Communications in Applied Analysis **xx** (200x), no. N, xxx–xxx

GENERALIZED GAP METRICS AND ROBUST STABILITY OF NONLINEAR SYSTEMS

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To Professor J. R. L. Webb on the occasion of his retirement

ABSTRACT. A gap metric of Georgiou and Smith (IEEE Trans. Auto. Control, 42(9):1200–1229, 1997), which does not need causal and surjective mapping between graphs to define, is studied and generalized based on the notion of biased norm, the corresponding robust stability theorem is presented in the notion of stability with bias terms. The obtained results are then applied to studied the stability of linear system realizations, semilinear systems with bounded nonlinearities and a nonlinear system with time delay in the inputs.

AMS (MOS) Subject Classification. 93D09, 93D25, 93C10.

1. Introduction

A basic property of stable feedback control systems is that they tolerate uncertainties which are sufficiently small in an appropriate sense, and reduce the effects of uncertainties. A significant approach to deal with uncertainties is the gap metric theory, it provides a measure of distance between dynamical systems which are not required to be stable in themselves, small distance between open-loop systems would correspond to small errors in norm in the closed loop.

In the context of nonlinear systems, a fundamental framework developed by Georgiou and Smith in [9] provides a generalization of linear gap metric([8, 16]) and associated robust stability results on the basis of a robust stability margin taken to be the inverse of the induced norm of a closed loop operator. Two types of gap metrics were introduced in [9], the $\vec{\delta}$ metric and the $\vec{\delta}_0$ metric as denoted in the paper. the first one is defined using causal and surjective mappings between graphs of open loop operators, but for the $\vec{\delta}_0$ metric, which is further studied in [11], no such mapping is involved in its definition. Under appropriate well posedness assumptions, the main robust stability theorems state that, given two plants P, P_1 and a controller K, if the closed loop $[P, K]$ is gain stable, i.e. the corresponding closed loop operator has a

Submitted November 3, 2010

finite induced norm, and if the gap (either $\vec{\delta}$ or $\vec{\delta}_0$) between P and P_1 is smaller than the robust stability margin, then the closed loop $[P_1, K]$ is also gain stable and its gain can be estimated in terms of the gap and the gain of the closed loop operator of $[P, K]$. This motivates many further studies on gap metric and its applications, see [1, 2, 5, 6, 10, 11, 12] and references therein.

The framework requires that both the plant and controller map zero inputs to zero outputs (i.e., $P0 = 0, K0 = 0$) and that the closed loop operator has an induced norm. However, there are important instances in which these sufficient conditions for robust stability generically fail; and yet for which robustness results should apply and for which, to date, either relatively ad-hoc methods have been utilized to establish robust stability, or no such such robust stability certificates have been established. Many such systems can be handled by developing a robust stability theory based on an underlying notion of stability which includes bias terms; for such notions of stability see [3, 15]. The second class of systems are those for which $P(0) = 0$, $K(0) = 0$ but whose closed loop operator is discontinuous at 0, thus precluding the existence of a (local) finite gain. Most adaptive controllers fall within this category [4]. A third class of examples includes systems which include inherent offsets, arising e.g. from quantization errors, sensors biases etc. Another such class of feedback systems include nonlinear high gain controller designs which attenuate the effects of unknown nonlinearities by high gain feedback, and which do not cancel the effect of the nonlinearities.

The notion of gain function stability given in [9] might be an alternative approach to those problems, but it is too general and fails to produce a clear description to the stability. So several generalized gap metrics and theorems have been introduced. In [10], based on shift operation, a theorem is given dealing with systems whose response depends on a non-zero initial condition, and which do not start at an equilibrium. In [5], notions of stability and gap matric with bias terms are presented under uniqueness assumption instead of the stronger well-posedness assumption. In [12], stability is defined via a biased norm but the gap metric remains the same as in [9]. All of those generalizations are based on the $\vec{\delta}$ gap metric which needs causal and surjective mappings between graphs. We note that, comparing with the $\vec{\delta}$ gap metric, the $\vec{\delta}_0$ gap metric has two advantages: firstly, it is smaller and therefore theoretically allows a wider range of perturbations; secondly, its definition does not needs causal and surjective mappings between graphs of which the causality is not easy to verify in applications (so an alternative causality is used in [2, 5, 6, 7]).

So, in this paper, we will generalize Georgiou and Smith's $\vec{\delta}$ gap metric and the corresponding robust stability theorem. In our setting, both the gap and the stability will be based on norm with bias, the systems do not need to have zero outputs for zero inputs, nor the system operator needs to be continuous at any point. This

generalization does not require causal mappings between graphs, but can be described using surjective mappings. Using the obtained results, we study the stability of a type of semilinear systems with bounded nonlinearities and linear system realizations. The robust stability of integrator system with saturation is also given to show the advantage of our results.

The paper is organized as follows. Section 2 introduces the basic signal spaces setting and closed loop systems. Generalized gap metrics and robust stability results are presented in Section 3. Stability of semilinear systems and linear system realizations are addressed in Sections 4 and 5, respectively, and in Section 6, we consider the integrator system with saturation.

2. The closed loop system

Let $\mathcal{T}$ denote either the discrete half-axis time set $\mathbb{N}$ or the continuous time counterpart, $\mathbb{R}_+$. In both cases $\mathcal{T} \cup \{\infty\}$ is totally ordered in the natural manner. For $\omega \in \mathcal{T} \cup \{\infty\}$, let $\mathcal{S}_\omega$ denote the set of all locally integrable maps $[0, \omega) \to \mathcal{X}$ where $\mathcal{X}$ is a nonempty set. For ease of notation define $\mathcal{S} := \mathcal{S}_\infty$. For $\tau \in \mathcal{T}$, $\omega \in \mathcal{T} \cup \{\infty\}$, $0 < \tau < \omega$ define a truncation operator T_τ and a restriction operator R_τ as follows:

$$T_\tau : \mathcal{S}_\tau \to \mathcal{S}\,, \quad v \mapsto T_\tau v : (T_\tau v)(t) = \begin{cases} v(t), & t \in [0, \tau) \\ 0, & \text{otherwise} \end{cases}$$

$$R_\tau : \mathcal{S}_\omega \to \mathcal{S}_\tau\,, \quad v \mapsto R_\tau v := (t \mapsto v(t), \quad t \in [0, \tau))\ .$$

Both operators are for considering signals over finite time intervals. The results of this paper will be based on the use of truncation, but remain true if it is replaced by restriction.

