Partial Equalization for MC–CDMA Systems in Non-Ideally Estimated Correlated Fading

Flavio Zabini, Barbara M. Masini, *Member, IEEE*, Andrea Conti, *Member, IEEE*, and Lajos Hanzo, *Fellow, IEEE*

***Abstract*—Multicarrier code-division multiple access (MC–CDMA) can support high data rates in next-generation multiuser wireless communication systems. Partial equalization (PE) is a low-complexity technique for combining the signals of subcarriers to improve the achievable performance of MC–CDMA systems in terms of their bit error probability (BEP) and bit error outage (BEO) in comparison with maximal ratio combining, orthogonality restoring combining, and equal-gain combining techniques. We analyze the performance of the multiuser MC–CDMA downlink and derive the optimal PE parameter expression, which minimizes the BEP. Realistic imperfect channel estimation and frequency-domain (FD) block-fading channels are considered. More explicitly, the analytical expression of the optimum PE parameter is derived as a function of the number of subcarriers, number of active users (i.e., the system load), mean signal-to-noise ratio (SNR), and variance of the channel-estimation errors for the aforementioned FD block-fading channel. We show that the choice of the optimal PE technique significantly increases the achievable system load for the given target BEP and BEO.**

***Index Terms*—Channel estimation, fading channel, multicarrier code-division multiple access (MC–CDMA), partial equalization (PE), performance evaluation.**

I. Introduction

Multicarrier code-division multiple access (MC–CDMA) systems harness the combination of orthogonal frequency-division multiplexing (OFDM) and code-division multiple access (CDMA) to efficiently combat frequency-selective fading and interference in high-rate multiuser communication [1]–[8]. Hence, they constitute promising candidates for next-generation mobile communications [9]. Multipath fading destroys the orthogonality of the users' spreading sequences, and thus, multiple-access interference (MAI) occurs. In the downlink (DL) of classical MC–CDMA systems, MAI mitigation can be accomplished at the receiver using low-complexity linear combining techniques [10]. Following the estimation of the channel-state information (CSI), the signals of different subcarriers are appropriately weighted and summed using equal-gain combining (EGC) [2], maximum ratio combining (MRC) [2], [11], orthogonality restoring combining (ORC; also known as zero forcing) [2], [11], or threshold-based ORC (TORC) [1], [2], [12]. The MRC technique represents the optimal choice when the system is noise limited; by contrast, when the system is interference limited, ORC may be employed to mitigate the MAI at the cost of enhancing the noise [13].[1] The minimum mean square error (MMSE) [14] criterion may also be used to derive the equalizer coefficients, whereas an even more powerful optimization criterion is the minimum bit error ratio (MBER) criterion [15]. However, although MRC, EGC, and ORC only require the CSI, the MMSE and MBER equalizers are more complex, because they exploit additional knowledge, e.g., the number of active users and the mean signal-to-noise ratio (SNR).[2]

As an alternative, the partial equalization (PE) technique in [17]–[19] weights the signal of the mth subcarrier by the complex gain of

$$G_m = \frac{H_m^\star}{|H_m|^{1+\beta}} \tag{1}$$

where H_m is the mth subcarriers gain, and β is a parameter with values in the range of $[-1, 1]$. It may be observed that (1) reduces to EGC, MRC, and ORC for $\beta = 0, -1,$ and 1, respectively. Again, MRC and ORC are optimum in the extreme cases of noise- and interference-limited systems, respectively, and for each intermediate situation, an optimum value of the PE parameter β can be found to optimize the performance [19]. Note that the PE scheme has the same complexity as the EGC, MRC, and ORC, but it is more robust to channel impairments and to MAI fluctuations. In [19], the bit error probability (BEP) of the MC–CDMA DL that employs PE has been analyzed in perfectly estimated uncorrelated Rayleigh fading channels. It was also shown that, despite its lower complexity, the PE can approximate the optimum MMSE scheme's performance.

In practical situations, the signals of adjacent subcarriers may experience correlated fading. A channel model, which

Manuscript received February 24, 2010; revised May 14, 2010; accepted June 13, 2010. Date of publication July 19, 2010; date of current version October 20, 2010. This work was supported in part by the FP7 European project OPTIMIX under Grant Agreement 214625, and in part by the Network of Excellence NEWCOM++. The review of this paper was coordinated by Dr. H. Lin.

F. Zabini and B. M. Masini are with the Wireless Communications Laboratory, Institute for Electronics, Information, and Telecommunications Engineering, Research Unit of Bologna, National Research Council, University of Bologna, 40136 Bologna, Italy (e-mail: flavio.zabini2@unibo.it; barbara.masini@unibo.it).

A. Conti is with the Engineering Department in Ferrara, University of Ferrara, 44100 Ferrara, Italy, and also with the WiLAB c/o University of Bologna, 40136 Bologna, Italy (e-mail: a.conti@ieee.org).

L. Hanzo is with the Department of Electronics and Computer Science, University of Southampton, SO17 1BJ Southampton, U.K. (e-mail: lh@.soton.ac.uk).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TVT.2010.2060217

[1]ORC is often improved with the aid of TORC, in which a threshold is introduced to cancel the contributions of the subcarriers gravely corrupted by the noise, hence facilitating good performance at low complexity.

[2]More complex nonlinear multiuser equalizers, e.g., interference cancellation (IC) and maximum likelihood (ML) detection, exploit the knowledge of the interfering users' spreading codes in the detection process at the expense of higher receiver complexity [14], [16].

enables us to account for both frequency-domain (FD) fading correlation and FD interleaving, is the FD block-fading channel (FDBFC) [20], [21].

In this paper, we analyze MC–CDMA systems using PE in FD-correlated fading modeled by the FDBFC and using realistic imperfect channel estimation. The mean BEP and bit error outage (BEO) [22], [23] are characterized as a function of the system parameters, e.g., the mean SNR, the number of users, the number of subcarriers, the channel-estimation errors, and the particular FDBFC model employed.

This paper is organized as follows. In Section II, the system model and our assumptions are presented, whereas the decision variable is derived in Section III. In Section IV, the BEP and BEO performance is characterized, and the optimum PE parameter is determined. In Section V, our analytical results are provided and compared to our simulations, whereas in Section VI, our conclusions are presented.

II. System Model and Assumptions

In this section, we present our system model and assumptions, followed by the characterization of the signals at the various processing stages.

A. Transmitted Signal

We consider the MC–CDMA architecture presented in [2], where FD spreading is performed using orthogonal Walsh–Hadamard (W–H) codes that have a spreading factor (SF), which is equal to the number of subcarriers. Hence, each data symbol is spread across all subcarriers and multiplied by the chip assigned to each particular subcarrier, as shown in Fig. 1(a).[3] For binary phase-shift keying (BPSK) modulation, the signal transmitted in the DL is given by

$$s(t)=\sqrt{\frac{2E_{\mathrm{b}}}{M}}\sum_{k=0}^{N_{\mathrm{u}}-1}\sum_{i=-\infty}^{+\infty}\sum_{m=0}^{M-1}c_m^{(k)}a^{(k)}[i]g(t-iT_{\mathrm{b}}) \times\cos(2\pi f_m t+\phi_m) \quad (2)$$

where E_{b} is the energy per bit, M is the number of subcarriers, the indices k, i, and m represent the user, data, and subcarrier indices, respectively, N_{u} is the number of active users, c_m is the mth spreading chip (taking values of ±1), $a^{(k)}[i]$ is the symbol transmitted during the ith interval, $g(t)$ is a rectangular signaling pulse waveform with duration $[[0,T]]$ and a unity energy, T_{b} is the bit duration of user k, f_m is the mth subcarrier frequency, and ϕ_m is the mth subcarrier phase. Furthermore, $T_{\mathrm{b}}=T+T_{\mathrm{g}}$ is the total MC–CDMA symbol duration, where the time guard T_{g} is inserted between consecutive multicarrier symbols to eliminate the intersymbol interference (ISI) due to the channel's delay spread.

[3]Note that we adopted a time-continuous representation for analysis. This approach is equivalent to practical implementation through the inverse fast Fourier transform (IFFT) at the transmitter and FFT at the receiver, without losses in generality.

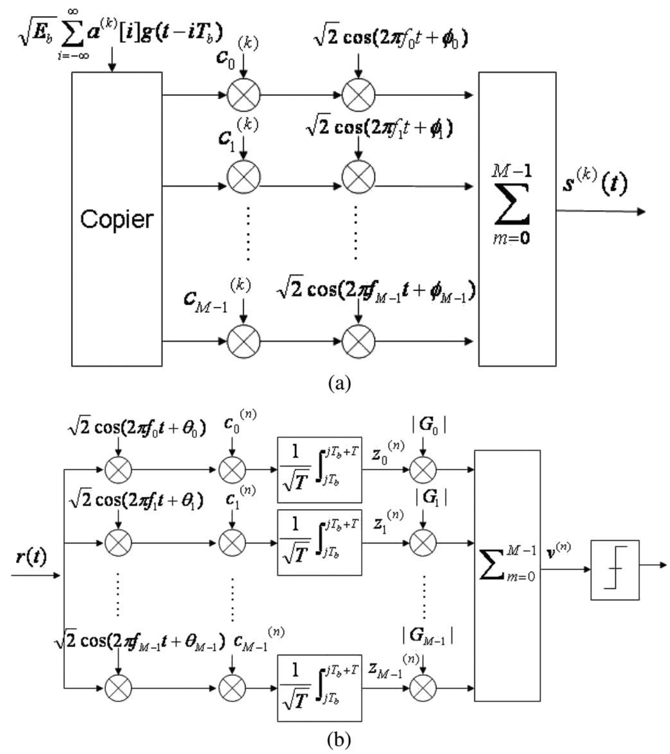

Fig. 1. Transmitter and receiver block schemes. (a) Transmitter. (b) Receiver.

