality, by regarding this coalgebra as an $\overline{F}$-coalgebra in $\mathsf{Kl}(\mathcal{P})$. The order-enrichedness of $\mathsf{Kl}(\mathcal{P})$ guaranteed the existence of a canonical choice for the infinite trace map.

Here we propose a notion of infinite trace that applies to coalgebraic signatures of the form $T \circ F$, with T a monad and F an endofunctor on a category C, related through a distributive law of T over F and subject to some additional constraints. Throughout this section, C denotes a category with countable limits, $F : \mathsf{C} \to \mathsf{C}$ is an endofunctor, $T : \mathsf{C} \to \mathsf{C}$ is a strong monad whose Kleisli category is *DCpo*-enriched, and $\lambda : F \circ T \Rightarrow T \circ F$ is a distributive law of T over F.

3.1 Possibly infinite traces

As in [10], the final F-coalgebra provides the potential infinite traces of elements of $T \circ F$-coalgebras. We work under the assumption that F preserves the limit of the following ω^{op}-chain

$$1 \xleftarrow{\;!\;} F1 \xleftarrow{\;F!\;} F^2 1 \xleftarrow{\;F^2!\;} \cdots$$

with 1 a final object in C and $! : F1 \to 1$ the unique such map[4]. Assuming the above, the carrier of the final F-coalgebra is obtained as the limit in C of the above ω^{op}-chain. We let $(Z, \zeta : Z \to FZ)$ denote a final F-coalgebra, and write $\pi_i : Z \to F^i 1$ with $i \in \omega$ for the corresponding projections. We begin by showing that, under some additional constraints on the monad T, a $T \circ F$-coalgebra $\gamma : X \to TFX$ induces a cone over the ω^{op}-chain:

$$T1 \xleftarrow{\;T!\;} TF1 \xleftarrow{\;TF!\;} TF^2 1 \xleftarrow{\;TF^2!\;} \cdots$$

To this end, we define an ω-indexed family of maps $(\gamma_i : X \to TF^i 1)_{i \in \omega}$ by:

- $\gamma_0 = \eta_1 \circ !_X : X \to T1$, where $!_X : X \to 1$ is the unique such map,
- $\gamma_{i+1} = \mu_{F^{i+1}1} \circ T\lambda_{F^i 1} \circ TF\gamma_i \circ \gamma : X \to TF^{i+1}1$ for $i \in \omega$.

That is, the maps γ_i arise by unfolding the coalgebra structure i times, and using the distributive law λ of T over F and the monad multiplication to discard inner occurrences of T from the codomain of the maps γ_i. As the elements of $F^i 1$ define finite approximations of potential infinite traces, the maps γ_i can be regarded as providing finite approximations of the infinite

[4] This assumption is weaker than requiring F to preserve the limits of *all* ω^{op}-chains, a condition that will not hold for certain instances of F considered later in the paper.

trace map for the $T \circ F$-coalgebra γ. It is also worth noting that one can alternatively define the γ_is as maps in $\mathsf{Kl}(T)$:

- $\gamma_0 = J!_X$,
- $\gamma_{i+1} = \overline{F}\gamma_i \circ \gamma$ for $i \in \omega$.

Lemma 3.1 *Let $!_{TF1} : TF1 \to 1$ be the only such map. If $\eta_1 \circ !_{TF1} = T!$, then the above γ_is define a cone over the ω^{op}-chain $(JF^i!)_{i\in\omega}$ in $\mathsf{Kl}(T)$.*

Proof. The hypothesis ensures that $\gamma_0 = J! \circ \gamma_1$. Now assuming $\gamma_i = JF^i! \circ \gamma_{i+1}$, we immediately obtain $\overline{F}\gamma_i = \overline{F}JF^i! \circ \overline{F}\gamma_{i+1} = JF^{i+1}! \circ \overline{F}\gamma_{i+1}$, where the last equality follows by Lemma 2.2. Precomposition with γ finally gives $\gamma_{i+1} = JF^{i+1}! \circ \gamma_{i+2}$.

We immediately observe that the hypothesis of the above result is *not* satisfied by either of the two monads identified earlier:

- for $T = \mathcal{P}$, $(\eta_1 \circ !_{TF1})(\emptyset) = 1 \neq \emptyset = (\mathcal{P}!)(\emptyset)$;
- for $T = \mathcal{G}$, $(\eta_1 \circ !_{TF1})(\nu_0) = \mu_1 \neq \mu_0 = (\mathcal{G}!)(\nu_0)$, where ν_0 is the subprobability measure on $F(1, \mathcal{P}1)$ [5] which assigns the value 0 to each measurable set, whereas μ_0 and μ_1 are the subprobability measures on $(1, \mathcal{P}1)$ given by $\mu_0(1) = 0$ and respectively $\mu_1(1) = 1$.

To remedy the situation, we will work with submonads of these two monads for which the hypothesis of Lemma 3.1 is true. To this end, we first note that if the monad T is such that $\eta_1 : 1 \to T1$ is an isomorphism, then the equality required by Lemma 3.1 is obtained immediately by finality. Strong monads with the above property are called *affine*, see e.g. [9] for an overview. Moreover, [9] shows how to construct, for any strong monad T, its *affine submonad* T_a, which is itself commutative whenever T is. This construction yields:

- the non-empty powerset monad $\mathcal{P}^+ : \mathsf{Set} \to \mathsf{Set}$ as the affine part of $\mathcal{P}$,
- the probability measure monad $\mathcal{G}_1 : \mathsf{Meas} \to \mathsf{Meas}$ (with $\mathcal{G}_1(X, \Sigma_X)$ containing only the probability measures on (X, Σ_X)) as the affine part of $\mathcal{G}$.

Thus, for $T = \mathcal{P}^+$ and $T = \mathcal{G}_1$, Lemma 3.1 applies. We also note that the canonical distributive laws of the original monads ($\mathcal{P}$, respectively $\mathcal{G}$) restrict to distributive laws of their affine submonads, and that the Kleisli categories of the affine submonads inherit an order-enriched structure from the Kleisli categories of the original monads. For the latter statement, one must verify that joins (taken in $\mathsf{Kl}(T)(X, Y)$) of directed sets in $\mathsf{Kl}(T_a)(X, Y)$ are themselves elements of $\mathsf{Kl}(T_a)(X, Y)$ and are preserved by arrow composition; this is straightforward in both cases. In fact, for $T = \mathcal{G}_1$, the inherited order on $\mathsf{Kl}(\mathcal{G}_1)(X, Y)$ is the equality. The former statement follows from a general result stating that any distributive law of a strong monad T over an endofunctor

[5] Note that $(1, \mathcal{P}1)$ is a final object in Meas.

