International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713393989

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First published on: 18 August 2010

To cite this Article Wu, Dongxiao , Wu, Jun and Chen, Sheng(2010) 'Robust stabilisation control for discrete-time networked control systems', International Journal of Control, 83: 9, 1885 — 1894, First published on: 18 August 2010 (iFirst)
To link to this Article: DOI: 10.1080/00207179.2010.500333
URL: http://dx.doi.org/10.1080/00207179.2010.500333

International Journal of Control
Vol. 83, No. 9, September 2010, 1885–1894

Taylor & Francis
Taylor & Francis Group

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(*Received 3 September 2008; final version received 8 June 2010*)

We consider the analysis and synthesis of discrete-time networked control systems (NCSs), where the plant has additive uncertainty and the controller is updated with the sensor information at stochastic time intervals. It is shown that the problem is linked to H_∞-control of linear discrete-time stochastic systems and a new small gain theorem is established. Based on this result, sufficient conditions are given for ensuring mean square stability of the NCS, and the genetic algorithm is utilised to design the controller of the NCS based on a linear matrix inequality technique. An illustrative example is given to demonstrate the effectiveness of our proposed method.

Keywords: networked control systems; robust control; H_∞-norm; mean square stability; stochastic systems

1. Introduction

Networked control systems (NCSs) are systems in which a control loop is closed via a shared communication network. The use of a shared network in the feedback path has several advantages, including low installation cost, reducing system wiring, simple system diagnosis and easy maintenance. However, some inherent shortcomings, such as bandwidth constraints, packet delays and packet dropout, will degrade performance of NCSs or even cause instability. NCSs have received much attention during the past decade; see, for example, Tipsuwan and Chow (2003), Antsaklis and Baillieul (2004), Matveev and Savkin (2005), Antsaklis and Baillieul (2007), Hespanha, Naghshtabrizi, and Xu (2007), Moyne and Tilbury (2007), Schenato, Sinopoli, Franceschetti, Poolla, and Sastry (2007), Zhang and Yu (2007), Ishii (2008) and Yu, Wang, Chu, and Xie (2008) and the references therein. Stability analysis of NCSs is investigated in Walsh, Ye, and Bushnell (1999), Beldiman, Walsh, and Bushnell (2000), Zhang, Branicky, and Phillips (2001), Montestruque and Antsaklis (2003, 2004) and Zhivoglyadov and Middleton (2003), and stabilising controllers are designed in Nilsson, Bernhardsson, and Wittenmark (1998) and Zhang, Shi, Chen, and Huang (2005). In the literature, stochastic approaches are typically adopted to deal with network packet dropout and to establish the stability of the NCS in the sense of mean square statistics (Ji, Chizeck, Feng, and Loparo 1991; Costa and Fragoso 1993; Costa and do Val 1996; Xiao, Hassibi, and How 2000; Seiler and Sengupta 2005; Wu and Chen 2007). The works of Seiler and Sengupta (2005), Elia (2005), Yue, Han, and Lam (2005) and Hu and Yan (2007) adopt robust control theory for the analysis and design of NCSs.

Most of the works in the NCS research utilise fixed controllers. Some exceptions are Montestruque and Antsaklis (2003, 2004) and Zhivoglyadov and Middleton (2003), which utilise more flexible controllers for NCSs where a network is located between the sensor and the controller. For NCSs, time periods frequently appear during which the controller cannot access sensor data due to network induced random delay and packet dropout. During these periods without sensor data, the underlying idea of Montestruque and Antsaklis (2003, 2004) and Zhivoglyadov and Middleton (2003) is that a nominal plant model is employed at the controller side to estimate the plant behaviour, and the estimated result is provided to the controller to replace the real plant behaviour information so that the computation of control signal can be executed in time. During time periods when the controller can access sensor data, the networked controllers in Montestruque and Antsaklis (2003, 2004) and Zhivoglyadov and Middleton (2003) perform the same feedback control as standard closed-loop control systems without network. This kind of control scheme for NCSs is referred to as model-based

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ISSN 0020–7179 print/ISSN 1366–5820 online

DOI: 10.1080/00207179.2010.500333
http://www.informaworld.com

networked control by Montestruque and Antsaklis (2003, 2004), where a model-based networked state-feedback control method and a model-based networked observed-state-feedback control method are presented. The so-called smart control of NCSs addressed in Zhivoglyadov and Middleton (2003) is another model-based networked observed-state-feedback control method. In the networked control schemes of Montestruque and Antsaklis (2003, 2004), the observer is included at the actuator side of the plant to be controlled, while for the networked control scheme of Zhivoglyadov and Middleton (2003) the observer is located at the controller side.

We notice that all the parameters of the plant are assumed to be known in the works of Montestruque and Antsaklis (2003, 2004) while the plant is assumed to be uncertainty free in Zhivoglyadov and Middleton (2003). But these assumptions on the plant are not met in most control practice. It is of great practical significance to remove these strict assumptions on the plant and to study model-based networked control with robustness considerations. We also notice that the study by Zhivoglyadov and Middleton (2003) on smart control does not consider the networks of random transmission with known probability, which belong to a most important class of networks in NCSs. The novel contribution of this article is that we study smart control for NCSs where discrete-time plants have additive perturbations and networks induce random delay and packet dropout. To our best knowledge, our key result, Theorem 4.4 which establishes a small gain theorem for linear discrete-time stochastic systems, was not seen previously in the literature. The remainder of this article is organised as follows. Section 2 gives notations and preliminary results, while the NCS problem is formulated in Section 3. Section 4 provides sufficient conditions for the NCS design solution based on linear stochastic system theory, and addresses the control synthesis method. A numerical design example is included in Section 5, and our conclusions are offered in Section 6.

2. Notations and preliminary results

We adopt the standard notation of $\mathbb{R}$ for real numbers and $\mathbb{N}$ for non-negative integers. $\mathbf{I}_n$ denotes the $n \times n$ identity matrix, while $\mathbf{I}$ and $\mathbf{0}$ represent the identity and zero matrices of appropriate dimensions, respectively. For $\mathbf{M} \in \mathbb{R}^{m\times n}$ and positive integer $p \geq m$, denote

$$\mathcal{H}_p(\mathbf{M}) = \begin{cases} \mathbf{M} \in \mathbb{R}^{p\times n}, & p = m, \\ \begin{bmatrix} \mathbf{M} \\ \mathbf{0} \end{bmatrix} \in \mathbb{R}^{p\times n}, & p > m. \end{cases} \tag{1}$$

Similarly, for $\mathbf{M} \in \mathbb{R}^{m\times n}$ and positive integer $q \geq n$, denote

$$\mathcal{W}_q(\mathbf{M}) = \begin{cases} \mathbf{M} \in \mathbb{R}^{m\times q}, & q = n, \\ \begin{bmatrix} \mathbf{M} & \mathbf{0} \end{bmatrix} \in \mathbb{R}^{m\times q}, & q > n. \end{cases} \tag{2}$$

For square matrix $\mathbf{S} \in \mathbb{R}^{m\times m}$, $\mathbf{S} > 0$ ($\mathbf{S} \geq 0$) indicates that $\mathbf{S}$ is a positive definite (semidefinite) matrix. For symmetric matrices $\mathbf{S}_1 \in \mathbb{R}^{m\times m}$ and $\mathbf{S}_2 \in \mathbb{R}^{m\times m}$, $\mathbf{S}_1 > \mathbf{S}_2$ means that $\mathbf{S}_1 - \mathbf{S}_2 > 0$.

For a discrete-time signal $\mathbf{r} = \{\mathbf{r}(t)\}_{t\in\mathbb{N}}$ with $\mathbf{r}(t) \in \mathbb{R}^r$, define

$$\|\mathbf{r}\|_2 \triangleq \sqrt{\sum_{t=0}^{\infty} \mathbf{r}^{\mathrm{T}}(t)\mathbf{r}(t)}\ . \tag{3}$$

Let ℓ_2^r be the set of $\mathbf{r}$s with $\|\mathbf{r}\|_2 < \infty$. A finite-dimensional linear time-invariant discrete-time system $\hat{G}$ can be written as

$$\begin{cases} \mathbf{x}_g(t+1) = \mathbf{A}_g\mathbf{x}_g(t) + \mathbf{B}_g\mathbf{u}_g(t), \\ \mathbf{y}_g(t) = \mathbf{C}_g\mathbf{x}_g(t) + \mathbf{D}_g\mathbf{u}_g(t), \end{cases} \quad t \in \mathbb{N}, \tag{4}$$

where $\mathbf{x}_g(t) \in \mathbb{R}^b$, $\mathbf{u}_g(t) \in \mathbb{R}^r$ and $\mathbf{y}_g(t) \in \mathbb{R}^d$ are state, input and output, respectively; $\mathbf{A}_g$, $\mathbf{B}_g$, $\mathbf{C}_g$ and $\mathbf{D}_g$ are constant real matrices of appropriate dimensions. The system $\hat{G}$ given by (4) with $\mathbf{u}_g(t) \equiv \mathbf{0}$ is said to be stable if $\forall \mathbf{x}_g(0) \in \mathbb{R}^b$, $\lim_{t\to\infty} \mathbf{x}_g^{\mathrm{T}}(t)\mathbf{x}_g(t) = 0$. Define

$$\|\hat{G}\|_\infty \triangleq \sup_{\substack{\mathbf{u}_g \in \ell_2^r \\ \|\mathbf{u}_g\|_2 \neq 0 \\ \mathbf{x}_g(0) = \mathbf{0}}} \frac{\|\mathbf{y}_g\|_2}{\|\mathbf{u}_g\|_2} \tag{5}$$

as the H_∞-norm of $\hat{G}$ which is stable with $\mathbf{u}_g(t) \equiv \mathbf{0}$. For $0 < \rho \in \mathbb{R}$, denote $\mathbb{D}_\rho^{d\times r}$ as the set of $\hat{G}$s which are stable with $\mathbf{u}_g(t) \equiv \mathbf{0}$ and $\|\hat{G}\|_\infty < 1/\rho$.

