Generalized Gap Metrics and Robust Stability of Nonlinear Systems

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To Prof. J.R.L. Webb on the occasion of his retirement

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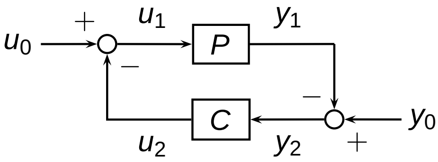
Introduction - system

- Consider the classical closed loop system:

$$[P, C] : y_1 = Pu_1,\ u_2 = Cy_2,\ u_0 = u_1 + u_2,\ y_0 = y_1 + y_2$$

as depicted in Figure 1:



- $u_0 \in \mathcal{U}, y_0 \in \mathcal{Y}$, $\quad$ $\mathcal{U}, \mathcal{Y}$: normed signal spaces,

$P : \mathcal{U}_a \to \mathcal{Y}_a$, $C : \mathcal{Y}_a \to \mathcal{U}_a$, $\quad$ $\mathcal{U}_a, \mathcal{Y}_a$: the ambient spaces,

e.g. $\quad$ if $U = L^2(\mathbb{R}_+, \mathbb{R}^n)$, $\quad$ then $\mathcal{U}_a = \bigcup_{\omega > 0} L^2_{loc}((0, \omega), \mathbb{R}^n)$.

Introduction - closed loop operators

- Closed loop operator:

$$H_{P//C}: \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \left(\begin{pmatrix} u_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \right),$$

assumed causal and well defined:
• well defined on $[0, \infty)$ – globally well posed;
• well defined on $[0, \omega)$ – locally well posed.

- Projections

$$\Pi_{P//C}: \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ y_1 \end{pmatrix}, \quad \Pi_{C//P}: \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} u_2 \\ y_2 \end{pmatrix}.$$

- Graphs

$$\mathrm{graph}(P) = \left\{ \begin{pmatrix} u_1 \\ Pu_1 \end{pmatrix} : u \in \mathcal{U}, Pu_1 \in \mathcal{Y} \right\}.$$

Introduction - stability and robustness

- Let $\|\Pi_{P//C}\| = \sup_{\tau>0, \|w\|_\tau \neq 0} \frac{\|\Pi_{P//C} w\|_\tau}{\|w\|_\tau}$,

  where $\|w\|_\tau = \|T_\tau w\|$ and T_τ is the truncation.
- $[P, C]$ is gain stable if $\|\Pi_{P//C}\| < \infty$ (or $\|\Pi_{C//P}\| < \infty$).
- Robust stability: if $[P, C]$ stable implies $[P_1, C]$ stable for any P_1 close to P.

Introduction - a theorem of Georgiou & Smith (1997)

- Gap metric: $\vec{\delta}_0(P, P_1) = \begin{cases} \inf_{\Phi \in \mathcal{O}} \|I - \Phi\| & \text{if } \mathcal{O} \neq \emptyset, \\ \infty & \text{if } \mathcal{O} = \emptyset \end{cases}$

$$\mathcal{O} = \left\{ \Phi : \mathrm{graph}(P) \to \mathrm{graph}(P_1) \;\; \begin{matrix} \Phi \text{ is surjective, causal,} \\ \text{gain stable and } \Phi 0 = 0 \end{matrix} \right\}.$$

- Under appropriate well posedness assumptions, if $[P, C]$ is gain stable and if

$$\vec{\delta}_0(P, P_1) < b_{P,C} := \|\Pi_{P//C}\|^{-1}$$

then $[P_1, C]$ is gain stable and

$$\|\Pi_{P_1//C}\| \leq \frac{1 + \vec{\delta}_0(P, P_1)}{1 - \vec{\delta}_0(P, P_1)\|\Pi_{P//C}\|} \|\Pi_{P//C}\|.$$