F restricts to a distributive law of T_a over F.

Proposition 3.2 *Let $\lambda : F \circ T \Rightarrow T \circ F$ be a distributive law of T over F. Then, λ restricts to a distributive law $\lambda : F \circ T_a \Rightarrow T_a \circ F$.*

Proof. As shown in [9], the action of the monad T_a on a C-object X is given by the following pullback diagram:

$$\begin{tikzcd} T_aX \arrow[r, dashed, "\iota_X"] \arrow[d, dashed, "!_{T_aX}"'] & TX \arrow[d, "T!_X"] \\ 1 \arrow[r, "\eta_1"] & T1 \end{tikzcd}$$

Thus, using that $!_{F1} \circ F!_X \; =!_{FX}$ (by finality of 1), the pullback diagram defining T_aFX can be written as

$$\begin{tikzcd} T_aFX \arrow[r, dashed, "\iota_{FX}"] \arrow[dd, dashed] & TFX \arrow[d, "TF!_X"] \\ & TF1 \arrow[d, "T!_{F1}"] \\ 1 \arrow[r, "\eta_1"] & T1 \end{tikzcd}$$

Next, note that the maps $\lambda_X \circ F\iota_X : FT_aX \to TFX$ and $!_{F1} \circ F!_{T_aX} : FT_aX \to 1$ define a cone over the diagram given by $T!_{F1} \circ TF!_X$ and η_1:

$$\begin{aligned} T!_{F1} \circ TF!_X \circ \lambda_X \circ F\iota_X = & \quad \text{(naturality of } \lambda) \\ T!_{F1} \circ \lambda_1 \circ FT!_X \circ F\iota_X = & \quad \text{(definition of } T_aX) \\ T!_{F1} \circ \lambda_1 \circ F\eta_1 \circ F!_{T_aX} = & \quad \text{(compatibility of } \lambda \text{ with monad structure)} \\ T!_{F1} \circ \eta_{F1} \circ F!_{T_aX} = & \quad \text{(naturality of } \eta) \\ \eta_1 \circ !_{F1} \circ F!_{T_aX} & \end{aligned}$$

The definition of T_aFX now yields a map $(\lambda_a)_X : FT_aX \to T_aFX$. The naturality of the resulting maps and their compatibility with the monad structure follow easily by diagram chasing.

For our two examples ($T = \mathcal{P}^+$ and $T = \mathcal{G}_1$), assuming that F is a shapely polynomial functor, one can simply work with the canonical distributive laws. An easy induction proof (not given here) shows that these coincide with the distributive laws given by the previous result. However, Proposition 3.2 shows how to obtain a distributive law of the affine submonad over an *arbitrary* endofunctor.

To motivate our definition of the infinite trace map of a $T \circ F$-coalgebra (X, γ), let us examine the case $T = \mathcal{P}^+$. Since the map γ_i takes a state of the coalgebra to a set of i-depth approximations of its possibly infinite traces, it seems natural to define the infinite trace map as a function $tr_\gamma : X \to \mathcal{P}^+Z$

sending a state s of the coalgebra to the set of possibly infinite traces whose i-depth approximation belongs to $\gamma_i(s)$. Such a trace map can be defined by exploiting the weak preservation of limits of ω^{op}-chains by J (which, in turn, follows from the weak preservation of such limits by $\mathcal{P}^+$). However, this property only guarantees the *existence* of a mediating map $tr_\gamma : X \to JZ$ in $\mathsf{Kl}(T)$. As shown in [10] for the case $T = \mathcal{P}$, a canonical choice for the infinite trace map is provided by the *largest* mediating map. Its existence is here guaranteed by the *DCpo*-structure of $\mathsf{Kl}(\mathcal{P}^+)(X, Z)$, together with the observation that in this particular case the mediating maps form a directed set. This justifies the following general definition of the infinite trace map.

Definition 3.3 Assume that the monad T is affine and that the functor J weakly preserves the limit $(Z, (\pi_i)_{i\in\omega})$ of the ω^{op}-chain $(F^i!)_{i\in\omega}$. For a $T \circ F$-coalgebra (X, γ), let $(X, (\gamma_i : X \to JF^i1)_{i\in\omega})$ be the induced cone over $(JF^i!)_{i\in\omega}$, and assume further that the corresponding mediating maps form a directed set. The *possibly infinite trace map* is the largest [6] mediating map $tr_\gamma : X \to JZ$ arising from the weak limiting property of $(JZ, (J\pi_i)_{i\in\omega})$ (regarded as a map in C).

In particular, Definition 3.3 can be applied to the non-empty powerset monad $\mathcal{P}^+ : \mathsf{Set} \to \mathsf{Set}$, as well as to the probability measure monad $\mathcal{G}_1 : \mathsf{Meas} \to \mathsf{Meas}$. The resulting notions of infinite trace are discussed in Sections 3.3 and 3.4. We also note that the affine submonad of the lift monad $1 + \mathsf{Id}$ on Set (as considered in [8]) is the identity monad, to which Definition 3.3 applies trivially. A treatment of monads that are *not* affine is outside the scope of this paper.

We conclude this section by proving some properties of the infinite trace map, similar to the *defining* properties of the infinite trace map in [10].

