# Computational Aspects of Extending the Shapley Value to Coalitional Games with Externalities[1]

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**Abstract.** Until recently, computational aspects of the Shapley value were only studied under the assumption that there are no *externalities from coalition formation*, i.e., that the value of any coalition is independent of other coalitions in the system. However, externalities play a key role in many real-life situations and have been extensively studied in the game-theoretic and economic literature. In this paper, we consider the issue of computing extensions of the Shapley value to coalitional games with externalities proposed by Myerson [21], Pham Do and Norde [23], and McQuillin [17]. To facilitate efficient computation of these extensions, we propose a new representation for coalitional games with externalities, which is based on *weighted logical expressions*. We demonstrate that this representation is *fully expressive* and, sometimes, *exponentially more concise* than the conventional partition function game model. Furthermore, it allows us to compute the aforementioned extensions of the Shapley value in time linear in the size of the input.

## 1 Introduction

The Shapley value is arguably the most important normative payoff division scheme in coalitional games. Whereas game theory focuses predominantly on the theoretical aspects of the Shapley value, computer science is equally concerned with the computational aspects involved. Given this, in this paper we consider the issue of computing the Shapley value extended to coalitional games that exhibit *externalities from coalition formation* (i.e., games in which the gain from forming a coalition may be affected by the formation of other coalitions). The issue of externalities has been extensively studied in the economic literature, as they play an important role in many real life problems (see, e.g., [4]). This issue has also been recently considered in the AI and multi-agent systems literature [18, 19, 25]. Here, externalities emerge in any situation where the utility of an agent or a coalition of agents is influenced by the functioning of coalitions created by other agents in the system.[2]

In the absence of externalities, Shapley [27] proposed to evaluate the role of each agent by averaging the marginal contributions of that agent to coalitions over all the possible permutations of agents in the game (see the formal definition in the next section). Assuming that the grand coalition, i.e. the coalition of all the agents in the game, is formed, the Shapley value identifies the *fair* division of the joint payoff. Here, fairness is defined by the following four desirable axioms: (i) *efficiency* — all the wealth available to agents is distributed among them; (ii) *symmetry* — any agents that have the same marginal contributions obtain the same payoff; (iii) *null player* — agents with zero marginal contributions receive zero payoff; and (iv) *additivity* — values of two games should sum up to the value computed for the sum of both games. This last axiom means that when agents divide the payoff from two different games, each agent's share does not depend on whether the two games are considered together or one after the other. The importance of Shapley value stems from the fact that it is the unique division scheme that meets all the above four fairness axioms.[3]

As for computational issues, the definition of the Shapley value is based on the characteristic function that assigns to every coalition a numerical value representing the quality of its performance, i.e., given $n$ agents, it considers $2^n$ coalitions. Such an exponential input is clearly computationally intractable for bigger systems (e.g., 100 or 1000 agents). Therefore, a number of works in the computer science literature proposed alternative representations which are sometimes more concise and have other interesting computational properties [8, 5, 6, 14, 22, 11]. For example, the representation of Ieong and Shoham [14], which is in many cases exponentially more concise than the characteristic function, allows us to compute the Shapley value in time linear in the size of the representation.

However, the issue of representing coalitional games is more challenging in the presence of externalities [19, 20]. In game theory, such games are represented using the *partition function game* representation. Here, the value of the coalitions depends on the *coalition structure* in which it is embedded, where a coalition structure is defined as a partition of all the agents into disjoint coalitions. Clearly, the partition function is much more computationally involved than the characteristic function. Furthermore, solution concepts developed for games with no ex-

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[2] See [19] for examples of settings with externalities.

[3] It should be noted that, in games with no externalities, the fourth axiom can be derived from the first three axioms.

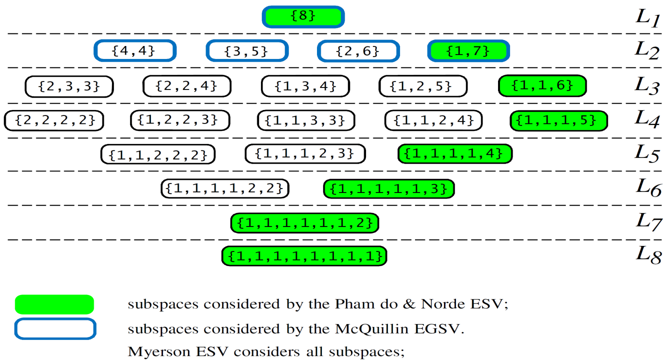


**Figure 1.** Integer partitions for 8 agents

ternalities, such as the Shapley value, have to be redefined (or *extended*) to allow for existence of externalities. To date, there have been a number of such extensions of the Shapley value in the game-theoretic literature, namely, those proposed by Myerson [21], Bolger [3], Potter [24], Maskin [16], Macho-Stadler et al. [15], Pham Do and Norde [23], Fujinaka [10], Hafalir [12], Hu and Yang [13], Albizuri et al. [1], DeClippel and Serrano [7] and McQuillin [17]. The main reason for this multiplicity of concepts is that the fairness axioms introduced by Shapley for games with no externalities are not sufficient to guarantee uniqueness of a division scheme in games with externalities. To attain this uniqueness, some extra axioms have to be introduced and this leads to a number of different extensions depending on the axioms that are being added. Although the Shapley value is one of the key solution concepts in coalition formation, none of its extensions to games with externalities has been considered in the computer science literature with the exception of the recent paper by Michalak et al. [20] which considered only one of those extensions.