B. Channel Model

In the DL, the signal of different users undergoes the same fading. We assume that the channel impulse response (CIR) $h(t)$ is time invariant for several MC–CDMA symbols, and we employ an FD channel-transfer function (FDCHTF) $H(f)$ characterized by

$$H(f)\simeq H(f_m),\qquad \text{for } |f-f_m|<\frac{W_{\mathrm{s}}}{2}\qquad \forall\, m \quad (3)$$

where H_m has real and imaginary parts of X_m and Y_m, respectively, whereas W_{s} is the bandwidth of each subcarrier. Because a nondispersive Dirac-shaped CIR represents a frequency flat-fading FDCHTF, this FDBFC assumption may loosely be interpreted in practical terms as having a low-dispersion CIR. We assume that, for each FD subchannel, the channel-induced spreading of the rectangular signaling pulse is such that the response to $g(t)$, i.e., $g'(t)$, still remains rectangular with a unity energy and duration of $T'\triangleq T+T_{\mathrm{d}}$, with T_{d} being the time dispersion, which is lower than the guard time T_{g} [3] (this approach will not limit the scope of our framework but simplifies the analysis).

Again, we consider a FDBFC [20], [21], [24], [25] across M subcarriers, which implies that the total number of subcarriers can be divided into L independent groups of $B=M/L$ subcarriers, as represented in Fig. 2, for which we have

$$H_{lB+1}=\widetilde{H}_l,\qquad \text{for } l=0,1,\ldots,L-1. \quad (4)$$

Hence, it is possible to describe the FDCTF using L rather than M coefficients $\widetilde{H}_l$. We assume that we have $\widetilde{H}_l=\alpha_l e^{j\vartheta_l}$

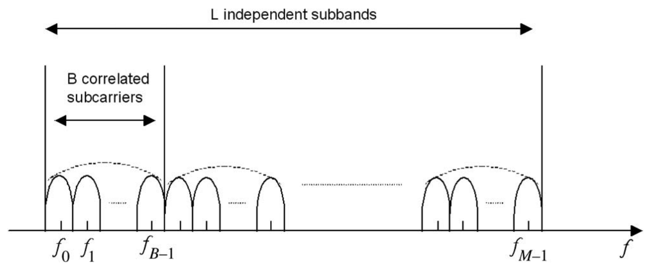

Fig. 2. Subcarrier spectrum in frequency BFC.

independent identically distributed (i.i.d.) random variables (RVs) with $\widetilde{H}_l = X_l + jY_l$ and $X_l, Y_l \sim \mathcal{N}(0, \sigma_{\rm H}^2)$.

C. Received Signal

The received signal is given by

$$r(t) = \sqrt{\frac{2E_{\rm b}}{M}} \sum_{k=0}^{N_{\rm u}-1} \sum_{i=-\infty}^{+\infty} \sum_{l=0}^{L-1} \sum_{b=0}^{B-1} \alpha_l c_{lB+b}^{(k)} a^{(k)}[i] g'(t - iT_{\rm b}) \times \cos(2\pi f_{lB+b} t + \vartheta_l) + n(t) \quad (5)$$

where $n(t)$ represents the additive white Gaussian noise (AWGN) with a double-sided power spectral density (PSD) of $N_0/2$.

D. Imperfect Channel Estimation

We analyze the performance with imperfect CSI by assuming a channel estimation error of E_l for each subcarriers block, which is complex Gaussian distributed with zero mean and a variance of $2\sigma_{\rm E}^2$; thus, $E_l \sim \mathcal{CN}(0, 2\sigma_{\rm E}^2)$.[4] Hence, the lth estimated complex FDCHTF coefficient is

$$\widehat{H}_l = H_l + E_l \quad (6)$$

where we have $\widehat{H}_l = \widehat{\alpha}_l e^{j\widehat{\vartheta}_l}$ and $E_l = X_{{\rm E}l} + jY_{{\rm E}l}$ so that $X_{{\rm E}l}$, $Y_{{\rm E}l} \sim \mathcal{N}(0, \sigma_{\rm E}^2)$. Note that $X_{{\rm E}l}$ and $Y_{{\rm E}l}$ are i.i.d.

E. Assumptions

We also stipulate the following common assumptions for the DL of a MC–CDMA system.

- The system is synchronous (different users and subcarriers experience the same delay, because their differences were perfectly compensated).
- The number of subcarriers is equal to the FD SF.
- The number of subcarriers is sufficiently high to enable the assessment of the BEP by exploiting the central limit theorem (CLT) and the law of large numbers (LLN).

The approximations that will be derived from the CLT and the LLN will all be verified by simulations. However, note that, in standardized systems, e.g., Worldwide Interoperability for Microwave Access (WiMAX) [31] and digital television broadcast service–terrestrial (DVB-T) [32], the number of subcarriers is sufficiently high (e.g., 2000 or 8000) to justify these assumptions.

[4]This result is the case of pilot-assisted channel estimation, where the $\sigma_{\rm E}^2$ depends on the number and energy of pilot symbols [26]–[30].

III. Decision Variable

The performance evaluation and the PE parameter optimization require the following analytical flow:

1) decomposition of the decision variable in useful, interfering, and noise terms;
2) statistical characterization of terms in item 1;
3) performance evaluation of conditioned and unconditioned BEP;
4) derivation of the optimum PE parameter;
5) BEO evaluation.

The signal of the qth subcarrier for the nth user at symbol instant j after the correlation receiver [see Fig. 1(b)] is

$$z_q^{(n)}[j] = \int_{jT_{\rm b}}^{jT_{\rm b}+T} \frac{r(t)}{\sqrt{T}} c_q^{(n)} \sqrt{2} \cos\left(2\pi f_q t + \widehat{\vartheta}_{\lfloor \frac{q}{B} \rfloor}\right) dt. \quad (7)$$

By substituting (5) in (7), we obtain (8), shown at the bottom of the page. Here, $n_q[j] \triangleq \int_{jT_{\rm b}}^{jT_{\rm b}+T} \sqrt{2}(c_q^{(n)}/\sqrt{T}) n(t) \cos(2\pi f_q t + \widehat{\vartheta}_{\lfloor (q/B) \rfloor}) dt$, and $u_{l,b,q}[j] \triangleq (1/T) \int_{jT_{\rm b}}^{jT_{\rm b}+T} 2\cos(2\pi f_{lB+b} t + \vartheta_l) \cos(2\pi f_q t + \widehat{\vartheta}_{\lfloor (q/B) \rfloor}) dt$. It can be shown that $n_q[j]$ is a zero-mean Gaussian RV (GRV) with a variance of $N_0/2$. In addition, $u_{lB+b,q}[j]$ is independent of index j and is given by

$$u_{lB+b,q}[j] = u_{l,b,q} = \begin{cases} \cos\left(\vartheta_l - \widehat{\vartheta}_{\lfloor \frac{q}{B} \rfloor}\right), & \text{for } lB+b = q \\ 0, & \text{otherwise.} \end{cases} \quad (9)$$

Hence, (8) becomes

$$z_q^{(n)}[j] = \sqrt{\frac{E_{\rm b}}{M}} \delta_{\rm d} \Bigg\{ \alpha_{\lfloor \frac{q}{B} \rfloor} \cos\left(\vartheta_{\lfloor \frac{q}{B} \rfloor} - \widehat{\vartheta}_{\lfloor \frac{q}{B} \rfloor}\right) + \sqrt{\frac{E_{\rm b}}{M}} \delta_{\rm d} \sum_{\substack{k=0 \\ k \neq n}}^{N_{\rm u}-1} \alpha_{\lfloor \frac{q}{B} \rfloor} c_q^{(k)} c_q^{(n)} \cos\left(\vartheta_{\lfloor \frac{q}{B} \rfloor} - \widehat{\vartheta}_{\lfloor \frac{q}{B} \rfloor}\right) \Bigg\} + n_q[j]$$

$$z_q^{(n)}[j] = \sqrt{\frac{E_{\rm b}}{M}\frac{T}{T'}} \left\{ \sum_{k=0}^{N_{\rm u}-1} \sum_{\substack{l=0 \\ \frac{q+1}{B}-1 < l < \frac{q}{B}}}^{L-1} \alpha_l c_q^{(k)} c_q^{(n)} a^{(k)}[j] u_{l,q-lB,q}[j] + \sum_{k=0}^{N_{\rm u}-1} \sum_{l=0}^{L-1} \sum_{\substack{b=0 \\ b \neq q-lB}}^{B-1} \alpha_l c_{lB+b}^{(k)} c_q^{(n)} a^{(k)}[j] u_{l,b,q}[j] + n_q[j] \right\} \quad (8)$$

where $\delta_{\rm d} \triangleq 1/(1+T_{\rm d}/T)$ represents the loss of energy caused by the channel-induced spreading of the rectangular signaling pulse. For simplicity, because the ISI is avoided by using a guard time, we will neglect the time index j in the following discussion.

The decision variable $v^{(n)}$ is obtained by a linear combination of the PE-based weighting of the signals gleaned from each subcarrier as

$$v^{(n)} = \sum_{q=0}^{M-1} |G_q| z_q^{(n)} \tag{10}$$

where G_q is the qth PE gain given by (1). In particular, for imperfect FDCHTF estimation and FDBFCs, we have

$$G_q = \frac{\widehat{H}^\star_{\lfloor \frac{q}{B} \rfloor}}{\left|\widehat{H}_{\lfloor \frac{q}{B} \rfloor}\right|^{1+\beta}} \tag{11}$$

with $|G_q| = |\widehat{H}_{\lfloor q/B \rfloor}|^{-\beta} = \widehat{\alpha}^{-\beta}_{\lfloor q/B \rfloor}$ and $\angle G_q = -\angle \widehat{H}_{\lfloor q/B \rfloor} = -\widehat{\vartheta}_{\lfloor q/B \rfloor}$. Note that, for $\beta = -1, 0,$ and 1, the coefficient in (11) leads to the cases of MRC, EGC, and ORC, respectively. Therefore, substituting (11) in (10), the decision variable becomes

$$v^{(n)} = U^{(n)} a^{(n)} + I^{(n)} + N^{(n)} \tag{12}$$

where $U^{(n)}$, $I^{(n)}$, and $N^{(n)}$ represent the useful, interfering, and noise terms, respectively, of user n and are given by

$$U^{(n)} = \sqrt{\frac{E_{\rm b}\delta_{\rm d}}{M}} \sum_{q=0}^{M-1} \Theta_q(\beta)$$

$$I^{(n)} = \sqrt{\frac{E_{\rm b}\delta_{\rm d}}{M}} \sum_{q=0}^{M-1} \sum_{\substack{k=0\\k\neq n}}^{N_{\rm u}-1} \Theta_q(\beta) c_q^{(n)} c_q^{(k)} a^{(k)}$$

$$N^{(n)} = \sum_{q=0}^{M-1} \widehat{\alpha}^{-\beta}_{\lfloor \frac{q}{B} \rfloor} n_q \tag{13}$$

$$\Theta_q(\beta) \triangleq \alpha_{\lfloor \frac{q}{B} \rfloor} \widehat{\alpha}^{-\beta}_{\lfloor \frac{q}{B} \rfloor} \cos\left(\vartheta_{\lfloor \frac{q}{B} \rfloor} - \widehat{\vartheta}_{\lfloor \frac{q}{B} \rfloor}\right). \tag{14}$$