Proposition 3.4 *Under the assumptions of Definition 3.3, the trace map $tr_\gamma : X \to JZ$ defines an* op-lax *$\overline{F}$-coalgebra morphism from (X, γ) to $(JZ, J\zeta)$, that is, $\overline{F}tr_\gamma \circ \gamma \sqsubseteq J\zeta \circ tr_\gamma$. Under the additional assumptions that $(JZ, (J\pi_i)_{i\in\omega})$ is a limit of $(JF^i!)_{i\in\omega}$, tr_γ defines an $\overline{F}$-coalgebra morphism, that is, $\overline{F}tr_\gamma \circ \gamma = J\zeta \circ tr_\gamma$.*

Proof. We begin by noting that the final F-coalgebra $\zeta : Z \to FZ$ satisfies $F\pi_i \circ \zeta = \pi_{i+1}$ for all $i \in \omega$, and hence, in $\mathsf{Kl}(T)$ we have $JF\pi_i \circ J\zeta = J\pi_{i+1}$ for all $i \in \omega$. Now recall that $(JFZ, (JF\pi_i)_{i\in\omega})$ is a weak limit of $(JF^{i+1}!)_{i\in\omega}$. Moreover, since $J\zeta$ is an isomorphism in $\mathsf{Kl}(T)$ (and hence admits an inverse), and since arrow composition in $\mathsf{Kl}(T)$ preserves directed joins, it follows that the map $J\zeta \circ tr_\gamma : X \to JFZ$ is the largest mediating map for the cone $(X, (\gamma_{i+1})_{i\in\omega})$ over the ω^{op}-chain $(JF^{i+1}!)_{i\in\omega}$. On the other hand, we have: $JF\pi_i \circ \overline{F}tr_\gamma \circ \gamma = \overline{F}J\pi_i \circ \overline{F}tr_\gamma \circ \gamma = \overline{F}\gamma_i \circ \gamma = \gamma_{i+1}$. Hence, since

[6] w.r.t. the order on $\mathsf{Kl}(T)(X, Z)$

$J\zeta \circ tr_\gamma : X \to JFZ$ is the largest mediating map for $(X, (\gamma_{i+1})_{i\in\omega})$, we obtain $\overline{F}tr_\gamma \circ \gamma \sqsubseteq J\zeta \circ tr_\gamma$. That is, tr_γ defines an op-lax $\overline{F}$-coalgebra morphism from (X,γ) to $(JZ, J\zeta)$. Under the stronger assumption that $(JZ, (J\pi_i)_{i\in\omega})$ is a limit of $(JF^i!)_{i\in\omega}$, uniqueness of a mediating arrow induced by the cone $(X, (\gamma_{i+1})_{i\in\omega})$ over $(JF\pi_i)_{i\in\omega}$ yields $\overline{F}tr_\gamma \circ \gamma = J\zeta \circ tr_\gamma$, that is, tr_γ defines an $\overline{F}$-coalgebra morphism.

In the case of the non-empty powerset monad, the above result only implies that the infinite trace map is an *op-lax* coalgebra morphism. This is weaker than the defining property of the infinite trace map in [10], which asks for a proper coalgebra morphism. The study of sufficient conditions for the infinite trace map to define a proper coalgebra morphism for an arbitrary (affine) monad T remains an open question, but we conjecture that the local continuity of the functor $\overline{F}$ [7] will be at least a necessary condition.

On the other hand, we will see later that the additional assumption of Proposition 3.4 which ensures that the trace map is a coalgebra morphism holds for the probability measure functor $\mathcal{G}_1$ on Meas, when taking certain shapely polynomial functors on Meas as instances of F.

3.2 Possibly infinite executions

To obtain a notion of possibly infinite execution of a state in a $T \circ F$-coalgebra, we use the approach in the previous section with a different choice of functor F. Similarly to Definition 2.3, given a $T \circ F$-coalgebra (X, γ), we consider the endofunctor $F_X : \mathsf{C} \to \mathsf{C}$ given by $F_X(Y) = X \times FY$ and the distributive law $\lambda_X : F_X \circ T \Rightarrow T \circ F_X$ given by $(\lambda_X)_Y = st_{X,FY} \circ (id_X \times \lambda_Y)$. We call an element of the carrier of the final F_X-coalgebra (Z_X, ζ_X) a potential *infinite execution*, or *computation path*.

Definition 3.5 Assume that T is affine and that J weakly preserves the limit $(Z_X, (\pi_i)_{i\in\omega})$ of the ω^{op}-chain $({F_X}^i!)_{i\in\omega}$. For a $T \circ F$-coalgebra (X,γ), let $(X, (\gamma_i : X \to J{F_X}^i 1)_{i\in\omega})$ be the cone over $(J{F_X}^i!)_{i\in\omega}$ induced by the $T \circ F_X$-coalgebra $(X, st_{X,FX} \circ \langle id_X, \gamma\rangle)$, and assume that the corresponding mediating maps form a directed set. The *possibly infinite execution map* $\mathsf{exec}_\gamma : X \to JZ_X$ of (X,γ) is the possibly infinite trace map of the $T \circ F_X$-coalgebra $(X, st_{X,FX} \circ \langle id_X, \gamma\rangle)$.

3.3 (Labelled) transition systems

These are modelled as $\mathcal{P}^+ \circ F$-coalgebras, with $F = \mathsf{Id}$ (respectively $F = A \times \mathsf{Id}$ for a fixed set A of labels). Our use of the non-empty powerset monad agrees

[7] A functor $F : \mathsf{C} \to \mathsf{D}$ between *DCpo*-enriched categories is *locally continuous* if it preserves suprema of directed joins in $\mathsf{C}(X,Y)$ for each X, Y. In enriched categorical terms, F is a *DCpo*-enriched functor.

with the standard constraint put on transition systems when defining computation paths. The next result ensures that the hypotheses of Definitions 3.3 and 3.5 are satisfied.

Lemma 3.6 *The (non-empty) powerset functor weakly preserves limits of ω^{op}-chains; hence, by Lemma 2.1, so does J. Moreover, the resulting mediating maps, regarded as arrows in $\mathsf{Kl}(\mathcal{P})$ (resp. $\mathsf{Kl}(\mathcal{P}^+)$) form a directed set.*

Proof. Let $(Z, (\pi_i)_{i\in\omega})$ denote the limit of an ω^{op}-chain $(f_i : Z_{i+1} \to Z_i)_{i\in\omega}$. For a cone $(\gamma_i : X \to \mathcal{P}Z_i)_{i\in\omega}$ over $(\mathcal{P}f_i : \mathcal{P}Z_{i+1} \to \mathcal{P}Z_i)_{i\in\omega}$, the map $m : X \to \mathcal{P}Z$ given by $m(x) = \{z \in Z \mid \pi_i(z) \in \gamma_i(x) \text{ for } i \in \omega\}$ for $x \in X$ is a mediating map. (If $X = \emptyset$, the existence of a mediating map is trivial.) The same applies when replacing $\mathcal{P}$ by $\mathcal{P}^+$. This time, one also has to show that the set defining $m(x)$ is non-empty. Using the axiom of choice one can construct, for each $x \in X$, a sequence $(z_i)_{i\in\omega}$ with $z_i \in \gamma_i(x)$ and $f_i(z_{i+1}) = z_i$ for $i \in \omega$; this, in turn, yields $z \in Z$ with $\pi_i(z) \in \gamma_i(x)$ for $i \in \omega$.