Against this background, in this paper:

- **We consider the computational aspects related to three extensions of the Shapley value to games with externalities**; namely, those proposed by Myerson, Pham Do and Norde, and McQuillin.
- To facilitate the efficient computation of the three extensions under consideration, we propose a **novel representation of coalitional games with externalities** that is based on weighted logical expressions. We show that our representation is fully expressive, i.e., it is able to represent any coalitional game with externalities, and is not restricted to any particular subclasses of these games. Furthermore, our representation can be much more concise compared to all available alternatives for games with externalities, namely the conventional *partition function game* representation, and the representations recently introduced by Michalak et al. [19, 20].[4]
- We show that, for all three extensions of the Shapley value, using our representation the **division of payoff is obtained in time linear in the number of logical rules** that model the coalitional game.

[4] In fact, it can be easily shown that the representation in [20], which is also based on logical expression, is only a very special case of our representation.

## 2 Notation and Basic Definitions

We denote by $A = \{a_1, \dots, a_{|A|}\}$ the set of agents participating in the game. A *characteristic function* $v : 2^A \to \mathbb{R}$ assigns to every coalition $C \subseteq A$ a real number representing the quality of its performance. A *characteristic function game* is then a tuple $(A, v)$. However, as common practice in the literature, we will simply denote it by $v$ alone. A *coalition structure*, denoted $\pi$, is a disjoint and exhaustive partition of the agents in $A$. That is, $\pi = \{C : C \subseteq A \wedge \bigcup_{C\in\pi} C = A \wedge \forall C, C' \in \pi, C \cap C' = \emptyset\}$. By $\Pi(A)$ we will denote the space of all coalition structures over $A$. In the spirit of [26], we will divide $\Pi(A)$ into sub-spaces such that every sub-space is represented by a unique integer partition of $|A|$, with the integers corresponding to coalition sizes. Figure 1 shows an example for $|A| = 8$, where all the possible integer partitions are divided into levels $L_1, L_2, \dots, L_8$ such that every level $L_i$ contains the integer partitions of size $i$. The integer partition $[1, 1, 6]$, for example, represents the subspace containing all the coalition structures within which two coalitions are of size 1, and one coalition is of size 6. For every coalition $C' \subseteq A$, we distinguish two coalition structures that contain it, denoted $\{C', singletons\}$ and $\{C', A \setminus C'\}$, as these will play an important role in the paper. Specifically:

(i) $\{C', singletons\} = \{C : C \subseteq A \wedge (C = C' \vee |C| = 1)\}$
(ii) $\{C', A \setminus C\} = \{C : C \subseteq A \wedge (C = C' \vee |C| = |A| - |C'|)\}$

In a coalitional game with externalities, given three disjoint coalitions: $C_1$,$C_2$,$C_3$, and given two coalition structures: $CS$,$CS'$ such that $C_1, C_2, C_3 \in CS$ and $C_1, (C_2 \cup C_3) \in CS'$, the value of $C_1$ may be different in $CS$ than in $CS'$ due to the merge of $C_2$ with $C_3$. Such effects are known as *externalities from coalition formation*. Games with externalities are conventionally modelled using a *partition function* that assigns a real value to any pair $(C, \pi)$ where $\pi \in \Pi(A)$ and $C \in \pi$. We will refer to such pairs as *embedded coalitions*, and the set of them will be denoted $E$. A *game in a partition function form* is then a tuple $(A, w)$. Again, for ease of notation we will denote such a game by $w$ alone. Following [7], we will call $w(C, \{C, singletons\})$ the *externality-free value* of coalition $C$, and denote it $v_{ef}(C)$; this is because $C$ is not subject to any externalities in $\{C, singletons\}$ (see [7] for more details).

Now, let $\omega \in \Omega(A)$ denote a permutation of agents in $A$, and let $C_\omega(i)$ denote the coalition made of all predecessors of agent $a_i$ in $\omega$. More formally, if we denote by $\omega(j)$ the location of $a_j$ in $\omega$, then: $C_\omega(i) = \{a_j \in \omega : \omega(j) < \omega(i)\}$. The Shapley value of $a_i$, denoted $SV_i(v)$, is then defined as the average marginal contribution of $a_i$ to coalition $C_\omega(i)$ over all $\omega \in \Omega$[27]:

$$SV_i(v) = \frac{1}{|A|!} \sum_{\omega\in\Omega} [v(C_\omega(i) \cup \{a_i\}) - v(C_\omega(i))]. \qquad (1)$$

Shapley provides the following intuition behind this formula: imagine that the players are to arrive at a meeting point in a random order, and that every player $a_i$ who arrives receives the marginal contribution that his arrival would bring to those already at the meeting point. Now if we average these contributions over all the possible orders of arrival, we obtain $SV_i(v)$, $a_i$'s payoff in the game. The formula in (1) can also be stated in the equivalent but computationally less involved form as:

$$SV_i(v) = \sum_{C\subseteq A\setminus\{a_i\}} \frac{|C|!(|A| - |C| - 1)!}{|A|!} [v(C \cup \{a_i\}) - v(C)]. \qquad (2)$$

## 3 Extensions of the Shapley Value

In this section we discuss three *extended Shapley values* (ESVs), i.e. extensions of the Shapley value to games with externalities. Each of these is based on a different axiomatic characterization and, thus, accommodates externalities in a different way. To better understand these differences, we show in Figure 1 the coalition structures that play a role in the computation process of every extension. These differences ultimately determine which extension is most suitable for a given application.