We define $\mathcal{V} \subset \mathcal{S}$ to be a *signal space* if, and only if, it is a vector space. Suppose additionally that $\mathcal{V}$ is a normed vector space and that the norm $\|\cdot\| = \|\cdot\|_{\mathcal{V}}$ is (also) defined for signals of the form $T_\tau v$, $v \in \mathcal{V}_\tau$, $\tau > 0$. We can define a norm $\|\cdot\|_\tau$ on $\mathcal{S}_\tau$ by $\|v\|_\tau = \|T_\tau v\|$, for $v \in \mathcal{S}_\tau$. We associate spaces as follows:

- $\mathcal{V}_e = \{v \in \mathcal{S} \,|\, \forall\ \tau > 0\ :\ T_\tau v \in \mathcal{V}\}$, *the extended space*;
- $\mathcal{V}_\omega = \{v \in \mathcal{S}_\omega \,|\, \forall\ \tau \in (0, \omega)\ :\ T_\tau v \in \mathcal{V}\}$, for $0 < \omega \leq \infty$; and
- $\mathcal{V}_a = \bigcup_{\omega \in (0,\infty]} \mathcal{V}_\omega$, *the ambient space.*

For example, in the case when $\mathcal{V} = L^p(\mathbb{R}_+, \mathbb{R}^n)$ with $1 \leq p \leq \infty$, we have $\mathcal{V}_e = L^p_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^n)$ and $\mathcal{V}_a = \bigcup_{\omega \in (0,\infty]} L^p_{\text{loc}}((0, \omega), \mathbb{R}^n)$. So the ambient space $\mathcal{V}_a$ includes signals with finite blow up.

For signal spaces $\mathcal{X}$, $\mathcal{V}$, define the following:

(i) An operator $Q\colon \mathcal{X}_a \to \mathcal{V}_a$ is called *causal* if, and only if,

$$\forall\ x, y \in \mathcal{U}_a,\ \forall\ \tau \in \operatorname{dom}(x) \cap \operatorname{dom}(y)\ :\ \ [T_\tau x = T_\tau y\ \Rightarrow\ T_\tau(Qx) = T_\tau(Qy)]\,. \quad (2.1)$$

(ii) An operator $Q\colon \mathcal{X}_a \to \mathcal{V}_a$ is called *stabilizable* if for all $x \in \mathcal{X}_a, v \in \mathcal{V}_a$ satisfying $Qx = v$ over $\mathrm{dom}(x) \cap \mathrm{dom}(v)$ and for all $\tau \in \mathrm{dom}(x) \cap \mathrm{dom}(v)$, there exists $\tilde{x} \in \mathcal{X}, \tilde{v} \in \mathcal{V}$ such that $Q\tilde{x} = \tilde{v}$ and $T_\tau(x, v)^\top = T_\tau(\tilde{x}, \tilde{v})^\top$.

(iii) A causal operator $Q\colon \mathcal{X}_a \to \mathcal{V}_a$ is called *gain stable* if $Q(\mathcal{X}) \subset \mathcal{V}$, $Q(0) = 0$ and

$$\|Q\| := \sup \left\{ \frac{\|T_\tau Qx\|_\tau}{\|T_\tau x\|_\tau} \ : \ x \in \mathcal{X}\,,\ \tau > 0\,,\ T_\tau x \neq 0 \right\} < \infty\,.$$

(iv) A causal operator $Q\colon \mathcal{X}_a \to \mathcal{V}_a$ is called (γ, β)-*gain stable*, with $\gamma, \beta \geq 0$, if $Q(\mathcal{X}) \subset \mathcal{V}$ and

$$\|T_\tau Qx\|_\tau \leq \gamma \|T_\tau x\|_\tau + \beta, \qquad \forall x \in \mathcal{X}\,,\ \tau > 0.$$

(v) A causal operator $Q\colon \mathcal{X}_a \to \mathcal{V}_a$ is called γ-*gain stable with bias* if, and only if, there exists $\beta \geq 0$ such that Q is (γ, β)-gain stable with bias.

We now consider the closed loop system

$$[P, K] \ : \quad y_1 = Pu_1,\ u_2 = Ky_2,\ u_0 = u_1 + u_2,\ y_0 = y_1 + y_2, \tag{2.2}$$

as depicted in Figure 1, where $P : \mathcal{U}_a \to \mathcal{Y}_a$ and $K : \mathcal{Y}_a \to \mathcal{U}_a$ are causal mappings

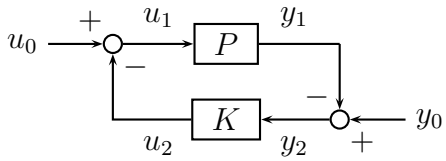

FIGURE 1. The closed-loop $[P, K]$.

representing the plant and the controller, respectively, and $\mathcal{U}, \mathcal{Y}$ are given two normed signal spaces. We write $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$ and define the norm in the product space $\mathcal{W}$ as

$$\|(u, y)^T\| = \max\{\|u\|, \|y\|\},\ \forall (u, y)^T \in \mathcal{W}.$$

For $w_0 = (u_0, y_0)^T \in \mathcal{W}$, a pair $(w_1, w_2) = ((u_1, y_1)^T, (u_2, y_2)^T) \in \mathcal{W}_a \times \mathcal{W}_a$, $\mathcal{W}_a := \mathcal{U}_a \times \mathcal{Y}_a$, is a solution to (2.2) if, and only if, (2.2) holds on $\mathrm{dom}(w_1, w_2) := \mathrm{dom}(w_1) \cap \mathrm{dom}(w_2)$. Let

$$\mathcal{X}_{w_0} := \{(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a \mid (w_1, w_2) \text{is a solution to (2.2)}\}$$

be the set of all solutions, which may be empty. The closed loop system $[P, K]$ is said to have the *existence property*, if $\mathcal{X}_{w_0} \neq \emptyset$ for all $w_0 \in \mathcal{W}$, and the *uniqueness property*, if

$$\forall\ w_0 \in \mathcal{W} \ : \ (\hat{w}_1, \hat{w}_2), (\tilde{w}_1, \tilde{w}_2) \in \mathcal{X}_{w_0}$$
$$\Longrightarrow\ (\hat{w}_1, \hat{w}_2) = (\tilde{w}_1, \tilde{w}_2) \quad \text{on} \quad \mathrm{dom}(\hat{w}_1, \hat{w}_2) \cap \mathrm{dom}(\tilde{w}_1, \tilde{w}_2)\,.$$

For each $w_0 \in \mathcal{W}$, define $\omega_{w_0} \in \mathcal{T} \cup \{\infty\}$ by the property

$$[0, \omega_{w_0}) := \bigcup_{(\hat{w}_1, \hat{w}_2) \in \mathcal{X}_{w_0}} \operatorname{dom}(\hat{w}_1, \hat{w}_2)$$

and define $(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a$, with $\operatorname{dom}(w_1, w_2) = [0, \omega_{w_0})$, by the property $R_t(w_1, w_2) \in \mathcal{X}_{w_0}$ for all $t \in [0, \omega_{w_0})$. This induces the operator

$$H_{P,K} : \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a, \quad w_0 \mapsto (w_1, w_2)$$

and the projection operators

$$\Pi_{P//K} \colon \mathcal{W} \to \mathcal{W}_a, \quad w_0 \mapsto w_1, \qquad \text{and} \qquad \Pi_{K//P} \colon \mathcal{W} \to \mathcal{W}_a, \quad w_0 \mapsto w_2.$$