For the second statement, note that the mediating map m defined previously is above any other mediating map (under the inclusion order), and thus the set of mediating maps is directed.

Remark 3.7 To see that neither $\mathcal{P}$ nor $\mathcal{P}^+$ preserve limits of ω^{op}-chains, consider the final sequence $(f_i : Z_{i+1} \to Z_i)_{i\in\omega}$ of the endofunctor $1 + A \times \mathsf{Id}$, with $Z_i = \bigcup_{0\le j\le i} A^i$, and with limit object $Z = A^* \cup A^\omega$. Now define a cone $(\gamma_i : 1 \to \mathcal{P}Z_i)_{i\in\omega}$ by letting $\gamma_i(*)$ consist only of the i-long sequence of a's, for some fixed $a \in A$. Then, both $m(*) = \{a\}^*$ and $m'(*) = \{a\}^* \cup \{a\}^\omega$ define mediating maps. (A similar example is discussed in [8, Section 4.2].)

As a result of Lemma 3.6, Definition 3.5 yields, for each state in a transition system, a set of infinite executions. As expected, this coincides with the set of computation paths from that state, as considered in the semantics of CTL*. For $F = A \times \mathsf{Id}$, the infinite execution map gives, for each state s, the set of labelled computation paths from s, as infinite sequences of the form $s = s_0 a_1 s_1 a_2 s_2 \ldots$ with $s_i \xrightarrow{a_i} s_{i+1}$ for $i \in \omega$, whereas the infinite trace map yields the sequences of labels that occur along such labelled computation paths.

One can also vary the functor F in order to model explicit termination. This is achieved by taking $F = 1 + \mathsf{Id}$ or $F = 1 + A \times \mathsf{Id}$, as in [8]. In these cases, the resulting possibly infinite trace (execution) maps capture both finite and infinite traces (respectively computation paths).

3.4 Probabilistic models

A large variety of discrete probabilistic models have been studied, see e.g. [17] for a coalgebraic account of such models. Among these, probabilistic transition systems (also called *Markov chains*) appear as coalgebras of the endofunctor

$\mathcal{D} = \mathcal{D} \circ \mathsf{Id}$ and are used to interpret the logic PCTL [7], while *generative probabilistic systems* coincide with $\mathcal{D}\circ(A\times\mathsf{Id})$-coalgebras. Here, $\mathcal{D} : \mathsf{Set} \to \mathsf{Set}$ denotes the *probability distribution monad*, a submonad of the subprobability distribution monad defined on objects by $\mathcal{D}X = \{\mu \in \mathcal{S}X \mid \sum_{x\in X} \mu(x) = 1\}$.

Unfortunately, although affine, the monad $\mathcal{D}$ does not satisfy the requirement of Definition 3.3 concerning the weak preservation of limits by the induced functor J. To see this, let $F : \mathsf{Set} \to \mathsf{Set}$ be given by $FX = \{a, b\} \times X$, and $\mu_i \in \mathcal{D}F^i1$ be given by $\mu_i(x) = \frac{1}{2^i}$ for $x \in \{a, b\}^i$, with $i \in \omega$. Thus, each μ_i defines a *finite* probability distribution over F^i1, and we have $(\mathcal{D}^i!)(\mu_{i+1}) = \mu_i$ for $i \in \omega$. However, there is no probability distribution μ on the final F-coalgebra (whose carrier, $\{a, b\}^\omega$, is uncountable) such that $(\mathcal{D}\pi_i)(\mu) = \mu_i$ for $i \in \omega$ – any such μ could only take non-zero values on countably-many elements of Z. Indeed, a state of a $\mathcal{D} \circ F$-coalgebra will in general have uncountably many infinite traces, and the emphasis when defining an infinite trace map should be on measuring *sets* of traces rather than individual traces.

A satisfactory treatment of infinite traces for discrete probabilistic models turns out to be possible by regarding such models as coalgebras of the probability measure monad $\mathcal{G}_1$. For technical reasons that will soon be made clear, we will work in a subcategory of Meas, namely the full subcategory SB of Meas whose objects are *standard Borel spaces* (spaces whose measurable sets arise as the Borel sets induced by a complete, separable metric, see e.g. [4]). A notable property of this category is that it is closed under countable coproducts and countable limits in Meas (see e.g. [16, Fact 1]). We also note that a discrete measurable space $(X, \mathcal{P}X)$ is standard Borel if and only if X is countable. As a result, we will only be able to define notions of infinite trace and infinite execution for $\mathcal{D} \circ F$-coalgebras *with countable carrier*. We will do so by lifting the functor F to a functor $\hat{F} : \mathsf{SB} \to \mathsf{SB}$, and regarding a $\mathcal{D} \circ F$-coalgebra on Set as a $\mathcal{G}_1 \circ \hat{F}$-coalgebra on SB.

We now proceed to define a restricted version of shapely polynomial functors on Meas. The restriction is driven by the need to work in the subcategory SB of Meas. Specifically, we call an endofunctor on Meas a *restricted shapely polynomial functor* if it is built from identity and constant functors $C_{(X,\Sigma_X)}$ with (X, Σ_X) a standard Borel space, using finite products and *countable* coproducts. Then, given a *restricted* shapely polynomial functor F on Set, that is, a functor built from identity and *countable* constant functors using finite products and *countable* coproducts, we write $\hat{F} : \mathsf{Meas} \to \mathsf{Meas}$ for its counterpart on measurable spaces, defined by structural induction on F:

- $\widehat{\mathsf{Id}}$ is the identity functor on Meas,
- $\widehat{C_X}$ is the constant functor $C_{(X,\mathcal{P}X)}$, for each countable set X,
- $\widehat{F_1 \times F_2} = \widehat{F_1} \times \widehat{F_1}$,

- $\widehat{\coprod_{i\in\omega} F_i} = \coprod_{i\in\omega} \widehat{F_i}$.

