Multidim Syst Sign Process
DOI 10.1007/s11045-010-0124-1

Lyapunov stability analysis of higher-order 2-D systems

C. Kojima · P. Rapisarda · K. Takaba

Received: 6 October 2009 / Revised: 8 June 2010 / Accepted: 15 June 2010

Abstract In this paper we prove a necessary and sufficient condition for the asymptotic stability of a 2-D system described by a system of higher-order linear partial difference equations. We show that asymptotic stability is equivalent to the existence of a vector Lyapunov functional satisfying certain positivity conditions together with its divergence along the system trajectories. We use the behavioral framework and the calculus of quadratic difference forms based on four-variable polynomial algebra.

Keywords 2-D system · Lyapunov function · Quadratic difference form · Polynomial Lyapunov equation

1 Introduction

Discrete- and continuous two-dimensional (in the following abbreviated as 2-D) systems have application in all those situations when the evolution of the system under study depends on two independent variables, for example time and (one-dimensional) space as it happens when analyzing the vibrations of structures, or in iterative learning control; or two spatial variables, for example in the case of digital image processing, in physics, etc.

C. Kojima (✉)
Department of Information Physics and Computing, Graduate School of Information Science and Technology, The University of Tokyo, Hongo, Bunkyo-Ku, 113-0033 Tokyo, Japan
e-mail: chiaki_kojima@ipc.i.u-tokyo.ac.jp

P. Rapisarda
Information: Signals, Images, Systems group, School of Electronics and Computer Science, University of Southampton, SO171BJ Southampton, UK
e-mail: pr3@ecs.soton.ac.uk

K. Takaba
Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, 606–8501 Kyoto, Japan
e-mail: takaba@amp.i.kyoto-u.ac.jp

Published online: 27 June 2010

In the past there has been considerable effort to define "standard" classes of mathematical models suitable for the analysis of 2-D systems. There is no question of providing useful or even adequate summaries of such results here; we point to (Attasi 1973; Fornasini and Marchesini 1978; Morf et al. 1977; Roesser 1975) for a thorough exposition. In this paper we adopt the behavioral framework pioneered by Willems in the 1-D case (see Polderman and Willems 1997), and extended to the 2-D case first by Rocha (1990) and subsequently adopted by many other authors, for example Valcher, Zerz, Lomadze, Napp-Avelli, Oberst, Pillai, Shankar and Wood. In this setting the main object of study is the behavior, the set consisting of all the trajectories admissible by the physical laws describing the system trajectories. The behavior can be described in many different ways, involving only the to-be-modeled variables (as in a so-called kernel representation), or also auxiliary variables (as in a so-called hybrid representation, of which the classical "state-space" models such as Givone-Roesser, and Fornasini-Marchesini are very special cases).

The notion of stability, because of its important consequences in the analysis and design of control systems and of filters, has attracted considerable interest also in the case of 2-D systems (e.g. Huang 1972; Bose 1982; Bistritz 2004; Ebihara et al. 2006). The issue of what a reasonable definition of stability is for this situation presents first and foremost the difficulty of extending the notion of "past" and "future", so self-evident in the 1-D framework, to the case of two independent variables, where there is no obvious such splitting. An eminently reasonable position is to let the laws describing the physical phenomenon themselves dictate what the direction is of the evolution of the system. This is the approach pioneered by Valcher (2000) and followed in this paper. Central in this framework are the definitions of characteristic set and characteristic cone of a behavior, which we review further in the paper; a linear, shift-invariant behavior which admits a nontrivial (or "proper") characteristic cone is called asymptotically stable if its trajectories go asymptotically to zero within the "future cone". In Valcher (2000) algebraic tests are given which, starting from the description of a system as the kernel of a polynomial operator in the shifts, determine whether a cone is characteristic for the system or not (see Proposition 2.9 of Valcher 2000); and whether a system is asymptotically stable or not. These tests are based on the location of the points of an algebraic variety associated with the polynomial matrix inducing a kernel representation of the system.

In this paper we present a necessary and sufficient condition for the asymptotic stability of 2-D systems based on Lyapunov functions. This idea is by no means original, having been applied already in Lu and Lee (1985), Fornasini and Marchesini (1980); however, those approaches relied entirely on a specific ("state-space", or first-order) type of representation of the system, while we deal with systems described in a general form, namely as the solutions of a system of higher-order partial difference equations. Moreover, the "generalized Bézoutian" introduced in Geronimo and Woerdeman (2004) is shown in this paper to be the scalar version of a generalized Bézoutian arising naturally as a Lyapunov function for 2-D systems.

The structure of the paper is as follows: Sects. 2 and 3 contain background material on 2-D systems and quadratic difference forms, respectively. Section 4 contains the main result of this paper, namely a stability criterion for higher-order systems of differential- or difference equations based on Lyapunov analysis. Section 5 discusses the current research directions being pursued.

In this paper, the concepts and tools of the behavioral approach and of quadratic difference- and differential forms will be put to strenuous use. The reader not familiar with them is referred to Pillai and Shankar (1999), Polderman and Willems (1997), Willems and Trentelman (1998) for a thorough exposition.

Notation We denote with $\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1,\xi_2]$ (respectively, $\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$) the set of all $\mathtt{r}\times\mathtt{w}$ matrices with entries in the ring $\mathbb{R}[\xi_1,\xi_2]$ of polynomials in two indeterminates, with real coefficients (respectively in the ring $\mathbb{R}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$ of Laurent polynomials in two indeterminates with real coefficients). Given a nonzero Laurent polynomial $p(\xi_1,\xi_2)=\sum_{\ell,m}p_{\ell,m}\xi_1^{\ell}\xi_2^{m}\in\mathbb{R}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$, the *Laurent variety* of p is defined as

$$\mathcal{V}_L(p):=\{(\alpha,\beta)\in\mathbb{C}\times\mathbb{C}\mid\alpha\beta\neq 0,\ p(\alpha,\beta)=0\}$$

This definition extends to sets $\mathcal{I}$ of Laurent polynomials, with $\mathcal{V}(\mathcal{I})$ being the intersection of the Laurent varieties of all polynomials in the set. Let $R\in\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$ have full column rank (as a rational matrix); then its *characteristic ideal* is the ideal of $\mathbb{R}[\xi_1,\xi_2]$ generated by the determinants of all $\mathtt{w}\times\mathtt{w}$ minors of R, and its *characteristic variety* is the set of roots common to all polynomials in the ideal. Further properties and definitions can be found in Fornasini and Valcher (1997).

A set $\mathcal{K}\subset\mathbb{R}\times\mathbb{R}$ is called a *cone* if $\alpha\mathcal{K}\subset\mathcal{K}$ for all $\alpha\geq 0$. A cone is *solid* if it contains an open ball in $\mathbb{R}\times\mathbb{R}$, and *pointed* if $\mathcal{K}\cap-\mathcal{K}=\{(0,0)\}$. A cone is *proper* if it is closed, pointed, solid, and convex. It is easy to see that a proper cone is uniquely identified as the set of nonnegative linear combinations of two linearly independent vectors $v_1,v_2\in\mathbb{R}^2$, called the *generating vectors* of the cone. In the following we will often consider the intersection of a cone $\mathcal{K}$ with $\mathbb{Z}\times\mathbb{Z}$; whenever it will be clear from the context, we will be denoting this set with $\mathcal{K}$ instead of with $\mathcal{K}\cap\mathbb{Z}\times\mathbb{Z}$.