**Extension of Pham Do and Norde.** The axiomatic characterization of this extension is similar to that of the standard Shapley value except for the null player and symmetry axioms. Whereas there are many ways of defining these axioms in games with externalities, in their definitions, Pham Do and Norde compute marginal contributions of an agent considering only those coalition structures in which this agent plays as a singleton (see [23] for more details). This uniquely determines the following ESV:

$$ESV_i^{\text{PDN}}(w) = \sum_{C \subseteq A \setminus \{a_i\}} \frac{|C|!(|A|-|C|-1)!}{|A|!}[v_{ef}(C \cup \{a_i\}) - v_{ef}(C)]. \tag{3}$$

In other words, the axioms proposed by Pham Do and Norde lead to an ESV that focuses solely on the *externality-free* value of every coalition $C$, which can be found in the coalition structure of the form $\{C, singletons\}$, i.e., it ignores the other values of $C$ (when $C$ is embedded in other coalition structures) as visible in Figure 1. This means that players are remunerated based on their performance when it is unaffected by externalities.

The procedural account for the $ESV^{PdN}$ is, in principle, similar to that of the conventional Shapley value, except for the fact that the agents who have not arrived yet at the meeting point, are taken into consideration as singletons.

**Extension by McQuillin.** This extension is related to the *problem of generalization* of the conventional Shapley value. This problem involves defining a fair division of the game's payoff under the assumption that the game is played not by agents but by coalitions in a certain, *a priori* known, coalition structure. Whereas the problem of generalization had been considered only in the context of games with no externalities, McQuillin analysed it in the presence of externalities [17]. Specifically, the author showed that the widely accepted solution to the problem of generalization forces a unique solution to the problem of extension, and he called this solution the *Extended, Generalized Shapley Value* ($EGSV$).

In order to formalize this concept, let us define for any set $T \subseteq \pi$ the operator $\lfloor T \rfloor := \bigcup_{C \in T} C$. For instance, if $T = \{\{a_1\}\{a_2\}\}$ then $\lfloor T \rfloor = \{a_1 a_2\}$. Now, given a game $w$ and $(C, \pi) \in E$, the $EGSV$ is defined as [17]:

$$EGSV_{(C,\pi)}(w) = \sum_{C \in T \subseteq \pi} \frac{(|T|-1)!(|\pi|-|T|)!}{|\pi|!} \Big( w_\pi(\lfloor T \rfloor) - w_\pi(\lfloor T \setminus \{C\} \rfloor) \Big) \tag{4}$$

where $w_\pi(C) := w(C, \{C, A \setminus C\})$. In other words, to compute the $EGSV$ of a coalition $C$ embedded in an *a priori* coalition structure $\pi$, a characteristic function game $(\pi, w_\pi)$, or $w_\pi$ for brevity, should be constructed in which the players are the coalitions from $\pi$ and the payoffs are given by: $w_\pi(T) := w\Big(\lfloor T \rfloor, \big\{\lfloor T \rfloor, \lfloor \pi \setminus T \rfloor\big\}\Big)$ for all $T \subseteq \pi$. Now, by computing the conventional Shapley value of player $C \in \pi$ in the game $w_\pi$, we obtain $EGSV_{(C,\pi)}(w)$.

McQuillin shows that apart from efficiency, symmetry, null-player, and linearity, the $EGSV$ meets also weak monotonicity, the rule of generalisation, strong linearity, cohesion, generalised null-player, and recursion axioms (see [17] for more details). The author proves that the extended values proposed by [7, 15, 21, 23, 24] asymptotically converge to the $EGSV$.

The procedural account for the $EGSV$ is, in principle, similar to that of the conventional Shapley value, but now marginal payoffs are calculated by assuming that agents, who have not arrived yet at the meeting point, form a coalition (which is exactly the opposite to the assumption made in $ESV^{PdN}$).

**Extension of Myerson.** The three axioms that uniquely characterize Myerson's extension are linearity, symmetry, and carrier, extended to games with externalities. This last axiom means that the value of the grand coalition should be divided only among the members of the carrier that is defined as:

**Definition 3.1**
*Let $(A, w)$ be a game with externalities. The coalition $C \subseteq A$ is called a carrier of $w$ if, for any embedded coalition $(C', \pi) \in E$, it holds that $w(C', \pi) = w(C \cap C', \pi \wedge \{C, A \setminus C\})$.*

This extended carrier axiom implies both the efficiency and the dummy-player concepts much stronger than in the original Shapley value as well as in $ESV^{PdN}$ and $EGSV$ (see [15, 17] for more details on this issue). Myerson's extension is then:

$$ESV_i^M(w) = \sum_{(C,\pi) \in E} (-1)^{|\pi|-1}(|\pi|-1)! \times \tag{5}$$
$$\times \Big( \frac{1}{|A|} - \sum_{\substack{\widetilde{C} \in \pi : \widetilde{C} \neq C \\ a_i \notin \widetilde{C}}} \frac{1}{(|\pi|-1)(|A|-|\widetilde{C}|)} \Big) w(C, \pi)$$

## 4 Weighted MC-Nets

We call our representation for games with externalities *weighted MC-nets* as we derive our inspiration from both:

- the various weighted-formula representations that have been used to represent preferences and valuation functions in other areas of AI [28]; and
- MC-nets — the representation of coalitional games with no externalities proposed by Ieong and Shoham [14].