We observe that α_l, $\widehat{\alpha}_l$, ϑ_l, and $\widehat{\vartheta}_l$ are i.i.d. for different subscripts l and the different Θ_{lB+b} values are also i.i.d. for different subscripts l, whereas we have $\Theta_{lB+b} = \Theta_{lB+b'}$ for the same l. To derive the BEP, we derive the distribution of $v^{(n)}$ from those of $U^{(n)}$, $I^{(n)}$, and $N^{(n)}$.[5]

A. Interference Term

The interference term of (13) can easily be rewritten as

$$I = \sqrt{\frac{E_{\rm b}\delta_{\rm d}}{M}} \sum_{\substack{k=0\\k\neq n}}^{N_{\rm u}-1} a^{(k)} \sum_{l=0}^{L-1} \Theta_{lB}(\beta) \sum_{b=0}^{B-1} c_{lB+b}^{(n)} c_{lB+b}^{(k)}. \tag{15}$$

By exploiting the properties of the W–H spreading matrices, it can be shown that, for all i integers and nonzero with values of i that satisfy $-n/B \le i \le -n/B + N_{\rm u} - 1$, we have

$$\sum_{b=0}^{B-1} c_{lB+b}^{(n)} c_{lB+b}^{(k)} = \begin{cases} 0, & k \neq n+iB \\ B, & k = n+iB, l \in \mathcal{L}_+ \\ -B, & k = n+iB, l \in \mathcal{L}_- \end{cases} \tag{16}$$

where $\mathcal{L}_+$ and $\mathcal{L}_-$ are appropriately chosen to ensure cardinalities $\#\mathcal{L}_+ = \#\mathcal{L}_- = L/2$ and $\mathcal{L}_+ \cup \mathcal{L}_- = 0, 1, \ldots, L-1$. Thus, the interference term of (15) becomes

$$I = \sqrt{\frac{E_{\rm b}\delta_{\rm d}}{M}} \sum_{\substack{i=-\lfloor \frac{n}{B} \rfloor\\ i\neq 0}}^{\lfloor \frac{-n+N_{\rm u}-1}{B} \rfloor} a^{(n+iB)} B \left(\overbrace{\sum_{l\in\mathcal{L}_+} \Theta_{lB}(\beta)}^{A_1^{(i)}} - \overbrace{\sum_{l\in\mathcal{L}_-} \Theta_{lB}(\beta)}^{A_2^{(i)}} \right). \tag{17}$$

For large values of L, we may apply the CLT to each of the internal sums in (17), obtaining $A_1^{(i)}$ and $A_2^{(i)} \sim \mathcal{N}(\mathbb{E}\{\Theta_{lB}(\beta)\}L/2, \zeta_\beta L/2)$, where ζ_β is the variance of $\Theta_{lB}^{1-\beta}$ defined as[6]

$$\zeta_\beta \triangleq \mathbb{E}\left\{\Theta_{lB}^2(\beta)\right\} - \mathbb{E}\left\{\Theta_{lB}(\beta)\right\}^2 \tag{18}$$

whose expression is evaluated in Appendix A. Therefore, $A^{(i)} \triangleq A_1^{(i)} - A_2^{(i)}$ is distributed as $\mathcal{N}(0, L\zeta_\beta)$. By exploiting the symmetry of the Gaussian probability density function (pdf), i.e., $a^{(n+iB)}A^{(i)} \sim \mathcal{N}(0, L\zeta_\beta)$, and capitalizing on the independence of the terms $a^{(n+iB)}$ in (17), which ensure that $\mathbb{E}\{(a^{(n+iB)}A^{(i)})(a^{(n+i'B)}A^{(i')})\} = 0, \forall\, i \neq i'$, and on the sum of uncorrelated, thus independent, GRVs, we infer that the interference term of (17) obeys the distribution $I \sim \mathcal{N}(0, \sigma_{\rm I}^2)$, where we have

$$\sigma_{\rm I}^2 = E_{\rm b}\delta_{\rm d} B \left\lfloor \frac{N_{\rm u}-1}{B} \right\rfloor \zeta_\beta. \tag{19}$$

B. Noise Term

The noise term of (12) is given by

$$N = \sum_{q=0}^{M-1} \widehat{\alpha}^{-\beta}_{\lfloor \frac{q}{B} \rfloor} n_q = \sum_{l=0}^{L-1} \widehat{\alpha}_l^{\beta} \overbrace{\sum_{b=0}^{B-1} n_{lB+b}}^{N_l}. \tag{20}$$

Because n_{lB+b} of (20) represents i.i.d. GRVs, $N_l \sim \mathcal{N}(0, (N_0/2)B)$ and N_l are also i.i.d. GRVs, where N consists of the sum of i.i.d. zero-mean RVs with a variance of $(N_0/2)B\mathbb{E}\{\widehat{\alpha}_l^{-2\beta}\}$. Based on the CLT, we can approximate the unconditioned noise term N as a zero-mean GRV with a variance of

$$\sigma_{\rm N}^2 = M\frac{N_0}{2}\mathbb{E}\left\{\widehat{\alpha}_l^{-2\beta}\right\}. \tag{21}$$

[5]To simplify the notation, we will neglect the index n in the following discussion without loss of generality.

[6]Note that ζ_β does not depend on index l, and it is an i.i.d. RV.

C. Useful Term

The useful term of (12) can be written as

$$U = \sqrt{\frac{E_\mathrm{b}\delta_\mathrm{d}}{M}}\sum_{l=0}^{L-1}\sum_{b=0}^{B-1}\Theta_{lB+b}(\beta) = \sqrt{\frac{E_\mathrm{b}\delta_\mathrm{d}}{M}}B\sum_{l=0}^{L-1}\Theta_{lB}(\beta). \tag{22}$$

By applying the CLT, U is assumed to be GRV with a mean and a variance, respectively, of

$$\mu_\mathrm{U} = \sqrt{ME_\mathrm{b}\delta_\mathrm{d}}\mathbb{E}\left\{\Theta_{lB}(\beta)\right\} \tag{23}$$

$$\sigma_\mathrm{U}^2 = E_\mathrm{b}\delta_\mathrm{d}B\zeta_\beta. \tag{24}$$

D. Independence of Interference, Noise, and Useful Terms

We now discuss the independence of the terms of Sections III-A, B, and C in (12). The independence of the data $a^{(k)}$ and the other variables (i.e., α_l, A, and n_l) guarantees that the interference term I is uncorrelated with the noise term N and with the useful term U. In addition, the independence between n_l and α_l guarantees that N is uncorrelated with U so that $\mathbb{E}\{NU\} = 0$. Similarly, I, N, and U are uncorrelated GRVs; hence, they are also independent.

IV. Performance Evaluation

We now evaluate the BEP and the BEO, and we derive the optimum PE parameter β.

A. BEP Evaluation

1) BEP: Given the decision variable (12) as $z = Ua + I + N$ and considering that $I + N$ is a zero-mean RV with a variance of $\sigma_\mathrm{I}^2 + \sigma_\mathrm{N}^2$, the BEP conditioned on the variable U becomes

$$P_\mathrm{b}|_U = Q\left(\frac{U}{\sqrt{\sigma_\mathrm{I}^2+\sigma_\mathrm{N}^2}}\right) \tag{25}$$

where $Q(x)$ is the Gaussian Q-function.

2) Unconditioned BEP: By applying the LLN (hence, U is substituted by its mean value μ_U), we obtain the approximation for the unconditioned BEP as follows:

$$P_\mathrm{b} \simeq Q\left(\frac{\mu_U}{\sqrt{\sigma_\mathrm{I}^2+\sigma_\mathrm{N}^2}}\right). \tag{26}$$

By substituting (19), (21), and (23) into (26), we obtain

$$P_\mathrm{b} \simeq Q\left(\sqrt{\frac{E_\mathrm{b}\delta_\mathrm{d}\left(\mathbb{E}\left\{\Theta_l(\beta)\right\}\right)^2}{E_\mathrm{b}\delta_\mathrm{d}\frac{B}{M}\left\lfloor\frac{N_\mathrm{u}-1}{B}\right\rfloor\zeta_\beta + \mathbb{E}\left\{\widehat{\alpha}_l^{-2\beta}\right\}\frac{N_0}{2}}}\right) \tag{27}$$

where the expression of $\mathbb{E}\{\widehat{\alpha}_l^{-2\beta}\}$, $\mathbb{E}\{\Theta_l(\beta)\}$, and ζ_β are given in Appendix A. By defining the mean SNR at the receiver as

$$\overline{\gamma} \triangleq \frac{2\sigma_\mathrm{H}^2 E_\mathrm{b}\delta_\mathrm{d}}{N_0} \tag{28}$$

we arrive at the BEP expression in (29), shown at the bottom of the page. Here, $\Gamma(\cdot)$ is the Euler gamma function [33], and

$$\Pi\left(\frac{\sigma_\mathrm{E}^2}{\sigma_\mathrm{H}^2}\right) \triangleq 1 - \frac{\frac{\sigma_\mathrm{E}^2}{\sigma_\mathrm{H}^2}}{\left(1-\frac{\sigma_\mathrm{E}^4}{\sigma_\mathrm{H}^4}\right)} \tag{30}$$

$$\Sigma\left(\frac{\sigma_\mathrm{E}^2}{\sigma_\mathrm{H}^2},\beta\right) \triangleq 1 - \frac{\frac{\sigma_\mathrm{E}^2}{\sigma_\mathrm{H}^2}}{\left(1-\frac{\sigma_\mathrm{E}^4}{\sigma_\mathrm{H}^4}\right)}\left[1+\frac{\left(\frac{\frac{1}{2}-\beta}{1-\beta}\right)}{\left(1+\frac{\sigma_\mathrm{E}^2}{\sigma_\mathrm{H}^2}\right)}\right]. \tag{31}$$

The BEP approximation provided by (29) is derived by applying the LLN to the unconditioned BEP expression given by (25). An exact evaluation of the BEP would require the averaging of (25) over the useful term. However, because we are not interested in the BEP exact expression and because (29) is a monotonic decreasing function with respect to its argument, the value of β that minimizes (29) represents the minimum also for the exact BEP given by (25), as will be verified in Section V through our simulations.