We denote with $\overline{\mathcal{P}_1}$ the closed unit polydisk:

$$\overline{\mathcal{P}_1}:=\{(\alpha,\beta)\in\mathbb{C}\times\mathbb{C}\mid|\alpha|\leq 1,|\beta|\leq 1\}$$

Given a set $\mathcal{S}\subset\mathbb{Z}\times\mathbb{Z}$, its *(discrete) convex hull* is the intersection of the convex hull of $\mathcal{S}$ (seen as a subset of $\mathbb{R}\times\mathbb{R}$) and of $\mathbb{Z}\times\mathbb{Z}$. In the following we will also refer to the (discrete) convex hull associated with an element $p\in\mathbb{R}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$, meaning the (discrete) convex hull of the *support of* p, i.e. the set

$$\mathrm{supp}(p):=\{(h,k)\in\mathbb{Z}\times\mathbb{Z}\mid\text{ the coefficient of }\xi_1^h\xi_2^k\text{ in }p(\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1})\text{ is }\neq 0\}$$

We denote with $\mathbb{W}^{\mathbb{T}}$ the set consisting of all trajectories from $\mathbb{T}$ to $\mathbb{W}$. We denote with $\sigma_1,\ \sigma_2$ the *shift operators* defined as

$$\begin{aligned}\sigma_i:\left(\mathbb{R}^{\mathtt{w}}\right)^{\mathbb{Z}\times\mathbb{Z}}&\rightarrow\left(\mathbb{R}^{\mathtt{w}}\right)^{\mathbb{Z}\times\mathbb{Z}}\quad i=1,2\\(\sigma_1 w)(k_1,k_2)&:=w(k_1-1,k_2)\\(\sigma_2 w)(k_1,k_2)&:=w(k_1,k_2-1)\end{aligned}$$

2 2-D behaviors

The purpose of this section is to introduce the reader to those concepts of 2-D behavioral system theory which are most relevant for the purposes of the paper; see Rocha (1990) for a thorough introductory treatment of the subject, and the other publications in the reference list for more detailed information on specific topics.

We call $\mathfrak{B}$ a *linear discrete-time complete 2-D behavior* if it is the solution set of a system of linear, constant-coefficient difference equations with two independent variables; more precisely, if $\mathfrak{B}$ is the subset of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}}$ consisting of all solutions to

$$R(\sigma_1, \sigma_2)w = 0 \tag{1}$$

where $R \in \mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}]$. We call (1) a *kernel representation* of $\mathfrak{B}$. We denote the set consisting of all linear discrete-time complete 2-D behaviors with $\mathtt{w}$ external variables with $\mathcal{L}_2^{\mathtt{w}}$.

$\mathfrak{B} \in \mathcal{L}_2^{\mathtt{w}}$ is *autonomous* if there exists a proper cone $\mathcal{K} \subset \mathbb{R} \times \mathbb{R}$ such that

$$\left[w_1, w_2 \in \mathfrak{B} \text{ and } w_{1|\mathcal{K}\cap\mathbb{Z}\times\mathbb{Z}} = w_{2|\mathcal{K}\cap\mathbb{Z}\times\mathbb{Z}}\right] \Longrightarrow [w_1 = w_2]$$

Such a cone $\mathcal{K}$ will be called a proper *characteristic* cone for $\mathfrak{B}$. Observe that if $\mathcal{K}$ is characteristic for $\mathfrak{B}$ and if $w \in \mathfrak{B}$ is such that $w_{|\mathcal{K}\cap\mathbb{Z}\times\mathbb{Z}} = 0$, then $w = 0$ (see Lemma 2.3 of Valcher 2000).

Proper characteristic cones play an important role in the definition of stability of a 2-D system according to Valcher, and we now proceed to characterize them algebraically, following closely reference Valcher (2000). The following result holds.

Theorem 1 *Let* $\mathfrak{B} \in \mathcal{L}_2^{\mathtt{w}}$ *be autonomous, and let* $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$ *for some* $R \in \mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}]$. *Assume there exist* $H \in \mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}]$ *right factor prime, and* $S \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}]$ *nonsingular, such that* $R = H \cdot S$.

Moreover, denote $\delta := \det(S) \in \mathbb{R}[\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}]$. *The following statements are equivalent:*

1. *The proper cone* $\mathcal{K}$ *is characteristic for* $\mathfrak{B}$*;*
2. *The proper cone* $\mathcal{K}$ *is characteristic for* $\ker S(\sigma_1, \sigma_2)$*;*
3. *The proper cone* $\mathcal{K}$ *is characteristic for* $\ker \delta(\sigma_1, \sigma_2)$*;*
4. *The discrete convex hull* $\mathcal{H}_\delta$ *of* δ *satisfies the following two conditions:*
   - *4a.* $-\mathcal{H}_\delta \subset \mathcal{K}$*;*
   - *4b.* $-\mathcal{H}_\delta \subset \mathcal{K}$ *intersects the generating lines of* $\mathcal{K}$ *only in* $(0, 0)$.

Proof The equivalence of statements (1) and (2) follows from Proposition 2.6 of Valcher (2000), and that between (1) and (3) follows from Proposition 2.8 of Valcher (2000). The equivalence of (3) and (4) follows from Proposition 2.9 of Valcher (2000). □

Remark 1 The factorization $R = H \cdot S$ in the statement of Theorem 1, and consequently the square factor S itself, is not unique. However, it can be shown that if $R = H' \cdot S'$ with H' right factor prime and S' square, then $\ker S(\sigma_1, \sigma_2) = \ker S'(\sigma_1, \sigma_2)$, i.e. the *behavior* associated with the square factor is the same for every factorization.

It can be shown (see Fornasini et al. 1993) that if $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$ for some right factor prime matrix $R \in \mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}]$, then $\mathfrak{B}$ is autonomous and finite-dimensional. It can be shown (see Lemma 2.4 of Valcher 2000) that in this case, every proper cone is characteristic for $\mathfrak{B}$.

If $\mathfrak{B}$ is autonomous, and $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$ for some square nonsingular Laurent matrix R, then $\mathfrak{B}$ is called a *square autonomous behavior*. Observe that the result of Theorem 1 shows that for any autonomous behavior $\mathfrak{B}$ whose kernel representation can be factored as HS with H right factor prime and S nonsingular, the characteristic cone is determined by its "square autonomous part" $\ker S(\sigma_1, \sigma_2)$.

We now discuss in some detail the concept of stability introduced by Valcher (2000). In order to do so, we need to distinguish the finite-dimensional and the square autonomous cases. In the former case, where each $w \in \mathfrak{B}$ is uniquely determined by its values in a finite subset of $\mathbb{Z} \times \mathbb{Z}$, the definition is as follows.

Definition 1 Let $\mathfrak{B} \in \mathcal{L}_2^{\mathtt{w}}$ be autonomous and finite-dimensional, and let $\mathcal{K}$ be any proper cone in $\mathbb{R} \times \mathbb{R}$. $\mathfrak{B}$ is $\mathcal{K}$-*stable* if

$$[w \in \mathfrak{B}] \Longrightarrow \left[\lim_{\substack{(i,j)\in\mathcal{K} \\ |i|+|j|\to+\infty}} \|w(i,j)\| = 0 \right]$$

The following algebraic characterization of finite-dimensional stable behaviors (see Theorem 3.3 p. 297 of Valcher 2000) holds. In order to avoid cumbersome details, we follow Valcher (2000), and only consider proper cones generated by unimodular integer matrices, which are then isomorphic to the first orthant of $\mathbb{Z} \times \mathbb{Z}$, in the sense that there exists a nonsingular square matrix $T : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ such that $T(\mathcal{K})$ is the first orthant.

