XXI Symposium
Electromagnetic Phenomena in Nonlinear Circuits
June 29 - July 2, 2010, Dortmund and Essen, Germany

# CRITICAL EVALUATION OF NUMERICAL TECHNIQUES FOR HIGHLY NON-LINEAR FIELD DIFFUSION MODELLING

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***Abstract* – Various numerical techniques have been applied to multidimensional field diffusion problems with front-type behaviour, moving boundaries and non-linear material properties. Advantages and implementation challenges of the methods are discussed with special attention paid to conservation properties of the algorithm and achieving accurate solutions close to the moving boundaries. The techniques are evaluated using analytical solutions of diffusion problems with cylindrical symmetry.**

## I. INTRODUCTION

It is increasingly common to encounter strong non-linearity in modern electrical engineering apparatus. A good example is devices utilising High Temperature Superconductors (HTS). Both electromagnetic field variation and heat flow can be formulated in terms of diffusion [1, 2] and distinctive regions with high and low losses ($JE$) can be observed (Fig. 1). Special consideration must be given to domains with intensive Joule heating which may require adaptive meshing [3, 4]. Additional difficulties arise when dealing with short circuit faults or other impulse loads. Such events require careful numerical modelling since electromagnetic and thermal parts of the problem are coupled via high sensitivity of HTS material properties to temperature changes [5, 6].

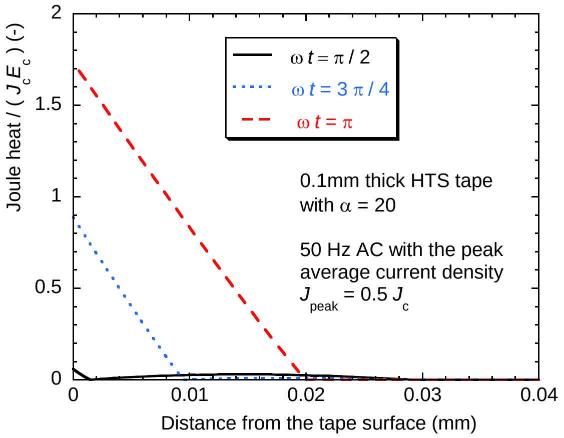


Fig. 1. Typical variation of Joule heat release in HTS tapes under AC load at different instants during the cycle.

A standard modelling approach on fixed grids can be utilised to simplify the design process. But such approach often fails to deliver appropriate balance between accuracy and efficiency, especially when modelling pulse events or shallow field penetration. Special methods – such as adaptive meshes, front fixing and level sets methods [7] – offer advantages in such applications but they have to be assessed and probably adapted for each particular problem. This study focuses on the analysis of the front fixing technique [7] since it requires only a small modification of the computational algorithm in comparison with models based on fixed grids [3, 8]. The major challenges include imposition of conservation laws and achieving accurate solutions close to the moving curved boundaries. Analytical solutions of common front type problems have been used to evaluate the performance of the numerical method [4].

## II. PROBLEM FORMULATION

### *A. Governing equation and material properties*

For HTS the governing equation takes the diffusion-like form [1, 2]

$$\text{curl}\,(\text{curl}\,\mathbf{E}) = -\mu_0 \frac{\partial \mathbf{J}}{\partial t} \qquad (1)$$

expressed in terms of the electric field $\mathbf{E}$ and current density $\mathbf{J}$. HTS materials exhibit strong flux creep $E{-}J$ behaviour often described by power law [9], $E_c^{-1}E = (J_c^{-1}J)^{\alpha}$, where the critical current density $J_c \approx 10^9\,\text{A m}^{-2}$ corresponds to a critical electric field $E_c \approx 10^{-4}\,\text{V m}^{-1}$. For practical HTS materials the power exponent $\alpha$ could be as high as 20 and represents very strong non-linearity.

### *B. Analytical solution: boundary and initial conditions*

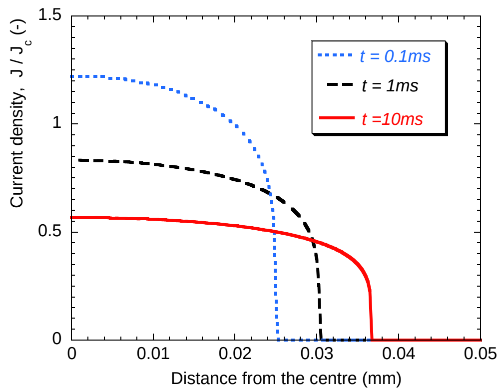


Fig. 2. Evolution of current density $J_z(r, t)$ in a HTS circular wire. The analytical prediction of electric field and current density for a wire with $I_0 = 2$A, $R$=0.5mm and $\alpha$=6 is shown at different instances of time.