B. System Load for a Target BEP

By fixing the BEP to a target value $P_\mathrm{b}^\star$, we now derive the system load, $s_\mathrm{L} \triangleq (1/L)\lfloor(N_\mathrm{u} - 1/B)\rfloor$, from (29) as a function of the other systems parameters, which is given by

$$s_\mathrm{L} = \frac{\frac{\Pi^2\left(\frac{\sigma_\mathrm{E}^2}{\sigma_\mathrm{H}^2}\right)}{\left[Q^{-1}\left(P_\mathrm{b}^\star\right)\right]^2}\Gamma^2\left(\frac{3-\beta}{2}\right) - \frac{\left(1+\frac{\sigma_\mathrm{E}^2}{\sigma_\mathrm{H}^2}\right)^{-1}}{2\overline{\gamma}}\Gamma(1-\beta)}{\Sigma\left(\frac{\sigma_\mathrm{E}^2}{\sigma_\mathrm{H}^2},\beta\right)\Gamma(2-\beta) - \Pi^2\left(\frac{\sigma_\mathrm{E}^2}{\sigma_\mathrm{H}^2}\right)\Gamma^2\left(\frac{3-\beta}{2}\right)}. \tag{32}$$

C. BEO

In wireless communications, where small-scale fading is superimposed on large-scale fading (i.e., shadowing), another important performance metric is given by the BEO [22], [23], [34], which is defined as the probability that the BEP exceeds the maximum tolerable level (i.e., the target BEP $P_\mathrm{b}^\star$) and given by

$$P_\mathrm{o} \triangleq \mathbb{P}\left\{P_\mathrm{b} > P_\mathrm{b}^\star\right\} = \mathbb{P}\left\{\overline{\gamma}_\mathrm{dB} < \overline{\gamma}_\mathrm{dB}^\star\right\} \tag{33}$$

$$P_\mathrm{b} \simeq Q\left(\sqrt{\frac{\overline{\gamma}\,\Pi^2\left(\frac{\sigma_\mathrm{E}^2}{\sigma_\mathrm{H}^2}\right)\Gamma^2\left(\frac{3-\beta}{2}\right)}{\frac{1}{L}\left\lfloor\frac{N_\mathrm{u}-1}{B}\right\rfloor\overline{\gamma}\left[\Sigma\left(\frac{\sigma_\mathrm{E}^2}{\sigma_\mathrm{H}^2},\beta\right)\Gamma(2-\beta) - \Pi^2\left(\frac{\sigma_\mathrm{E}^2}{\sigma_\mathrm{H}^2}\right)\Gamma^2\left(\frac{3-\beta}{2}\right)\right] + \frac{1}{2}\left(1+\frac{\sigma_\mathrm{E}^2}{\sigma_\mathrm{H}^2}\right)^{-1}\Gamma(1-\beta)}}\right) \tag{29}$$

where $\overline{\gamma}_{\mathrm{dB}} = 10\log_{10}\overline{\gamma}$, and $\overline{\gamma}^{\star}_{\mathrm{dB}}$ is the SNR (in decibels), which ensures that $P_{\mathrm{b}}(\overline{\gamma}^{\star}) = P^{\star}_{\mathrm{b}}$. We consider the case of a shadowing environment in which $\overline{\gamma}$ is log-normal distributed with parameters of μ_{dB} and σ^2_{dB} (i.e., $\overline{\gamma}_{\mathrm{dB}}$ is a GRV with a mean of μ_{dB} and variance of σ^2_{dB}) [35]. Hence, the BEO is given by

$$P_{\mathrm{o}} = Q\left(\frac{\mu_{\mathrm{dB}} - \overline{\gamma}^{\star}_{\mathrm{dB}}}{\sigma_{\mathrm{dB}}}\right). \tag{34}$$

By inverting (29), we can derive the required SNR $\overline{\gamma}^{\star}$, enabling the derivation of the optimal β for a target BEP and a given system load as[7]

$$\overline{\gamma}^{\star} = \frac{\Gamma(1-\beta)\left(1+\frac{\sigma^2_{\mathrm{E}}}{\sigma^2_{\mathrm{H}}}\right)^{-1}\Pi^{-2}\left(\frac{\sigma^2_{\mathrm{E}}}{\sigma^2_{\mathrm{H}}}\right)\Gamma^{-2}\left(\frac{3-\beta}{2}\right)}{\frac{2}{\left[invQ\left(P^{\star}_{\mathrm{b}}\right)\right]^2} - 2s_{\mathrm{L}}\left[\frac{\Sigma\left(\frac{\sigma^2_{\mathrm{E}}}{\sigma^2_{\mathrm{H}}},\beta\right)\Gamma(2-\beta)}{\Pi^2\left(\frac{\sigma^2_{\mathrm{E}}}{\sigma^2_{\mathrm{H}}}\right)\Gamma^2\left(\frac{3-\beta}{2}\right)} - 1\right]}. \tag{35}$$

Given the target BEP and BEO, we obtain the required value of μ_{dB} from (35) and (34) (i.e., the median value of the SNR) that can be used for wireless system design, because it is strictly related to the link budget when the path-loss law is known.

D. Optimum PE Parameter

We aim at finding the optimum value of the PE parameter $\beta^{(\mathrm{opt})}$ defined as that particular value of β within the range [−1, 1], which minimizes the BEP in (29) as

$$\beta^{(\mathrm{opt})} \triangleq \arg\min_{\beta}\left\{P_{\mathrm{b}}\left(\beta, \overline{\gamma}, \frac{\sigma^2_{\mathrm{E}}}{\sigma^2_{\mathrm{H}}}\right)\right\}. \tag{36}$$

Because the BEP is monotonically decreasing as a function of β, we obtain (37), shown at the bottom of the page.[8] It will be shown in Section V that, although the adoption of the CLT and the LLN may lead to a less-accurate BEP expression for a low number of subcarriers and users, it still results in an accurate value for the optimum β.

Setting the derivative of the argument in (37) with respect to β to zero, we can derive the optimum value of β as the implicit solution of (38), shown at the bottom of the page (for details refer to Appendix B). Here, the parameter ξ quantifies the degree that the system is noise limited (low values) or interference limited (high values) and is defined as

$$\xi \triangleq \overline{\gamma}\frac{2}{L}\left\lfloor\frac{N_{\mathrm{u}}-1}{B}\right\rfloor \tag{39}$$

$$\chi\left(\frac{\sigma^2_{\mathrm{E}}}{\sigma^2_{\mathrm{H}}}\right) \triangleq \frac{1}{2}\frac{\frac{\sigma^2_{\mathrm{E}}}{\sigma^2_{\mathrm{H}}}}{\left(1-\frac{\sigma^4_{\mathrm{E}}}{\sigma^4_{\mathrm{H}}}\right)\left(1+\frac{\sigma^2_{\mathrm{E}}}{\sigma^2_{\mathrm{H}}}\right)}. \tag{40}$$

E. Case of Ideal Channel Estimation

In the case of ideal CSI (i.e., $\sigma^2_{\mathrm{E}}/\sigma^2_{\mathrm{H}}$ approaching zero) and for channels with uncorrelated FDCHTFs over the subcarriers, it is easy to verify that $\Pi(0) = 1$ and $\chi(0) = 0$, and then, (38) becomes

$$\xi = \left(\frac{1}{\Psi\left(\frac{3-\beta}{2}\right) - \Psi(1-\beta)} + \beta - 1\right)^{-1} \tag{41}$$

confirming the results obtained in [19] for the ideal conditions, which are used as a benchmark.

F. Coded Systems

Note that (29) is the BEP of an uncoded system. Thus, a question may arise: Can the methodology for obtaining the optimum β in uncoded systems also be applied to coded system? By remembering that we are not interested in the value of the BEP itself but in the value of the PE parameter β, which minimizes the BEP, we may assert that, for coded systems where the codeword error probability is a monotonic function of the uncoded BEP, the derivation of the optimum value of β that minimizes

[7] $invQ$ denotes the inverse of the Gaussian Q-function.

[8] Because the optimization is based on the derivation of the BEP, $P_{\mathrm{b}}(\beta, \overline{\gamma}, \sigma^2_{\mathrm{E}}/\sigma^2_{\mathrm{H}})$, with respect to β (and not with respect to $\overline{\gamma}$, which is considered a parameter), nothing would change in the analysis if we assume a channel-estimation process based on the training sequence of M symbols. In this case, the normalized estimation error variance would result in $\sigma^2_{\mathrm{E}}/\sigma^2_{\mathrm{H}} = 1/M\overline{\gamma}$, and it could easily be exploited by simply substituting its value.

$$\beta^{(\mathrm{opt})} = \arg\max_{\beta}\left\{\frac{\overline{\gamma}\Pi^2\left(\frac{\sigma^2_{\mathrm{E}}}{\sigma^2_{\mathrm{H}}}\right)\Gamma^2\left(\frac{3-\beta}{2}\right)}{\frac{1}{L}\left\lfloor\frac{N_{\mathrm{u}}-1}{B}\right\rfloor\overline{\gamma}\left[\Sigma\left(\frac{\sigma^2_{\mathrm{E}}}{\sigma^2_{\mathrm{H}}},\beta\right)\Gamma(2-\beta) - \Pi^2\left(\frac{\sigma^2_{\mathrm{E}}}{\sigma^2_{\mathrm{H}}}\right)\Gamma^2\left(\frac{3-\beta}{2}\right)\right] + \frac{1}{2}\left(1+\frac{\sigma^2_{\mathrm{E}}}{\sigma^2_{\mathrm{H}}}\right)^{-1}\Gamma(1-\beta)}\right\} \tag{37}$$

$$\xi = \frac{\left[\frac{\Pi\left(\sigma^2_{\mathrm{E}}/\sigma^2_{\mathrm{H}}\right) + \frac{\chi\left(\sigma^2_{\mathrm{E}}/\sigma^2_{\mathrm{H}}\right)}{\beta-1}\left(1-\frac{2\beta-1}{1+\sigma^2_{\mathrm{E}}/\sigma^2_{\mathrm{H}}}\right)}{\Psi\left(\frac{3-\beta}{2}\right)-\Psi(1-\beta)} + \Pi\left(\sigma^2_{\mathrm{E}}/\sigma^2_{\mathrm{H}}s\right)(\beta-1) - \frac{\chi\left(\sigma^2_{\mathrm{E}}/\sigma^2_{\mathrm{H}}\right)}{\left(1+\sigma^2_{\mathrm{E}}/\sigma^2_{\mathrm{H}}\right)}(2\beta-1)\right]^{-1}}{\left(1+\sigma^2_{\mathrm{E}}/\sigma^2_{\mathrm{H}}\right)} \tag{38}$$