Theorem 2 *Let* $\mathfrak{B} = \ker H(\sigma_1, \sigma_2)$*, with*

$$H(\xi_1, \xi_2) = \sum_{\ell,m} H_{\ell,m} \xi_1^{\ell} \xi_2^{m} \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}]$$

right-factor prime, and let $\mathcal{K}$ *be a proper cone isomorphic to the first orthant. Denote the transformation mapping* $\mathcal{K}$ *to the first orthant with* T*, and with* $(t_1(\ell,m), t_2(\ell,m))$ *the image of* $(\ell,m) \in \mathbb{Z} \times \mathbb{Z}$ *under* T*. Define*

$$H_T(\xi_1, \xi_2) := \sum_{\ell,m} H_{\ell,m} \xi_1^{t_1(\ell,m)} \xi_2^{t_2(\ell,m)}$$

Then the following two statements are equivalent:

1. $\mathfrak{B}$ *is* $\mathcal{K}$*-stable;*
2. *Every* (α, β) *in the Laurent variety of the maximal order minors of* H_T *satisfies* $|\alpha| > 1$ *and* $|\beta| > 1$.

The definition of stability in the square autonomous case takes into account the fact that since the set of points in which a trajectory can be freely assigned is infinite, it may happen that particular choices of the "initial conditions" correspond to trajectories of the behavior which do not die out within a proper characteristic cone $\mathcal{K}$. In order to state the definition of stability for the square case, we need to introduce the following notation: given a proper cone $\mathcal{K}$, we denote with $\delta(-\mathcal{K})$ the *boundary* of $-\mathcal{K}$, i.e. the generating lines of $-\mathcal{K}$. Moreover, we denote with $(\delta(-\mathcal{K}))^n$ the set consisting of the points of $\mathbb{Z} \times \mathbb{Z}$ whose distance from $\delta(-\mathcal{K})$ is not greater than n:

$$(\delta(-\mathcal{K}))^n := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid \min\{|i-h| + |j-k| \text{ with } (h,k) \in \delta(-\mathcal{K})\} \leq n\}$$

The definition of $\mathcal{K}$-stable square autonomous behavior is as follows.

Definition 2 Let $\mathcal{K}$ be a proper cone such that $-\mathcal{K}$ is characteristic for a square autonomous behavior $\mathfrak{B} \in \mathcal{L}_2^{\mathtt{w}}$. $\mathfrak{B}$ is $\mathcal{K}$-stable if there exists some positive integer n such that

$$\left[w \in \mathfrak{B},\, w \text{ bounded in } (\delta(-\mathcal{K}))^n\right] \Longrightarrow \left[\lim_{\substack{(i,j)\in\mathcal{K} \\ |i|+|j|\to+\infty}} \|w(i,j)\| = 0 \right]$$

The following is an algebraic characterization of $\mathcal{K}$-stability (see Theorem 3.6 of Valcher (2000) for a proof).

Theorem 3 *Let* $\mathfrak{B} = \ker S(\sigma_1, \sigma_2)$ *be a square autonomous behavior, and let* $\mathcal{K}$ *be a proper cone for* $\mathfrak{B}$ *which is* T*-isomorphic to the first orthant. Denote* $\delta := \det(S)$*, and assume w.l.o.g. that* $\mathcal{H}_\delta \subset \mathcal{K}$ *and that* $\mathcal{H}_\delta \cap \delta\mathcal{K} = \{(0,0)\}$*. Denote with* $(t_1(\ell, m), t_2(\ell, m))$ *the image of* $(\ell, m) \in \mathbb{Z} \times \mathbb{Z}$ *under* T*. Define*

$$S_T(\xi_1, \xi_2) := \sum_{\ell,m} S_{\ell,m} \xi_1^{t_1(\ell,m)} \xi_2^{t_2(\ell,m)}$$

Then the following two statements are equivalent:

1. $\mathfrak{B}$ *is* $\mathcal{K}$*-stable;*
2. *The Laurent variety of* $\det\ S_T$ *does not intersect the closed unit polydisk* $\overline{\mathcal{P}_1}$.

In Sect. 4 we will establish an equivalent characterization of $\mathcal{K}$ stability for the square autonomous case, which will be useful for the purposes of computing Lyapunov functions for a given behavior. We proceed in the next section to review some important concepts related to quadratic difference forms.

3 Bilinear- and quadratic difference forms for 2-D systems

In many modeling and control problems for linear systems it is necessary to study bilinear- and quadratic functionals of the system variables and their shifts (or their derivatives in the continuous-time case). For finite-dimensional continuous-time linear systems, an efficient representation for such functionals by means of two-variable polynomial matrices was introduced in Willems and Trentelman (1998); in order to represent bilinear- and quadratic functionals of the variables of continuous-time 2-D-systems, four-variable polynomial matrices are used (see Pillai and Willems 2002).

In the 1-D discrete-time case, *quadratic difference forms* have been introduced in Kaneko and Fujii (2000). We now examine the extension of quadratic difference forms to the 2-D discrete setting; some preliminary results in this sense have been obtained in Kojima and Takaba (2006).

In order to simplify the notation, define the multi-indices $\mathbf{k} := (k_1, k_2)$, $\mathbf{l} := (l_1, l_2)$, and the notation $\zeta := (\zeta_1, \zeta_2)$ and $\eta := (\eta_1, \eta_2)$, and define $\zeta^{\mathbf{k}}\eta^{\mathbf{l}} := \zeta_1^{k_1}\zeta_2^{k_2}\eta_1^{l_1}\eta_2^{l_2}$.

Let $\mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$ denote the set of real $\mathtt{w}_1 \times \mathtt{w}_2$ polynomial matrices in the four indeterminates ζ_i and η_i, $i = 1, 2$; that is, an element of $\mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$ is of the form

$$\Phi(\zeta, \eta) = \sum_{\mathbf{k},\mathbf{l}} \Phi_{\mathbf{k},\mathbf{l}} \zeta^{\mathbf{k}} \eta^{\mathbf{l}}$$

where $\Phi_{\mathbf{k},\mathbf{l}} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$; the sum ranges over the nonnegative multi-indices $\mathbf{k}$ and $\mathbf{l}$, and is assumed to be finite. Such matrix induces a *bilinear difference form* (*BDF* in the following) L_Φ

$$L_\Phi : (\mathbb{R}^{\mathtt{w}_1})^{\mathbb{Z}\times\mathbb{Z}} \times (\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}\times\mathbb{Z}} \longrightarrow (\mathbb{R})^{\mathbb{Z}\times\mathbb{Z}}$$
$$L_\Phi(v, w) := \sum_{\mathbf{k},\mathbf{l}} (\sigma^{\mathbf{k}} v)^\top \Phi_{\mathbf{k},\mathbf{l}} (\sigma^{\mathbf{l}} w)$$

where the $\mathbf{k}$-th shift operator $\sigma^{\mathbf{k}}$ is defined as $\sigma^{\mathbf{k}} := \sigma_1^{k_1}\sigma_2^{k_2}$, and analogously for $\sigma^{\mathbf{l}}$.

A four-variable polynomial matrix $\Phi(\zeta_1, \zeta_2, \eta_1, \eta_2) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ is called *symmetric* if $\Phi(\zeta_1, \zeta_2, \eta_1, \eta_2) = \Phi(\eta_1, \eta_2, \zeta_1, \zeta_2)^\top$, concisely written as $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top$. In this case, Φ induces also a quadratic functional

$$\begin{aligned} Q_\Phi &: (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}} \longrightarrow (\mathbb{R})^{\mathbb{Z}\times\mathbb{Z}} \\ Q_\Phi(w) &:= L_\Phi(w, w) \end{aligned}$$

We will call Q_Φ the *quadratic difference form* (in the following abbreviated with QDF) associated with the four-variable polynomial matrix Φ.