Although there are a number of alternative representations of games with no externalities in the literature [2, 5, 6, 8, 14, 22], we find MC-nets to be the most suitable starting point of our computational analysis of various extensions of Shapley value for games with externalities. Specifically, in addition to being fully expressive and, for many games, concise, this representation also facilitates a very efficient way of computing the conventional Shapley value. In MC-nets, a game with no externalities is represented with a set of *simple* rules $\mathcal{R}$, where each rule is of the form $(\mathcal{B}, v)$, with $\mathcal{B}$ being a Boolean expression over $A$, and $v \in \mathbb{R}$.[5] Such a rule is interpreted as follows: the value of any coalition $C$ is increased by $v$ if that coalition *meets* the expression $\mathcal{B}$, i.e., if $\mathcal{B}$ evaluates to `true` when every Boolean variable corresponding to an agent in $C$ is set to `true`, and every one corresponding to an agent in $A \setminus C$ is set to `false`. In this case, we write $C \models \mathcal{B}$. Similarly to [14], our computational results are derived for the special case where all expressions are conjunctions of literals. In this context, within any expression,

[5] In [14], the notation $\mathcal{B} \longrightarrow v$ is used instead of $(\mathcal{B}; v)$.

an agent will be called a *negative literal* if it is preceded with the sign "$\neg$", and will be called a *positive literal* otherwise.[6] We assume that any expression contains at least one positive literal.

The basic idea behind our *weighted MC-nets* representation is to generalize the aforementioned rules such that Boolean expressions are matched against coalition structures rather than just coalitions. Specifically, in our representation, a rule is of the following form:

$$(\mathcal{B}_1^1; v_1^1) \ldots (\mathcal{B}_{r_1}^1; v_{r_1}^1) | \ldots | (\mathcal{B}_1^s; v_1^s) \ldots (\mathcal{B}_{r_s}^s; v_{r_s}^s) \quad (6)$$

Any rule of this form will be called a *weighted rule*, and will be denoted as $WR \in \mathcal{WR}$. A coalition structure $\pi$ is said to *meet* such a rule if $\pi$ can be divided into disjoint non-empty sets of coalitions $\pi_1, \ldots, \pi_s$ such that $\pi_1 \cup \cdots \cup \pi_s = \pi$ and every expression $\mathcal{B}_k^l$ for a given $l \in \{1, \ldots, s\}$ and $k \in \{1, \ldots, r_l\}$ is met by at least one coalition in $\pi_l$. This, as well as the assumption that every expression $\mathcal{B}_k^l$ contains at least one positive literal, imply that $\mathcal{B}_k^l$ does not meet any coalition in $\pi_{l' \neq l}$. In this case, we write $\pi \models \mathcal{WR}$. A coalitional game with externalities is then represented as a tuple $(A, \mathcal{WR})$, where the value of any embedded coalition $(C, \pi)$ is computed as follows:

$$w(C, \pi) = \sum_{WR \in \mathcal{WR} : \pi \models WR} \ \sum_{(\mathcal{B}_k^l, v_k^l) \in WR : C \models \mathcal{B}_k^l} v_k^l$$

When convenient, an expression $\mathcal{B}_k^l$ of the form in (6) will be denoted as a conjunction of positive and negative literals. For example, the rule $(a_1 \wedge a_2; 15)(a_3 \wedge \neg a_5; 10)|(a_7; 20)$ will be denoted as $(p_1^1; 15)(p_2^1 \wedge \neg n_2^1; 10)|(p_1^2; 20)$, where $p_1^1 = a_1 \wedge a_2$, $p_2^1 = a_3$, $\neg n_2^1 = \neg a_5$, and $p_1^2 = a_7$. Furthermore, we will denote by $P_k^l$ and $N_k^l$ the sets containing the agents in $p_k^l$ and $\neg n_k^l$, respectively. For example, in the above rule we have $P_1^1 = \{a_1, a_2\}$, $P_2^1 = \{a_3\}$, $N_2^1 = \{a_5\}$, and $P_1^2 = \{a_7\}$.

Having introduced our representation, we will now evaluate its properties, starting with expressiveness:

**Proposition 4.1 (Expressiveness)**
*Every coalitional game with externalities that is represented with a partition function can be expressed using weighted MC-nets.*

Proof: To prove this, it suffices to note that given a coalition structure $\pi = \{C_1, \ldots, C_{|\pi|}\}$ we can define a "canonical" weighted rule $WR$ such that $\forall (C, \pi) \in E, \pi' \models WR$ iff $\pi' = \pi$. This rule is $(\mathcal{B}_1; w(C_1, \pi))| \ldots |(\mathcal{B}_{|A|}; w(C_{|\pi|}, \pi))$, where the expressions $\mathcal{B}_1, \ldots, \mathcal{B}_{|\pi|}$ are composed of positive literals corresponding to the agents in $C_1, \ldots, C_{|\pi|}$ respectively. □

**Corollary 4.2 (Conciseness)**
*Weighted MC-nets are at least as concise as the partition function game representation.*

**Proposition 4.3 (Conciseness w.r.t. certain games)**
*Weighted MC-nets are exponentially more concise than the partition function game representation for certain games.*

Sketch of Proof: This follows from the well-known result in Boolean algebra that, with a set of Boolean formulas, one can sometimes express the information in an exponentially more concise manner compared to the extensive representation such as the partition function. □

Finally, it is easy to show that weighted MC-nets are at least as concise and sometimes much more concise than the representations introduced by Michalak et al. in [19, 20].