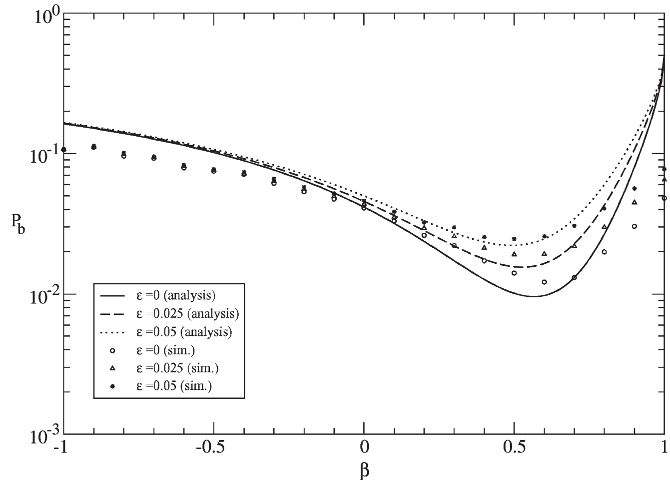

Fig. 3. BEP as a function of β for different values of $\varepsilon \triangleq \sigma_{\rm E}^2/\sigma_{\rm H}^2$ when $N_{\rm u} = M = 1024$, $L = M/16$, and $\overline{\gamma} = 10$ dB.

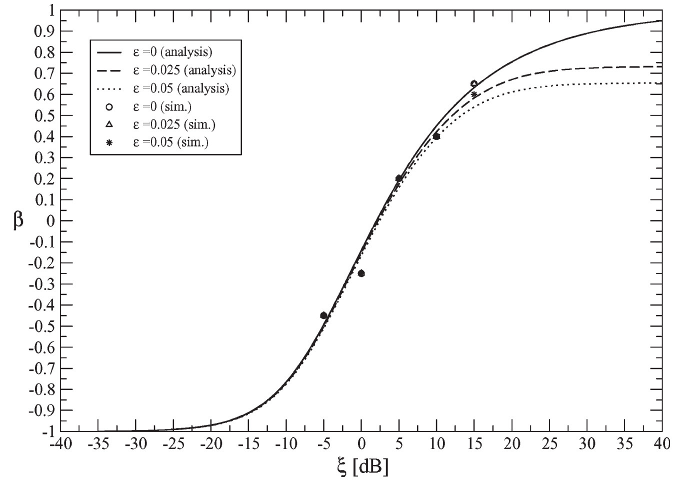

Fig. 4. Optimum value of β as a function of ξ (in decibels) for different values of normalized estimation error $\varepsilon \triangleq \sigma_{\rm E}^2/\sigma_{\rm H}^2$.

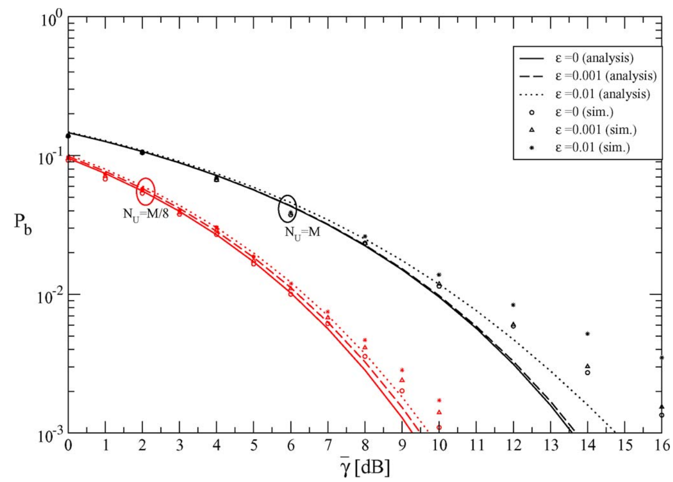

Fig. 5. BEP versus the mean SNR $\overline{\gamma}$ adopting the optimum β in case of perfect CSI, varying the normalized estimation errors $\varepsilon \triangleq \sigma_{\rm E}^2/\sigma_{\rm H}^2$. Comparison between $N_{\rm U} = M = 1024$ and $N_{\rm U} = M/8 = 128$, with $L = M/16$.

the uncoded BEP is equivalent to finding the value of β that minimizes the codeword error probability for the SNR value that accounts for the code rate. One example of the application of this relation and its relative proof are given in Appendix C. Hence, the aforementioned framework can be applied to the coded systems of interest. For further investigation on coded MC–CDMA systems, see [25] and [36]–[38].

V. Numerical Results

In this section, we report numerical results on the BEP and the BEO for the DL of a MC–CDMA system that employs PE. Our results are also compared with those of other combining techniques. Both ideal and nonideal channel estimation are considered in FDBFCs. The FDBFC estimation errors are taken into account in terms of the normalized estimation error $\varepsilon \triangleq \sigma_{\rm E}^2/\sigma_{\rm H}^2$. The value of $\sigma_{\rm h}^2$ is considered to be equal to 1/2.[9] We set the number of subcarrier to $M = 1024$, and $L = 64$ is for the FDBFC.[10]

In Fig. 3, the BEP given by (29) is shown as a function of the PE parameter β for different values of the normalized estimation error ε when the system is fully loaded $(N_{\rm u} = M = 1024)$ and $\overline{\gamma} = 10$ dB. The impact of channel estimation errors on the optimum value of β that minimizes the BEP can be observed. In particular, we note that, as the estimation error increases, the optimum value of β shifts to the left (i.e., toward a less interference-limited situation). In fact, to be effective, ORC (i.e., $\beta = 1$) requires accurate CSI; when this condition is not guaranteed, the ORC does not perform close to the optimal solution. The analytical results are also compared to our Monte Carlo simulations in Fig. 3. It is evident that, although the BEP approximation becomes less accurate for $\beta < 0$ (due to the adoption of the LLN), a good agreement can be observed for the optimum values of β, confirming that the method adopted is valid for deriving the PE parameter $\beta^{(\rm opt)}$. In Fig. 4, the optimum value of β is plotted as a function of ξ (in decibels) as defined in (39), i.e., as a function of different combinations of $\overline{\gamma}$, $N_{\rm u}$, B, and L. It can be observed that, for high values of ξ, increasing the estimation errors shifts down the curves, thus requiring a reduction of β, which means that, in interference-limited situations (high ξ), as the estimation error increases, having $\beta \simeq 1$, i.e., using the ORC, is no longer optimal. In fact, the accuracy of CSI has a substantial impact on the ORC $(\beta = 1)$ rather than on the EGC $(\beta = 0)$ and MRC $(\beta = -1)$. Monte Carlo simulation results are also provided in Fig. 4, showing a good agreement with respect to the choice of the optimum β.

Fig. 5 shows the BEP as a function of the mean SNR $\overline{\gamma}$ for different levels of estimation errors and system loads $(N_{\rm u} = M$ and $N_{\rm u} = M/8)$. The results were plotted for the optimum value of β in conjunction with perfect CSI [i.e., for each SNR, the value of β is derived from (38)]. The analytical results evaluated from (29) are compared with our simulation results, again showing an agreement in the region of interest

[9]Thus, the mean channel gain is normalized to 1 for each subcarrier.

[10]This means that each group consists of $B = 16$ totally correlated subcarriers.

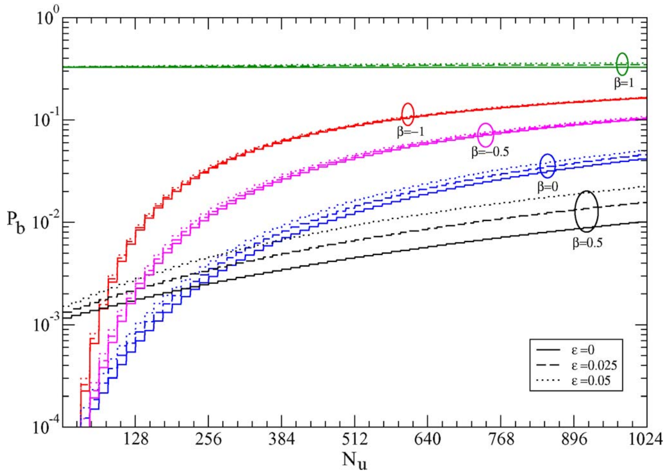

Fig. 6. BEP versus the number of active users, varying the normalized estimation errors $\varepsilon \triangleq \sigma_{\mathrm{E}}^2/\sigma_{\mathrm{H}}^2$, when $\overline{\gamma} = 10$ dB. Comparison among different combining techniques (i.e., different values of β).

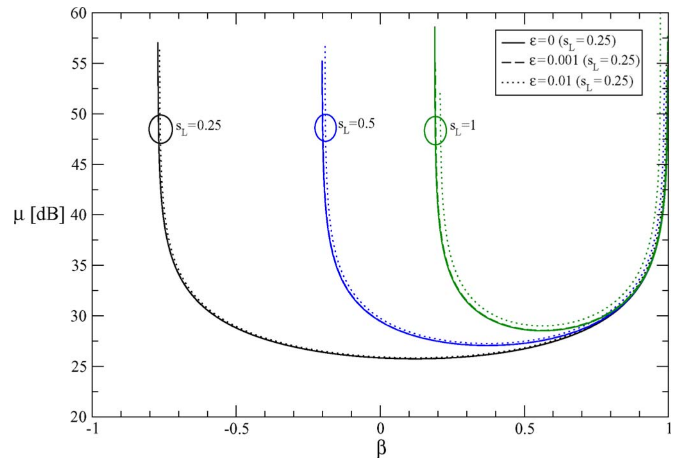

Fig. 8. Median SNR versus β, giving $P_{\mathrm{b}}^{\star} = 10^{-2}$ and $P_{\mathrm{o}}^{\star} = 10^{-2}$ for different estimation errors $\varepsilon \triangleq \sigma_{\mathrm{E}}^2/\sigma_{\mathrm{H}}^2$ and system loads.