In this paper we also consider "vectors" of four-variable polynomial matrices $\Psi \in (\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta, \eta])^2$, i.e.

$$\Psi(\zeta, \eta) = \begin{bmatrix} \Psi_1(\zeta, \eta) \\ \Psi_2(\zeta, \eta) \end{bmatrix} =: \mathrm{col}(\Psi_i(\zeta, \eta))_{i=1,2}$$

with $\Psi_i \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta, \eta]$ and with $\mathrm{col}(A_i)_{i=1,2}$ the matrix obtained by stacking the two matrices A_i, both with the same number of columns, on top of each other. Such Ψ induces a *vector bilinear difference form* (abbreviated with *VBDF*), defined as

$$\begin{aligned} L_\Psi &: (\mathbb{R}^{\mathtt{w}_1})^{\mathbb{Z}\times\mathbb{Z}} \times (\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}\times\mathbb{Z}} \longrightarrow (\mathbb{R}^2)^{\mathbb{Z}\times\mathbb{Z}} \\ L_\Psi(v, w) &:= \begin{bmatrix} L_{\Psi_1}(v, w) \\ L_{\Psi_2}(v, w) \end{bmatrix} = \mathrm{col}(L_{\Psi_i}(v, w))_{i=1,2}. \end{aligned}$$

Finally, we introduce the notion of (discrete) divergence of a VBDF. Given a VBDF $L_\Psi = \mathrm{col}(L_{\Psi_1}, L_{\Psi_2})$, we define its *divergence* as the BDF defined by

$$\begin{aligned} (\nabla L_\Psi)(w_1, w_2) := &\left(L_{\Psi_1}(w_1, w_2) - \sigma_1(L_{\Psi_1}(w_1, w_2))\right) \\ &+ \left(L_{\Psi_2}(w_1, w_2) - \sigma_2(L_{\Psi_2}(w_1, w_2))\right) \end{aligned} \tag{2}$$

for all w_1, w_2. If L_Φ is the divergence of $L_\Psi = \mathrm{col}(L_{\Psi_1}, L_{\Psi_2})$, it is straightforward to verify that in terms of the four-variable polynomial matrices associated with the BDF's, their relationship is

$$\Phi(\zeta_1, \zeta_2, \eta_1, \eta_2) = (1 - \zeta_1\eta_1)\Psi_1(\zeta_1, \zeta_2, \eta_1, \eta_2) + (1 - \zeta_2\eta_2)\Psi_2(\zeta_1, \zeta_2, \eta_1, \eta_2);$$

which we write in shorthand as

$$\Phi = \mathrm{div}\ \mathrm{col}(\Psi_1, \Psi_2).$$

In order to characterize those BDFs which are the divergence of some VBDF, we need to introduce the "del" operator, defined as

$$\begin{aligned} \partial &: \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta_1, \zeta_2, \eta_1, \eta_2] \longrightarrow \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}] \\ \partial\Phi(\xi_1, \xi_2) &:= \Phi(\xi_1^{-1}, \xi_2^{-1}, \xi_1, \xi_2). \end{aligned}$$

The following result holds true.

Proposition 1 *A BDF L_Φ is the divergence of some VBDF L_Ψ if and only if*

$$\partial\Phi(\xi_1, \xi_2) = 0.$$

Proof That the condition $\partial\Phi(\xi_1, \xi_2) = 0$ is necessary follows immediately from the definition of discrete divergence, and its expression in terms of four-variable polynomial matrices. We now prove sufficiency. Observe first that the polynomials $1 - \zeta_1\eta_1$ and $1 - \zeta_2\eta_2$ form a Gröbner basis for the ideal generated by them (see Becker and Weispfenning 1993 for a thorough introduction to Gröbner bases). Now let $p \in \mathbb{R}[\zeta_1, \zeta_2, \eta_1, \eta_2]$, and consider that the normal form of p modulo $1-\zeta_1\eta_1$ and $1-\zeta_2\eta_2$ only involves linear combinations of the terms $\zeta_k,\ \eta_k,\ k=1,2$, and $\zeta_i\eta_k$, for $i,k=1,2$ with $i \neq k$. Observe that the image under ∂ of this normal form is zero if and only if the coefficients of the linear combination are all zero. Conclude that if $p \in \mathbb{R}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ is such that $\partial p = 0$, then necessarily its normal form modulo $1-\zeta_1\eta_1$ and $1-\zeta_2\eta_2$ is zero, i.e. there exist polynomials $\psi_i \in \mathbb{R}[\zeta_1, \zeta_2, \eta_1, \eta_2]$, $i=1,2$, such that $p(\zeta_1, \zeta_2, \eta_1, \eta_2) = (1-\zeta_1\eta_1)\psi_1(\zeta_1, \zeta_2, \eta_1, \eta_2) + (1-\zeta_2\eta_2)\psi_2(\zeta_1, \zeta_2, \eta_1, \eta_2)$. This argument can be extended entrywise to polynomial matrices. This concludes the proof. □

The definition and properties described above can be adapted to a vector quadratic difference form (VQDF) in an obvious manner.

We now introduce the notion of positivity of a QDF. We define a QDF Q_Δ induced by a four-variable polynomial matrix $\Delta \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ to be *nonnegative* if $Q_\Delta(w(k_1,k_2)) \geq 0$ for all $(k_1,k_2) \in \mathbb{Z}\times\mathbb{Z}$ and for all $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}}$. This will be denoted with $Q_\Delta \geq 0$ or $\Delta(\zeta,\eta) \geq 0$. We call Q_Δ positive, denoted $Q_\Delta > 0$ or $\Delta(\zeta,\eta) > 0$, if $Q_\Delta \geq 0$ and $Q_\Delta(w(k_1,k_2)) = 0$ for all (k_1,k_2) implies $w=0$. Often in the following we will also consider QDFs induced by matrices of the form $\Delta(e^{-i\omega}, \zeta_2, e^{i\omega}, \eta_2)$, i.e. matrices in the indeterminates ζ_2, η_2 with coefficients being polynomials in $e^{i\omega}$ for some $\omega \in \mathbb{R}$. The definition of nonnegativity and positivity in this case is readily adapted from the above definition.

Finally, we define the equivalence of QDFs along a behavior. Let $\mathfrak{B} \in \mathcal{L}_2^{\mathtt{w}}$ and $\Phi_i \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$, $i=1,2$. Then Q_{Φ_1} is *equivalent modulo* $\mathfrak{B}$ *to* Q_{Φ_2}, denoted $Q_{\Phi_1} \stackrel{\mathfrak{B}}{=} Q_{\Phi_2}$, if $Q_{\Phi_1}(w) = Q_{\Phi_2}(w)$ for all $w \in \mathfrak{B}$. Now let $\mathfrak{B} = \ker R(\sigma_1,\sigma_2)$; then it can be shown that $Q_{\Phi_1} \stackrel{\mathfrak{B}}{=} Q_{\Phi_2}$ if and only if there exists $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ such that

$$\begin{aligned}\Phi_1(\zeta_1, \zeta_2, \eta_1, \eta_2) = \Phi_2(\zeta_1, \zeta_2, \eta_1, \eta_2)& \\ + R^\top(\zeta_1, \zeta_2)X(\zeta_1, \zeta_2, \eta_1, \eta_2)& \\ + X^\top(\eta_1, \eta_2, \zeta_1, \zeta_2)R(\eta_1, \eta_2)&\end{aligned}$$

(see Proposition 10 in Kojima et al. 2007). In this case we also write

$$\Phi_1(\zeta_1, \zeta_2, \eta_1, \eta_2) = \Phi_2(\zeta_1, \zeta_2, \eta_1, \eta_2) \bmod R,$$

or $\Phi_1(\zeta_1, \zeta_2, \eta_1, \eta_2) - \Phi_2(\zeta_1, \zeta_2, \eta_1, \eta_2) = 0 \bmod R$.