Clearly,

$$H_{P,K} = \left(\Pi_{P//K}, \Pi_{K//P}\right) \quad \text{and} \quad \Pi_{P//K} + \Pi_{K//P} = I. \tag{2.3}$$

The *graph*s of the plant P and the controller K, denoted by $\mathcal{G}_P$ and $\mathcal{G}_K$ (or $\operatorname{graph}(P)$ and $\operatorname{graph}(K)$) respectively, are defined as follows:

$$\mathcal{G}_P = \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} \;\middle|\; u \in \mathcal{U},\, Pu \in \mathcal{Y} \right\}, \quad \mathcal{G}_K = \left\{ \begin{pmatrix} Ky \\ y \end{pmatrix} \;\middle|\; Ky \in \mathcal{U},\, y \in \mathcal{Y} \right\}.$$

The closed loop system $[P, K]$ given by (2.2), is said to be:

(i) *locally well posed* if, and only if, it has the existence and uniqueness properties and the operator $H_{P,K} : \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a,\ w_0 \mapsto (w_1, w_2)$, is causal;

(ii) *globally well posed* if, and only if, it is locally well posed and $H_{P,K}(\mathcal{W}) \subset \mathcal{W}_e \times \mathcal{W}_e$;

(iii) *regularly well posed* if it is locally well posed and for all $w_0 \in \mathcal{W}$ with $\omega_{w_0} < \infty$, we have

$$\|H_{P,K} w_0\|_\sigma \to \infty \text{ as } \sigma \to \omega_{w_0}.$$

(iv) BIBO *stable* if, and only if, it is globally well posed and $H_{P,K}(\mathcal{W}) \subset \mathcal{W} \times \mathcal{W}$.

(v) *gain stable,* (γ, β)*-gain stable* or γ*-gain stable with bias* if, and only if, $\Pi_{P//K}$ is gain stable, (γ, β)-gain stable or γ-gain stable with bias respectively.

We remark that in the above definitions, the operator $H_{P//K}$ can be replaced by $\Pi_{P//K}$ or $\Pi_{K//P}$ due to the relations between the three operators given in (2.3).

3. Robustness

In this section, we deal with robustness of globally gain stability with bias of feedback systems in the sense that if, for given nominal plant P and controller K, $\Pi_{P//K}$ is (γ, β)-gain or γ-gain stable with bias, then $\Pi_{P_1//K}$ is also (γ, β)-gain or γ-gain stable with bias for a suitable perturbation P_1. The allowed perturbations are measured by distances which are generalizations of a gap metric given in [9]. It is

proved that if the distance is smaller than the inverse of the gain γ, then the feedback stability is preserved.

Let $\mathcal{U}, \mathcal{Y}$ be two signal spaces and let $P, P_1 : \mathcal{U}_a \to \mathcal{Y}_a$ be the input-to-output operators of two control systems. Given a number $\beta \geq 0$, we define the following four gap metrics from P to P_1:

$$\vec{\delta}_\beta(P, P_1) = \limsup_{\tau\to\infty} \inf \left\{ \alpha > 0 : \begin{array}{l} \text{for any } y \in \mathcal{G}_{P_1}, \text{ there exists } x \in \mathcal{G}_P \\ \text{such that } \|y - x\|_\tau \leq \alpha\|x\|_\tau + \beta \end{array} \right\};$$

$$\vec{\delta}(P, P_1) = \limsup_{\tau\to\infty} \inf \left\{ \alpha > 0 : \begin{array}{c} \text{there exists } \beta \geq 0 \text{ such that, for any} \\ y \in \mathcal{G}_{P_1}, \text{ there exists } x \in \mathcal{G}_P \text{ satisfying} \\ \|y - x\|_\tau \leq \alpha\|x\|_\tau + \beta \end{array} \right\};$$

$$\vec{d}_\beta(P, P_1) = \sup_{\tau>0} \inf \left\{ \alpha > 0 : \begin{array}{l} \text{for any } y \in \mathcal{G}_{P_1}, \text{ there exists } x \in \mathcal{G}_P \\ \text{such that } \|y - x\|_\tau \leq \alpha\|x\|_\tau + \beta \end{array} \right\};$$

$$\vec{d}(P, P_1) = \sup_{\tau>0} \inf \left\{ \alpha > 0 : \begin{array}{c} \text{there exists } \beta \geq 0 \text{ such that, for any} \\ y \in \mathcal{G}_{P_1}, \text{ there exists } x \in \mathcal{G}_P \text{ satisfying} \\ \|y - x\|_\tau \leq \alpha\|x\|_\tau + \beta \end{array} \right\}.$$

Clearly, $\vec{\delta}(P, P_1) \leq \vec{\delta}_\beta(P, P_1)$, $\vec{d}(P, P_1) \leq \vec{d}_\beta(P, P_1)$, $\vec{\delta}(P, P_1) \leq \vec{d}(P, P_1)$ and $\vec{\delta}_\beta(P, P_1) \leq \vec{d}_\beta(P, P_1)$ for any $\beta > 0$. In the case when $\mathcal{G}_P = \mathcal{G}_{P_1}$, we have $\vec{\delta}(P, P_1) = \vec{\delta}_\beta(P, P_1) = \vec{d}(P, P_1) = \vec{d}_\beta(P, P_1) = 0$.

It is noted that in the case when $\beta = 0$, we have

$$\vec{\delta}_0(P, P_1) = \limsup_{\tau\to\infty} \sup_{\substack{y\in\mathcal{G}_{P_1} \\ \|y\|_\tau\neq 0}} \inf_{\substack{x\in\mathcal{G}_P \\ \|x\|_\tau\neq 0}} \frac{\|y - x\|_\tau}{\|x\|_\tau},$$

which is exactly the alternative gap metric studied in [9] (where the setting is slightly different but won't affect the results) and its generalizations in [11]. So the two δ-gap metrics above and the corresponding robust stability theorems below are direct generalizations of those in [9, 11].

To describe the gap metrics alternatively, we denote by

$$\mathcal{O}(P, P_1; \mathcal{D}) = \left\{ \Phi : \begin{array}{l} \Phi : \mathcal{D} \subset \mathcal{G}_P \to \mathcal{G}_{P_1} \text{ is a surjective} \\ \text{set-valued mapping with bounded values} \end{array} \right\},$$

and for any mapping $Q : \mathcal{X} \subset \mathcal{W}_a \to \mathcal{W}_a$ and number $\beta \geq 0$, we let:

$$\|Q\|_{(\beta)} = \limsup_{\tau\to\infty} \inf\{\gamma > 0 : \|T_\tau Q x\|_\tau \leq \gamma\|T_\tau x\|_\tau + \beta, \quad \forall x \in \mathcal{X}\},$$
$$\|Q\|^{(\beta)} = \sup_{\tau>0} \inf\{\gamma > 0 : \|T_\tau Q x\|_\tau \leq \gamma\|T_\tau x\|_\tau + \beta, \quad \forall x \in \mathcal{X}\},$$

where, in the case when Q is set-valued, $\|T_\tau Q x\|_\tau = \sup\{\|T_\tau y\|_\tau : y \in Qx\}$. Note, both $\|Q\|_{(\beta)}$ and $\|Q\|^{(\beta)}$ are generalizations of the induced operator norm $\|Q\|$ and $\|Q\|^{(\beta)}$ is called the β-gain in [2] or biased gain in [12].