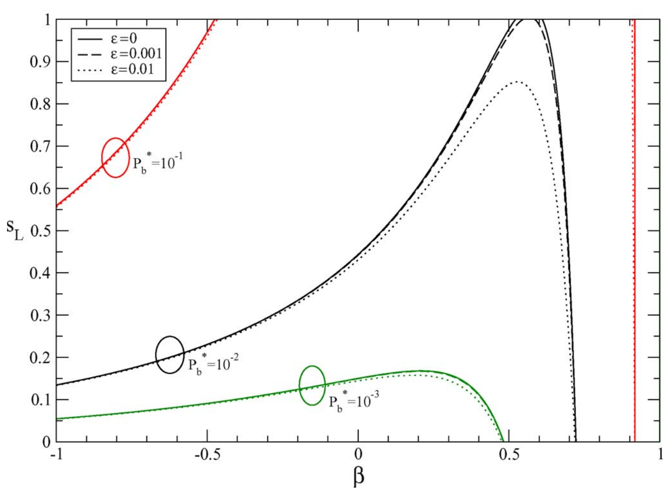

Fig. 7. System load versus β, giving the target BEP $P_{\mathrm{b}}^{\star} = 10^{-1}$ (red), $P_{\mathrm{b}}^{\star} = 10^{-2}$ (black), and $P_{\mathrm{b}}^{\star} = 10^{-3}$ (green) for different normalized estimation errors $\varepsilon \triangleq \sigma_{\mathrm{E}}^2/\sigma_{\mathrm{H}}^2$.

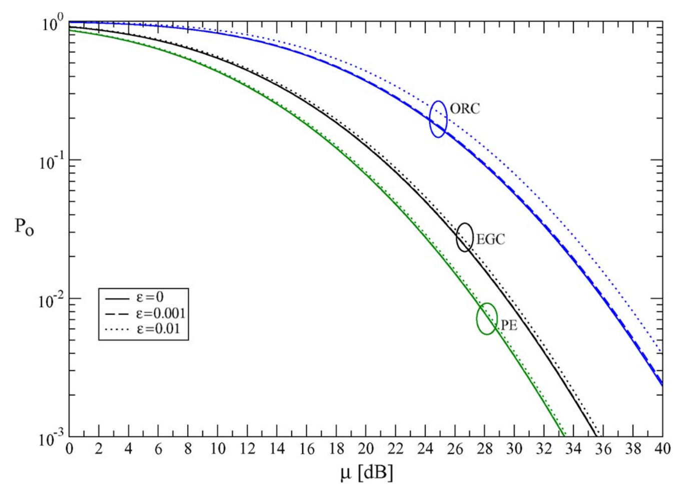

Fig. 9. BEO versus μ_{dB} for different estimation errors $\varepsilon \triangleq \sigma_{\mathrm{E}}^2/\sigma_{\mathrm{H}}^2$ and $P_{\mathrm{b}}^{\star} = 10^{-2}$. Comparison among different combining techniques.

for uncoded systems (i.e., $P_{\mathrm{b}} \in [10^{-2}, 10^{-1}]$).[11] However, we remark that the goal of this paper is not the exact derivation of an analytical formula for the BEP itself but, rather, the specific value of β for which the BEP is minimum. In Fig. 6, the BEP is shown as a function of the number of active users N_{u} for different values of β while varying the normalized channel-estimation error and considering $\overline{\gamma} = 10$ dB. Note that the choice of $\beta = 0.5$ results in a better performance for almost any system load, except for very low system loads, for which the optimum combiner is the EGC ($\beta = 0$).

In Fig. 7, the maximum achievable system load that results in a specific target BEP is plotted for different normalized estimation errors as a function of β according to (32). It can be observed that the closer β is to the optimum according to (38), the higher the attainable system load becomes. The presence of the estimation error decreases the maximum achievable system load and, as previously observed, slightly shifts the optimal value of β to the left toward -1 (MRC).

In Fig. 8, the median SNR μ_{dB}, maintaining the target BEO of $P_{\mathrm{o}} = 10^{-2}$,[12] is shown as a function of β for different system loads s_{L}. Note that the higher the system load, the narrower the range of β values that satisfy the target BEO. Finally, in Fig. 9, the BEO is presented as a function of μ_{dB} for $P_{\mathrm{b}}^{\star} = 10^{-2}$ and for a half-loaded system when the EGC, ORC, and PE using $\beta = 0.5$ are adopted.[13] Note that the PE-associated $\beta = 0.5$ outperforms both the ORC and EGC. Moreover, the BEO is less affected by the presence of estimation errors compared with the classic estimation techniques, confirming that a suitable choice of the PE parameter facilitates a performance improvement with respect to classical combining techniques while maintaining the same complexity.

[11] Note, in fact, that although the noise term N in (20) is a weighted sum of GRVs, the term I in (15) is constituted by a weighted sum of non-GRVs, thus the adoption of the CLT to assert their independence, because Gaussian and uncorrelated lead to an approximation.

[12] The target BEO is defined with respect to a target BEP equal to 10^{-2} according to (35).

[13] Note that $\beta = 0.5$ is close to the optimum value in terms of BEO when half-loaded systems are considered.

VI. Conclusion

In this paper, we have analyzed the DL performance of a MC–CDMA system that adopts PE at the receiver with non-ideal channel estimation conditions and correlated FDBFC. We derived the optimum value of the PE parameter that minimizes the BEP, showing a beneficial performance improvement over the traditional linear combining techniques, e.g., EGC, MRC, and ORC. We have demonstrated that the optimum value of the PE parameter does not significantly change in the presence of less accurate CSI, implying that a system designer may adopt the optimum value of the PE parameter determined for perfect CSI conditions, despite having channel-estimation errors. We also compared the analytical results with our simulation results to confirm the validity of the analytical framework.

Appendix A

In this Appendix, we evaluate the expression of the following values: 1) $\mathbb{E}\{\widehat{\alpha}_l^{-2\beta}\}$; 2) $\mathbb{E}\{\Theta_l(\beta)\}$; 3) $\mathbb{E}\{\Theta_l^2(\beta)\}$; and 4) $\zeta_\beta(\alpha) = \mathbb{E}\{\Theta_l^2(\beta)\} - (\mathbb{E}\{\Theta_l(\beta)\}^2$. Here, $\widehat{\alpha}_l$ are Rayleigh distributed with a pdf of $p_{\widehat{\alpha}_l}(x) = (x/\sigma_{\rm H}^2 + \sigma_{\rm E}^2)\exp[-(x^2/2(\sigma_{\rm H}^2+\sigma_{\rm E}^2))]$. It is known that [33]

$$\int_0^{+\infty} x^{a-1}\exp[-px^2]\,dx = \frac{1}{2}p^{-\frac{a}{2}}\Gamma\left(\frac{a}{2}\right), \qquad a>0 \tag{42}$$

where $\Gamma(z)$ represents the Euler gamma function [33]. Hence, we have

$$\begin{aligned}\mathbb{E}\left\{\widehat{\alpha}_l^{-2\beta}\right\} &= \int_0^{+\infty} x^{-2\beta}\frac{x}{\sigma_{\rm H}^2+\sigma_{\rm E}^2}e^{-\frac{x^2}{2\left(\sigma_{\rm H}^2+\sigma_{\rm E}^2\right)}}\,dx\\ &= \left(2\sigma_{\rm H}^2\right)^{-\beta}\left(1+\frac{\sigma_{\rm E}^2}{\sigma_{\rm H}^2}\right)^{-\beta}\Gamma(1-\beta).\end{aligned} \tag{43}$$

Based on (14) and by neglecting the index l (because we are studying i.i.d. RVs), we arrive at

$$\begin{aligned}\Theta_l(\beta) = &\left[(X+X_{\rm E})^2+(Y+Y_{\rm E})^2\right]^{-(\beta+1)/2}\\ &\times\left[X(X+X_{\rm E})+Y(Y+Y_{\rm E})\right].\end{aligned}$$

We define the auxiliary variables $\widehat{X}=X+X_{\rm E}$, $\widehat{X}\sim\mathcal{N}(0,\sigma_{\rm H}^2+\sigma_{\rm E}^2)$, and $\widehat{Y}\sim\mathcal{N}(0,\sigma_{\rm H}^2+\sigma_{\rm E}^2)$. By exploiting the independence and zero mean of X, Y, $X_{\rm E}$, and $Y_{\rm E}$, we can write $\mathbb{E}\{\widehat{X}\widehat{Y}\}=0$, $\mathbb{E}\{\widehat{X}X_{\rm E}\}=\sigma_{\rm E}^2$, and $\mathbb{E}\{\widehat{Y}Y_{\rm E}\}=\sigma_{\rm E}^2$. Hence

$$\mathbb{E}\left\{\Theta_l(\beta)\right\} = L_1 - L2 \tag{44}$$

where

$$\begin{aligned}L_1 &= \mathbb{E}\left\{\left[\widehat{X}^2+\widehat{Y}^2\right]^{\frac{1-\beta}{2}}\right\}\\ L_2 &= \mathbb{E}\left\{\left[\widehat{X}^2+\widehat{Y}^2\right]^{-\frac{1+\beta}{2}}\left(\widehat{X}X_{\rm E}+\widehat{Y}Y_{\rm E}\right)\right\}.\end{aligned}$$