4 Necessary and sufficient Lyapunov conditions for stability of 2-D systems

In this section we establish the main result of this paper, a necessary and sufficient condition for an autonomous square behavior $\mathfrak{B} \in \mathcal{L}_2^{\mathtt{w}}$ to be asymptotically stable expressed in the language of Lyapunov functions. Strengthened by the result of Theorem 3, which allows us to bring $\mathcal{K}$-stability for a general proper cone $\mathcal{K}$ back to stability on the first orthant, in this and the remaining sections of this paper we concentrate on stability with respect to the proper cone consisting of the first orthant of $\mathbb{Z}\times\mathbb{Z}$. We will denote this set with $\mathcal{K}_0$ in the following. Moreover, in this paper, we concentrate on the case of square autonomous systems; the case of finite dimensional 2-D systems will be treated elsewhere.

We begin this section with a straightforward but important refinement of Proposition 3.5 of Valcher (2000).

Proposition 2 *Let $\mathfrak{B} \in \mathcal{L}_2^{\mathtt{w}}$ be square and autonomous, and let $\mathfrak{B} = \ker S(\sigma_1, \sigma_2)$ with $S \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1, \xi_2]$ nonsingular. Assume that $\delta := \det S$ is such that $\mathcal{H}_\delta$ is a subset of $\mathcal{K}_0$, the first orthant of $\mathbb{Z} \times \mathbb{Z}$, that intersects the coordinate axes only in the origin. Then the following statements are equivalent:*

1. *$\mathfrak{B}$ is $\mathcal{K}_0$-stable;*
2. *For all $\omega \in \mathbb{R}$, the polynomial $\delta(e^{j\omega}, \xi_2)$ has all its roots outside of the closed unit disk $\{z_2 \in \mathbb{C} \mid |z_2| \geq 1\}$, and the polynomial $\delta(\xi_1, e^{j\omega})$ has all its roots outside of the closed unit disk $\{z_1 \in \mathbb{C} \mid |z_1| \geq 1\}$.*

Proof The proof follows from Theorem 3 and from the equivalence of statements i) and iv) in Proposition 3.1 of Geronimo and Woerdeman (2006). □

The result of Proposition 2 shows that in order to check the stability of a square autonomous behavior, the stability of two families of complex polynomials depending on the parameter $\omega \in \mathbb{R}$ needs to be tested. In Geronimo and Woerdeman (2006) it has been proposed to perform this test using an ω-dependent family of complex Hermitian matrices analogous to the Bézoutian used in the case of univariate polynomials (see Chap. 8 of Fuhrmann 1996). We now state an equivalent condition in terms of a pair of quadratic difference forms satisfying a Lyapunov-type equation. In order to do this, we introduce first some notation; in the following we denote with $\mathrm{Per}_2 \subset (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}}$ the set consisting of all trajectories $v \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}}$ such that the restriction of v to the lines $\{(i, j) \mid j \in \mathbb{Z}\}$ is periodic for all $i \in \mathbb{Z}$, i.e.

$$\mathrm{Per}_2 := \left\{ v \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}} \mid v(i, \cdot) \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}} \text{ is periodic for all } i \in \mathbb{Z} \right\}$$

and analogously we define

$$\mathrm{Per}_1 := \left\{ v \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}} \mid v(\cdot, j) \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}} \text{ is periodic for all } j \in \mathbb{Z} \right\}.$$

Moreover, we need to introduce the concept of *characteristic variety of a behavior* $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$. Let $(\lambda, \mu) \in \mathbb{C}^2$ be in the characteristic variety of R, denoted $\mathcal{C}(R)$. Observe that $\mathcal{C}(R) = \mathcal{C}(R')$ for any polynomial matrix R' inducing a kernel representation of $\mathfrak{B}$; consequently, it is also correct to speak about the characteristic variety of $\mathfrak{B}$, denoted with $\mathcal{C}(\mathfrak{B})$.

The main result of this section is the following.

Theorem 4 *Let $\mathfrak{B}$ be a 2-D square autonomous linear behavior, and let $\mathfrak{B} = \ker R(\sigma_1, \sigma_2)$. Then the following statements are equivalent:*

1. *$\mathfrak{B}$ is $\mathcal{K}_0$-stable.*
2. *There exists a VQDF $Q_\Phi = \mathrm{col}(Q_{\Phi_1}, Q_{\Phi_2})$ and a QDF Q_Δ such that*
   - *(2a) $\nabla Q_\Phi \overset{\mathfrak{B}}{=} -Q_\Delta$;*
   - *(2b) $Q_{\Phi_1}(w), Q_\Delta(w) > 0$ for all $w \in \mathfrak{B} \cap \mathrm{Per}_2$, and $Q_{\Phi_2}(w), Q_\Delta(w) > 0$ for all $w \in \mathfrak{B} \cap \mathrm{Per}_1$.*
3. *There exist $\Phi = \mathrm{col}(\Phi_1, \Phi_2)$ and Δ, with $\Phi_1, \Phi_2, Y \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$, $\Delta \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ such that*
   - *(3a)* $$(1 - \zeta_1\eta_1)\Phi_1(\zeta_1, \zeta_2, \eta_1, \eta_2) + (1 - \zeta_2\eta_2)\Phi_2(\zeta_1, \zeta_2, \eta_1, \eta_2) = -\Delta(\zeta_1, \zeta_2, \eta_1, \eta_2) + R(\zeta_1, \zeta_2)^\top Y(\zeta_1, \zeta_2, \eta_1, \eta_2) + Y(\eta_1, \eta_2, \zeta_1, \zeta_2)^\top R(\eta_1, \eta_2);$$
   - *(3b)* $$\Phi_1(\zeta_1, \zeta_2, \eta_1, \eta_2) \overset{\mathfrak{B}\cap\mathrm{Per}_2}{>} 0,\ \Phi_2(\zeta_1, \zeta_2, \eta_1, \eta_2) \overset{\mathfrak{B}\cap\mathrm{Per}_1}{>} 0,\ \Delta(\zeta_1, \zeta_2, \eta_1, \eta_2) \overset{\mathfrak{B}\cap\mathrm{Per}_i}{>} 0,\ i = 1, 2.$$

We refer to the VQDF $Q_\Phi = \mathrm{col}(Q_{\Phi_1}, Q_{\Phi_2})$ satisfying (2a) and (2b) as "a Lyapunov function of $\mathfrak{B}$".

Proof Using the calculus of QDFs it is easy to see that the statements (2) and (3) are equivalent. We will consequently prove the equivalence of (3) and (1).

We first prove the implication (3) $\Longrightarrow$ (1). In the following it will be convenient to consider behaviors $\mathfrak{B}$ whose trajectories take values in $\mathbb{C}^{\mathrm{w}}$, obtained e.g. by complexification of a real behavior $\mathfrak{B}' \in \mathcal{L}_2^{\mathrm{w}}$:

$$[w \in \mathfrak{B}] \iff \left[\text{both real and imaginary part of } w \text{ belong to } \mathfrak{B}'\right].$$

Let $(\lambda, \mu) \in \mathbb{C}^2$ be in the characteristic variety of $\mathfrak{B}$; then there exists a vector $v \in \mathbb{C}^{\mathrm{w}}$ depending on λ and μ such that the trajectory w defined by $w(k_1, k_2) := v\, \lambda^{k_1}\, \mu^{k_2}$ belongs to $\mathfrak{B}$. It is easy to see that v is such that $R(\lambda, \mu)v = 0$, i.e. $v \in \ker R(\lambda, \mu)$.