Lemma 3.8 *If $F : \mathsf{Set} \to \mathsf{Set}$ is a restricted shapely polynomial functor, then so is $\widehat{F} : \mathsf{Meas} \to \mathsf{Meas}$. Moreover, $\widehat{F}$ preserves (discrete) SB-spaces.*

Proof. The first statement is immediate. Preservation of SB-spaces by $\widehat{F}$ follows from results in [16], whereas preservation of discrete spaces follows by induction on the structure of F:

- For $F = C_X$ with X countable, $\widehat{F}(Y, \mathcal{P}Y) = (X, \mathcal{P}X) = (FX, \mathcal{P}FX)$ for all Y.
- For $F = \mathsf{Id}$, $\widehat{Id}(X, \mathcal{P}X) = (X, \mathcal{P}X) = (FX, \mathcal{P}FX)$ for all X.
- For $F = F_1 \times F_2$, $\widehat{F}(X, \mathcal{P}X) = \widehat{F_1}(X, \mathcal{P}X) \times \widehat{F_2}(X, \mathcal{P}X) = (F_1X, \mathcal{P}F_1X) \times (F_2X, \mathcal{P}F_2X) = (F_1X \times F_2X, \mathcal{P}(F_1X \times F_2X))$, where the last equality follows from finite products of discrete SB-spaces being themselves discrete SB-spaces.
- The case $F = \coprod_{i\in\omega} F_i$ is treated similarly.

As a result, we immediately obtain

Proposition 3.9 *A $\mathcal{D} \circ F$-coalgebra (X, γ) with countable carrier yields a $\mathcal{G}_1 \circ \widehat{F}$-coalgebra $((X, \mathcal{P}X), \widehat{\gamma})$, such that the cone $(\gamma_i)_{i\in\omega}$ in $\mathsf{Kl}(\mathcal{D})$:*

$$\begin{tikzcd}
X \arrow[d, "\gamma_0"'] \arrow[dr, "\gamma_1"] \arrow[drr, "\gamma_2"] & & & \\
J1 & JF1 \arrow[l, "J!"] & JF^2 1 \arrow[l, "JF!"] & \cdots \arrow[l, "JF^2!"]
\end{tikzcd}$$

with the γ_is being as in Section 3.1, defines a cone in $\mathsf{Kl}(\mathcal{G}_1)$:

$$\begin{tikzcd}
(X, \mathcal{P}X) \arrow[d, "\gamma_0"'] \arrow[dr, "\gamma_1"] \arrow[drr, "\gamma_2"] & & & \\
J'(1, \mathcal{P}1) & J'\widehat{F}(1, \mathcal{P}1) \arrow[l, "J'!"] & J'\widehat{F}^2(1, \mathcal{P}1) \arrow[l, "J'\widehat{F}!"] & \cdots \arrow[l, "J'\widehat{F}^2!"]
\end{tikzcd}$$

where $J : \mathsf{Set} \to \mathsf{Kl}(\mathcal{D})$ and $J' : \mathsf{Meas} \to \mathsf{Kl}(\mathcal{G}_1)$ are as in Section 2.

The coalgebra map $\widehat{\gamma} : (X, \mathcal{P}X) \to \mathcal{G}_1\widehat{F}(X, \mathcal{P}X) = \mathcal{G}_1(FX, \mathcal{P}FX)$ yields, for each state $x \in X$, the probability measure on $(FX, \mathcal{P}FX)$ induced by the probability distribution $\gamma(x)$ on FX. Since $(1, \mathcal{P}1)$ is final in Meas, the latter of the above cones is over the image under J' of the final sequence of $\widehat{F}$. As a result, we can use the existence of trace maps of $\mathcal{G}_1 \circ \widehat{F}$-coalgebras to define trace maps for $\mathcal{D} \circ F$-coalgebras.

The next lemma ensures that $\widehat{F}$ and $\widehat{F_X}$ preserve the limit of the initial ω^{op}-segment of their respective final sequences, as required by Definitions 3.3 and 3.5.

Lemma 3.10 ([16]) *Restricted shapely polynomial functors on* Meas *preserve surjective* SB*-morphisms and limits of* ω^{op}*-chains of surjective* SB*-morphisms.*

Proof. It was proved in [16, Proposition 3] that the class of endofunctors on Meas that preserve surjective SB-morphisms and limits of ω^{op}-chains of surjective SB-morphisms is closed under countable coproducts and countable limits. The conclusion then follows after noting that the identity functor and constant functors $C_{(X,\Sigma_X)}$ with (X,Σ_X) a standard Borel space belong to this class.

The required property of $\widehat{F}$ now follows, since $! : \widehat{F}(1,\mathcal{P}1) \to (1,\mathcal{P}1)$ is a surjective SB-morphism (assuming that F is non-trivial, i.e. $F1 \neq \emptyset$). As a result, for every restricted shapely polynomial functor F on Set, the final sequence of $\widehat{F}$ belongs to SB, stabilises at ω, and its limit is the carrier of a final $\widehat{F}$-coalgebra, itself in SB. Moreover, if X is a countable set, the above also applies to the functor $F_X : \mathsf{Set} \to \mathsf{Set}$ defined by $F_X Y = X \times FY$. The restriction to countable carriers is necessary to ensure applicability of Definition 3.5. This is precisely the reason for working with the category SB.

Recall from Section 2 that commutative monads on any category with products and coproducts admit canonical distributive laws over shapely polynomial functors. This applies in particular to the monad $\mathcal{G}_1$ and any restricted shapely polynomial functor on Meas. Then, to be able to apply Definition 3.1 to the functors $\widehat{F}$ and $\widehat{F_X}$, with $F : \mathsf{Set} \to \mathsf{Set}$ a restricted shapely polynomial functor and X a countable set, all that remains to verify is that the functor $\mathcal{G}_1$ weakly preserves the limits of the final sequences of $\widehat{F}$ and $\widehat{F_X}$. In fact, a stronger result holds:

Lemma 3.11 ([16]) *The functor* $\mathcal{G}_1 : \mathsf{Meas} \to \mathsf{Meas}$ *preserves limits of* ω^{op}*-chains of surjective* SB*-morphisms.*

We note that the result in [16] refers to the subprobability measure functor $\mathcal{G}$, but a similar proof can be given for the probability measure functor.

As a consequence, we obtain probabilistic trace and execution maps for $\mathcal{D} \circ F$-coalgebras with countable carrier, with $F : \mathsf{Set} \to \mathsf{Set}$ as above.