Lemma 2.1 (Zhou, Doyle, and Glover 1995)**:** *The system $\hat{G} \in \mathbb{D}_\rho^{d\times r}$ if and only if there exists a $0 < \mathbf{P} \in \mathbb{R}^{b\times b}$ such that*

$$\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\rho^2}\mathbf{I}_r \end{bmatrix} - \begin{bmatrix} \mathbf{A}_g & \mathbf{B}_g \\ \mathbf{C}_g & \mathbf{D}_g \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_d \end{bmatrix} \begin{bmatrix} \mathbf{A}_g & \mathbf{B}_g \\ \mathbf{C}_g & \mathbf{D}_g \end{bmatrix} > 0. \tag{6}$$

For a discrete-time stochastic signal $\bar{\mathbf{r}} = \{\bar{\mathbf{r}}(t)\}_{t\in\mathbb{N}}$ with $\bar{\mathbf{r}}(t)$ a $\mathbb{R}^r$-valued random variable, define

$$\|\bar{\mathbf{r}}\|_{2s} \triangleq \sqrt{\sum_{t=0}^{\infty} E(\bar{\mathbf{r}}^{\mathrm{T}}(t)\bar{\mathbf{r}}(t))}, \tag{7}$$

where $E(\cdot)$ denotes the expectation. Let ℓ_{2s}^r be the set of $\bar{\mathbf{r}}$s with $\|\bar{\mathbf{r}}\|_{2s} < \infty$. For positive integer M, denote

$\mathcal{M}=\{1,\dots,M\}$. Consider the following stochastic system, denoted as $\hat{F}$:

$$\begin{cases} \mathbf{x}_f(t+1) = \mathbf{A}_f(\theta_t)\mathbf{x}_f(t) + \mathbf{B}_f(\theta_t)\mathbf{u}_f(t), \\ \mathbf{y}_f(t) = \mathbf{C}_f(\theta_t)\mathbf{x}_f(t) + \mathbf{D}_f(\theta_t)\mathbf{u}_f(t), \end{cases} \quad t \in \mathbb{N}, \qquad (8)$$

where θ_ts are independently identically distributed (i.i.d.) $\mathcal{M}$-valued random variables; $\mathbf{A}_f$, $\mathbf{B}_f$, $\mathbf{C}_f$ and $\mathbf{D}_f$ are mappings from $\mathcal{M}$ to $\mathbb{R}^{b\times b}$, $\mathbb{R}^{b\times r}$, $\mathbb{R}^{d\times b}$ and $\mathbb{R}^{d\times r}$, respectively. The probability mass function of θ_t is given by $q_j=\Pr(\theta_t=j)$ with $j\in\mathcal{M}$. Clearly, θ_t can be regarded as a special Markov chain (Karlin and Taylor 1975). The following results from Ji et al. (1991) and Seiler and Sengupta (2005) define the stochastic stability of $\hat{F}$.

Definition 1: The system $\hat{F}$ given by (8) with $\mathbf{u}_f(t)\equiv\mathbf{0}$ is said to be stochastically stable if $\forall \mathbf{x}_f(0)\in\mathbb{R}^b$, $\|\mathbf{x}_f\|_{2s}^2 < \infty$.

Lemma 2.2: *The system $\hat{F}$ with $\mathbf{u}_f(t)\equiv\mathbf{0}$ is stochastically stable if and only if $\forall \mathbf{x}_f(0)\in\mathbb{R}^b$, $\lim_{t\to\infty} E(\mathbf{x}_f^{\mathrm{T}}(t)\mathbf{x}_f(t)) = 0$.*

Definition 2: The matrix $\mathbf{A}_f(\theta_t)$ in (8) is said to be stochastically stable if there exists a $0<\mathbf{P}\in\mathbb{R}^{b\times b}$ such that $\mathbf{P}-\sum_{j=1}^{M} q_j\mathbf{A}_f^{\mathrm{T}}(j)\mathbf{P}\mathbf{A}_f(j)>0$.

Lemma 2.3: *The system $\hat{F}$ with $\mathbf{u}_f(t)\equiv\mathbf{0}$ is stochastically stable if and only if $\mathbf{A}_f(\theta_t)$ is stochastically stable.*

Definition 3: $(\mathbf{C}_f(\theta_t), \mathbf{A}_f(\theta_t))$ is said to be stochastically detectable if there exists an $\mathbf{H}_f(\theta_t)$ mapping from $\mathcal{M}$ to $\mathbb{R}^{b\times d}$ such that $\mathbf{A}_f(\theta_t)-\mathbf{H}_f(\theta_t)\mathbf{C}_f(\theta_t)$ is stochastically stable.

From Theorem 4.8 in Dragan and Morozan (2006), we have the following lemma.

Lemma 2.4: *Suppose that $(\mathbf{C}_f(\theta_t), \mathbf{A}_f(\theta_t))$ is stochastically detectable. Then the following are equivalent*:

(1) *$\mathbf{A}_f(\theta_t)$ is stochastically stable.*
(2) *The discrete-time backward difference equations*

$$\mathbf{X}_t(j) = \mathbf{A}_f^{\mathrm{T}}(j)\sum_{l=1}^{M} q_l\mathbf{X}_{t+1}(l)\mathbf{A}_f(j) + \mathbf{C}_f^{\mathrm{T}}(j)\mathbf{C}_f(j), \qquad (9)$$

where $t\in\mathbb{N}$ and $j\in\mathcal{M}$, have a bounded solution $\{[\overline{\mathbf{X}}_t^{\mathrm{T}}(1)\ \cdots\ \overline{\mathbf{X}}_t^{\mathrm{T}}(M)]^{\mathrm{T}}\}_{t\in\mathbb{N}}$ such that $\forall t\in\mathbb{N}$, $\forall j\in\mathcal{M}$,

$$0 \le \overline{\mathbf{X}}_t(j) \in \mathbb{R}^{b\times b}. \qquad (10)$$

Replacing the backward difference equations with algebraic equations and further applying Theorem 3.5 in Dragan and Morozan (2006) lead to Corollary 2.5.

Corollary 2.5: *Suppose that $(\mathbf{C}_f(\theta_t), \mathbf{A}_f(\theta_t))$ is stochastically detectable. Then the following are equivalent*:

(1) *$\mathbf{A}_f(\theta_t)$ is stochastically stable.*
(2) *There exist $0\le\mathbf{X}(j)\in\mathbb{R}^{b\times b}$, $j\in\mathcal{M}$, satisfying*

$$\mathbf{X}(j) = \mathbf{A}_f^{\mathrm{T}}(j)\sum_{l=1}^{M} q_l\mathbf{X}(l)\mathbf{A}_f(j) + \mathbf{C}_f^{\mathrm{T}}(j)\mathbf{C}_f(j). \qquad (11)$$

Define

$$\|\hat{F}\|_{\infty s} \triangleq \sup_{\substack{\mathbf{u}_f\in\ell_{2s}^r \\ \|\mathbf{u}_f\|_{2s}\neq 0 \\ \mathbf{x}_f(0)=\mathbf{0} \\ \theta_0\in\mathcal{M}}} \frac{\|\mathbf{y}_f\|_{2s}}{\|\mathbf{u}_f\|_{2s}} \qquad (12)$$

as the H_∞-norm of $\hat{F}$ which is stochastically stable with $\mathbf{u}_f(t)\equiv\mathbf{0}$. For $0<\rho\in\mathbb{R}$, denote $\mathbb{D}_{\rho s}^{d\times r}$ as the set of $\hat{F}$s which are stochastically stable with $\mathbf{u}_f(t)\equiv\mathbf{0}$ and $\|\hat{F}\|_{\infty s}<1/\rho$.