[6] For convenience, with a slight abuse of conventional notation, by $\neg n$, where, for instance, $n = a_1 \wedge a_2$, we mean $\neg a_1 \wedge \neg a_2$.

## 5 Computing ESVs with weighted MC-nets

The key role in our algorithms is played by the additivity axiom, which is met by all three ESVs and allows for computing these values by considering every $WR \in \mathcal{WR}$ as a separate sub-game.[7]

**Lemma 5.1**
*Let $w$ be the game represented by $(A, \mathcal{WR})$, and let $w_z$ be the game represented by $(A, \{WR_z\})$, where $WR_z \in \mathcal{WR}$. This means $w_z$ is represented with a single weighted rule. The $ESV^{PdN}$, $ESV^M$ and $EGSV$ for $w$ are equal to the sum of the $ESV^{PdN}$s, $ESV^M$s and $EGSV$s, respectively, computed for every $w_z : WR_z \in \mathcal{WR}$.*

We will say that a set of expressions $\{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$ is *compatible*, which we denote as $\oplus\{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$, if there exists at least one coalition that meets all these expressions. Formally, $\oplus\{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$ if $\exists C \subseteq A : C \models \mathcal{B}_k \forall k \in \{1, \ldots, m\}$. This happens if $\{\bigcup_{k=1}^m P^k\} \cap \{\bigcup_{k=1}^m N^k\} = \emptyset$. We will denote a set of *incompatible* expressions as $\ominus\{\mathcal{B}_1, \ldots, \mathcal{B}_m\}$. In what follows we assume that every weighted rule $WR \in \mathcal{WR}$ that is not correctly defined, i.e. $\forall \pi \in \Pi(A), \pi \not\models WR$, is omitted from $\mathcal{WR}$ (as it does not influence the game at all).

### 5.1 Computing $ESV^{PdN}$

Since $ESV^{PdN}$ is computed in a similar way to the standard Shapley value but using the externality-free values, only the rules influencing these values (i.e., influencing the value of a coalition $C$ in $\{C, singletons\}$) should be taken into account. An algorithm is, therefore, needed to (i) identify those weighted rules and (ii) transform each of them into the corresponding simple rule(s). This transformation has to be done very carefully in order to preserve other conditions affecting $C$ that are present in each weighted rule! For example, the weighted rule $(a_1 \wedge a_2, 5)(a_4 \wedge \neg a_2, 6)|(a_3, 7)$ is met by some coalition structures of the form $\{C, singletons\}$, where $C$ contains $a_1$ and $a_2$. However, this only happens under the condition that both $a_3$ and $a_4$ do not belong to $C$.

**Theorem 5.2**
*Let $(A, \mathcal{WR})$ represent $w$. Algorithm 1 transforms $\mathcal{WR}$ into a corresponding set of simple rules $\mathcal{R}$ (from which $ESV_i^{PdN}(w)$ can be computed for all $a_i \in A$ in time linear in $|\mathcal{R}|$ as shown in [14]). Furthermore, for each $WR \in \mathcal{WR}$ the running time of Algorithm 1 is $O(|WR|^2|)$. Therefore, it is linear in the size of the representation $|\mathcal{WR}|$. Finally, it holds that $|\mathcal{R}| \leq |A| \times |\mathcal{WR}|$.*

Proof: We denote by $\mathcal{B}^*$ an interim expression which we use in the process of building a simple rule. For every $WR \in \mathcal{WR}$:

(i) If for more than one $l$ there exist $\mathcal{B}_k^l : |P_k^l| > 1$ then $WR$ cannot be met by any coalition structure of the form $\{C, singletons\}$; thus, $WR$ is disregarded;

(ii) If for exactly one $l$ there exists $\mathcal{B}_k^l : |P_k^l| > 1$ then we need to ensure that, for this $l$, all expressions $\mathcal{B}_k^l : |P_k^l| > 1$ are compatible as they have to be met by the same coalition $C$. Thus:

  (a) If $\ominus\{\mathcal{B}_k^l : |P_k^l| > 1\}$ then $WR$ is disregarded;

[7] For an elaboration on this argument for simple MC-nets see [14].

**Algorithm 1 :** $f(\mathcal{WR})$

{**Note:** Given $WR$ as in (6), unless stated differently, we assume that $l \in \{1, \ldots, s\}$ and, given $l$, $k \in \{1, \ldots, r_l\}$}