We now prove that if μ lies on the unit circle, i.e. $\mu = e^{i\omega}$ for some $\omega \in \mathbb{R}$, then $|\lambda| > 1$. Once this will have been established, statement (1) follows from Proposition 2.

Let $\zeta_1 = \overline{\lambda},\ \eta_1 = \lambda,\ \zeta_2 = \overline{\mu} = e^{-i\omega},\ \eta_2 = \mu = e^{i\omega}$ in (3*a*), and multiply the resulting expression on the left by $v^\top$ and on the right by v. It follows from the fact that $v \in \ker R(\lambda, \mu)$ that

$$(1 - \overline{\lambda}\lambda)\ v^\top \Phi_1(\overline{\lambda}, e^{-i\omega}, \lambda, e^{i\omega})v = -v^\top \Delta(\overline{\lambda}, e^{-i\omega}, \lambda, e^{i\omega})v$$

The right-hand side of this equation is strictly negative; on the left-hand side it holds that $v^\top \Phi_1(\overline{\lambda}, e^{-i\omega}, \lambda, e^{i\omega})v > 0$, and consequently it follows that $1 - \overline{\lambda}\lambda < 0$. An analogous argument is used when $w(k_1, k_2) = v\, e^{i\omega k_1} \mu^{k_2}$. This proves the claim.

The proof of implication (1) $\Longrightarrow$ (3) is established by producing matrices $\Phi_i \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$, $i = 1, 2$, and $\Delta \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ such that $(3a) - (3b)$ hold. In order to do this, we reduce ourselves to the scalar case as follows.

Denote with $\mathrm{Adj}(R)(\xi_1, \xi_2)$ the adjoint matrix of $R(\xi_1, \xi_2)$, i.e.

$$\mathrm{Adj}(R)(\xi_1, \xi_2)\ R(\xi_1, \xi_2) = d(\xi_1, \xi_2)\ I_{\mathrm{w}}, \tag{3}$$

with $d(\xi_1, \xi_2) := \det(R(\xi_1, \xi_2))$. Now define $\mathfrak{B}' := \ker d(\sigma_1, \sigma_2)\ I_{\mathrm{w}}$, and note that because of (3), $\mathfrak{B}' \supset \mathfrak{B}$. Note also that $w' \in \mathfrak{B}'$ if and only if each of the components w_i' of w' satisfies the scalar difference equation $d(\sigma_1, \sigma_2)w_i' = 0,\ i = 1, \ldots, \mathrm{w}$. Consequently, because of Proposition 2, $\mathfrak{B}'$ is $\mathcal{K}_0$-stable if and only if so $\mathfrak{B}$ is. Moreover, since $\mathfrak{B}' \supset \mathfrak{B}$ any pair of functionals satisfying statement (3) for $\mathfrak{B}'$ also satisfy it for $\mathfrak{B}$.

We now consider the scalar square behavior $\ker d(\sigma_1, \sigma_2)$, and construct a pair of QDFs satisfying statement (3); the case $\mathrm{w} > 1$, i.e. the case of $\mathfrak{B}'$, will follow in a straightforward manner.

Write $d(\xi_1, \xi_2) = \sum_{i=0}^{L_1} d_i(\xi_2)\xi_1^{L_1} = \sum_{i=0}^{L_2} d_i'(\xi_1)\xi_2^{L_2}$, where L_i is the highest power of ξ_i in d, $i = 1, 2$. Define the four-variable polynomial

$$\gamma(\zeta_1, \zeta_2, \eta_1, \eta_2) := d(\zeta_1, \zeta_2)d(\eta_1, \eta_2) - \zeta_1^{L_1}\zeta_2^{L_2}\eta_1^{L_1}\eta_2^{L_2} d(\eta_1^{-1}, \eta_2^{-1})d(\zeta_1^{-1}, \zeta_2^{-1}), \tag{4}$$

and

$$y(\xi_1, \xi_2) := \frac{1}{2} d(\xi_1, \xi_2)$$
$$\delta(\zeta_1, \zeta_2, \eta_1, \eta_2) := \zeta_1^{L_1}\eta_1^{L_1}\zeta_2^{L_2}\eta_2^{L_2} d(\eta_1^{-1}, \eta_2^{-1})d(\zeta_1^{-1}, \zeta_2^{-1});$$

then it is immediate to verify that

$$\gamma(\zeta_1, \zeta_2, \eta_1, \eta_2) = -\delta(\zeta_1, \zeta_2, \eta_1, \eta_2) + y(\zeta_1, \zeta_2)d(\eta_1, \eta_2) + d(\zeta_1, \zeta_2)y(\eta_1, \eta_2).$$

From (4) it follows that $\partial\gamma = 0$; conclude from Proposition 1 that there exists $\phi = \mathrm{col}(\phi_1, \phi_2) \in \mathbb{R}^{2\times 1}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ such that $\mathrm{div}\,\phi(\zeta_1, \zeta_2, \eta_1, \eta_2) = \gamma(\zeta_1, \zeta_2, \eta_1, \eta_2)$. This proves (3*a*).

In order to prove (3*b*) we proceed as follows. Observe that

$$(1-\zeta_1\eta_1)\phi_1(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega}) = d(\zeta_1, e^{-i\omega})y(\eta_1, e^{i\omega}) + y(\zeta_1, e^{-i\omega})d(\eta_1, e^{i\omega}) \\ -\delta(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega}). \tag{5}$$

Following Kojima and Takaba (2005) (see Eq. 4 therein) we call (5) a *ω-dependent 1-D two-variable polynomial Lyapunov equation*.

From Proposition 2 it follows that since $\mathfrak{B}$ is $\mathcal{K}_0$-stable, for all $\omega \in \mathbb{R}$ the polynomial $d(\xi_1, e^{i\omega})$ is anti-Schur, i.e. all its roots have modulus greater than one. Now from the fact that

$$\delta(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega}) = \zeta_1^{L_1}\eta_1^{L_1} d(\zeta_1^{-1}, e^{-i\omega}) d(\eta_1^{-1}, e^{i\omega})$$

is "square", it follows that $\delta(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega}) \geq 0$ for all $\omega \in \mathbb{R}$. From the fact that $\det\ \xi_1^{L_1} R(\xi_1^{-1}, e^{i\omega})$ is Schur, conclude that $\delta(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega}) \overset{\mathfrak{B}_d\cap \mathrm{Per}_2'}{>} 0$, where $\mathfrak{B}_d = \ker d(\sigma_1, \sigma_2)$ and

$$\mathrm{Per}_2' := \left\{ v \in (\mathbb{R})^{\mathbb{Z}\times\mathbb{Z}} \mid v(i, \cdot) \in (\mathbb{R})^{\mathbb{Z}} \text{ is periodic for all } i \in \mathbb{Z} \right\}.$$

Now apply Theorem 1 of Kojima and Takaba (2005) to conclude that $\phi_1(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega}) \overset{\mathfrak{B}_d\cap \mathrm{Per}_2'}{\geq} 0$. In order to prove that $\phi_1(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega}) \overset{\mathfrak{B}_d\cap \mathrm{Per}_2'}{>} 0$, assume by contradiction that there exists a trajectory in $\mathfrak{B}_d \cap \mathrm{Per}_2'$ along which the QDF induced by $\phi_1(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega})$ is zero; then from (5) it follows that also the QDF induced by $\delta(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega})$ is zero along the same trajectory, a contradiction with the result $\delta(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega}) \overset{\mathfrak{B}_d\cap \mathrm{Per}_2'}{>} 0$ established previously. This proves half of the claim (3*b*).