Lemma 2.6 (Seiler and Sengupta 2005)**:** *The system $\hat{F}\in\mathbb{D}_{\rho s}^{d\times r}$ if there exists a $0<\mathbf{P}\in\mathbb{R}^{b\times b}$ such that*

$$\begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\rho^2}\mathbf{I}_r \end{bmatrix} - \sum_{j=1}^{M} q_j \begin{bmatrix} \mathbf{A}_f(j) & \mathbf{B}_f(j) \\ \mathbf{C}_f(j) & \mathbf{D}_f(j) \end{bmatrix}^{\mathrm{T}} \times \begin{bmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_d \end{bmatrix} \begin{bmatrix} \mathbf{A}_f(j) & \mathbf{B}_f(j) \\ \mathbf{C}_f(j) & \mathbf{D}_f(j) \end{bmatrix} > 0. \qquad (13)$$

Consider a special case of $\hat{F}$ described by

$$\begin{cases} \mathbf{x}_f(t+1) = \mathbf{A}_f(\theta_t)\mathbf{x}_f(t) + \mathbf{B}_{f1}(\theta_t)\mathbf{u}_{f1}(t) \\ \qquad\qquad + \mathbf{B}_{f2}(\theta_t)\mathbf{u}_{f2}(t), \\ \mathbf{y}_f(t) = \mathbf{C}_f(\theta_t)\mathbf{x}_f(t) + \mathbf{D}_{f1}(\theta_t)\mathbf{u}_{f1}(t), \end{cases} \quad t\in\mathbb{N}, \qquad (14)$$

where $\mathbf{B}_{f1}$, $\mathbf{B}_{f2}$ and $\mathbf{D}_{f1}$ are mappings from $\mathcal{M}$ to $\mathbb{R}^{b\times r_1}$, $\mathbb{R}^{b\times r_2}$ and $\mathbb{R}^{d\times r_1}$, respectively. Set

$$\mathbf{u}_{f1}(t) = -\mathbf{K}_{f1}(\theta_t)\mathbf{x}_f(t), \qquad (15)$$

where $\mathbf{K}_{f1}$ is a mapping from $\mathcal{M}$ to $\mathbb{R}^{r_1\times b}$. A closed-loop stochastic system $\hat{F}_c$ is formed as

$$\begin{cases} \mathbf{x}_f(t+1) = (\mathbf{A}_f(\theta_t) - \mathbf{B}_{f1}(\theta_t)\mathbf{K}_{f1}(\theta_t))\mathbf{x}_f(t) \\ \qquad\qquad + \mathbf{B}_{f2}(\theta_t)\mathbf{u}_{f2}(t), \\ \mathbf{y}_f(t) = (\mathbf{C}_f(\theta_t) - \mathbf{D}_{f1}(\theta_t)\mathbf{K}_{f1}(\theta_t))\mathbf{x}_f(t), \end{cases} \quad t\in\mathbb{N}. \qquad (16)$$

The following result for $\hat{F}_c$ is due to the main theorem in Costa and do Val (1996).

Lemma 2.7: *Assume that there exist feedback gains $\mathbf{K}_{f1}(\theta_t)$ such that the corresponding closed-loop system $\hat{F}_c$ lies in $\mathbb{D}_{\rho s}^{d\times r_2}$, $(\mathbf{C}_f(\theta_t), \mathbf{A}_f(\theta_t))$ is stochastically detectable and*

$$\begin{cases} \mathbf{C}_f^{\mathrm{T}}(j)\mathbf{D}_{f1}(j) = \mathbf{0}, \\ \mathbf{D}_{f1}^{\mathrm{T}}(j)\mathbf{D}_{f1}(j) = \mathbf{I}, \end{cases} \quad j\in\mathcal{M}. \qquad (17)$$

Then there exist $0 \le \mathbf{X}_j \in \mathbb{R}^{b\times b}$ *with* $j \in \mathcal{M}$, *which satisfy*, $\forall j \in \mathcal{M}$,

(1) $\mathbf{I} - \rho^2 \mathbf{B}_{f2}^{\mathrm{T}}(j) \sum_{l=1}^{M} q_l \mathbf{X}_l \mathbf{B}_{f2}(j) > 0$;
(2) $\mathbf{X}_j = \mathbf{C}_f^{\mathrm{T}}(j)\mathbf{C}_f(j) + \mathbf{R}_{f4}^{\mathrm{T}}(j) \sum_{l=1}^{M} q_l \mathbf{X}_l \mathbf{R}_{f4}(j) + \mathbf{R}_{f2}^{\mathrm{T}}(j)\mathbf{R}_{f2}(j) - \mathbf{R}_{f3}^{\mathrm{T}}(j)\mathbf{R}_{f3}(j)$ *and*
(3) $\mathbf{R}_{f4}(\theta_t)$ *is stochastically stable*

where

$$\mathbf{R}_{f1}(j) = \mathbf{I} - \rho^2 \mathbf{B}_{f2}(j)\mathbf{B}_{f2}^{\mathrm{T}}(j) \sum_{l=1}^{M} q_l \mathbf{X}_l,$$

$$\mathbf{R}_{f2}(j) = \left(\mathbf{I} + \mathbf{B}_{f1}^{\mathrm{T}}(j) \sum_{l=1}^{M} q_l \mathbf{X}_l \mathbf{R}_{f1}^{-1}(j)\mathbf{B}_{f1}(j)\right)^{-1} \times \mathbf{B}_{f1}^{\mathrm{T}}(j) \sum_{l=1}^{M} q_l \mathbf{X}_l \mathbf{R}_{f1}^{-1}(j)\mathbf{A}_f(j),$$

$$\mathbf{R}_{f3}(j) = \left(\mathbf{I} - \rho^2 \mathbf{B}_{f2}^{\mathrm{T}}(j) \sum_{l=1}^{M} q_l \mathbf{X}_l \mathbf{B}_{f2}(j)\right)^{-1} \times \rho \mathbf{B}_{f2}^{\mathrm{T}}(j) \sum_{l=1}^{M} q_l \mathbf{X}_l (\mathbf{A}_f(j) - \mathbf{B}_{f1}(j)\mathbf{R}_{f2}(j)),$$

$$\mathbf{R}_{f4}(j) = \mathbf{A}_f(j) - \mathbf{B}_{f1}(j)\mathbf{R}_{f2}(j) + \rho \mathbf{B}_{f2}(j)\mathbf{R}_{f3}(j).$$

Finally, consider $\hat{F} \in \mathbb{D}_{\rho s}^{d\times r}$ with $\mathbf{D}_f(\theta_t) \equiv \mathbf{0}$, which is obviously equivalent to the system

$$\begin{cases} \mathbf{x}_f(t+1) = \mathbf{A}_f(\theta_t)\mathbf{x}_f(t) + \mathbf{B}_f(\theta_t)\mathbf{u}_f(t), \\ \begin{bmatrix} \mathbf{y}_f(t) \\ 0\mathbf{x}_f(t) \end{bmatrix} = \begin{bmatrix} \mathbf{C}_f(\theta_t) \\ 0\mathbf{I}_b \end{bmatrix} \mathbf{x}_f(t), \end{cases} \quad t \in \mathbb{N}. \quad (18)$$

The latter system belongs to $\mathbb{D}_{\rho s}^{(d+b)\times r}$, as it can be viewed as

$$\begin{cases} \mathbf{x}_f(t+1) = \mathbf{A}_f(\theta_t)\mathbf{x}_f(t) + 0\mathbf{I}_b \mathbf{w}_f(t) + \mathbf{B}_f(\theta_t)\mathbf{u}_f(t), \\ \begin{bmatrix} \mathbf{y}_f(t) \\ \mathbf{w}_f(t) \end{bmatrix} = \begin{bmatrix} \mathbf{C}_f(\theta_t) \\ 0\mathbf{I}_b \end{bmatrix} \mathbf{x}_f(t) + \begin{bmatrix} 0\mathbf{C}_f(\theta_t) \\ \mathbf{I}_b \end{bmatrix} \mathbf{w}_f(t), \end{cases} \quad t \in \mathbb{N}, \quad (19)$$

by connecting $\mathbf{w}_f(t) = -\mathbf{K}_f(\theta_t)\mathbf{x}_f(t)$ with $\mathbf{K}_f(\theta_t) \equiv 0\mathbf{I}_b$. It is easy to check

$$\begin{bmatrix} \mathbf{C}_f(\theta_t) \\ 0\mathbf{I}_b \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0\mathbf{C}_f(\theta_t) \\ \mathbf{I}_b \end{bmatrix} = \mathbf{0}, \quad (20)$$

$$\begin{bmatrix} 0\mathbf{C}_f(\theta_t) \\ \mathbf{I}_b \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0\mathbf{C}_f(\theta_t) \\ \mathbf{I}_b \end{bmatrix} = \mathbf{I}_b. \quad (21)$$

Moreover, $\hat{F} \in \mathbb{D}_{\rho s}^{d\times r}$ implies $\mathbf{A}_f(\theta_t)$ is stochastically stable and hence $([\mathbf{C}_f^{\mathrm{T}}(\theta_t) \;\; 0\mathbf{I}_b]^{\mathrm{T}}, \mathbf{A}_f(\theta_t))$ is stochastically detectable. Thus, applying Lemma 2.7 to (18) leads to the following corollary.