1: $\mathcal{R} \leftarrow \emptyset$;
2: **for** $WR \in \mathcal{WR}$ **do**
3: **if** $\exists l : (|P_k^l| > 1) \wedge (\forall l' \neq l, |P_k^{l'}| = 1) \wedge (\oplus\{\mathcal{B}_k^l : |P_k^l| > 1\})$ **then**
4: $\mathcal{B}^* \leftarrow \bigwedge_{k:|P_k^l|>1} B_k^l$;
5: **for** $(l' \in \{1, \ldots, s\} \setminus \{l\}) \wedge (k' = \{1, \ldots, r_{l'}\})$ **do**
6: $\mathcal{B}^* \leftarrow \mathcal{B}^* \wedge \neg p_{k'}^{l'}$;
7: **for** $k' \in \{1, \ldots, r_l\} : |P_{k'}^l| = 1$ **do**
8: **if** $\ominus\{\mathcal{B}^*, \mathcal{B}_{k'}^l\}$ **then**
9: $\mathcal{B}^* \leftarrow \mathcal{B}^* \wedge \neg p_{k'}^l$;
10: $\mathcal{R} \leftarrow \{\mathcal{R}, (\mathcal{B}^*; \sum_{k:|P_k^l|>1} v_k^l)\}$;
11: **for** $k' \in \{1, \ldots, r_l\} : |P_{k'}^l| = 1$ **do**
12: **if** $\oplus\{\mathcal{B}^*, \mathcal{B}_{k'}^l\}$ **then**
13: $\mathcal{R} \leftarrow \{\mathcal{R}, (\mathcal{B}^* \wedge \mathcal{B}_{k'}^l; v_{k'}^l)\}$;
14: **if** $\forall l, |P_k^l| = 1$ **then**
15: **for** $(l \in \{1, \ldots, s\}) \wedge (k \in \{1, \ldots, r_l\})$ **do**
16: $\mathcal{B}^* \leftarrow B_k^l$;
17: **for** $(l' \in \{1, \ldots, s\} \setminus \{l\}) \wedge (k' = \{1, \ldots, r_{l'}\})$ **do**
18: $\mathcal{B}^* \leftarrow \mathcal{B}^* \wedge \neg p_{k'}^{l'}$;
19: **for** $k' \in \{1, \ldots, r_l\} \setminus \{k\}$ **do**
20: **if** $\ominus\{\mathcal{B}^*, \mathcal{B}_{k'}^l\}$ **then**
21: $\mathcal{B}^* \leftarrow \mathcal{B}^* \wedge \neg p_{k'}^l$;
22: $\mathcal{R} \leftarrow \{\mathcal{R}, (\mathcal{B}^*; v_k^l)\}$;

(b) Otherwise, $\exists C \subseteq A : (\{C, singletons\} \models WR) \wedge (C \models \mathcal{B}^*)$, where $\mathcal{B}^* = \bigwedge_{k:|P_k^l|>1} \mathcal{B}_k^l)$. Now, what is left is to ensure that the other conditions in $WR$ that affect $C$ are preserved. As for $\mathcal{B}_{k'}^{l'} : l' \neq l$, where $l' \in \{1, \ldots, s\} \setminus \{l\}$ and $k' \in \{1, \ldots, r_{l'}\}$, the only conditions that these expressions place on $C$ is that $C \cap P_{k'}^{l'} = \emptyset$. Thus, $\mathcal{B}^* \leftarrow \mathcal{B}^* \wedge \neg p_{k'}^{l'}$. As for $\mathcal{B}_{k'}^l : k' \in \{1, \ldots, r_l\} \wedge |P_{k'}^l| = 1$, whenever $\ominus\{\mathcal{B}^*, \mathcal{B}_{k'}^l\}$ it places a condition on $C$ that $C \not\models \mathcal{B}_{k'}^l$ since $C \models \mathcal{B}^*$. At this point, we have incorporated in $\mathcal{B}^*$ all the conditions necessary for $C$ to meet; thus, $(\mathcal{B}^*, \sum_{k:|P_k^l|>1} v_k^l)$ becomes our first simple rule. However, if there exist $\mathcal{B}_{k'}^l$ such that $\oplus\{\mathcal{B}^*, \mathcal{B}_{k'}^l\}$ then they contribute to the value of $C$ if $C \models \mathcal{B}^* \wedge \mathcal{B}_{k'}^l$ (see Steps 15-16).

(iii) Otherwise, every $B_k^l$ such that $l \in \{1, \ldots, s\}$ and $k \in \{1, \ldots, r_l\}$ has exactly one positive literal. While in (ii,b) we focused on $\mathcal{B}_k^l : |P_k^l| > 1$, here the focus is on every $B_k^l$ (as coalition $C$ in $\{C, singletons\}$ can, in principle, meet any of them). Similar reasoning applies.

It is clear that in both cases $(ii)$ and $(iii)$ the maximum number of simple rules that can be created from a single $WR \in \mathcal{WR}$ is $|A|$; thus, $|\mathcal{R}| \leq |A| \times |\mathcal{WR}|$. The running time comes from the fact that every $WR$ contains at most $|A|$ expressions. □

## 5.2 Computing EGSV

In contrast to the $ESV^{PdN}$, the $EGSV$ is computed for an *a priori* coalition $\pi = \{C_1, \ldots, C_{|\pi|}\}$ using coalition structures that are of the form $\{C, A \setminus C\}$ and, at the same time, satisfy $\exists \pi_1, \pi_2 : (\pi_1 \cup \pi_2 = \pi) \wedge (\cup \pi_1 = C) \wedge (\cup \pi_2 = A \setminus C)$. This latter condition identifies those $\{C, A \setminus C\}$ that are relevant to the game $w_\pi$. Recall that in this game there are no externalities and every coalition in $\pi$ is considered to be a single player. We denote these players as $A_\pi = \{a_{C_1}, \ldots, a_{C_{|\pi|}}\}$. The following theorem holds:

**Theorem 5.3**
*Let $(A, \mathcal{WR})$ represent $w$, and assume that either $s > 1$ or $(s = 1) \wedge (r_s = 1)$ for every $WR \in \mathcal{WR}$. For a given a priori coalition structure $\pi$, Algorithm 2 transforms $\mathcal{WR}$ into a corresponding set of simple rules $\mathcal{R}$ (from which the $EGSV_{(C,\pi)}(w)$ can be computed in time linear in $|\mathcal{R}|$ as shown in [14]). Furthermore, for each $WR \in \mathcal{WR}$ the running time of Algorithm 2 is $O(max(|WR| + |\pi|))$. Therefore, it is linear in the size of the representation $|\mathcal{WR}|$. Finally, it holds that $|\mathcal{R}| \leq 2 \times |\mathcal{WR}|$.*