Consider a HTS wire with a circular cross-section of radius $R$ with embedded electrically insulated very fine conductor in the centre. If the current pulse $J_z(r,t) = I_0\delta(\hat{\mathbf{r}})\delta(t-t_0)$ is applied to the conductor along the $z$ axis at an instant $t$=$t_0$; the dimensionless solution for (1) in the case of cylindrical symmetry can be derived as shown in [10]. Although the case looks somewhat artificial it is useful as a robust test since the solution has distinctive features common to HTS, see Fig. 2, as the field and the current are zero outside the front region. The electric field and the current gradually spread from the centre of the wire towards the edges and there is a sharp interface between the region with non-zero field and the outside part of the wire.

It is helpful to conduct tests in at least 2D geometry. The existence of an analytical solution in cylindrical coordinates provides an opportunity to evaluate the technique's performance on curved boundaries using the Cartesian coordinate system.

## III. COMPUTATIONAL RESULTS

Commercial finite element software COMSOL [11], together with an in-house finite volume code [3, 12, 13], were used to access efficiency of the computational methods. A very high gradient of the electric field originates during the impulse of current. It spreads very quickly in all directions. It forces COMSOL to use extremely small time steps, ~$10^{-30}$ s, otherwise Newton iterations do not converge. The in-house code, with more robust algorithms adjusted to particular problem, like Jacobi iterations or directional splitting [14, 15], allow computations with 10 μs time steps.

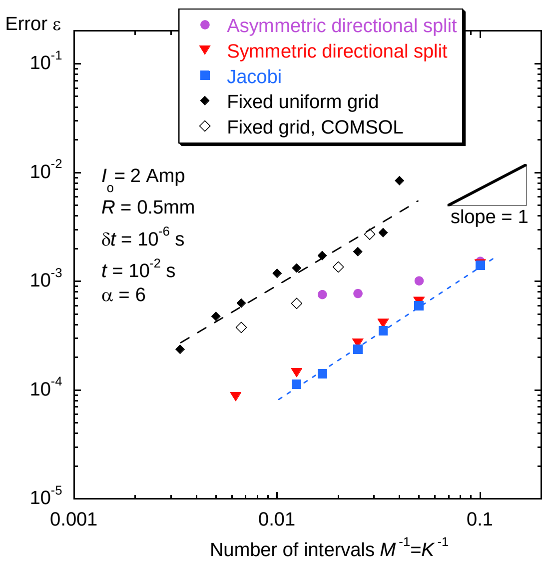


Fig. 3. Numerical prediction of an electric field $E_z(x,y)$ for a wire with $I_0$ = 2A, $R$=0.5mm and α=6 after $t$ = 10 ms: the mesh size effects. Errors in predictions reveal an approximately first order of accuracy $O(M^{-1})$.

Predictions from fixed grid calculations and the front-fixing method are summarized in Fig. 3, where the variations of errors with an increasing number of space intervals for a rectangular mesh $M$=$K$ in $x$- and $y$- directions are plotted. For a triangular mesh $M = K = (number\ of\ nodes)^{1/2}$ is assumed. The error ε is estimated in a continuous **C** norm

$$\varepsilon = \max_{m,k=0}^{M,K} \left| e_{m,k}^{\text{analytical}} - e_{m,k}^{\text{model}} \right| \qquad (2)$$

A slope of the $M^{-1} - \varepsilon$ curve in log-log scale indicates only the 1st order space approximation. This is true for both fixed grid and front-fixing approaches and is not affected by various splitting steps introduced in the algorithm. Generally the asymmetric technique has a large directional bias, see [3, 4]. The bias cannot be eliminated by mesh refinement and errors start to saturate at relatively high level. A symmetric version of the split has almost the same accuracy as the Jacobi method but requires significantly less computational effort. In fact the symmetric split also has a directional bias which is an inherent feature of any split technique [16]; it is just moved to the $x$=$y$ plane. Even the totally symmetric Jacobi method has a slight bias around the $x$=$y$ plane, although hardly noticeable.

## III. CONCLUSIONS

The paper will review and evaluate numerical techniques for modelling diffusion problems under highly-nonlinear conditions with special emphasis on the front fixing method.

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