Corollary 2.8: *Suppose that* $\hat{F} \in \mathbb{D}_{\rho s}^{d\times r}$ *with* $\mathbf{D}_f(\theta_t) \equiv \mathbf{0}$. *Then there exists* $0 \le \mathbf{X}_j \in \mathbb{R}^{b\times b}$ *with* $j \in \mathcal{M}$, *which satisfy*, $\forall j \in \mathcal{M}$,

(1) $\mathbf{I} - \rho^2 \mathbf{B}_f^{\mathrm{T}}(j) \sum_{l=1}^{M} q_l \mathbf{X}_l \mathbf{B}_f(j) > 0$;
(2) $\mathbf{X}_j = \mathbf{C}_f^{\mathrm{T}}(j)\mathbf{C}_f(j) + \mathbf{S}_f^{\mathrm{T}}(j) \sum_{l=1}^{M} q_l \mathbf{X}_l \mathbf{S}_f(j) - \mathbf{R}_f^{\mathrm{T}}(j)\mathbf{R}_f(j)$ *and*
(3) $\mathbf{S}_f(\theta_t)$ *is stochastically stable*

where

$$\mathbf{R}_f(j) = \left(\mathbf{I} - \rho^2 \mathbf{B}_f^{\mathrm{T}}(j) \sum_{l=1}^{M} q_l \mathbf{X}_l \mathbf{B}_f(j)\right)^{-1} \times \rho \mathbf{B}_f^{\mathrm{T}}(j) \sum_{l=1}^{M} q_l \mathbf{X}_l \mathbf{A}_f(j), \quad (22)$$

$$\mathbf{S}_f(j) = \mathbf{A}_f(j) + \rho \mathbf{B}_f(j)\mathbf{R}_f(j). \quad (23)$$

3. Problem formulation

The NCS $\hat{P}_K$ of Figure 1 contains a linear time-invariant discrete-time plant $\hat{P}$ and a discrete-time controller $\hat{K}$. The plant $\hat{P}$ consists of a nominal plant model $\hat{P}_0$ and an additive perturbation $\hat{\Delta}$, as shown in Figure 1. $\hat{P}_0$ is described by

$$\begin{cases} \mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{y}_0(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t), \end{cases} \quad t \in \mathbb{N}, \quad (24)$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$ and $\mathbf{D} \in \mathbb{R}^{p\times m}$ are given matrices. $\hat{\Delta} \in \mathbb{D}_{\gamma}^{p\times m}$ with a given $0 < \gamma \in \mathbb{R}$. $\hat{\Delta}$ and $\hat{P}_0$ have the same input $\mathbf{u}(t)$ which is also the input of $\hat{P}$. The output of $\hat{\Delta}$ is $\mathbf{w}(t)$ which is added with $\mathbf{y}_0(t)$ to form the output of $\hat{P}$

$$\mathbf{y}(t) = \mathbf{y}_0(t) + \mathbf{w}(t). \quad (25)$$

The plant $\hat{P}$ and the controller $\hat{K}$ are connected via a shared communication network through which the sensor transmits data to the controller. The controller is collocated with the actuator. At each instant $t \in \mathbb{N}$, the sensor tries to transmit $\mathbf{y}(t)$ to $\hat{K}$. After the attempt, $\mathbf{y}(t)$ is discarded by the sensor. Each transmission has

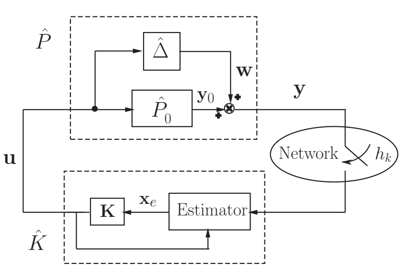

Figure 1. Networked control system $\hat{P}_K$.

two alternative outcomes: one is that the transmission succeeds and $\hat{K}$ receives $\mathbf{y}(t)$ at t; the other is that the transmission fails due to packet dropout by the network and thus $\hat{K}$ misses $\mathbf{y}(t)$. A successful packet transmission time through the network is assumed to be negligible. Note that a packet arriving late due to network induced random delay has the same effect as packet dropout and is treated as a transmission failure. Those instants at which transmissions succeed are denoted by t_k, $k \in \mathbb{N}$, in ascending order, and $t_0 = 0$ is assumed without loss of generality. The time instant t_k is referred to as update instant. After t_k, $\mathbf{y}(t_k)$ remains to be the newest information for $\hat{K}$ until t_{k+1} when $\mathbf{y}(t_{k+1})$ arrives.

It is clear that $\hat{P}_K$ is in the mode of open loop when $t \neq t_k$ and in the mode of closed loop when $t = t_k$. A smart control mechanism, similar to the one in Zhivoglyadov and Middleton (2003), is adopted for $\hat{K}$ as

$$\begin{cases} \mathbf{x}_e(t+1) = \mathbf{A}\mathbf{x}_e(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{L}(\mathbf{C}\mathbf{x}_e(t) \\ \qquad + \mathbf{D}\mathbf{u}(t) - \mathbf{y}(t)), & t = t_k, \\ \mathbf{x}_e(t+1) = \mathbf{A}\mathbf{x}_e(t) + \mathbf{B}\mathbf{u}(t), & t \neq t_k, \\ \mathbf{u}(t) = \mathbf{K}\mathbf{x}_e(t), & t \in \mathbb{N}, \end{cases} \tag{26}$$

where the state feedback gain matrix $\mathbf{K} \in \mathbb{R}^{m \times n}$ and the observer gain matrix $\mathbf{L} \in \mathbb{R}^{n \times p}$. Clearly, $\hat{K}$ expressed by (26) is an estimator-based controller. When $\mathbf{y}(t)$ is available at $t = t_k$, a standard observer law is employed to estimate $\mathbf{x}(t)$ using $\mathbf{x}_e(t)$, while when $\mathbf{y}(t)$ is unavailable at $t \neq t_k$, an imitation law is employed to estimate $\mathbf{x}(t)$ with $\mathbf{x}_e(t)$. Define the update interval

$$h_k \triangleq t_{k+1} - t_k, \ k \in \mathbb{N}. \tag{27}$$

Let the maximal update interval be N, and denote $\mathcal{N} = \{1, \ldots, N\}$. The value of N can be viewed as a network service quality measure. When the network is very busy, experiencing long delay and a large number of packet dropouts, N will be very large. By contrast, a small N shows that the network is offering good-quality service. The update intervals h_ks are assumed to be i.i.d. $\mathcal{N}$-valued random variables. The probability mass function of h_k is denoted by $p_i = \Pr(h_k = i)$ with $i \in \mathcal{N}$. Let the order of $\hat{\Delta}$ be q. Then $\hat{\Delta}$ can be described in a state-space form as

$$\begin{cases} \mathbf{x}_\delta(t+1) = \mathbf{A}_\delta \mathbf{x}_\delta(t) + \mathbf{B}_\delta \mathbf{u}(t), \\ \mathbf{w}(t) = \mathbf{C}_\delta \mathbf{x}_\delta(t) + \mathbf{D}_\delta \mathbf{u}(t), \end{cases} \quad t \in \mathbb{N}, \tag{28}$$

where the matrices $\mathbf{A}_\delta \in \mathbb{R}^{q \times q}$, $\mathbf{B}_\delta \in \mathbb{R}^{q \times m}$, $\mathbf{C}_\delta \in \mathbb{R}^{p \times q}$ and $\mathbf{D}_\delta \in \mathbb{R}^{p \times m}$. Define the state of $\hat{P}_K$ as

$$\mathbf{z}(t) \triangleq [\mathbf{x}^{\mathrm{T}}(t) \ \mathbf{x}_e^{\mathrm{T}}(t) \ \mathbf{x}_\delta^{\mathrm{T}}(t)]^{\mathrm{T}}. \tag{29}$$

As $\{h_k\}$ is a sequence of random variables, $\mathbf{z}(t)$ is actually a random process. For $\hat{P}_K$, we define the stability in the sense of mean square statistics.

Definition 4: $\hat{P}_K$ is mean square stable if $\forall \mathbf{z}(0) \in \mathbb{R}^{2n+q}$, $\lim_{t\to\infty} E(\mathbf{z}^{\mathrm{T}}(t)\mathbf{z}(t)) = 0$.

Our NCS design problem can now be stated: given $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, γ, N and $p_i \ \forall i \in \mathcal{N}$, determine $\mathbf{K}$ and $\mathbf{L}$ such that $\forall \hat{\Delta} \in \mathbb{D}_\gamma^{p \times m}$, $\hat{P}_K$ is mean square stable.

4. Theoretical analysis and design method

We now study the dynamic response of $\hat{P}_K$ by oversampling it at each update instant. Since h_k is a $\mathcal{N}$-valued random variable, the dimension of the input (output) of the oversampled system is also random. To tackle this difficulty, we use the auxiliary systems of $\hat{P}_0$, $\hat{\Delta}$ and $\hat{K}$ by augmenting them up to the constant dimension Nm. The auxiliary system $\hat{P}_{0s}$ of $\hat{P}_0$ is constructed as

$$\begin{cases} \overline{\mathbf{x}}(k+1) = \mathbf{A}^{h_k}\overline{\mathbf{x}}(k) + \mathcal{W}_{Nm}([\mathbf{A}^{h_k-1}\mathbf{B} \cdots \mathbf{B}])\overline{\mathbf{u}}(k), \\ \overline{\mathbf{y}}_0(k) = \mathbf{C}\overline{\mathbf{x}}(k) + \mathcal{W}_{Nm}(\mathbf{D})\overline{\mathbf{u}}(k), \end{cases} \quad k \in \mathbb{N}. \tag{30}$$

The auxiliary system $\hat{\Delta}_s$ of $\hat{\Delta}$ is constructed as

$$\begin{cases} \overline{\mathbf{x}}_\delta(k+1) = \mathbf{A}_\delta^{h_k}\overline{\mathbf{x}}_\delta(k) \\ \qquad + \mathcal{W}_{Nm}([\mathbf{A}_\delta^{h_k-1}\mathbf{B}_\delta \ \cdots \ \mathbf{B}_\delta])\overline{\mathbf{u}}(k), & k \in \mathbb{N}. \\ \overline{\mathbf{w}}(k) = \mathbf{C}_\delta\overline{\mathbf{x}}_\delta(k) + \mathcal{W}_{Nm}(\mathbf{D}_\delta)\overline{\mathbf{u}}(k), \end{cases} \tag{31}$$