Proposition 3.1.

$$\vec{\delta}_\beta(P,P_1) = \inf\left\{\|(\Phi - I)|_{\mathcal{D}}\|_{(\beta)} : \Phi \in \mathcal{O}(P,P_1;\mathcal{D}), \mathcal{D} \subset \mathcal{G}_P\right\}, \tag{3.1}$$

$$\vec{d}_\beta(P,P_1) = \inf\left\{\|(\Phi - I)|_{\mathcal{D}}\|^{(\beta)} : \Phi \in \mathcal{O}(P,P_1;\mathcal{D}), \mathcal{D} \subset \mathcal{G}_P\right\}. \tag{3.2}$$

Proof. We only prove (3.1) since the proof for (3.2) is similar.

For convenience, we denote the right hand sides of (3.1) and (3.2) by $\vec{\delta}^\beta(P,P_1)$ and $\vec{d}^\beta(P,P_1)$, respectively.

Let $\Phi : \mathcal{D} \subset \mathcal{G}_P \to \mathcal{G}_{P_1}$ is a surjective mapping. Then for any $y \in \mathcal{G}_{P_1}$, there exists $x_y \in \mathcal{G}_P$ such that $y \in \Phi x_y$ and, therefore

$$\begin{aligned}
\vec{\delta}_\beta(P,P_1) &= \limsup_{\tau\to\infty} \inf\left\{\alpha > 0 : \ \|y - x_y\|_\tau \le \alpha\|x_y\|_\tau + \beta \text{ for all } y \in \mathcal{G}_{P_1}\right\} \\
&\le \limsup_{\tau\to\infty} \inf\left\{\alpha > 0 : \ \|(\Phi - I)x_y\|_\tau \le \alpha\|x_y\|_\tau + \beta \text{ for all } y \in \mathcal{G}_{P_1}\right\} \\
&\le \limsup_{\tau\to\infty} \|(\Phi - I)|_{\mathcal{D}}\|_{(\beta)},
\end{aligned}$$

which $\vec{\delta}_\beta(P,P_1) \le \vec{\delta}^\beta(P,P_1)$.

To show the reverse inequality, we let $\varepsilon > 0$. Then for any $y \in \mathcal{G}_{P_1}$, there exists at least one $x \in \mathcal{G}_P$ such that

$$\|y - x\|_\tau \le (\vec{\delta}_\beta(P,P_1) + \varepsilon)\|x\|_\tau + \beta, \quad \text{for large } \tau > 0. \tag{3.3}$$

Let $\mathcal{D} = \{x \in \mathcal{G}_P : \text{ there exists } y \in \mathcal{G}_{P_1} \text{ such that } x, y \text{ satisfy (3.3)}\}$ and define a mapping $\Phi : \mathcal{D} \subset \mathcal{G}_P \to \mathcal{G}_{P_1}$ as:

$$\Phi(x) = \{y : x, y \text{ satisfy (3.3)}\} \text{ for } x \in \mathcal{D}.$$

Clearly, Φ is surjective mapping from $\mathcal{G}_P$ to $\mathcal{G}_{P_1}$ with bounded values and

$$\|(\Phi - I)x\|_\tau \le (\vec{\delta}_\beta(P,P_1) + \varepsilon)\|x\|_\tau + \beta$$

for all $x \in \mathcal{D}$ and large $\tau > 0$. So $\vec{\delta}^\beta(P,P_1) \le \|(\Phi - I)|_{\mathcal{D}}\|_{(\beta)} \le \vec{\delta}_\beta(P,P_1) + \varepsilon$. Since ε is arbitrary, we see $\vec{\delta}^\beta(P,P_1) \le \vec{\delta}_\beta(P,P_1)$. This completes the proof. □

We remark that $\vec{\delta}^\beta(P,P_1)$ is a generalization of the main gap metric of [9] where, and for all of other generalizations, the mappings Φ in $\mathcal{O}(P,P_1;\mathcal{D})$ are required to be causal. In the case when the mappings Φ are required causal, we can only have the inequality $\vec{\delta}_\beta(P,P_1) \le \vec{\delta}^\beta(P,P_1)$. So, the $\vec{\delta}$-gap metric defined in this paper is the smaller than those defined using causal and surjective mappings between graphs and, therefore, better in theory and applications.

The following theorems generalize the standard results from both linear and nonlinear robust control, see [9, 11] and the references therein.

Theorem 3.2. *Consider the feedback system described in Figure 1. Let $P, P_1 \colon \mathcal{U}_a \to \mathcal{Y}_a$, $K \colon \mathcal{Y}_a \to \mathcal{U}_a$ be stabilizable. Let $\beta \geq 0, \gamma > 0$ and $[P, K]$ be globally well-posed and $[P_1, K]$ is either globally or regularly well-posed. If $[P, K]$ is (γ, r)-gain stable and*

$$\vec{\delta}_\beta(P, P_1) < \gamma^{-1},$$

then $[P_1, K]$ is globally well-posed and (γ_1, r_1)-gain stable with

$$\gamma_1 = \frac{(1 + \vec{\delta}_\beta(P, P_1))\gamma}{1 - \vec{\delta}_\beta(P, P_1)\gamma}, \ r_1 = \frac{(1 + \vec{\delta}_\beta(P, P_1))(\gamma r + \beta)}{1 - (\vec{\delta}_\beta(P, P_1))\gamma} + \beta.$$

The same conclusion holds if the gap metric $\vec{\delta}_\beta(P, P_1)$ is replaced by $\vec{\delta}(P, P_1)$, $\vec{d}_\beta(P, P_1)$ or $\vec{d}(P, P_1)$.

Proof. Let $\rho = \vec{\delta}_\beta(P, P_1)$. By assumption, there exists a $\varepsilon_0 > 0$ such that $(\rho + \varepsilon_0)r < 1$.