**Algorithm 2 :** $f(\mathcal{WR}, \pi)$

1: **for** $WR \in \mathcal{WR}$ **do**
2: $R \leftarrow \emptyset$;
3: **if** $(s = 2) \wedge (\oplus\{\mathcal{B}_k^1 : k \in \{1, ..., r_1\}\}) \wedge (\oplus\{\mathcal{B}_k^2 : k \in \{1, ..., r_2\}\}) \wedge ((\bigcup_{k=1}^{r_1} N_k^1) \cap (\bigcup_{k=1}^{r_2} N_k^2))$ **then**
4: $\Gamma_1 \leftarrow \bigcup_{k=1}^{r_1} P_k^1 \cup \bigcup_{k=1}^{r_2} N_k^2$;
5: $\Gamma_2 \leftarrow \bigcup_{k=1}^{r_1} N_k^1 \cup \bigcup_{k=1}^{r_2} P_k^2$;
6: **if** $\forall a_C \in A_\pi : (a_C \cap \Gamma_1 \neq \emptyset \implies a_C \cap \Gamma_2 = \emptyset) \wedge (a_C \cap \Gamma_2 \neq \emptyset \implies a_C \cap \Gamma_1 = \emptyset)$ **then**
7: $\forall m \in \{1, 2\}, \mathbf{p}^m \leftarrow a_C \in A_\pi : C \cap \{\bigcup_{k=1}^{r_m} P_k^m\} \neq \emptyset$;
8: $\forall m \in \{1, 2\}, \mathbf{n}^m \leftarrow a_C \in A_\pi : C \cap \{\bigcup_{k=1}^{r_m} N_k^m\} \neq \emptyset$;
9: $\mathcal{R} \leftarrow \{\mathcal{R}, (\bigwedge_{k=1}^{r_2} \mathcal{B}_k^1 \wedge \mathbf{n}^2 \wedge \neg \mathbf{p}^2; \sum_{k=1}^{r_1} v_k^1)$ ;
10: $\mathcal{R} \leftarrow \{\mathcal{R}, (\bigwedge_{k=1}^{r_2} \mathcal{B}_k^2 \wedge \mathbf{n}^1 \wedge \neg \mathbf{p}^1; \sum_{k=1}^{r_2} v_k^2)$ ;
11: **if** $(s = 1) \wedge (r_s = 1) \wedge \forall a_C \in A_\pi : (a_C \cap P_1^1 \neq \emptyset \implies a_C \cap N_1^1 = \emptyset) \wedge (a_C \cap N_1^1 \neq \emptyset \implies a_C \cap P_1^1 = \emptyset)$ **then**
12: $\mathcal{R} \leftarrow \{\mathcal{R}, (\mathcal{B}_1^1, v_1^1)$;

Proof: As we are only interested in any coalition structure of the form $\{C, A \setminus C\}$, all rules for which $s > 2$ should be disregarded. Furthermore, for any $WR \in \mathcal{WR}$ such that $s = 2$, it may still happen that $WR$ is not met by any such coalition structure. To ensure that $\exists C \subseteq A : \{C, singletons\} \models WR$, it is sufficient to ensure that the following two conditions are satisfied: (i) expressions in $\{\mathcal{B}_k^1 : k \in \{1, ..., r_1\}\}$ have to be compatible, as well as those in $\{\mathcal{B}_k^2 : k \in \{1, ..., r_2\}\}$; and (ii) $((\bigcup_{k=1}^{r_1} N_k^1) \cap (\bigcup_{k=1}^{r_2} N_k^2))$. If so then all that is left to check is whether agents in $\Gamma_1$ and $\Gamma_2$ (defined as in Steps 4 and 5) can be replaced with players in $A_\pi$. This last issue is important as we are not interested in all structures $\{C, A \setminus C\}$ but only in those which determine the values of the game $w_\pi$ with players $A_\pi$. Finally, Steps 13-14 cover the special case where the weighted rule is actually a simple rule. Finally, it is easy to see from the algorithm that $|\mathcal{R}| \leq 2 \times |\mathcal{WR}|$ and $O(max(|WR| + |\pi|))$. □

## 5.3 Computation of $ESV^M$

The starting point of our algorithm is the following property of $ESV^M$ [21]:

**Proposition 5.4**
*Let $w_{(C,\pi)}$ be the partition form game such that $w_{(C,\pi)}(\widetilde{C}, \widetilde{\pi}) = 1$ if and only if $(\widetilde{C}, \widetilde{\pi}) \geq (C, \pi)$, otherwise $w_{(C,\pi)}(\widetilde{C}, \widetilde{\pi}) = 0$.[8] Then, $ESV_i^M(w) = \frac{1}{|C|}$ for $i \in C$ and 0 otherwise.*

Let us define the function $f : \mathbb{Z}^2 \to \mathbb{R}$ by:

$$f(a, b) = \sum_{0 \leq j \leq b} (-1)^j \binom{b}{j} \frac{1}{a + j}.$$

**Theorem 5.5**
*Let $(A, \mathcal{WR})$ represent $w$, and assume that $(s = 1) \wedge (r_s = 1)$ for every $WR \in \mathcal{WR}$. Algorithm 3 computes the $ESV_i^M$ for all $a_i \in A$. Furthermore, for each $WR \in \mathcal{WR}$, the running time of Algorithm 3 is $O(|A| \times |WR|)$. Therefore, it is linear in $|\mathcal{WR}|$.*