The auxiliary system $\hat{K}_s$ of $\hat{K}$ is constructed as

$$\begin{cases} \overline{\mathbf{x}}_e(k+1) = \boldsymbol{\Lambda}_0^{h_k-1}\boldsymbol{\Lambda}_1\overline{\mathbf{x}}_e(k) - \boldsymbol{\Lambda}_0^{h_k-1}\mathbf{L}\overline{\mathbf{y}}_0(k) \\ \qquad - \boldsymbol{\Lambda}_0^{h_k-1}\mathbf{L}\overline{\mathbf{w}}(k), & k \in \mathbb{N}, \\ \overline{\mathbf{u}}(k) = \boldsymbol{\Upsilon}_1(h_k)\overline{\mathbf{x}}_e(k) - \boldsymbol{\Upsilon}_2(h_k)\overline{\mathbf{y}}_0(k) - \boldsymbol{\Upsilon}_2(h_k)\overline{\mathbf{w}}(k), \end{cases} \tag{32}$$

where $\boldsymbol{\Lambda}_0 = \mathbf{A} + \mathbf{BK}$, $\boldsymbol{\Lambda}_1 = \mathbf{A} + \mathbf{BK} + \mathbf{LC} + \mathbf{LDK}$,

$$\boldsymbol{\Upsilon}_1(h_k) = \begin{cases} \mathcal{H}_{Nm}(\mathbf{K}), & h_k = 1, \\ \mathcal{H}_{Nm}([\mathbf{K}^{\mathrm{T}} \ (\mathbf{K}\boldsymbol{\Lambda}_1)^{\mathrm{T}} \ \cdots (\mathbf{K}\boldsymbol{\Lambda}_0^{h_k-2}\boldsymbol{\Lambda}_1)^{\mathrm{T}}]^{\mathrm{T}}), & h_k > 1, \end{cases} \tag{33}$$

$$\boldsymbol{\Upsilon}_2(h_k) = \begin{cases} \mathcal{H}_{Nm}(\mathbf{0KL}), & h_k = 1, \\ \mathcal{H}_{Nm}([(\mathbf{0KL})^{\mathrm{T}} \ (\mathbf{KL})^{\mathrm{T}} \ \cdots (\mathbf{K}\boldsymbol{\Lambda}_0^{h_k-2}\mathbf{L})^{\mathrm{T}}]^{\mathrm{T}}), & h_k > 1. \end{cases} \tag{34}$$

Combining $\hat{P}_{0s}$, $\hat{\Delta}_s$ and $\hat{K}_s$ forms the auxiliary stochastic system $\hat{P}_{Ks}$, depicted in Figure 2, of $\hat{P}_K$. Define the state of $\hat{P}_{Ks}$ as

$$\overline{\mathbf{z}}(k) = [\overline{\mathbf{x}}^{\mathrm{T}}(k) \ \overline{\mathbf{x}}_e^{\mathrm{T}}(k) \ \overline{\mathbf{x}}_\delta^{\mathrm{T}}(k)]^{\mathrm{T}}. \tag{35}$$

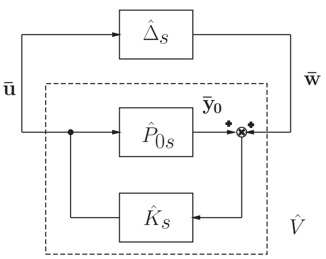

Figure 2. Auxiliary system $\hat{P}_{Ks}$ for $\hat{P}_K$.

From (24) to (35), it is easy to see the following relationships between $\hat{P}_{Ks}$ and $\hat{P}_K$. Given $t_0=0$ and $\bar{\mathbf{z}}(0)=\mathbf{z}(t_0)$, $\forall k\in\mathbb{N}$,

$$\bar{\mathbf{z}}(k)=\mathbf{z}(t_k), \tag{36}$$

$$\bar{\mathbf{y}}_0(k)=\mathbf{y}_0(t_k), \tag{37}$$

$$\bar{\mathbf{w}}(k)=\mathbf{w}(t_k), \tag{38}$$

$$\bar{\mathbf{u}}(k)=\mathcal{H}_{Nm}([\mathbf{u}^{\mathrm{T}}(t_k)\ \cdots\ \mathbf{u}^{\mathrm{T}}(t_k+h_k-1)]^{\mathrm{T}}). \tag{39}$$

These results imply that $\lim_{k\to\infty}E(\bar{\mathbf{z}}^{\mathrm{T}}(k)\bar{\mathbf{z}}(k))=0$ if $\lim_{t\to\infty}E(\mathbf{z}^{\mathrm{T}}(t)\mathbf{z}(t))=0$. On the other hand, as h_k is bounded by N, there exists a constant real scalar η independent of k and τ such that $\forall k\in\mathbb{N}$, $\mathbf{z}^{\mathrm{T}}(t_k+\tau)\times\mathbf{z}(t_k+\tau)<\eta\mathbf{z}^{\mathrm{T}}(t_k)\mathbf{z}(t_k)$ for any $\tau\in\{1,\ldots,h_k\}$. Then (36) implies that $\lim_{t\to\infty}E(\mathbf{z}^{\mathrm{T}}(t)\mathbf{z}(t))=0$ if $\lim_{k\to\infty}E(\bar{\mathbf{z}}^{\mathrm{T}}(k)\bar{\mathbf{z}}(k))=0$. Thus, we have the following proposition.

Proposition 4.1: *$\hat{P}_K$ is mean square stable if and only if $\hat{P}_{Ks}$ is stochastically stable.*

Next, we discuss the relationship between $\hat{\Delta}$ and $\hat{\Delta}_s$.

Proposition 4.2: *For any $\hat{\Delta}\in\mathbb{D}_{\gamma}^{p\times m}$, its auxiliary system $\hat{\Delta}_s\in\mathbb{D}_{\gamma s}^{p\times(Nm)}$.*

Proof: From $\hat{\Delta}\in\mathbb{D}_{\gamma}^{p\times m}$ and Lemma 2.1, there exist $0<\mathbf{P}_\delta\in\mathbb{R}^{q\times q}$ such that

$$\begin{bmatrix}\mathbf{P}_\delta & \mathbf{0}\\ \mathbf{0} & \dfrac{1}{\gamma^2}\mathbf{I}_m\end{bmatrix}-\begin{bmatrix}\mathbf{A}_\delta & \mathbf{B}_\delta\\ \mathbf{C}_\delta & \mathbf{D}_\delta\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\mathbf{P}_\delta & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_p\end{bmatrix}\begin{bmatrix}\mathbf{A}_\delta & \mathbf{B}_\delta\\ \mathbf{C}_\delta & \mathbf{D}_\delta\end{bmatrix}>0. \tag{40}$$

Noticing (28), inequality (40) means that $\forall t\in\mathbb{N}$ and $\forall[\mathbf{x}_\delta^{\mathrm{T}}(t)\quad\mathbf{u}^{\mathrm{T}}(t)]^{\mathrm{T}}\in\mathbb{R}^{q+m}$,

$$\begin{aligned}&\mathbf{x}_\delta^{\mathrm{T}}(t)\mathbf{P}_\delta\mathbf{x}_\delta(t)+\frac{1}{\gamma^2}\mathbf{u}^{\mathrm{T}}(t)\mathbf{u}(t)\\ &\quad\geq\mathbf{w}^{\mathrm{T}}(t)\mathbf{w}(t)+\mathbf{x}_\delta^{\mathrm{T}}(t+1)\mathbf{P}_\delta\mathbf{x}_\delta(t+1).\end{aligned} \tag{41}$$

Equality holds in (41) if and only if $[\mathbf{x}_\delta^{\mathrm{T}}(t)\quad\mathbf{u}^{\mathrm{T}}(t)]^{\mathrm{T}}=\mathbf{0}$. Now $\forall t\in\mathbb{N}$, $\forall i\in\mathcal{N}$ and $\forall[\mathbf{x}_\delta^{\mathrm{T}}(t)\ \mathbf{u}^{\mathrm{T}}(t)\ \cdots\ \mathbf{u}^{\mathrm{T}}(t+i-1)]^{\mathrm{T}}\in\mathbb{R}^{q+im}$,

$$\begin{aligned}&\mathbf{x}_\delta^{\mathrm{T}}(t)\mathbf{P}_\delta\mathbf{x}_\delta(t)+\frac{1}{\gamma^2}\sum_{l=0}^{i-1}\mathbf{u}^{\mathrm{T}}(t+l)\mathbf{u}(t+l)\\ &\quad\geq\mathbf{w}^{\mathrm{T}}(t)\mathbf{w}(t)+\mathbf{x}_\delta^{\mathrm{T}}(t+1)\mathbf{P}_\delta\mathbf{x}_\delta(t+1)\\ &\qquad+\frac{1}{\gamma^2}\sum_{l=1}^{i-1}\mathbf{u}^{\mathrm{T}}(t+l)\mathbf{u}(t+l)\\ &\quad\geq\sum_{l=0}^{i-1}\mathbf{w}^{\mathrm{T}}(t+l)\mathbf{w}(t+l)+\mathbf{x}_\delta^{\mathrm{T}}(t+i)\mathbf{P}_\delta\mathbf{x}_\delta(t+i)\\ &\quad\geq\mathbf{w}^{\mathrm{T}}(t)\mathbf{w}(t)+\mathbf{x}_\delta^{\mathrm{T}}(t+i)\mathbf{P}_\delta\mathbf{x}_\delta(t+i).\end{aligned} \tag{42}$$