Introduction - motivations

- Requires $\Pi_{P//C}(0) = 0$, i.e. $P(0) = 0, C(0) = 0$;
- Examples in adaptive control show that even $\Pi_{P//C}(0) = 0$ is not sufficient for finite gain, but system has been proved robustly stable;
- Strong causality:

$$T_\tau \Phi T_\tau w = T_\tau \Phi w \text{ for all } \tau > 0, w \in \mathrm{dom}(\Phi)$$

  causing signals must discontinuous, plants memoryless.
- There are gap metrics without Φ between graphs, e.g.:

$$\vec{\delta}_1(P, P_1) = \limsup_{\tau \to \infty} \sup_{y \in \mathrm{graph}(P_1)} \inf_{x \in \mathrm{graph}(P)} \frac{\|y - x\|_\tau}{\|x\|_\tau}.$$

Generalizations - definitions

- Generalized gap metric:

$$\vec{\delta}(P,P_1) = \limsup_{\tau\to\infty} \inf \left\{ k>0 : \begin{array}{c} \text{there exists } \beta \geq 0 \text{ such that} \\ \forall\, y \in \mathrm{graph}(P_1), \exists\, x \in \mathrm{graph}(P) \\ \text{s.t. } \|y-x\|_\tau \leq \alpha \|x\|_\tau + \beta \end{array} \right\}.$$

- Equivalent expression:

$$\vec{\delta}(P,P_1) = \begin{cases} \inf_{\substack{\omega\in\mathcal{U}\times\mathcal{Y} \\ \Phi\in\Theta_\omega}} \|\Phi - I\|_{(\infty)} & \text{if } \Theta_w \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

where

$$\|Q\|_{(\infty)} = \limsup_{\tau\to\infty} \inf \left\{ k>0 : \begin{array}{c} \text{there exists } \beta \geq 0 \text{ s.t.} \forall\, \tau > 0 \\ \|Qu\|_\tau \leq \alpha \|u\|_\tau + \beta,\ \forall\, u \in \mathcal{U}_a \end{array} \right\}.$$

$$\Theta_w = \left\{ \Phi : \begin{array}{l} \Phi : \mathrm{graph}(P) \to \mathrm{graph}(P_1) + w \text{ is set-valued,} \\ \text{surjective and gain stable in } \|\cdot\|_{(\infty)} \text{ sense} \end{array} \right\}.$$

Generalizations - an alternative

- Let $\beta > 0$ be given. Define

$$\vec{\beta}(P, P_1) = \limsup_{\tau \to \infty} \inf \left\{ \alpha > 0 : \begin{array}{l} \forall\, y \in \mathrm{graph}(P_1), \exists\, x \in \mathrm{graph}(P) \\ \text{s.t. } \|y - x\|_\tau \leq \alpha \|x\|_\tau + \beta \end{array} \right\}$$

- Similar equivalence expression involving set-valued mappings between graphs.

Generalization - gain stability.

- An operator $Q : \mathcal{X} \to \mathcal{V}_a$ is said gain stable with bias if $Q(\mathcal{X}) \subset \mathcal{V}$ and $\|Q\|_{(\infty)} < \infty$;
- Q is said β-gain stable with bias if $Q(\mathcal{X}) \subset \mathcal{V}$ and

$$\|Q\|_{(\beta)} =: \limsup_{\tau\to\infty} \inf\{k > 0 : \|Qx\|_\tau \leq k\|x\|_\tau + \beta,\ \forall x \in \mathcal{X}\} < \infty.$$

- Closed loop $[P, C]$ is gain stable (resp. β-gain stable) with bias if $\Pi_{P,C}$ is (or if $\Pi_{C//P}$ is).

Robust stability theorems

- Suppose $[P, C]$ is globally well posed, $[P_1, C]$ is locally well posed. If $[P, C]$ is gain stable with bias and if

$$\vec{\delta}(P, P_1) < \|\Pi_{P//C}\|_{(\infty)}^{-1},$$

then $[P_1, C]$ is gain stable with bias on and

$$\|\Pi_{P_1//C}\|_{(\infty)} \leq \frac{1 + \vec{\delta}(P, P_1)}{1 - \vec{\delta}(P, P_1)\|\Pi_{P//C}\|_{(\infty)}} \|\Pi_{P//C}\|_{(\infty)}.$$

- Similar conclusion holds for β-gain stability: $[P, C]$ is β-gain stable implies $[P_1, C]$ is β_1-gain stable.