Proof: Following Lemma 5.1, we focus on the case when $|\mathcal{WR}| = 1$ and rewrite the weighted rule as $k = 1, \ldots, r_1$ weighted rules of the form:

$$(\mathcal{B}_1^1; 0) \ldots (\mathcal{B}_{k-1}^1; 0)(\mathcal{B}_k^1; v)(\mathcal{B}_{k+1}^1; 0) \ldots (\mathcal{B}_{r_1}^1; 0) \qquad (7)$$

[8] See Myerson [21, p. 24] for more details.

**Algorithm 3 :** $f(\mathcal{WR})$

$\forall a_i \in A\ ESV_i^M \leftarrow 0;$
**for** $WR \in \mathcal{WR}$ **do**
  **for** $k \in \{1, \dots r_1\}$ **do**
    **for** $a_i \in A$ **do**
      **if** $(a_i \in P_j^1) \wedge \left(\bigcup_{j' \neq j} N_{j'}^1 = \emptyset\right)$ **then**
        $ESV_i^M(w) \leftarrow ESV_i^M(w) + f(|P_j^1|, |N_j^1|)v_j^1;$
      **if** $(a_i \in N_j^1) \wedge \left(\bigcup_{j' \neq j} N_{j'}^1 = \emptyset\right)$ **then**
        $ESV_i^M(w) \leftarrow ESV_i^M(w) - |P_j^1|(|N_j^1|)^{-1} f(|P_j^1|, |P_j^1|)v_j^1;$

Now, the game can be represented by the following rules which consist of only positive literals:

$$(p_1^1 \wedge n_1'^1; 0) \dots (p_k^1 \wedge n_k'^1; (-1)^{|N_1'^1| + \dots + |N_{r_1}'^1|} . v) \dots (p_{r_1}^1 \wedge n_{r_1}'^1; 0) \quad (8)$$

for $N_j'^1 \subseteq N_j^1$ and $j = 1, \dots, r_1$. This follows from the following formula $\sum_{0 \leq l \leq r_1} (-1)^l \binom{r_1}{l} = 0$. According to Proposition 5.4, for each $k$, $ESV_i^M(w)$ is:

$$\sum_{(N_1'^1, \dots, N_{r_1}'^1): N_j'^1 \subseteq N_j^1} (-1)^{|N_1'^1| + \dots + |N_{r_1}'^1|} \cdot \frac{v}{|P_k^1| + |N_k'^1|}$$

$$= \sum_{N_k'^1 \subseteq N_k^1} (-1)^{|N_1'^1|} \cdot \frac{v}{|P_k^1| + |N_k'^1|} \times \sum_{(N_1'^1, \dots, N_{r_1}'^1): N_{j \neq k}'^1 \subseteq N_j^1} (-1)^{|N_{1 \neq k}'^1| + \dots + |N_{r_1 \neq k}'^1|}$$

The sum $\sum_{(N_1'^1, \dots, N_{r_1}'^1): N_{j \neq k}'^1 \subseteq N_j^1}$ has non zero value iff $\forall j \neq k$, $N_j^1 = \emptyset$. Now, let us assume that $N_j^1 = \emptyset$ for $j \neq k$. Then, the sum $\sum_{(N_1'^1, \dots, N_{r_1}'^1): N_{j \neq k}'^1 \subseteq N_j^1} (-1)^{|N_{1 \neq k}'^1| + \dots + |N_{r_1 \neq k}'^1|} = 1$; thus

$$ESV_i^M(w) = \sum_{N_k'^1 \subseteq N_k^1} (-1)^{|N_k'^1|} \cdot \frac{v}{|P_k^1| + |N_k'^1|} = f(|P_k^1|, |N_k^1|).$$

From (8) we have that $ESV_i^M(w) = 0$ for $a_i \notin P_k^1$. The agents in $N_k^1$ are equally valued by $ESV^M$ due to symmetry axiom. Since $A$ is always a carrier, we have that $\sum_{a_i} ESV_i^M(w) = w(A, \{A\})$; thus $ESV_i^M(w) = -\frac{|P_k^1|}{|N_k^1|} f(|P_k^1|, |N_k^1|)$ for $a_i \in N_k^1$. The running time comes from the fact that, in every rule, $|WR|$ expressions have to be checked against every $a_i \in A$. □

## 6 Conclusions

If coalitional games with externalities are modelled using a conventional partition function game representation, the computation of the three different extensions of Shapley value considered in this paper requires an exponential number of operations. To tackle this problem, we propose a representation of coalitional games with externalities based on weighted logic formulas. We demonstrate that it is fully expressive and at least as concise as the conventional partition function game representation. However, it can be exponentially more concise. Finally, we show that all three extensions of the Shapley value considered in this paper can be computed in the time linear in the size of the input (which is the number of weighted rules).

Our work can be extended in several directions. Firstly, the computational aspects of remaining ESVs are still to be analysed. Secondly, more involved Boolean expression can be considered (as it is done in Elkind [9] for basic MC-nets). Finally, it is interesting to consider the properties of the weighted MC-nets with respect to other coalitional games solution concepts such as the core.

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