Equality holds in (42) if and only if $[\mathbf{x}_\delta^{\mathrm{T}}(t)\ \mathbf{u}^{\mathrm{T}}(t)\ \cdots\ \mathbf{u}^{\mathrm{T}}(t+i-1)]^{\mathrm{T}}=\mathbf{0}$. Since

$$\begin{bmatrix}\mathbf{x}_\delta(t+i)\\ \mathbf{w}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{A}_\delta^i & \mathbf{A}_\delta^{i-1}\mathbf{B}_\delta & \cdots & \mathbf{B}_\delta\\ \mathbf{C}_\delta & \mathbf{D}_\delta & & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}_\delta(t)\\ \mathbf{u}(t)\\ \vdots\\ \mathbf{u}(t+i-1)\end{bmatrix}, \tag{43}$$

inequality (42) means that $\forall i\in\mathcal{N}$,

$$\begin{aligned}&\begin{bmatrix}\mathbf{P}_\delta & \mathbf{0}\\ \mathbf{0} & \dfrac{1}{\gamma^2}\mathbf{I}_{im}\end{bmatrix}-\begin{bmatrix}(\mathbf{A}_\delta^{\mathrm{T}})^i & \mathbf{C}_\delta^{\mathrm{T}}\\ \mathbf{B}_\delta^{\mathrm{T}}(\mathbf{A}_\delta^{\mathrm{T}})^{i-1} & \mathbf{D}_\delta^{\mathrm{T}}\\ \vdots & \mathbf{0}\\ \mathbf{B}_\delta^{\mathrm{T}} & \end{bmatrix}\\ &\quad\times\begin{bmatrix}\mathbf{P}_\delta & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_p\end{bmatrix}\begin{bmatrix}\mathbf{A}_\delta^i & \mathbf{A}_\delta^{i-1}\mathbf{B}_\delta & \cdots & \mathbf{B}_\delta\\ \mathbf{C}_\delta & \mathbf{D}_\delta & & \mathbf{0}\end{bmatrix}>0.\end{aligned} \tag{44}$$

Hence

$$\begin{bmatrix}\mathbf{P}_\delta & \mathbf{0}\\ \mathbf{0} & \dfrac{1}{\gamma^2}\mathbf{I}_{Nm}\end{bmatrix}-\sum_{i=1}^{N}p_i\mathbf{\Phi}_i^{\mathrm{T}}\begin{bmatrix}\mathbf{P}_\delta & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_p\end{bmatrix}\mathbf{\Phi}_i>0 \tag{45}$$

with

$$\mathbf{\Phi}_i=\begin{bmatrix}\mathbf{A}_\delta^i & \mathcal{W}_{Nm}([\mathbf{A}_\delta^{i-1}\mathbf{B}_\delta\ \ \cdots\mathbf{B}_\delta])\\ \mathbf{C}_\delta & \mathcal{W}_{Nm}(\mathbf{D}_\delta)\end{bmatrix}. \tag{46}$$

Applying Lemma 2.6 to (45) completes the proof. □

From Propositions 4.1 and 4.2, it is easy to see the following proposition.

Proposition 4.3: *Suppose that $\forall\hat{\Delta}_s\in\mathbb{D}_{\gamma s}^{p\times(Nm)}$, $\hat{P}_{Ks}$ is stochastically stable. Then $\forall\hat{\Delta}\in\mathbb{D}_{\gamma}^{p\times m}$, $\hat{P}_K$ is mean square stable.*

The above proposition shows that our NCS design problem can be tackled by solving the corresponding problem for $\hat{P}_{Ks}$. Let $\hat{P}_{Ks}$ be divided into an unknown part $\hat{\Delta}_s$ and a known part $\hat{V}$, as shown in Figure 2. $\hat{V}$ is the closed-loop system formed by $\hat{P}_{0s}$ and $\hat{K}_s$, and is described as

$$\begin{cases} \begin{bmatrix} \bar{\mathbf{x}}(k+1) \\ \bar{\mathbf{x}}_e(k+1) \end{bmatrix} = \mathbf{A}_v(h_k)\begin{bmatrix} \bar{\mathbf{x}}(k) \\ \bar{\mathbf{x}}_e(k) \end{bmatrix} + \mathbf{B}_v(h_k)\bar{\mathbf{w}}(k), \\ \bar{\mathbf{u}}(k) = \mathbf{C}_v(h_k)\begin{bmatrix} \bar{\mathbf{x}}(k) \\ \bar{\mathbf{x}}_e(k) \end{bmatrix} + \mathbf{D}_v(h_k)\bar{\mathbf{w}}(k), \end{cases} \quad k \in \mathbb{N}, \tag{47}$$

where

$$\mathbf{A}_v(h_k) = \begin{bmatrix} \mathbf{A} & \mathbf{BK} \\ \mathbf{0} & \mathbf{\Lambda}_0 \end{bmatrix}^{h_k-1} \begin{bmatrix} \mathbf{A} & \mathbf{BK} \\ -\mathbf{LC} & \mathbf{\Lambda}_0 + \mathbf{LC} \end{bmatrix}, \tag{48}$$

$$\mathbf{B}_v(h_k) = \begin{bmatrix} \mathbf{A} & \mathbf{BK} \\ \mathbf{0} & \mathbf{\Lambda}_0 \end{bmatrix}^{h_k-1} \begin{bmatrix} \mathbf{0} \\ -\mathbf{L} \end{bmatrix}, \tag{49}$$

$$\mathbf{C}_v(h_k) = \begin{bmatrix} -\mathbf{\Upsilon}_2(h_k)\mathbf{C} & \mathbf{\Upsilon}_3(h_k) \end{bmatrix}, \tag{50}$$

$$\mathbf{D}_v(h_k) = -\mathbf{\Upsilon}_2(h_k), \tag{51}$$

with

$$\mathbf{\Upsilon}_3(h_k) = \begin{cases} \mathcal{H}_{Nm}(\mathbf{K}), & h_k = 1, \\ \mathcal{H}_{Nm}([\mathbf{K}^{\mathrm{T}} \ (\mathbf{K}(\mathbf{\Lambda}_0 + \mathbf{LC}))^{\mathrm{T}} \cdots \\ \quad (\mathbf{K}\mathbf{\Lambda}_0^{h_k-2}(\mathbf{\Lambda}_0 + \mathbf{LC}))^{\mathrm{T}}]^{\mathrm{T}}), & h_k > 1. \end{cases} \tag{52}$$

Let $\xi = 1/\gamma$. Then our main result can be presented.

Theorem 4.4: *Suppose that $\hat{V} \in \mathbb{D}_{\xi s}^{(Nm)\times p}$. Then $\forall \hat{\Delta}_s \in \mathbb{D}_{\gamma s}^{p\times(Nm)}$, $\hat{P}_{Ks}$ is stochastically stable.*

Proof: See the Appendix. □

We refer to Theorem 4.4 as the small gain theorem of *discrete-time stochastic* systems. This result has not been presented previously in the literature. Note that the small gain theorem of Zhou et al. (1995) is valid for deterministic systems while the small gain theorem of Dragan, Halanay, and Stoica (1997) is derived for continuous-time stochastic systems.

According to Proposition 4.3 and Theorem 4.4, any pair of **K** and **L** ensuring $\hat{V} \in \mathbb{D}_{\xi s}^{(Nm)\times p}$ is a solution to our NCS design problem. Since **A**, **B**, **C**, **D**, γ, N and p_i, $\forall i \in \mathcal{N}$, are known, $\mathbf{A}_v(i)$, $\mathbf{B}_v(i)$, $\mathbf{C}_v(i)$ and $\mathbf{D}_v(i)$ with $i \in \mathcal{N}$ in (48) to (51) are functions of **K** and **L**. Therefore, we can denote

$$\mathbf{F}_i(\mathbf{K}, \mathbf{L}) = \begin{bmatrix} \mathbf{A}_v(i) & \mathbf{B}_v(i) \\ \mathbf{C}_v(i) & \mathbf{D}_v(i) \end{bmatrix}, \quad i \in \mathcal{N}. \tag{53}$$

Further define

$$\underline{\alpha}(\mathbf{K}, \mathbf{L}) = \inf_{\substack{0 < \mathbf{Q} \in \mathbb{R}^{(2n)\times(2n)} \\ \alpha \in \mathbb{R}}} \{\alpha \mid \alpha \mathbf{U}(\mathbf{Q}, \gamma) > \sum_{i=1}^{N} p_i \mathbf{F}_i^{\mathrm{T}}(\mathbf{K}, \mathbf{L}) \mathbf{S}(\mathbf{Q}) \mathbf{F}_i(\mathbf{K}, \mathbf{L})\} \tag{54}$$

with

$$\mathbf{U}(\mathbf{Q}, \gamma) = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \gamma^2 \mathbf{I}_p \end{bmatrix} \quad \text{and} \quad \mathbf{S}(\mathbf{Q}) = \begin{bmatrix} \mathbf{Q} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{Nm} \end{bmatrix}. \tag{55}$$

For given $\mathbf{K} \in \mathbb{R}^{m\times n}$ and $\mathbf{L} \in \mathbb{R}^{n\times p}$, $\underline{\alpha}(\mathbf{K}, \mathbf{L})$ can be computed conveniently by a combination of linear matrix inequality (LMI) technique (Boyd, El Ghaoui, Feron, and Balakrishan 1994) and bisection search (Quarteroni, Sacco, and Saleri 2000). From Proposition 4.3 and Theorem 4.4 as well as Lemma 2.6, the following result is plain.