Because $\widehat{X}$ and $\widehat{Y}$ are uncorrelated and, thus, independent GRVs, by defining $r\triangleq\sqrt{\widehat{X}^2+\widehat{Y}^2}$ and $r_{\rm E}\triangleq\sqrt{X_{\rm E}^2+Y_{\rm E}^2}$, they are Rayleigh distributed, and $\phi\triangleq\angle\widehat{X}+j\widehat{Y}$, as well as $\phi_{\rm E}\triangleq\angle X_{\rm E}+jY_{\rm E}$, are uniformly distributed in $[0,2\pi[$. Thus, L_1 becomes

$$\begin{aligned}L_1 = \mathbb{E}\{r^{1-\beta}\} &= \int_0^{+\infty} r^{1-\beta}\frac{r}{\sigma_{\rm H}^2+\sigma_{\rm E}^2}e^{-\frac{r^2}{2\left(\sigma_{\rm H}^2+\sigma_{\rm E}^2\right)}}\,dr\\ &= \left(2\sigma_{\rm H}^2\right)^{\frac{1-\beta}{2}}\left(1+\frac{\sigma_{\rm E}^2}{\sigma_{\rm H}^2}\right)^{\frac{1-\beta}{2}}\Gamma\left(\frac{3-\beta}{2}\right)\end{aligned} \tag{45}$$

and for the joint pdf of GRVs in polar coordinates (i.e., for $\widehat{X}=r\cos\phi$, $\widehat{Y}=r\sin\phi$, $X_{\rm E}=r_{\rm E}\cos\phi_{\rm E}$, $Y_{\rm E}=r_{\rm E}\sin\phi_{\rm E}$), L_2 becomes

$$\begin{aligned}L_2 = &\int_0^\infty\int_0^\infty\int_0^{2\pi}\int_0^{2\pi} r^{-(1+\beta)}rr_{\rm E}\cos(\phi-\phi_{\rm E})\\ &\times\frac{\exp\left\{-\left[\frac{r^2}{2\sigma_{\rm H}^2}+\left(\frac{1}{2\sigma_{\rm H}^2}+\frac{1}{2\sigma_{\rm E}^2}\right)r_{\rm E}^2\right]\right\}}{4\pi^2\sigma_{\rm E}^2\left(\sigma_{\rm H}^2-\sigma_{\rm E}^2\right)}\\ &\times\exp\left[\frac{rr_{\rm E}\cos(\phi-\phi_{\rm E})}{\sigma_{\rm H}^2}\right]|rr_{\rm E}|\,d\phi\,d\phi_{\rm E}\,dr\,dr_{\rm E}.\end{aligned} \tag{46}$$

By exploiting the properties of periodic functions, it can be shown that

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(\phi-\phi_{\rm E})\exp\left[\frac{rr_{\rm E}}{\sigma_{\rm H}^2}\cos(\phi-\phi_{\rm E})\right]d\phi = I_1\left(\frac{rr_{\rm E}}{\sigma_{\rm H}^2}\right)$$

where $I_1(z)$ is the modified Bessel function of the first order.[14] Consider that, for $a,b>0$, $\int_0^{+\infty}t^2I_1(bt)\exp[-at^2]dt = (b/4a^2)e^{b^2/4a}$ [33], L_2 results in

$$\begin{aligned}L_2 = &\int_0^\infty\frac{r^{1-\beta}}{\sigma_{\rm E}^2\left(\sigma_{\rm H}^2-\sigma_{\rm E}^2\right)}\exp\left[-\frac{r^2}{2\sigma_{\rm H}^2}\right]\\ &\times\frac{r\,\exp\left[\frac{r^2}{4\sigma_{\rm H}^4\left(\frac{1}{2\sigma_{\rm H}^2}+\frac{1}{2\sigma_{\rm E}^2}\right)}\right]}{4\sigma_{\rm H}^2\left(\frac{1}{2\sigma_{\rm H}^2}+\frac{1}{2\sigma_{\rm E}^2}\right)^2}\,dr\\ =&\frac{\frac{\sigma_{\rm E}^2}{\sigma_{\rm H}^2}}{\left(1-\frac{\sigma_{\rm E}^4}{\sigma_{\rm H}^4}\right)}\left(2\sigma_{\rm H}^2\right)^{\frac{1-\beta}{2}}\left(1+\frac{\sigma_{\rm E}^2}{\sigma_{\rm H}^2}\right)^{\frac{1-\beta}{2}}\Gamma\left(\frac{3-\beta}{2}\right).\end{aligned} \tag{47}$$

Consequently, (44) becomes

$$\mathbb{E}\left\{\Theta_l(\beta)\right\} = \left(2\sigma_{\rm H}^2+2\sigma_{\rm E}^2\right)^{\frac{1-\beta}{2}}\left[1-\frac{\frac{\sigma_{\rm E}^2}{\sigma_{\rm H}^2}}{\left(1-\frac{\sigma_{\rm E}^4}{\sigma_{\rm H}^4}\right)}\right]\Gamma\left(\frac{3-\beta}{2}\right). \tag{48}$$

[14]The modified Bessel function of the nth order is defined as $I_n(z) = (1/\pi)\int_0^\pi\cos n\theta e^{z\cos\theta}d\theta$.

Following the same methodology, we derive

$$\begin{aligned}\mathbb{E}\left\{\Theta_l^2(\beta)\right\} &= L_3 - 2L_4 + L_5\\ L_3 &= \mathbb{E}\left\{(\widehat{X}^2+\widehat{Y}^2)^{1-\beta}\right\}\\ L_4 &= \mathbb{E}\left\{(\widehat{X}^2+\widehat{Y}^2)^{-\beta}(\widehat{X}X_{\mathrm{E}}+\widehat{Y}Y_{\mathrm{E}})\right\}\\ L_5 &= \mathbb{E}\left\{(\widehat{X}^2+\widehat{Y}^2)^{-(\beta+1)}(\widehat{X}X_{\mathrm{E}}+\widehat{Y}Y_{\mathrm{E}})^2\right\} \quad (49)\end{aligned}$$

where

$$\begin{aligned}L_3 &= \int_0^{+\infty} r^{2-2\beta}\frac{r}{(\sigma_{\mathrm{H}}^2+\sigma_{\mathrm{E}}^2)}\exp\left[-\frac{r^2}{2(\sigma_{\mathrm{H}}^2+\sigma_{\mathrm{E}}^2)}\right]dr\\ &= \left(2\sigma_{\mathrm{H}}^2\right)^{1-\beta}\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)^{1-\beta}\Gamma(2-\beta) \quad (50)\\ L_4 &= (2\sigma_{\mathrm{H}})^{1-\beta}\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)^{1-\beta}\frac{\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}}{\left(1-\frac{\sigma_{\mathrm{E}}^4}{\sigma_{\mathrm{H}}^4}\right)}\Gamma(2-\beta) \quad (51)\\ L_5 &= \int_0^{\infty}\int_0^{\infty} r^{1-2\beta}r_{\mathrm{E}}^3\frac{\exp\left\{-\left[\frac{r^2}{2\sigma_H^2}+\left(\frac{1}{2\sigma_{\mathrm{H}}^2}+\frac{1}{2\sigma_{\mathrm{E}}^2}\right)r_E^2\right]\right\}}{\sigma_{\mathrm{E}}^2(\sigma_{\mathrm{H}}^2-\sigma_{\mathrm{E}}^2)}\\ &\quad\times\left(\frac{R_1(r,r_{\mathrm{E}})+R_2(r,r_{\mathrm{E}})}{2}\right)dr\,dr_{\mathrm{E}}. \quad (52)\end{aligned}$$

Here

$$\begin{aligned}R_1(r,r_{\mathrm{E}}) &\triangleq \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\exp\left[\frac{rr_{\mathrm{E}}}{\sigma_H^2}\cos(\phi-\phi_{\mathrm{E}})\right]d\phi\,d\phi_{\mathrm{E}}\\ &= I_0\left(\frac{rr_{\mathrm{E}}}{\sigma_{\mathrm{H}}^2}\right) \quad (53)\\ R_2(r,r_{\mathrm{E}}) &\triangleq \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\cos\left[2(\phi-\phi_{\mathrm{E}})\right]\\ &\quad\times\exp\left[\frac{rr_{\mathrm{E}}}{\sigma_H^2}\cos(\phi-\phi_{\mathrm{E}})\right]d\phi\,d\phi_{\mathrm{E}}\\ &= I_2\left(\frac{rr_{\mathrm{E}}}{\sigma_{\mathrm{H}}^2}\right) \quad (54)\end{aligned}$$

where $I_0(z)$ and $I_2(z)$ are the modified Bessel functions of orders 0 and 2, respectively. By substituting (53) and (55) in (52) and considering that $\int_0^{+\infty}(1/2)[I_0(bt)+I_2(bt)]t^3\exp[-at^2]=((2a+b^2)/8a^3)\exp[(b^2/4a)]$, for a, $b>0$ [33], L_5 results in

$$\begin{aligned}L_5 &= \int_0^{\infty} r^{1-2\beta}\frac{\exp\left[-\frac{r^2}{2\sigma_{\mathrm{H}}^2}\right]}{\sigma_{\mathrm{E}}^2(\sigma_{\mathrm{H}}^2-\sigma_{\mathrm{E}}^2)}\\ &\quad\times\left\{\frac{2\left(\frac{1}{2\sigma_{\mathrm{H}}^2}+\frac{1}{2\sigma_{\mathrm{E}}^2}\right)+\left(\frac{r}{\sigma_{\mathrm{H}}^2}\right)^2}{8\left(\frac{1}{2\sigma_{\mathrm{H}}^2}+\frac{1}{2\sigma_{\mathrm{E}}^2}\right)^3}\exp\left[\frac{\left(\frac{r}{\sigma_{\mathrm{H}}^2}\right)^2}{4\left(\frac{1}{2\sigma_{\mathrm{H}}^2}+\frac{1}{2\sigma_{\mathrm{E}}^2}\right)}\right]\right\}dr\\ &= \frac{\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\left(Q_1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}Q_2\right)}{\left(1-\frac{\sigma_{\mathrm{E}}^4}{\sigma_{\mathrm{H}}^4}\right)\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)} \quad (55)\end{aligned}$$

with

$$\begin{aligned}Q_1 &\triangleq \int_0^{+\infty} r^{1-2\beta}\exp\left[-\frac{1}{2\sigma_{\mathrm{H}}^2}\left(\frac{1}{1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}}\right)r^2\right]dr\\ &= \frac{1}{2}\left(2\sigma_{\mathrm{H}}^2\right)^{1-\beta}\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)^{1-\beta}\Gamma(1-\beta) \quad (56)\\ Q_2 &\triangleq \int_0^{+\infty}\frac{r^{3-2\beta}}{\sigma_{\mathrm{H}}^2\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)}\exp\left[-\frac{1}{2\sigma_{\mathrm{H}}^2}\left(\frac{1}{1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}}\right)r^2\right]dr\\ &= \left(2\sigma_{\mathrm{H}}^2\right)^{1-\beta}\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)^{1-\beta}\Gamma(2-\beta). \quad (57)\end{aligned}$$