Definition 3.12 Let $F : \mathsf{Set} \to \mathsf{Set}$ be a restricted shapely polynomial functor, let (X,γ) be a $\mathcal{D} \circ F$-coalgebra with countable carrier, and let $(\gamma_i : (X,\mathcal{P}X) \to J'\widehat{F}^i(1,\mathcal{P}1))_{i\in\omega}$ denote the cone over $(J'\widehat{F}^i!)_{i\in\omega}$ induced by the $\mathcal{G}_1 \circ \widehat{F}$-coalgebra $\widehat{\gamma} : (X,\mathcal{P}X) \to \mathcal{G}_1\widehat{F}(X,\mathcal{P}X)$. The *probabilistic trace map* $tr_\gamma : X \to JZ$ is defined as the underlying function of the unique measurable map arising from the limiting property of $J'(Z,\Sigma_Z)$, where (Z,Σ_Z) is the carrier of a final $\widehat{F}$-coalgebra.

Since limits in Meas are constructed from the underlying limits in Set (see e.g. [15]), the state space Z of the final $\widehat{F}$-coalgebra is the carrier of a final

F-coalgebra, and thus the probabilistic trace map yields, as expected, for each state of a $\mathcal{D} \circ F$-coalgebra, a probability measure over (Z, Σ_Z).

Returning to the example of Markov chains ($F = \mathsf{Id}$), the resulting notion of probabilistic execution gives, for each state in a Markov chain, a probability measure over its computation paths. Similarly, in the case of generative probabilistic systems ($F = A \times \mathsf{Id}$), the notion of probabilistic execution gives, for each state, a probability measure over its labelled computation paths. Finally, explicit termination can be modelled by taking $F = 1 + \mathsf{Id}$ or $F = 1 + A \times \mathsf{Id}$, as in [8], and the resulting notions of possibly infinite execution also incorporate finite (labelled) computation paths.

4 Path-Based Coalgebraic Temporal Logics

We now introduce CTL*-like coalgebraic temporal logics whose semantics is defined in terms of possibly infinite executions. Throughout this section, we fix a monad $T : \mathsf{C} \to \mathsf{C}$, a functor $F : \mathsf{C} \to \mathsf{C}$, and a $T \circ F$-coalgebra (X, γ) together with a map $\mathsf{exec}_\gamma : X \to TZ_X$ obtained using the approach in Section 3, where (Z_X, ζ_X) is a final F_X-coalgebra. We note in passing that the temporal languages defined in this section can also be interpreted by using the finite execution map $\mathsf{fexec}_\gamma : X \to TI_X$ with (I_X, ι_X) an initial $(X \times F)$-algebra, as given by Definition 2.3, instead of the infinite execution map – the forthcoming definitions do not rely on the finality of (Z_X, ζ_X). However, this is only useful when $F0 \neq 0$, with 0 an initial object in C, as otherwise the initial F_X-algebra has empty carrier. In particular, modelling explicit termination via functors such as $F = 1 + \mathsf{Id}$ or $F = 1 + A \times \mathsf{Id}$ yields non-trivial finite execution maps to which the definitions in this section can be applied.

The temporal logics that we define are parameterised by sets Λ_F and Λ of monotone predicate liftings for the functors F and respectively T. The category C will be instantiated to Set and Meas.

We recall that the definition of predicate liftings requires functors $U : \mathsf{C} \to \mathsf{Set}$ and $P : \mathsf{C} \to \mathsf{Set}^{\mathrm{op}}$ such that P is a subfunctor of $\hat{\mathcal{P}} \circ U$. In addition, defining the semantics of path-based temporal logics will at least require that PX is closed under countable (including finite) unions and intersections, for each C-object X.

4.1 Path-based fixpoint logics

We first consider the case when PX is a complete lattice for each X. Under this assumption, which holds e.g. when $\mathsf{C} = \mathsf{Set}$ and $P = \hat{\mathcal{P}}$, we are able to define path-based coalgebraic *fixpoint* logics. These logics are two-sorted, with *path formulas* denoted by $\varphi^F, \psi^F, \ldots$ expressing properties of possibly infinite executions, and *state formulas* denoted by $\varphi, \psi, \ldots$ expressing properties of states of $T \circ F$-coalgebras.

The language $\mu\mathcal{L} ::= \mu\mathcal{L}_{\Lambda}^{\Lambda_F}(\mathcal{V}_F, \mathcal{V})$ over a 2-sorted set $(\mathcal{V}_F, \mathcal{V})$ of propositional variables (with sorts for paths and respectively states) is defined by the grammar

$$\begin{aligned} \mu\mathcal{L}_F \ni \varphi^F &::= \mathsf{tt} \mid \mathsf{ff} \mid p^F \mid \varphi \mid \varphi^F \wedge \varphi^F \mid \varphi^F \vee \varphi^F \mid [\lambda_F]\varphi^F \mid \eta p^F.\varphi^F \\ \mu\mathcal{L} \ni \varphi &::= \mathsf{tt} \mid \mathsf{ff} \mid p \mid [\lambda]\varphi^F \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \end{aligned}$$

where $p^F \in \mathcal{V}_F$, $p \in \mathcal{V}$, $\eta \in \{\mu, \nu\}$, $\lambda_F \in \Lambda_F$ and $\lambda \in \Lambda$. Thus, path formulas are constructed from propositional variables and state formulas using positive boolean operators, modal operators $[\lambda_F]$ and fixpoint operators, whereas state formulas are constructed from atomic propositions and modal formulas $[\lambda]\varphi_F$ with φ_F a *path* formula, using positive boolean operators. The modal operators $[\lambda_F]$ and $[\lambda]$ with $\lambda_F \in \Lambda_F$ and $\lambda \in \Lambda$ are thus both applied to path formulas, to obtain new path formulas and respectively *state* formulas. They are, however, of very different natures: while the operators $[\lambda_F]$ quantify over the *one-step* behaviour of computation paths, the operators $[\lambda]$ quantify over the (suitably structured) *long-term* computation paths from particular states. This is made precise in the formal semantics of $\mu\mathcal{L}_{\Lambda}^{\Lambda_F}(\mathcal{V}_F, \mathcal{V})$, as defined below.