Let $w \in \mathcal{W}, \varepsilon \in (0, \varepsilon_0)$ and $0 < \tau < \omega_w$. By the well-posedness assumption, we may suppose $w = w_1 + w_2$ with unique $w_1 = (u_1, y_1)^\top, w_2 = (u_2, y_2)^\top \in \mathcal{W}_a$ such that $y_1 = P_1 u_1, u_2 = K y_2$. This tells $\Pi_{P_1 // K} w = w_1$. By the stabilization assumptions, there exist $w_1' \in \mathcal{G}_{P_1}, w_2' \in \mathcal{G}_K$ such that $T_\tau w_1 = T_\tau w_1', T_\tau w_2 = T_\tau w_2'$. By the definition of $\vec{\delta}_\beta(P, P_1)$, there exists $x \in \mathcal{G}_P$ such that

$$\|w_1' - x\|_\tau \leq (\rho + \epsilon)\|x\|_\tau + \beta. \tag{3.4}$$

Write $\bar{w} = x + w_2'$. Since $[P, K]$ is well-posed, we see

$$\Pi_{P // K} \bar{w} = x.$$

Therefore, by (3.4) and the (γ, r)-gain stability of $\Pi_{P // K}$, we see

$$\begin{aligned} \|\Pi_{P_1 // K} w\|_\tau = \|w_1\|_\tau = \|w_1'\|_\tau &\leq (1 + \rho + \varepsilon)\|x\|_\tau + \beta \\ &= (1 + \rho + \varepsilon)\|\Pi_{P // K}\bar{w}\|_\tau + \beta \\ &\leq (1 + \rho + \varepsilon)\gamma\|\bar{w}\|_\tau + (1 + \rho + \varepsilon)r + \beta. \end{aligned} \tag{3.5}$$

Again by (3.4), there exists $z_w \in \mathcal{W}$ such that $\|z_w\|_\tau \leq 1$ and

$$\begin{aligned} T_\tau w &= T_\tau w_1' + T_\tau w_2' = T_\tau x + T_\tau w_2' + ((\rho + \varepsilon)\|x\|_\tau + \beta) T_\tau z_w \\ &= T_\tau \bar{w} + ((\rho + \varepsilon)\|x\|_\tau + \beta) T_\tau z_w, \end{aligned}$$

from which it follows:

$$\begin{aligned} \|\bar{w}\|_\tau &\leq \|w\|_\tau + (\rho + \varepsilon)\|x\|_\tau + \beta = \|w\|_\tau + (\rho + \varepsilon)\|\Pi_{P // K}\bar{w}\|_\tau + \beta \\ &\leq \|w\|_\tau + (\rho + \varepsilon)\gamma\|\bar{w}\|_\tau + (\rho + \varepsilon)r + \beta \end{aligned}$$

and therefore

$$\|\bar{w}\|_\tau \leq \frac{\|w\|_\tau}{1 - (\rho + \varepsilon)\gamma} + \frac{(\rho + \varepsilon)r + \beta}{1 - (\rho + \varepsilon)\gamma}.$$

Substituting into (3.5) to obtain

$$\|\Pi_{P_1//K}w\|_\tau \le \frac{(1+\rho+\varepsilon)\gamma}{1-(\rho+\varepsilon)\gamma}\|w\|_\tau + \frac{(1+\rho+\varepsilon)(\gamma r+\beta)}{1-(\rho+\varepsilon)\gamma} + \beta.$$

Since $w \in \mathcal{W}$, the right hand side is uniformly bound for all $0 < \tau < \omega_w$, it follows that if $[P_1, K]$ is regularly well-posed, it is also globally well-posed. By letting $\varepsilon \to 0$, we obtain $\|\Pi_{P_1//K}w\|_\tau \le \gamma_1\|w\|_\tau + r_1$, i.e., $[P_1, K]$ is (γ_1, r_1)-gain stable.

If the gap metric $\vec{\delta}_\beta(P, P_1)$ is replaced by $\vec{\delta}(P, P_1)$, then the assumption $\vec{\delta}(P, P_1) < \gamma^{-1}$ implies that there exists $\beta > 0$ such that $\vec{\delta}_\beta(P, P_1) < \gamma^{-1}$. Therefore, the same conclusion holds.

The proofs for the cases when $\vec{\delta}_\beta(P, P_1)$ is replaced by the $\vec{d}$-gap metrics are similar. □

To close this section, we next present an estimate to gap metric for operators with certain graph representations. This estimate will be useful when the above theorem is applied to study the robust stability of some special systems in the rest of this paper.

Proposition 3.3. *Let $P, P_1 : \mathcal{U}_a \to \mathcal{Y}_a$ be given. Suppose that there exist operators $M, \triangle M : \mathcal{D} \subset \mathcal{U} \to \mathcal{Y}$ and $N, \triangle N : \mathcal{D} \subset \mathcal{U} \to \mathcal{U}$ such that*

$$\mathcal{G}_P = \left\{ \begin{pmatrix} M \\ N \end{pmatrix} v : v \in \mathcal{D} \right\},\ \mathcal{G}_{P_1} = \left\{ \begin{pmatrix} M+\triangle M \\ N+\triangle N \end{pmatrix} v : v \in \mathcal{D} \right\}. \tag{3.6}$$

If there exist $k \ge 0, \beta \ge 0$ such that

$$\left\| \begin{pmatrix} \triangle M \\ \triangle N \end{pmatrix} u \right\|_\tau \le k \left\| \begin{pmatrix} M \\ N \end{pmatrix} u \right\|_\tau + \beta \quad \text{for all } u \in \mathcal{D} \text{ and large } \tau > 0, \tag{3.7}$$

then $\vec{\delta}_\beta(P, P_1) \le k$. If, in addition, $k < 1$, then $\vec{\delta}_\beta(P, P_1) \le \frac{k}{1-k}$.

If (3.7) are satisfied for all $\tau > 0$, then the same conclusions hold for the gap metric $\vec{d}_\beta(P_1, P)$ as well.

Proof. For any $y = \begin{pmatrix} M+\triangle M \\ N+\triangle N \end{pmatrix} u \in \mathcal{G}_P$ with $u \in \mathcal{D}$, let $x = \begin{pmatrix} M \\ N \end{pmatrix} u \in \mathcal{G}_P$. By the assumptions, we see that

$$\|y - x\|_\tau = \left\| \begin{pmatrix} \triangle M \\ \triangle N \end{pmatrix} u \right\|_\tau \le k \left\| \begin{pmatrix} M \\ N \end{pmatrix} u \right\|_\tau + \beta$$

for large $\tau > 0$, which implies that $\vec{\delta}_\beta(P, P_1) \le k$.

If $k < 1$, then for any $u \in \mathcal{D}$, (3.7) implies

$$\left\| \begin{pmatrix} \triangle M \\ \triangle N \end{pmatrix} u \right\|_\tau \le k \left\| \begin{pmatrix} M+\triangle M \\ N+\triangle N \end{pmatrix} u \right\|_\tau + \left\| \begin{pmatrix} \triangle M \\ \triangle N \end{pmatrix} u \right\|_\tau + \beta$$

and therefore

$$\|x-y\|_\tau \le \frac{k}{1-k}\|y\|_\tau + \beta$$

which implies $\vec{\delta}_\beta(P_1, P) \le \frac{k}{1-k}$.

If (3.7) is satisfied for all $\tau > 0$, using the same method, we can show the same conclusions hold for the gap metric $\vec{d}_\beta(P_1, P)$. □

Note that the operators $M, N, \triangle M$ and $\triangle N$ are not required to be stable nor invertible. Hence, (M, N) (resp. $(M+\triangle M, N+\triangle N)$) is not necessarily the coprime factorisation for P (resp. P_1) (see [1, 13, 14]). However, if the operators admit (right) coprime factorisations, then the graph representations (3.6) can be achieved with $(M, N), (M+\triangle M, N+\triangle N)$ the coprime factorisations. Interestingly, when $(\triangle M, \triangle N)$ is uniformly bounded, k may be taken as 0 and we will have $\vec{\delta}_\beta(P, P_1) = 0$.