An analogous argument based on the same considerations and on the fact that $d(e^{i\omega}, \xi_2)$ is anti-Schur for all $\omega \in \mathbb{R}$ can be used for proving the remaining half of (3*b*).

Now in order to prove (3*a*)-(3*b*) for $\mathfrak{B}'$, define $\Phi_1 := \phi_1\ I_{\mathrm{w}},\ \Phi_2 := \phi_2\ I_{\mathrm{w}},\ \Delta = \delta\ I_{\mathrm{w}}$. The inclusion $\mathfrak{B}' \supset \mathfrak{B}$ implies that $\Phi_1 := \phi_1\ I_{\mathrm{w}},\ \Phi_2 := \phi_2\ I_{\mathrm{w}},\ \Delta = \delta\ I_{\mathrm{w}}$ also satisfy $(3a) - (3b)$ for $\mathfrak{B}$. This concludes the proof of the claim.

We now illustrate the result of Theorem 4 with an example.

Example 1 Consider the system described in kernel form by the matrix

$$R(\xi_1, \xi_2) = \begin{bmatrix} 1 + \frac{\xi_1}{2} + \frac{\xi_2}{4} & 1 + \frac{\xi_1}{2} + \frac{\xi_2}{4} \\ 1 - \frac{\xi_1}{4} - \frac{\xi_2}{2} & 2 - \frac{\xi_1}{2} - \xi_2 \end{bmatrix}$$

Define the matrices

$$\Phi_1(\zeta_1, \zeta_2, \eta_1, \eta_2) := \begin{bmatrix} \frac{1}{4}(-\eta_2 - \zeta_2 + 8\eta_2\zeta_2) & \frac{1}{16}(3 - 12\eta_2 - 12\zeta_2 + 48\eta_2\zeta_2) \\ \frac{1}{16}(3 - 12\eta_2 - 12\zeta_2 + 48\eta_2\zeta_2) & \frac{1}{16}(9 - 28\eta_2 - 28\zeta_2 + 80\eta_2\zeta_2) \end{bmatrix}$$

$$\Phi_2(\zeta_1, \zeta_2, \eta_1, \eta_2) := \begin{bmatrix} \frac{1}{4}(8 + \eta_1 + \zeta_1) & \frac{45}{16} \\ \frac{45}{16} & \frac{1}{16}(71 - 8\eta_1 - 8\zeta_1) \end{bmatrix}$$

and observe that $(1-\zeta_1\eta_1)\Phi_1(\zeta_1, \zeta_2, \eta_1, \eta_2) + (1-\zeta_2\eta_2)\Phi_2(\zeta_1, \zeta_2, \eta_1, \eta_2)$ is equivalent along $\mathfrak{B}$ to

$$-\Delta(\zeta_1, \zeta_2, \eta_1, \eta_2) := -\zeta_1\eta_1\zeta_2\eta_2 R(\zeta_1^{-1}, \eta_1^{-1})^\top R(\zeta_2^{-1}, \eta_2^{-1})$$

Observe that

$$\Phi_1(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega}) = \begin{bmatrix} 2-\frac{\cos(\omega)}{2} & \frac{51}{16}-3\frac{\cos(\omega)}{2} \\ \frac{51}{16}-3\frac{\cos(\omega)}{2} & \frac{89}{16}-7\frac{\cos(\omega)}{2} \end{bmatrix}$$

$$\Phi_2(e^{-i\omega}, \zeta_2, e^{i\omega}, \eta_2) = \begin{bmatrix} 2+\frac{\cos(\omega)}{2} & \frac{45}{16} \\ \frac{45}{16} & \frac{71}{16}-\cos(\omega) \end{bmatrix}$$

which are easily seen to be positive for all $\omega \in \mathbb{R}$. It can be shown using Gröbner bases computations that $\Delta(\zeta_1, \zeta_2, \eta_1, \eta_2)$ is equivalent along $\mathfrak{B}$ respectively to

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^\top \begin{bmatrix} \frac{1}{64}(4+13\eta_2+4\eta_2^2)(4+13\zeta_2+4\zeta_2^2) & 0 \\ 0 & \frac{1}{16}(8-19\eta_2+8\eta_2^2)(8-19\zeta_2+8\zeta_2^2) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^\top \begin{bmatrix} \frac{1}{16}(8+19\eta_1+8\eta_1^2)(8+19\zeta_1+8\zeta_1^2) & 0 \\ 0 & \frac{1}{64}(4-13\eta_1+4\eta_1^2)(4-13\zeta_1+4\zeta_1^2) \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

It is easily verified that the first matrix is positive definite when $\zeta_2 = e^{-i\omega}$, $\eta_2 = e^{i\omega}$; and that the second one is positive when $\zeta_1 = e^{-i\omega}$, $\eta_1 = e^{i\omega}$. This proves that $\mathrm{col}(Q_{\Phi_1}, Q_{\Phi_2})$ is indeed a Lyapunov function for $\mathfrak{B}$.

The QDFs ϕ_i $i = 1, 2$ given in the proof of Theorem 4 are discrete-time ω-parametrized versions of the classical Bézoutian used in analyzing stability of 1-D continuous-time systems $\ker\ p(\frac{d}{dt})$, see for example Lev-Ari et al. (1991). In the single-variable (i.e. $\mathtt{w} = 1$) case, stability conditions based on the positivity of the coefficient matrix of an ω-dependent Bézoutian have been obtained in Geronimo and Woerdeman (2004, 2006). Of course, there are more Lyapunov functions than only the Bézoutian, as the following example shows.

Example 2 Consider the system described in kernel form by the polynomial

$$p(\zeta_1, \zeta_2, \eta_1, \eta_2) := 1 + \frac{1}{2}\xi_1 + \frac{1}{2}\xi_2 + \frac{1}{2}\xi_1\xi_2$$

The Bézoutian is

$$\begin{aligned} B(\zeta_1, \zeta_2, \eta_1, \eta_2) &= p(\zeta_1, \zeta_2)p(\eta_1, \eta_2) - \zeta_1\eta_1\zeta_2\eta_2\, p(\zeta_1^{-1}, \zeta_2^{-1})p(\eta_1^{-1}, \eta_2^{-1}) \\ &= \frac{3}{4} + \frac{1}{4}\eta_1 + \frac{1}{4}\eta_2 + \frac{1}{4}\zeta_1 - \frac{1}{4}\eta_1\eta_2\zeta_1 + \frac{1}{4}\zeta_2 - \frac{1}{4}\eta_1\eta_2\zeta_2 \\ &\quad -\frac{1}{4}\eta_1\zeta_1\zeta_2 - \frac{1}{4}\eta_2\zeta_1\zeta_2 - \frac{3}{4}\eta_1\eta_2\zeta_1\zeta_2 \\ &= (1-\zeta_1\eta_1)\underbrace{\frac{1}{2}(\eta_2+\zeta_2+3\eta_2\zeta_2)}_{=:\Psi_1(\zeta_1,\zeta_2,\eta_1,\eta_2)} + (1-\zeta_2\eta_2)\underbrace{\frac{1}{4}(3+\eta_1+\zeta_1)}_{=:\Psi_2(\zeta_1,\zeta_2,\eta_1,\eta_2)} \end{aligned}$$

It is easy to see that

$$\Psi_1(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega}) = \frac{1}{4}(3+2\cos(\omega)) = \Psi_2(e^{-i\omega}, \zeta_2, e^{i\omega}, \eta_2) > 0$$

for all $\omega \in \mathbb{R}$, and consequently the system is stable.