Given a $T \circ F$-coalgebra (X, γ) and a 2-sorted valuation $V : (\mathcal{V}_F, \mathcal{V}) \to (PZ_X, PX)$ (interpreting path and state variables as sets of computation paths and respectively of states), the semantics $(\!|\varphi^F|\!)_{\gamma,V} \in PZ_X$ of path formulas $\varphi^F \in \mu\mathcal{L}_F$ and $[\![\varphi]\!]_{\gamma,V} \in PX$ of state formulas $\varphi \in \mu\mathcal{L}$ is defined inductively on the structure of φ^F and φ by:

$$\begin{aligned} (\!|p^F|\!)_{\gamma,V} &= V(p^F) \\ (\!|\varphi|\!)_{\gamma,V} &= P(\pi_1 \circ \zeta_X)([\![\varphi]\!]_{\gamma,V}) \\ (\!|[\lambda_F]\varphi^F|\!)_{\gamma,V} &= (P(\pi_2 \circ \zeta_X) \circ (\lambda_F)_{Z_X})((\!|\varphi^F|\!)_{\gamma,V}) \\ (\!|\mu p^F.\varphi^F|\!)_{\gamma,V} &= lfp((\varphi^F)_{p^F}^{\gamma,V}) \\ (\!|\nu p^F.\varphi^F|\!)_{\gamma,V} &= gfp((\varphi^F)_{p^F}^{\gamma,V}) \\ [\![p]\!]_{\gamma,V} &= V(p) \\ [\![[\lambda]\varphi^F]\!]_{\gamma,V} &= (P\mathsf{exec}_\gamma \circ \lambda_{Z_X})((\!|\varphi^F|\!)_{\gamma,V}) \end{aligned}$$

and the usual clauses for the boolean operators, where, for $p^F \in \mathcal{V}_F$, $(\varphi^F)_{p^F}^{\gamma,V} : PX \to PX$ denotes the monotone map defined by $(\varphi^F)_{p^F}^{\gamma,V}(Y) = (\!|\varphi^F|\!)_{\gamma,V'}$ with $V'(p^F) = Y$ and $V'(q) = V(q)$ for $q \neq p^F$, whereas $lfp(\_)$ and $gfp(\_)$ construct least and respectively greatest fixpoints. We note that the monotonicity of the predicate liftings in Λ^F and Λ together with the absence of negation in either path or state formulas ensure that the maps $(\varphi^F)_{p^F}^{\gamma,V} : PX \to PX$ are monotone, and hence, by the Knaster-Tarski theorem, admit least and greatest fixpoints. Let us now examine the definition of the semantics of $\mu\mathcal{L}_{\Lambda}^{\Lambda_F}(\mathcal{V}_F, \mathcal{V})$

in more detail:

- To define $(\!|\varphi|\!)_{\gamma,V} \in PZ_X$ from $[\![\varphi]\!]_{\gamma,V} \in PX$, one uses the inverse image of the map $\pi_1 \circ \zeta_X$ which extracts the first state of a computation path in Z_X:

$$Z_X \xrightarrow{\zeta_X} X \times FZ_X \xrightarrow{\pi_1} X$$

  This formalises the idea that a state formula φ (regarded as a path formula) holds in a path precisely when it holds in the first state of that path.

- To define $(\!|[\lambda_F]\varphi^F|\!)_{\gamma,V} \in PZ_X$ from $(\!|\varphi^F|\!)_{\gamma,V} \in PZ_X$, one first applies the relevant component of the predicate lifting λ_F to obtain a set of one-step F-observations of computation paths, and then uses the inverse image of the map $\pi_2 \circ \zeta_X$

$$Z_X \xrightarrow{\zeta_X} X \times FZ_X \xrightarrow{\pi_2} FZ_X$$

  (which extracts the one-step F-observation of a computation path in Z_X) to obtain a set of computation paths again. This is the standard interpretation of the modal operator $[\lambda_F]$ in the F-coalgebra $\pi_2 \circ \zeta_X$.

- Finally, to define $[\![[\lambda]\varphi^F]\!]_{\gamma,V} \in PX$ from $(\!|\varphi^F|\!)_{\gamma,V} \in PZ_X$, one first applies the relevant component of the predicate lifting λ to obtain a set of suitably-structured computation paths, and then uses the inverse image of the execution map to obtain a set of states:

$$PZ_X \xrightarrow{(\lambda)_{Z_X}} PTZ_X \xrightarrow{P\mathsf{exec}_\gamma} PX$$

Example 4.1 We are now able to recover the negation-free fragment of the logic CTL*[8] as a fragment of the path-based fixpoint logic obtained by taking $T = \mathcal{P}^+$, $F = \mathsf{Id}$, $\Lambda = \{\Box, \Diamond\}$ and $\Lambda_F = \{\bigcirc\}$, where the predicate liftings $\lambda_\Box, \lambda_\Diamond : \hat{\mathcal{P}} \Rightarrow \hat{\mathcal{P}} \circ \mathcal{P}^+$ and $\lambda_\bigcirc : \hat{\mathcal{P}} \Rightarrow \hat{\mathcal{P}} \circ \mathsf{Id}$ associated to these modalities are given by:

$$\begin{aligned}
(\lambda_\Box)_X(Y) &= \{Z \in \mathcal{P}^+X \mid Z \subseteq Y\},\\
(\lambda_\Diamond)_X(Y) &= \{Z \in \mathcal{P}^+X \mid Z \cap Y \neq \emptyset\},\\
(\lambda_\bigcirc)_X(Y) &= Y.
\end{aligned}$$

The choice of $\lambda_\Box$ and $\lambda_\Diamond$ as predicate liftings for $\mathcal{P}^+$ captures precisely the path quantifiers **A** and **E** of CTL*, whereas the $\bigcirc$ modality captures the **X** operator on paths. The remaining path operators of CTL* (**F**, **G** and **U**) can be encoded as fixpoint formulas. For example, the CTL* path formula $\varphi\,\mathbf{U}\,\psi$ can be encoded as $\mu X.(\psi \vee (\varphi \wedge \bigcirc X))$.

[8] The entire language can also be obtained, using an approach similar to that of Section 4.2.