4. Semilinear systems with bounded nonlinearities

In the rest of this paper, as applications of results in the last section, we will address some stability problems in nonlinear feedback control regarding. We will estimate the gap metrics between certain specific nominal plants and their perturbations, and study the robust stability of the underlying control systems.

First of all, in this section, we consider the effect of initial state value to the stability of a control system. There has always been a creative tension in control theory between state space and input-output methods, but the original formulation of gap metric approach precludes the case of non-zero initial conditions as the assumptions require both plant and controller map zero to zero. Although this condition has been relaxed later and, particularly, recently in the biased notions [2, 5, 10], a problem that is not addressed within the gap metric framework is: if a controller stabilizes a system with a given initial state condition, does it stabilize the system when the initial state condition changes? We will prove that under our framework, for semiliner systems with bounded nonlinearities, changing initial state condition does not change the robust stability and the stability margin.

The systems concerned are of the following form, denoted by $\Xi(f, x_0)$

$$\Xi(f, x_0) : u \mapsto y : \begin{array}{l} x'(t) = Ax(t) + f(t, x(t)) + Bu(t), x(0) = x_0 \in Y \\ y(t) = Cx(t) + Du(t) \end{array} \tag{4.1}$$

where U, Y are normed vector spaces, $A : \mathrm{dom}(A) \subset Y \to Y$ is a linear operator, $C : Y \to Y$, $B, D : U \to Y$ are bounded linear operators and $f(t, x) : \mathbb{R} \times Y \to Y$ is measurable in t and Lipschitz in x. Let $\mathcal{U} = L^\infty(\mathbb{R}_+, U), \mathcal{Y} = L^\infty(\mathbb{R}_+, Y)$ be the signal spaces. We assume the following two assumptions.

Assumption 1. $\mathcal{D} \subset \mathcal{U}$ and for each $u \in \mathcal{D}$, (4.1) has a solution in $\mathcal{Y}$.

Assumption 2. There exists a linear operator $F : Y \to U$ such that $A + BF$ generates a c_0-semigroup of bounded linear operators $S(t)$ and, for each $v \in \mathcal{U}$, the equation

$$x'(t) = (A + BF)x(t) + f(t, x(t)) + Bv(t), \quad x(0) = x_0. \tag{4.2}$$

has unique solution $x \in \mathcal{Y}$.

We first give a graph representation for $\Xi(f, x_0)$, which is a generalization of the corresponding results in [13, 14].

Lemma 4.1.

$$\text{graph}(\Xi(f, x_0)) = \left\{ \begin{pmatrix} M \\ N \end{pmatrix} v : v \in \mathcal{U} \right\}$$

with

$$\begin{aligned} Mv(t) &= FS(t)x_0 + F\int_0^t S(t-s)[Bv(s) + f(s, x(s))]ds + v(t), \\ Nv(t) &= (C + DF)S(t)x_0 \\ &\quad + (C + DF)\int_0^t S(t-s)[Bv(s) + f(s, x(s))]ds + Dv(t), \end{aligned}$$

where x is the solution to (4.2) corresponding to input v.

Proof. It is known that $M : v \mapsto u$ and $N : v \mapsto y$ are the input-output mappings to the closed loops

$$x'(t) = (A + BF)x(t) + f(t, x(t)) + Bv(t), \quad x(0) = x_0 \tag{4.3}$$

$$u(t) = Fx(t) + v(t) \tag{4.4}$$

$$y(t) = (C + DF)x(t) + Dv(t). \tag{4.5}$$

By our assumptions, both M and N are well-defined operators. Furthermore, M is invertible with the inverse $M^{-1} : u \mapsto v$ given by:

$$x'(t) = Ax(t) + f(t, x(t)) + Bu(t), \quad x(0) = x_0 \tag{4.6}$$

$$v(t) = u(t) - Fx(t). \tag{4.7}$$

By Assumption 1, we see that $\text{dom}(M^{-1}) \supset \mathcal{D}$. If $u \in \text{dom}(M^{-1})$, then $v = M^{-1}u \in \mathcal{Y}$ and the x satisfying (4.6)-(4.7) is the solution to (4.2). From Assumption 2, it follows $x \in \mathcal{Y}$ and therefore $\text{dom}(M^{-1}) = \mathcal{D}$. Hence, the composition NM^{-1} is the input-out mapping associated with

$$x'(t) = Ax(t) + f(t, x(t)) + Bu(t), \quad x(0) = x_0 \tag{4.8}$$

$$v(t) = u(t) - Fx(t), \tag{4.9}$$

$$z'(t) = (A + BF)z(t) + f(t, z(t)) + Bv, \quad z(0) = x_0 \tag{4.10}$$

$$y(t) = (C + DF)z(t) + Dv(t). \tag{4.11}$$

Substituting (4.9) into (4.10) and subtracted by (4.8), we have

$$z' - x' = (A + BF)(z - x) + f(t, z(t)) - f(t, x(t)).$$

Since $S(t)$ is a c_0-semigroup, there exist $\omega, r \geq 0$ such that $\|S(t)\| \leq re^{\omega t}$ for all $t \geq 0$. Since f is Lipschitz and x, z have the same initial value, we see

$$\begin{aligned} |z(t) - x(t)| &= \left| \int_0^t S(t-s)[f(s, z(s)) - f(s, x(s))]ds \right| \\ &\leq cr \int_0^t e^{\omega(t-s)} |z(s) - x(s)|ds \quad \text{with some } c > 0 \end{aligned}$$

and

$$e^{-\omega t}|z(t) - x(t)| \leq cr \int_0^t e^{-\omega s}|z(s) - x(s)|ds, \text{ for all } t \geq 0.$$

By Gronwall's Inequality, $x \equiv z$. Therefore, $NM^{-1} = \Xi(f, x_0)$ and

$$\text{graph}(\Xi(f, x_0)) = \left\{ \begin{pmatrix} u \\ \Xi(f, x_0)u \end{pmatrix} : u \in \mathcal{D} \right\} = \left\{ \begin{pmatrix} Mv \\ Nv \end{pmatrix} : v \in \mathcal{U} \right\}.$$

□

Remark: since no more information is given about the left invertibility of $(M, N)^\top$, (M, N) is not necessarily the coprime factorisation of $\Xi(f, x_0)$.

We now in the position to estimate the gap between $\Xi(f, x_0)$ and $\Xi(f, \tilde{x}_0)$ with $x_0, \tilde{x}_0 \in Y$ and $x_0 \neq \tilde{x}_0$.