We now compute another Lyapunov function for $\ker\ p(\sigma_1, \sigma_2)$. Define first the two-variable polynomial

$$\Delta'(\zeta_1, \zeta_2, \eta_1, \eta_2) := 1 + \frac{1}{4}(\zeta_1 + \eta_1 + \zeta_2 + \eta_2 + \zeta_1\eta_1 + \zeta_2\eta_2).$$

Since $\Delta'(\zeta_1, \zeta_2, \eta_1, \eta_2)$ can be rewritten as

$$\Delta'(\zeta_1, \zeta_2, \eta_1, \eta_2) = \frac{1}{2} + \frac{1}{4}(1+\zeta_1)(1+\eta_1) + \frac{1}{4}(1+\zeta_2)(1+\eta_2),$$

we have

$$\Delta'(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega}) > 0 \text{ and } \Delta'(e^{-i\omega}, \zeta_2, e^{i\omega}, \eta_2) > 0$$

for all $\omega \in \mathbb{R}$. Now define

$$\begin{aligned}\Phi_1'(\zeta_1, \zeta_2, \eta_1, \eta_2) &:= \frac{1}{4}(1+\zeta_2)(1+\eta_2) + \frac{1}{4}\\ \Phi_2'(\zeta_1, \zeta_2, \eta_1, \eta_2) &:= \frac{1}{4}(1+\zeta_1)(1+\eta_1) + \frac{1}{4}\end{aligned}$$

and observe that

$$\begin{aligned}\Phi_1'(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega}) &= \frac{1}{4} \mid 1 + e^{i\omega} \mid^2 + \frac{1}{4}\\ &= \Phi_2'(e^{-i\omega}, \zeta_2, e^{i\omega}, \eta_2) > 0\end{aligned}$$

for all $\omega \in \mathbb{R}$. It is a matter of straightforward verification to check that with these positions,

$$\frac{1}{4}\begin{bmatrix}(1+\zeta_2)(1+\eta_2)+1\\ (1+\zeta_1)(1+\eta_1)+1\end{bmatrix}$$

is a Lyapunov function for $\mathfrak{B} = \ker\, p(\sigma_1, \sigma_2)$ with divergence equal to $-\Delta'(\zeta_1, \zeta_2, \eta_1, \eta_2)$ along $\mathfrak{B}$.

We conclude this section with a discussion on the 2-D polynomial Lyapunov equation, which we now introduce. Consider the set $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2] \times \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ consisting of all $2\mathtt{w} \times \mathtt{w}$ VQDFs, and the div map defined as

$$\begin{aligned}\text{div} : \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2] \times \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2] &\to \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]\\ \text{div}\ \text{col}(\Psi_1(\zeta_1, \zeta_2, \eta_1, \eta_2), \Psi_2(\zeta_1, \zeta_2, \eta_1, \eta_2)) &:= (1-\zeta_1\eta_1)\Psi_1(\zeta_1, \zeta_2, \eta_1, \eta_2)\\ &\quad +(1-\zeta_2\eta_2)\Psi_2(\zeta_1, \zeta_2, \eta_1, \eta_2).\end{aligned}$$

It is easy to see that div is linear, and that in order for the co-domain of div to be $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$, the VQDF must satisfy the following relation:

$$\begin{aligned}&(1-\zeta_1\eta_1)(\Psi_1(\zeta_1, \zeta_2, \eta_1, \eta_2) - \Psi_2(\eta_1, \eta_2, \zeta_1, \zeta_2))\\ &\quad +(1-\zeta_2\eta_2)(\Psi_2(\zeta_1, \zeta_2, \eta_1, \eta_2) - \Psi_1(\eta_1, \eta_2, \zeta_1, \zeta_2)) = 0 \bmod R\end{aligned}$$

Now observe that condition (3a) of Theorem 4 can be rewritten using div as

$$\text{div}\ \text{col}(\Psi_1(\zeta_1, \zeta_2, \eta_1, \eta_2), \Psi_2(\zeta_1, \zeta_2, \eta_1, \eta_2)) = -\Delta(\zeta_1, \zeta_2, \eta_1, \eta_2) \bmod R; \qquad (6)$$

we call (6) the 2-D *polynomial Lyapunov equation* (*PLE* in the following), by analogy with the equation studied in the 1-D case in Sect. 4 of Willems and Trentelman (1998) and in Peeters and Rapisarda (2001). It follows from this discussion that a fourth condition equivalent with those stated in Theorem 4 can be given for the $\mathcal{K}_0$-stability of a square autonomous behavior $\mathfrak{B}$, namely:

4. *There exist a VQDF* $\Phi = \text{col}(\Phi_1, \Phi_2)$ *with* $\Phi_1, \Phi_2 \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$*, and* $\Delta \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ *such that*

(4a) *The 2-D PLE (6) is satisfied;*

(4b) $\Phi_1(\zeta_1, \zeta_2, \eta_1, \eta_2) \overset{\mathfrak{B}\cap \mathrm{Per}_2}{>} 0,\ \Phi_2(\zeta_1, \zeta_2, \eta_1, \eta_2) \overset{\mathfrak{B}\cap \mathrm{Per}_1}{>} 0,$
$\Delta(\zeta_1, \zeta_2, \eta_1, \eta_2) \overset{\mathfrak{B}\cap \mathrm{Per}_i}{>} 0,\ i = 1, 2.$

The issue of how to efficiently solve the general 2-D PLE is a matter of ongoing research. In the finite-dimensional case, a procedure similar to that illustrated in Peeters and Rapisarda (2001) can be devised to solve the equation. Such ramification of the results presented here will be pursued elsewhere.

5 Conclusions

The main result of this paper is Theorem 4, which states necessary and sufficient conditions for the asymptotic stability of a 2-D behavior in the sense of Valcher in terms of the existence of a Lyapunov-type functional. Current research efforts are directed at studying the 2-D Lyapunov Eq. (6) and at devising algorithms for solving it in an efficient way.

Acknowledgments The second author gratefully acknowledges the financial support of *The Great Britain Sasakawa Foundation* for financially supporting the visit to Kyoto University during which the results presented in this paper were obtained. The authors thank the reviewers for their useful comments and for their corrections, which contributed greatly to improving the paper. The first author also acknowledges that this work is partly supported by Grant-in-Aid for Young Scientists (Start-up) No. 20860025 of Japan Society for the Promotion of Science.

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Author Biographies

C. Kojima He received the B.S. degree in science from the Department of Mathematics of the Kyoto University, Japan, in 2002, and received the M.S. and Ph.D. degrees in informatics from the Department of Applied Mathematics and Physics of the same university in 2004 and 2007, respectively. In 2007, he joined the Department of Information Physics and Computing, the University of Tokyo as Assistant Professor. His current research interests are in dissipation theory, behavioral system theory, and multi-dimensional system theory. He is a member of IEEE, SICE, ISCIE, and JSIAM.

P. Rapisarda got a Laurea (M.Sc.) degree in Computer Science at the University of Udine, Italy; and a Ph.D. in Mathematics at the University of Groningen, The Netherlands, working under the supervision of Jan C. Willems and Harry L. Trentelman. He has worked as a Lecturer at the Department of Electrical, Computer and Electronics Engineering of the University of Trieste, Italy; and at the Department of Mathematics of the University of Maastricht, The Netherlands. Currently he is Senior Lecturer at the Information: Signals, Images, Systems group of the School of Electronics and Computer Science of the University of Southampton, United Kingdom.

K. Takaba He received his B.Eng., M.Eng., and Dr.Eng. degrees all from Kyoto University, Japan, in 1989, 1991, and 1996, respectively. In 1991, he joined the faculty of Kyoto University, where he is currently an Associate Professor of Department of Applied Mathematics and Physics. His research interests include behavioral sytem theory, robust control, and constrained control systems. He is a member of IEEE, ISCIE and SICE.