Moreover, by varying the functor F to $A \times \mathsf{Id}$, we obtain an interesting variant of CTL* interpreted over labelled transition systems. For this, we take $\Lambda_F = \{a \mid a \in A\} \cup \{\bigcirc\}$, where the predicate liftings $\lambda_a : 1 \Rightarrow \hat{\mathcal{P}} \circ (A \times \mathsf{Id})$ with $a \in A$ and $\lambda_{\bigcirc} : \hat{\mathcal{P}} \Rightarrow \hat{\mathcal{P}} \circ (A \times \mathsf{Id})$ are given by:

$$(\lambda_a)_X(*) = \{a\} \times X,$$
$$(\lambda_{\bigcirc})_X(Y) = A \times Y.$$

The resulting temporal language can easily express the property "a occurs along every computation path", namely as $\Box \mu X.(a \vee \bigcirc X)$. The reader should compare this to the formulation of the same property in the language obtained by adding fixpoints to the negation-free variant of Hennessy-Milner logic, namely as $\mu X.(\langle \_ \rangle \mathsf{tt} \wedge [-a]X)$. Here, the formulas $\langle \_ \rangle \varphi$ and $[-a]\varphi$ should be read as "there exists a successor state (reachable by *some* label) satisfying φ" and respectively "all states reachable by labels other than a satisfy φ". It is easy to see that, as the required nesting depth of fixpoint operators increases, the encodings of path properties in the latter language become complex very quickly, making the path-based language a better alternative.

4.2 Path-based temporal logics with Until operators

We now return to the more general situation when PX is only closed under countable unions and intersections. This is for instance the case when $\mathsf{C} = \mathsf{Meas}$ and $P(X, \Sigma_X) = \Sigma_X$. In this case, least or greatest fixpoints of monotone maps on PX do not necessarily exist, and we must restrict ourselves to temporal operators for which we are able to provide a well-defined semantics. In what follows we only consider Until operators similar to the ones of CTL* and PCTL, however, our approach supports more general temporal operators. In particular, a suitable choice of temporal operators can be used to obtain the full language of CTL* without resorting to arbitrary fixpoints.

Before defining the general syntax of path-based temporal logics with Until operators, we observe that the structure of the functor F may result in the associated notions of trace and execution involving some branching (as is for instance the case when $FX = A \times X \times X$). In such cases, Until operators must take into account the branching. Due to space limitations, here we only consider existential versions of branching Until operators, and refer the reader to [2] for their universal counterparts.

Path-based temporal logics with Until operators are obtained by discarding propositional variables $\mathcal{V}_F$ from the path formulas of $\mu\mathcal{L}_F$, and replacing fixpoint formulas $\eta p^F.\varphi^F$ with $\eta \in \{\mu, \nu\}$ by formulas $\varphi^F U_L \psi^F$, with $L \subseteq \Lambda_F$ a subset of (typically disjunction-preserving) predicate liftings. Furthermore, one can add negation to the syntax of both path and state formulas, and discard the requirement that only *monotone* predicate liftings should be considered in Λ and Λ_F, since no appeal to the Knaster-Tarski theorem is needed to interpret Until operators. Instead, the semantics of Until operators is defined by

$$(\!|\varphi^F U_L \psi^F|\!)_{\gamma,V} = \bigcup_{t\in\omega} (\!|\varphi^F U_L^{\leq t} \psi^F|\!)_{\gamma,V}$$

where the formulas $\varphi^F U_L^{\leq t} \psi^F$ with $t \in \omega$ are defined inductively by:

$$\begin{aligned} \varphi^F U_L^{\leq 0} \psi^F &::= \psi^F \\ \varphi^F U_L^{\leq t+1} \psi^F &::= \psi^F \vee (\varphi^F \wedge \bigvee_{\lambda^F \in L} [\lambda^F](\varphi^F U_L^{\leq t} \psi^F) \end{aligned}$$

The semantics of state formulas remains as before.

Example 4.2 One can recover the logic PCTL [7] as a fragment of the temporal logic obtained by taking $T = \mathcal{G}_1$ and $F = \mathsf{Id}$ on Meas. Predicate liftings for endofunctors $F : \mathsf{Meas} \to \mathsf{Meas}$ were considered in [15], as natural transformations of type $P \Rightarrow P \circ F$ with $P : \mathsf{Meas} \to \mathsf{Set}$ given by $P(X, \Sigma_X) = \Sigma_X$. In particular, the identity natural transformation defines a predicate lifting for $F = \mathsf{Id}$, and we write $\bigcirc$ for the associated modality. Also, for $q \in \mathbb{Q} \cap [0,1]$, the natural transformation $\lambda_q : P \Rightarrow P \circ \mathcal{G}_1$ given by $(\lambda_q)_{(X,\Sigma_X)}(Y) = \{\mu \in \mathcal{M}_1(X, \Sigma_X) \mid \mu(Y) \geq q\}$ for $Y \in \Sigma_X$ defines a predicate lifting for $T = \mathcal{G}_1$, and we write L_q for the associated modality. The logic PCTL (interpreted over measurable spaces) is now obtained by letting $\Lambda_F = \{\bigcirc\}$ and $\Lambda = \{L_q \mid q \in \mathbb{Q} \cap [0,1]\}$, and further simplifying the syntax of path formulas to

$$\varphi^F ::= \bigcirc\varphi \mid \varphi U_{\{\bigcirc\}} \varphi$$

Its interpretation over Markov chains with countable state spaces is then obtained by regarding each such Markov chain as a discrete measurable space. For example, the path formula $\varphi\, \mathbf{U}^{\leq\infty}\, \psi$ of PCTL is encoded as $\varphi U_{\{\bigcirc\}} \psi$.

Moreover, by varying the transition type to $F = \mathsf{Id}$ or $F = 1 + A \times \mathsf{Id}$, one automatically obtains variants of PCTL interpreted over generative probabilistic systems, possibly with explicit termination.

We conclude this section by noting that the full language of CTL* can be recovered using a similar approach, i.e. by defining the CTL* path operators directly rather than through fixpoint operators.

5 Concluding Remarks

We have provided a general account of possibly infinite traces and executions in systems modelled as coalgebras. The notion of infinite execution has subsequently been used to give semantics to generic path-based coalgebraic temporal logics, instances of which subsume known path-based logics such as CTL* and PCTL. Moreover, we have shown that by simply varying the transition type, interesting variants of known logics can be obtained with very little effort.

Future work will generalise these results to arbitrary (non-affine) monads. Apart from the powerset, lift and subprobability measure monads, a non-affine monad of interest is the multiset monad, due to its relevance to graded temporal logic. The study of the relationship between finite and possibly infinite traces constitutes another direction for future work.

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