Corollary 4.5: *A pair of $\mathbf{K} \in \mathbb{R}^{m\times n}$ and $\mathbf{L} \in \mathbb{R}^{n\times p}$ guarantee that $\hat{P}_K$ is mean square stable for any $\hat{\Delta} \in \mathbb{D}_{\gamma}^{p\times m}$, if $\underline{\alpha}(\mathbf{K}, \mathbf{L}) < 1$.*

According to Corollary 4.5, we can design **K** and **L** by solving the nonlinear optimisation problem

$$\mu = \inf_{\substack{\mathbf{K} \in \mathbb{R}^{m\times n} \\ \mathbf{L} \in \mathbb{R}^{n\times p}}} \underline{\alpha}(\mathbf{K}, \mathbf{L}). \tag{56}$$

We solve this optimisation problem using the genetic algorithm (GA) (Goldberg 1989; Man, Tang, and Kwong 1998) to obtain a pair of $\mathbf{K}^*$ and $\mathbf{L}^*$ such that $\underline{\alpha}(\mathbf{K}^*, \mathbf{L}^*) < 1$. Note that in some cases we may be unable to achieve $\underline{\alpha}(\mathbf{K}, \mathbf{L}) < 1$, even though the NCS design problem does have solutions. This is because Corollary 4.5 only provides a sufficient condition. If $\mathbf{K}^*$ and $\mathbf{L}^*$ are not found by the GA to meet $\underline{\alpha}(\mathbf{K}^*, \mathbf{L}^*) < 1$, we can rearrange some conditions of the NCS design problem, for example by increasing the value of γ, to ease the design problem.

5. A numerical example

On the basis of the method presented in the previous section, a MATLAB program for NCS design was developed where the *feasp* and *ga* functions of MATLAB were used to solve LMI and to implement GA, respectively. We considered an unstable third-order $\hat{P}_0$ of (24) with the parameters

$$\mathbf{A} = \begin{bmatrix} -1.05 & 0 & 0 \\ -2 & 0.75 & 0 \\ 0 & 1.05 & 0.5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}, \quad \mathbf{D} = 0.2.$$

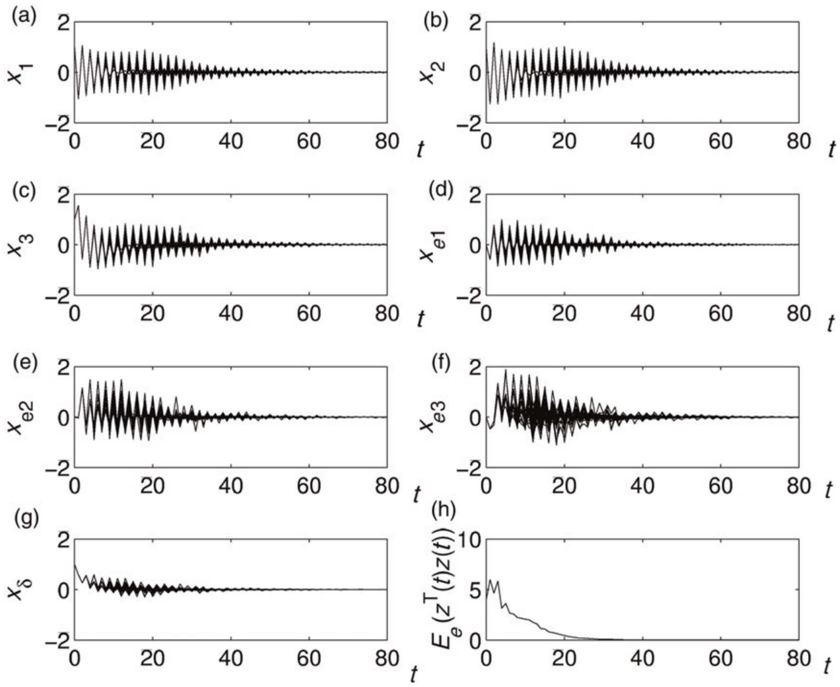

Figure 3. (a)–(g): State trajectories of the plant $\hat{P}_k$ for 50 simulations, and (h): $E_e(\mathbf{z}^T(t)\mathbf{z}(t))$ calculated by averaging $\mathbf{z}^T(t)\mathbf{z}(t)$ over these 50 simulations.

The value of N was set to $N=10$ and $p_i=0.1$ were assumed for $i\in\mathcal{N}\triangleq\{1,\ldots,10\}$, while $\gamma=0.4$ was set. Applying the MATLAB program to this NCS design problem yielded

$$\mathbf{K}^* = \begin{bmatrix} -0.2741 & 0.5791 & 0.0260 \end{bmatrix},$$

$$\mathbf{L}^* = \begin{bmatrix} 0.2657 & 0.1480 & -0.1718 \end{bmatrix}^T$$

with $\underline{\alpha}(\mathbf{K}^*,\mathbf{L}^*)=0.9872<1$.

The NCS with the designed $\mathbf{K}^*$ and $\mathbf{L}^*$ was simulated in the MATLAB platform for 50 times. In the simulation, a stable first-order $\hat{\Delta}$ of (28) was specified by $\mathbf{A}_\delta=0.5833$, $\mathbf{B}_\delta=1$, $\mathbf{C}_\delta=1$ and $\mathbf{D}_\delta=0.1$ with $\|\hat{\Delta}\|_\infty=2.4998<1/\gamma$. The initial state was chosen to be

$$\mathbf{x}(0)=[x_1(0)\ x_2(0)\ x_3(0)]^T=[1\ 1\ 1]^T,$$

$$\mathbf{x}_e(0)=[x_{e1}(0)\ x_{e2}(0)\ x_{e3}(0)]^T=[0\ 0\ 0]^T,\quad \mathbf{x}_\delta(0)=1.$$

Figure 3(a)–(g) depicts the 50 trajectories of each state element, respectively. These trajectories display our NCS behaviour under the 50 different realisations of $\{h_k\}$. For any $t\in\mathbb{N}$, we obtained 50 observations of the random variable $\mathbf{z}^T(t)\mathbf{z}(t)$. The first sample moment of the observations, denoted by $E_e(\mathbf{z}^T(t)\mathbf{z}(t))$, was computed. According to the standard statistics theory (Devore 2000), $E_e(\mathbf{z}^T(t)\mathbf{z}(t))$ is a confident estimation of $E(\mathbf{z}^T(t)\mathbf{z}(t))$ when the observation number is large. Figure 3(h) shows the trajectory of $E_e(\mathbf{z}^T(t)\mathbf{z}(t))$ where it can be seen that $E_e(\mathbf{z}^T(t)\mathbf{z}(t))$ converged to zero.

6. Conclusions

We have studied discrete-time NCSs where the plant has additive uncertainty and a smart controller is updated with the sensor information at stochastic time intervals. We have shown that the issue is linked to H_∞-control of linear stochastic systems. Under the condition that update intervals are i.i.d. $\mathcal{N}$-valued random variables, a new small gain theorem has been derived for discrete-time stochastic systems. Based on this result, sufficient conditions have been established for guaranteeing the mean square stability of NCSs and a design method for smart controller has been provided. A numerical example has been used to illustrate the proposed design method.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (Grant Nos. 60774001, 60736021 and 60721062), the 973 program of China (Grant No. 2009CB320603), the program for New Century Excellent Talents in University of China (NCET-04-0547) and the 111 Project of China (Grant No. B07031). S. Chen wishes to thank the support of the United Kingdom Royal Society and Royal Academy of Engineering.