Corollary 4.2. *Consider the system given by equation* (4.1). *Let* $x_0, \tilde{x}_0 \in Y$ *and* $x_0 \neq \tilde{x}_0$. *Let* $P = \Xi(f, x_0)$ *be the nominal plant and* $P_1 = \Xi(f, \tilde{x}_0)$ *the perturbation. Suppose* F *is a bounded operator and the semigroup of linear operators* $S(t)$ *generated by* $A + BF$ *is such that*

$$\|S(t)\| \leq re^{-\omega t} \text{ with } r \geq 1, \omega > 0 \text{ for all } t > 0. \tag{4.12}$$

If either

$$\|f(t, x)\| \leq c \text{ with } c \geq 0 \text{ for all } t \geq 0, x \in Y, \tag{4.13}$$

or there exists $d \in [0, \omega/r)$ *such that*

$$\|f(t, x_1) - f(t, x_2)\| \leq d\|x_1 - x_2\| \text{ for all } t \geq 0, x_1, x_2 \in Y, \tag{4.14}$$

then $\vec{\delta}(P, P_1) = 0$.

Proof. As shown in Lemma 4.1, there exist operators $N, N_1 : \mathcal{Y} \to \mathcal{Y}$ and $M, M_1 : \mathcal{U} \to \mathcal{Y}$ such that

$$\text{graph}(P) = \left\{ \begin{pmatrix} M \\ N \end{pmatrix} u : u \in \mathcal{U} \right\}, \quad \text{graph}(P_1) = \left\{ \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} u : u \in \mathcal{U} \right\}$$

and

$$\begin{aligned}\triangle Mu(t) =: (M_1 - M)u(t) &= FS(t)(\tilde{x}_0 - x_0) \\ &+ F\int_0^t S(t-s)\big[f(s,\tilde{x}(s)) - f(s,x(s))\big]ds, \qquad (4.15)\\ \triangle Nu(t) =: (N_1 - N)u(t) &= (C+DF)S(t)(\tilde{x}_0 - x_0) \\ &+ (C+DF)\int_0^t S(t-s)[f(s,\tilde{x}(s)) - f(s,x(s))]ds. \qquad (4.16)\end{aligned}$$

Here x is the solution to equations (4.2) and $\tilde{x}$ is the solution to

$$x'(t) = (A+BF)x(t) + f(t,x(t)) + Bu(t), \quad x(0) = \tilde{x}_0.$$

Therefore

$$\tilde{x}'(t) - x'(t) = (A+BF)(\tilde{x}(t) - x(t)) + f(t,\tilde{x}(t)) - f(t,x(t)). \qquad (4.17)$$

If condition (4.13) hold, then, for any $\tau > 0$, we have

$$\begin{aligned}\|\triangle Mu\|_\tau &\le \|FS(t)(\tilde{x}_0 - x_0)\|_\tau + 2c\|F\|\int_0^t e^{-\omega(t-s)}ds \\ &\le r\|F\|\|\tilde{x}_0 - x_0\| + 2c\|F\|\omega^{-1}, \\ \|\triangle Nu\|_\tau &\le \|(C+DF)S(t)(\tilde{x}_0 - x_0)\|_\tau + 2c\|C+DF\|\int_0^t e^{-\omega(t-s)}ds \\ &\le r\|C+DF\|\|\tilde{x}_0 - x_0\| + 2c\|C+DF\|\omega^{-1}\end{aligned}$$

for all $u \in \mathcal{U}$, and by Proposition 3.3 with $k = 0$, $\vec{\delta}(P, P_1) = 0$.

If condition (4.14) hold, then, for any $\tau > 0$, from (4.17), it follows

$$\begin{aligned}\|\tilde{x}(t) - x(t)\| &\le \|S(t)(\tilde{x}_0 - x_0)\| + \int_0^t \|S(t-s)f(s,\tilde{x}(s)) - f(s,x(s))\|ds \\ &\le re^{-\omega t}\|\tilde{x}_0 - x_0\| + rd\int_0^t e^{-\omega(t-s)}\|\tilde{x}(s) - x(s)\|ds.\end{aligned}$$

This gives $\|\tilde{x}(t) - x(t)\| \le r\|\tilde{x}_0 - x_0\|e^{-(\omega - rd)t} \le r\|\tilde{x}_0 - x_0\|$ and, therefore

$$\|f(t,\tilde{x}(t)) - f(t,x(t))\| \le rd\|\tilde{x}_0 - x_0\|.$$

Using the same argument as used above, we see $\vec{\delta}(P, P_1) = 0$. □

By Theorem 3.2, we conclude:

Corollary 4.3. *Consider the system given by equation* (4.1). *Under the assumptions of Corollary, 4.2, if a controller K is such that $[\Xi(f, x_0), K]$ is γ-gain stable with bias, then $[\Xi(f, \tilde{x}_0), K]$ is also γ-gain stable with bias for any $\tilde{x}_0$.*

The existence of operator F satisfying (4.12) indicates that the nominal plant $\Xi(f, x_0)$ is stabilizable with the feedback controller $u = Fx + v$. If the system is of finite dimension, it is equivalent to that $A + BF$ is Hurwitz and, in that case, $S(t) = \exp[(A+BF)t]$ and $-\omega$ is the maximal eigenvalue of $A+BF$. However, if the nominal plant is not stabilizable, it is known that the initial state value is essential.

The next result shows that if a linear system is perturbed by a uniformly bounded (nonlinear) function, its stability remains the same.

Corollary 4.4. *Consider the system given by equation* (4.1)*. Under the assumptions of Corollary 4.2 and condition* (4.13)*, we have* $\vec{\delta}(\Xi(0, x_0), \Xi(f, x_0)) = 0$.

Proof. As shown in Lemma 4.1, there exist operators $N, N_1 : \mathcal{Y} \to \mathcal{Y}$ and $M, M_1 : \mathcal{U} \to \mathcal{Y}$ such that

$$\text{graph}(\Xi(0, x_0)) = \left\{ \begin{pmatrix} M \\ N \end{pmatrix} u : u \in \mathcal{U} \right\}, \ \text{graph}(\Xi(f, x_0)) = \left\{ \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} u : u \in \mathcal{U} \right\}$$

and

$$\begin{pmatrix} \triangle M \\ \triangle N \end{pmatrix} v(t) =: \begin{pmatrix} M_1 - M \\ N_1 - N \end{pmatrix} v(t) = \begin{pmatrix} F \int_0^t S(t-s) f(s, x(s)) ds \\ (C + DF) \int_0^t S(t-s) f(s, x(s)) ds \end{pmatrix},$$

where x is the solution to the equation (4.2). Since f is uniformly bounded, as shown in Corollary 4.2, there exists a constant $c_f > 0$ such that

$$\left\| \begin{pmatrix} \triangle M \\ \triangle N \end{pmatrix} v \right\|_\tau \leq c_f \quad \text{for all } v \in \mathcal{U}, \tau > .$$

By Proposition 3.3 with $k = 0$, $\vec{\delta}(\Xi(0, x_0), \Xi(f, x_0)) = 0$. □

5. Stability of realizations

It is known that, for a given transfer function, there are possibly infinite many different state space realizations of which the stability may vary. We now use Corollary 4.2 to show that the gap between any two realizations which are stabilizable is zero and, therefore, by Theorem 3.2, a controller stabilizing one realization also stabilizes the other one.

We first present a lemma regarding the estimate of gap metric.