$\beta_1 = \beta_1(\beta, \|\Pi_{P//C}\|, \vec{\delta}(P, P_1)).$

$\beta_1 = \beta$ provided $\vec{\delta}(P, P_1) = 0.$

Examples - semilinear systems (1/2)

Consider the system $P(f, x_0)$ given by

$$P(f, x_0): \; u \mapsto y: \; \begin{array}{l} x' = Ax + f(t,x) + Bu, \; x(0) = x_0, \\ y = C_1 x + Dy \end{array}$$

with C_1, B, D bounded linear operators, A a linear operator and f Lipschitz w.r.t. x. If there exists a linear operator F such that $A + BF$ generates a c_0-semigroup s.t.

$$x' = (A + BF)x + f(t,x) + Bu, x(0) = x_0$$

has unique solution for each x_0, then

- $\vec{\delta}(P(f, x_0), P(f, \tilde{x}_0)) = 0$ for any initials $x_0, \tilde{x}_0$,
- $\vec{\delta}(P(0, x_0), P(f, x_0)) = 0$ if f is uniformly bounded.

Examples - semilinear systems (2/2)

- $\vec{\beta}\big(P(f, x_0), P(f, \tilde{x}_0)\big) = 0$, with a suitable $\beta = \beta(x_0, \tilde{x}_0)$
- If a smaller β is chosen, then $\vec{\beta}\big(P(f, x_0), P(f, \tilde{x}_0)\big) \neq 0$.
- Similar conclusion for $\vec{\delta}\big(P(0, x_0), P(f, x_0)\big)$.

Examples - system realization

- Each linear system has a transfer function $G(s)$, more than one systems may have the same transfer function.
- Let $G(s)$ be a transfer function, having two linear realizations P and P_1. If $[P, C]$ is stable, how about $[P_1, C]$?
- If both are stabilizable, then $\vec{\delta}(P, P_1) = 0$.
- Controller can be designed based on either realization.

Examples - time delay systems

- Consider the systems

$$P : u \to x \ \text{ with } \ \dot{x}(t) = f(x(t), u(t)), \ x(0) = x_0$$
$$P_1 : u \to x \ \text{ with } \ \dot{x}(t) = f(x(t), u(t-h)), \ x(0) = x_0.$$

If $|f(x,u)| \leq c + k(|x| + |u|)$, then

$$\vec{\delta}(P, P_1) \leq kh, \quad \vec{\beta}(P, P_1) \leq kh \ \text{ with some } \beta > 0.$$

- A typical example

$$\dot{x}(t) = \mathrm{sat}(u(t)) =: \begin{cases} |u| & \text{if } |u| \leq 1 \\ \mathrm{sign}(u) & \text{otherwise} \end{cases}, \ x(0) = x_0$$

for which $\vec{\delta}(P, P_1) = 0$. $[P_1, -1]$ is gain stable for any $h > 0$. $\vec{\beta}(P, P_1) = 0$ for some $\beta > 0$, and $\vec{\beta}(P, P_1) \to h$ as $\beta \to 0$.

Further generalizations

In $L^2(\mathbb{R}_+)$ setting, consider

$$P(\theta) : u_1 \to y_1 \text{ with } \dot{y}_1 = \theta y_1 + u_1, y_1(0) = 0$$
$$C : u_2 = -ky_2(t) \text{ with } \dot{k}(t) = \frac{\alpha}{(n+1)k^n} y_2^2, k(0) = 0.$$

Then

$$\|\Pi_{P//C} w\| \leq \gamma(r)\|w\| + \beta, \text{ for each } w \in B(0, r),$$

where $\gamma(r)$ is a nonlinear function, tends to linear as n increase.

For this type of problems, we need to introduce the notion of regional gain stability to deal with the nonlinear growth.

Conclusions and comments

- Generalized the robust stability theory of Georgious and Smith, allows stability with bias term;
- Applied to stability of linear system with perturbations, realizations and systems with larger time delay;
- Sacrifice with system bahaviours at transition period.
- Local stability can be given;
- What will happen if the stability is defined via nonlinear growth?
- Tracking problems with nonlinear controllers?

Thank you