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Appendix. Proof of Theorem 4.4

Based on $\hat{V}$ and $\hat{\Delta}_s$, construct a stochastic system $\hat{W}$ as

$$\begin{cases} \mathbf{x}_w(k+1) = \mathbf{A}_{h_k}\mathbf{x}_w(k) + \mathbf{B}_{h_k}\mathbf{u}_w(k), \\ \mathbf{y}_w(k) = \mathbf{C}_{h_k}\mathbf{x}_w(k) + \mathbf{D}_{h_k}\mathbf{u}_w(k), \end{cases} \quad k \in \mathbb{N}, \tag{A1}$$

where

$$\mathbf{A}_{h_k} = \begin{bmatrix} \mathbf{A}_\delta^{h_k} & \mathcal{W}_{Nm}([\mathbf{A}_\delta^{h_k-1}\mathbf{B}_\delta \ \cdots \ \mathbf{B}_\delta])\mathbf{C}_v(h_k) \\ \mathbf{0} & \mathbf{A}_v(h_k) \end{bmatrix}, \tag{A2}$$

$$\mathbf{B}_{h_k} = \begin{bmatrix} \mathcal{W}_{Nm}([\mathbf{A}_\delta^{h_k-1}\mathbf{B}_\delta \ \cdots \ \mathbf{B}_\delta])\mathbf{D}_v(h_k) \\ \mathbf{B}_v(h_k) \end{bmatrix}, \tag{A3}$$

$$\mathbf{C}_{h_k} = [\mathbf{C}_\delta \ \mathcal{W}_{Nm}(\mathbf{D}_\delta)\mathbf{C}_v(h_k)], \tag{A4}$$

$$\mathbf{D}_{h_k} = \mathcal{W}_{Nm}(\mathbf{D}_\delta)\mathbf{D}_v(h_k). \tag{A5}$$

Since $\hat{V}$ and $\hat{\Delta}_s$ are stochastically stable, from Definition 2 and Lemma 2.3, there exist $0 < \mathbf{P}_v \in \mathbb{R}^{(2n)\times(2n)}$ and $0 < \mathbf{P}_s \in \mathbb{R}^{q\times q}$ satisfying $\forall i \in \mathcal{N}$,

$$\mathbf{P}_v - \sum_{i=1}^{N} p_i \mathbf{A}_v^{\mathrm{T}}(i)\mathbf{P}_v\mathbf{A}_v(i) > 0, \tag{A6}$$

$$\mathbf{P}_s - \sum_{i=1}^{N} p_i (\mathbf{A}_\delta^i)^{\mathrm{T}}\mathbf{P}_s\mathbf{A}_\delta^i > 0. \tag{A7}$$

It can be seen easily that for sufficiently large $0 < \kappa \in \mathbb{R}$

$$\begin{bmatrix} \mathbf{P}_s & \mathbf{0} \\ \mathbf{0} & \kappa\mathbf{P}_v \end{bmatrix} - \sum_{i=1}^{N} p_i \mathbf{A}_i^{\mathrm{T}} \begin{bmatrix} \mathbf{P}_s & \mathbf{0} \\ \mathbf{0} & \kappa\mathbf{P}_v \end{bmatrix} \mathbf{A}_i > 0, \quad \forall i \in \mathcal{N}. \tag{A8}$$

This implies that $\hat{W}$ is stochastically stable. Denoting

$$\mathbf{v}_w(k) = [\mathbf{0} \quad \mathbf{C}_v(h_k)]\mathbf{x}_w(k) + \mathbf{D}_v(h_k)\mathbf{u}_w(k), \tag{A9}$$

it can be seen that $\hat{W}$ is the tandem connection of $\hat{V}$ and $\hat{\Delta}_s$, i.e. $\mathbf{v}_w = \hat{V}\mathbf{u}_w, \mathbf{y}_w = \hat{\Delta}_s\mathbf{v}_w$. Therefore

$$\|\hat{W}\|_{\infty s} = \sup_{\substack{\mathbf{u}_w \in \ell_{2s}^p \\ \|\mathbf{u}_w\|_{2s} \neq 0 \\ \mathbf{x}_w(0)=\mathbf{0} \\ h_0 \in \mathcal{N}}} \frac{\|\mathbf{y}_w\|_{2s}}{\|\mathbf{u}_w\|_{2s}} \le \sup_{\substack{\mathbf{v}_w \in \ell_{2s}^{Nm} \\ \|\mathbf{v}_w\|_{2s} \neq 0 \\ \mathbf{x}_w(0)=\mathbf{0} \\ h_0 \in \mathcal{N}}} \frac{\|\mathbf{y}_w\|_{2s}}{\|\mathbf{v}_w\|_{2s}} \sup_{\substack{\mathbf{u}_w \in \ell_{2s}^p \\ \|\mathbf{u}_w\|_{2s} \neq 0 \\ \mathbf{x}_w(0)=\mathbf{0} \\ h_0 \in \mathcal{N}}} \frac{\|\mathbf{v}_w\|_{2s}}{\|\mathbf{u}_w\|_{2s}}$$
$$= \|\hat{\Delta}_s\|_{\infty s}\|\hat{V}\|_{\infty s} < \left(\frac{1}{\gamma}\right)\left(\frac{1}{\xi}\right) = 1. \tag{A10}$$

In addition, from (1), (2), (34), (51) and (A5), $\mathbf{D}_{h_k} \equiv \mathbf{0}$. Thus, applying Corollary 2.8 to $\hat{W}$, there exist $0 \le \mathbf{Y}_i \in \mathbb{R}^{(2n+q)\times(2n+q)}$ with $i \in \mathcal{N}$, which satisfy $\forall i \in \mathcal{N}$,

$$\mathbf{I} - \mathbf{B}_i^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l \mathbf{B}_i > 0, \tag{A11}$$

$$\mathbf{Y}_i = \mathbf{C}_i^{\mathrm{T}}\mathbf{C}_i + (\mathbf{A}_i + \mathbf{B}_i\mathbf{R}_i)^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l (\mathbf{A}_i + \mathbf{B}_i\mathbf{R}_i) - \mathbf{R}_i^{\mathrm{T}}\mathbf{R}_i, \tag{A12}$$

$$\mathbf{R}_i = \left(\mathbf{I} - \mathbf{B}_i^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l \mathbf{B}_i\right)^{-1} \mathbf{B}_i^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l \mathbf{A}_i, \tag{A13}$$

and $\mathbf{A}_{h_k} + \mathbf{B}_{h_k}\mathbf{R}_{h_k}$ is stochastically stable.

Substituting (A13) into (A12), we get $\forall i \in \mathcal{N}$,

$$\begin{aligned} \mathbf{Y}_i &= \mathbf{C}_i^{\mathrm{T}}\mathbf{C}_i + \mathbf{A}_i^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l \mathbf{A}_i + \mathbf{A}_i^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l \mathbf{B}_i \\ &\quad \times \left(\mathbf{I} - \mathbf{B}_i^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l \mathbf{B}_i\right)^{-1} \mathbf{B}_i^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l \mathbf{A}_i \\ &= \widetilde{\mathbf{C}}_i^{\mathrm{T}}\widetilde{\mathbf{C}}_i + \widetilde{\mathbf{A}}_i^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l \widetilde{\mathbf{A}}_i \end{aligned} \tag{A14}$$

with

$$\widetilde{\mathbf{A}}_i = \mathbf{A}_i + \mathbf{B}_i\mathbf{C}_i, \tag{A15}$$

$$\widetilde{\mathbf{C}}_i = \left(\mathbf{I} - \mathbf{B}_i^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l \mathbf{B}_i\right)^{-1/2} \left(\mathbf{B}_i^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l \widetilde{\mathbf{A}}_i - \mathbf{C}_i\right). \tag{A16}$$

Let

$$\widetilde{\mathbf{H}}_i = -\mathbf{B}_i \left(\mathbf{I} - \mathbf{B}_i^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l \mathbf{B}_i\right)^{-1/2}, \quad i \in \mathcal{N}. \tag{A17}$$

Then,

$$\begin{aligned} \widetilde{\mathbf{A}}_{h_k} - \widetilde{\mathbf{H}}_{h_k}\widetilde{\mathbf{C}}_{h_k} &= \widetilde{\mathbf{A}}_{h_k} + \mathbf{B}_{h_k}\left(\mathbf{I} - \mathbf{B}_{h_k}^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l \mathbf{B}_{h_k}\right)^{-1} \\ &\quad \times \left(\mathbf{B}_{h_k}^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l \widetilde{\mathbf{A}}_{h_k} - \mathbf{C}_{h_k}\right) \\ &= \mathbf{A}_{h_k} + \mathbf{B}_{h_k}\left(\mathbf{I} - \mathbf{B}_{h_k}^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l \mathbf{B}_{h_k}\right)^{-1} \\ &\quad \times \mathbf{B}_{h_k}^{\mathrm{T}} \sum_{l=1}^{N} p_l \mathbf{Y}_l \mathbf{A}_{h_k} \\ &= \mathbf{A}_{h_k} + \mathbf{B}_{h_k}\mathbf{R}_{h_k} \end{aligned} \tag{A18}$$

is stochastically stable. Thus, $(\widetilde{\mathbf{C}}_{h_k}, \widetilde{\mathbf{A}}_{h_k})$ is stochastically detectable. Using Corollary 2.5, we conclude that $\widetilde{\mathbf{A}}_{h_k}$ is stochastically stable.

Since $\hat{P}_{Ks}$ is the closed-loop system of $\hat{V}$ and $\hat{\Delta}_s$, $\hat{P}_{Ks}$ can be viewed as the unity-feedback control system of $\hat{W}$ which is the tandem connection of $\hat{V}$ and $\hat{\Delta}_s$. In other words, $\hat{P}_{Ks}$ can be written as (A1) with

$$\mathbf{u}_w(t) = \mathbf{y}_w(t). \tag{A19}$$

Combining (A1) and (A19) with $\mathbf{D}_{h_k} \equiv \mathbf{0}$, the system $\hat{P}_{Ks}$, which can be written as

$$\mathbf{x}_w(k+1) = (\mathbf{A}_{h_k} + \mathbf{B}_{h_k}\mathbf{C}_{h_k})\mathbf{x}_w(k) = \widetilde{\mathbf{A}}_{h_k}\mathbf{x}_w(k), \quad k \in \mathbb{N}, \tag{A20}$$

is a stochastically stable system.