Lemma 5.1. *Let* $P_1, P_2, P_3 : \mathcal{U}_a \to \mathcal{Y}_a$ *and* $\vec{\delta}(P_1, P_2) \leq k_1, \vec{\delta}(P_2, P_3) \leq k_2$ *with* $k_1, k_2 < \infty$*. Then* $\vec{\delta}(P_1, P_3) \leq k_1 + k_2 + k_1 k_2$.

The same results hold if the gap metric $\vec{\delta}$ *is replaced by* $\vec{d}$.

Proof. Suppose $\varepsilon > 0$. By the definition of the gap metric, there are $\beta_1 > 0, \beta_2 > 0$ such that for each $w_3 \in \text{graph}(P_3)$, there exist $w_2 \in \text{graph}(P_2)$ and, therefore, $w_1 \in \text{graph}(P_1)$ satisfying:

$$\|w_3 - w_2\|_\tau \le (k_2 + \varepsilon)\|w_2\|_\tau + \beta_2, \ \|w_2 - w_1\|_\tau \le (k_1 + \varepsilon)\|w_1\|_\tau + \beta_1$$

for all $\tau > 0$. So

$$\begin{aligned}\|w_3 - w_1\|_\tau &\le \|w_3 - w_2\|_\tau + \|w_2 - w_1\|_\tau \\ &\le (k_2 + \varepsilon)\big[\|w_2 - w_1\|_\tau + \|w_1\|_\tau\big] + \beta_2 + (k_1 + \varepsilon)\|w_1\|_\tau + \beta_1 \\ &\le (k_2 + \varepsilon)\big[(1 + k_1 + \varepsilon)\|w_1\|_\tau + \beta_1\big] + \alpha_2 + (k_1 + \varepsilon)\|w_1\|_\tau + \beta_1.\end{aligned}$$

So $\vec{\delta}(P_1, P_3) \le (k_2 + \varepsilon)(1 + k_1 + \varepsilon) + k_1 + \varepsilon$. By let $\varepsilon \to 0$ we obtain

$$\vec{\delta}(P_1, P_3) \le k_1 + k_2 + k_1 k_2.$$

□

Corollary 5.2. *Suppose that $G(s)$ is a transfer matrix having the following two realizations of the same dimension*

$$\begin{aligned}P_0(x_0) : x'(t) &= Ax(t) + Bu(t), x(0) = x_0, \\ y(t) &= Cx(t) + Du(t), \\ P_1(x_1) : z'(t) &= A_1 z(t) + B_1 u(t), z(0) = x_1, \\ y(t) &= C_1 z(t) + D_1 u(t),\end{aligned}$$

where A, B, C, D and A_1, B_1, C_1, D_1 are matrices of finite dimensions. If both $P_0(x_0)$ and $P_1(x_1)$ are stabilizable, then $\vec{\delta}(P_0(x_0), P_1(x_1)) = 0$. Consequently, if K is a suitable controller such that $[P_0(x_0), K]$ is γ-gain stable, then $[P_1(x_1), K]$ is γ-gain stable.

Proof. By our assumptions and Corollary 4.2, we see

$$\vec{\delta}(P_0(x_0), P_0(0)) = 0 \ \text{ and } \ \vec{\delta}(P_1(x_1), P_1(0)) = 0.$$

By Lemma 5.1

$$\begin{aligned}\vec{\delta}(P_0(x_0), P_1(x_1)) &\le \vec{\delta}(P_0(x_0), P_0(0)) + \vec{\delta}(P_0(0), P_1(x_1)) \\ &\quad + \vec{\delta}(P_0(x_0), P_0(0))\vec{\delta}(P_0(0), P_1(x_1)) \\ &= \vec{\delta}(P_0(0), P_1(x_1)) \\ &\le \vec{\delta}(P_0(0), P_1(0)) + \vec{\delta}(P_1(0), P_1(x_1)) \\ &\quad + \vec{\delta}(P_0(0), P_1(0))\vec{\delta}(P_1(0), P_1(x_1)) \\ &= \vec{\delta}(P_0(0), P_1(0)).\end{aligned}$$

As $P_0(0)$ and $P_1(0)$ have the same graph given by $\{(u,y)^\top : y = Gu\}$, we have $\vec{\delta}(P_0(0), P_1(0)) = 0$. The γ-gain stability of $[P_1(x_1), K]$ follows from Theorem 3.2. This completes the proof. □

6. A nonlinear system with time delay

In this section, we consider the stability of integrator system with saturation and delay in the input. We let the nominal plant be the system without delay and the perturbation be the system with delay, then applying the results in Section 3 to show the stability of systems with delay.

Let $\mathcal{U} = \mathcal{Y} = L^\infty(\mathbb{R}_+, \mathbb{R})$. The nominal and perturbed plants P, P_1 are described, respectively, by

$$\begin{aligned} P: &\quad y_1(t) = x(t), \quad x_1'(t) = \mathrm{sat}(u_1(t)), &\quad x(0) = x_0,\\ P_1: &\quad y_1(t) = x(t), \quad x_1'(t) = \mathrm{sat}(u_1(t-h)), &\quad x(0) = x_0, \end{aligned}$$

where $\mathrm{sat}(u_1) = u_1$ when $|u_1| \le 1$ and is equal to $\mathrm{sign}(u_1)$ when $|u_1| > 1$.

It is known that

$$\mathrm{graph}(P) = \left\{ \begin{pmatrix} M \\ N \end{pmatrix} u : u \in \mathrm{dom}(P) \right\}, \quad \mathrm{graph}(P_1) = \left\{ \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} u : u \in \mathrm{dom}(P) \right\},$$

where $M = M_1 = I$ (the identity), $N = P$ and $N_1 = P_1$. Moreover

$$\begin{pmatrix} \triangle M \\ \triangle N \end{pmatrix} u(t) =: \begin{pmatrix} M_1 - M \\ N_1 - N \end{pmatrix} u(t) = \begin{pmatrix} 0 \\ x(t-h) - x(t) \end{pmatrix}.$$

Since

$$|x(t-h) - x(t)| \le \sup_{s\in[t-h,t]} |x'(s)|h \le \sup_{s\in[t-h,t]} |\mathrm{sat}(u(s))|h,$$

we have

$$\left\| \begin{pmatrix} \triangle M \\ \triangle N \end{pmatrix} u \right\|_\tau \le h\|\mathrm{sat}(u)\|_\tau \le h$$

for all $\tau > 0$. Hence, by Proposition 3.3, $\vec{\delta}_h(P, P_1) = \vec{\delta}(P, P_1) = 0$.

Choose the feedback controller to be $K = -1$. Then both $[P, K]$ and $[P_1, K]$ are well-posed and $[P, K]$ is 4-gain stable with zero bias (see [9]). By Theorem 3.2, $[P_1, K]$ is 4-gain stable with bias for all $h > 0$.

However, if we apply the $\vec{\delta}_0$-gap metric, since $\|\mathrm{sat}(u)\| \le \|u\|$, we would have $\vec{\delta}_0(P, P_1) \le h$ and $[P_1, K]$ is 4-gain stable with zero bias for $h < 1/4$.

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