By substituting (56) and (57) in (55), we obtain

$$\begin{aligned}L_5 &= \left(2\sigma_{\mathrm{H}}^2\right)^{1-\beta}\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)^{1-\beta}\frac{\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}}{\left(1-\frac{\sigma_{\mathrm{E}}^4}{\sigma_{\mathrm{H}}^4}\right)}\\ &\quad\times\left[1-\frac{\left(\frac{\frac{1}{2}-\beta}{1-\beta}\right)}{\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)}\right]\Gamma(2-\beta). \quad (58)\end{aligned}$$

Now, by substituting (50), (51), and (58) into (49), we find that

$$\mathbb{E}\left\{\Theta_l^2(\beta)\right\} = \left(2\sigma_{\mathrm{H}}^2\right)^{1-\beta}\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)^{1-\beta}\Sigma\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2},\beta\right)\Gamma(2-\beta) \quad (59)$$

where

$$\Sigma\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2},\beta\right) \triangleq 1-\frac{\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}}{\left(1-\frac{\sigma_{\mathrm{E}}^4}{\sigma_{\mathrm{H}}^4}\right)}\left[1+\frac{\left(\frac{1}{2}-\beta1-\beta\right)}{\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)}\right]. \quad (60)$$

By exploiting (48) and (59), we finally arrive at

$$\begin{aligned}\zeta_\beta(\alpha) &= \left(2\sigma_{\mathrm{H}}^2\right)^{1-\beta}\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)^{1-\beta}\\ &\quad\times\left[\Sigma\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2},\beta\right)\Gamma(2–\beta)–\Pi^2\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)\Gamma^2\left(\frac{3–\beta}{2}\right)\right] \quad (61)\end{aligned}$$

where

$$\Pi\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right) \triangleq 1-\frac{\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}}{\left(1-\frac{\sigma_{\mathrm{E}}^4}{\sigma_{\mathrm{H}}^4}\right)}. \quad (62)$$

Appendix B

We derive the optimum value of the PE parameter $\beta^{(\mathrm{opt})}$, defined as that particular value of β within the range [−1, 1], which minimizes the BEP. Because the BEP is monotonically decreasing as a function of β, we have (37).

Setting to zero the derivative of the argument in (37) with respect to β and defining $\Gamma'(x) \triangleq d\Gamma(x)/dx$, as well as remembering that $\Gamma((3-\beta)/2) \neq 0$ for $-1 \leq \beta \leq 1$, we obtain

$$\begin{aligned} &-\Gamma'\left(\frac{3-\beta}{2}\right)\Gamma\left(\frac{3-\beta}{2}\right) \\ &\times\left\{\frac{1}{L}\left\lfloor\frac{N_{\mathrm{u}}-1}{B}\right\rfloor\overline{\gamma}\right. \\ &\times\left[\Sigma\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2},\beta\right)\Gamma(2-\beta)-\Pi^2\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)\Gamma^2\left(\frac{3-\beta}{2}\right)\right] \\ &\left.+\frac{1}{2}\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)^{-1}\Gamma(1-\beta)\right\} \\ &=\Gamma^2\left(\frac{3-\beta}{2}\right) \\ &\times\left\{\frac{1}{L}\left\lfloor\frac{N_{\mathrm{U}}-1}{B}\right\rfloor\overline{\gamma}\right. \\ &\times\left[\Sigma'\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2},\beta\right)\Gamma(2-\beta)-\Sigma\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2},\beta\right)\Gamma'(2-\beta)\right. \\ &\left.+\Pi^2\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)\Gamma\left(\frac{3-\beta}{2}\right)\Gamma'\left(\frac{3-\beta}{2}\right)\right] \\ &\left.-\frac{1}{2}\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)^{-1}\Gamma'(1-\beta)\right\} \end{aligned} \tag{63}$$

where

$$\Sigma'\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2},\beta\right) \triangleq \frac{\partial}{\partial\beta}\Sigma\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2},\beta\right) = \frac{\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}}{\left(1-\frac{\sigma_{\mathrm{E}}^4}{\sigma_{\mathrm{H}}^4}\right)\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)}\frac{1}{2(1-\beta)^2}. \tag{64}$$

Because $\Gamma'(x) = \Psi(x)\Gamma(x)$, where $\Psi(x)$ is the logarithmic derivative of the Gamma function (the so-called Digamma function) defined as $\Psi(x) \triangleq d\ln(\Gamma(x))/dx$ [33] and after some further mathematical manipulations, we obtain

$$\begin{aligned} &\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)^{-1}\Gamma(1-\beta)\left[-\Psi\left(\frac{3-\beta}{2}\right)+\Psi(1-\beta)\right] \\ &=\overline{\gamma}\frac{2}{L}\left\lfloor\frac{N_{\mathrm{U}}-1}{B}\right\rfloor\left\{\Sigma'\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2},\beta\right)\Gamma(2-\beta)\right. \\ &\left.+\Sigma\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2},\beta\right)\Gamma(2-\beta)\left[\Psi\left(\frac{3-\beta}{2}\right)-\Psi(2-\beta)\right]\right\}. \end{aligned} \tag{65}$$

By exploiting that $\Gamma(x+1) = x\Gamma(x)$ and $\Psi(x+1) = \Psi(x) + 1/x$ [33], considering that $\Gamma(1-\beta) \neq 0$ for $-1 \leq \beta \leq 1$ and $1-\beta \neq 0$, $\beta < 1$, and through (40), we obtain

$$\begin{aligned} &\left[\Psi\left(\frac{3-\beta}{2}\right)-\Psi(1-\beta)\right] = \overline{\gamma}\frac{2}{L}\left\lfloor\frac{N_{\mathrm{U}}-1}{B}\right\rfloor\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right) \\ &\left\{\frac{\chi\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)}{\beta-1}+\left[\Pi\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)(\beta-1)-\frac{\chi\left(\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)}{\left(1+\frac{\sigma_{\mathrm{E}}^2}{\sigma_{\mathrm{H}}^2}\right)}(2\beta-1)\right]\right. \\ &\left.\times\left[\Psi\left(\frac{3-\beta}{2}\right)-\Psi(1-\beta)+\frac{1}{\beta-1}\right]\right\}. \end{aligned} \tag{66}$$

Based on the definition of the parameter ξ in (39), we can derive the optimum value of β as the implicit solution of (38).

Appendix C

Here, we aim to demonstrate that the framework for obtaining the PE parameter β that minimizes the uncoded BEP is also valid when hard-decision binary Bose–Chaudhuri–Hocquenghem (BCH) codes are employed. To this purpose, we show that the codeword error probability P_{e} is a monotonic increasing function of the uncoded BEP P_{b}. The averaged codeword error probability is related to the averaged uncoded BEP as follows:

$$P_{\mathrm{e}}(n,k,t,\overline{\gamma},\beta) = F\left[n,t,P_{\mathrm{b}}\left(\frac{k}{n}\overline{\gamma},\beta\right)\right] \tag{67}$$

where n is the codeword length, k is the number of information bits, t is the number of correctable errors, with the errors assumed independent of each other, and

$$F(n,t,x) \triangleq 1-\sum_{i=0}^{t}\binom{n}{i}x^i(1-x)^{n-i}. \tag{68}$$

By deriving (67) with respect to β, we obtain

$$\frac{\partial P_{\mathrm{e}}(n,k,t,\overline{\gamma},\beta)}{\partial\beta} = \frac{\partial F(n,t,x)}{\partial x}\frac{\partial P_{\mathrm{b}}\left(\frac{k}{n}\overline{\gamma},\beta\right)}{\partial\beta}. \tag{69}$$

Hence, if $F(n,t,x)$ is a monotonic function of x, then the sign of the derivative in (69) only depends on the sign of $\partial P_{\mathrm{b}}((k/n)\overline{\gamma},\beta)/\partial\beta$ (i.e., the value of β that minimizes the uncoded P_{b} for an equivalent average SNR fixed to $\overline{\gamma}' \triangleq (k/n)\overline{\gamma}$ also minimizes the coded error probability). We aim to prove that

$$f(x) \triangleq \frac{\partial F(n,t,x)}{\partial x} \geq 0 \qquad \forall n,k \in \mathcal{N}. \tag{70}$$

By substituting (68) in (70), we have

$$f(x) = n(1-x)^{n-1} - \sum_{i=1}^{t}\binom{n}{i}(i-nx)x^{i-1}(1-x)^{n-i-1} \tag{71}$$

and $f(x) \geq 0$ if and only if

$$\sum_{i=1}^{t}\binom{n}{i}\left(\binom{i}{nx}-1\right)\left(\binom{x}{1-x}\right)^i \leq 1. \tag{72}$$

Assuming $x \leq 1/n$, with $i \geq 1$, we have $i/(nx) \geq 1/(nx) \geq 1$; thus, $((i/nx) - 1) \geq 0$, and $(x/(1-x)) \geq 0$. Therefore, (72) is the sum of positive terms, and because $t \leq n$, we have

$$\sum_{i=1}^{t} \binom{n}{i} \binom{i}{nx} - 1 \left(\frac{x}{1-x}\right)^i \leq \sum_{i=1}^{n} \binom{n}{i} \binom{i}{nx} - 1 \left(\frac{x}{1-x}\right)^i. \tag{73}$$

Hence, the proof is concluded if

$$\sum_{i=1}^{n} \binom{n}{i} \left(\frac{i}{nx} - 1\right) \left(\frac{x}{1-x}\right)^i = 1. \tag{74}$$

We can write

$$\sum_{i=1}^{n} \binom{n}{i} \left(\frac{i}{nx} - 1\right) \left(\frac{x}{1-x}\right)^i = \frac{1}{nx} \sum_{i=1}^{n} \binom{n}{i} i \left(\frac{x}{1-x}\right)^i - \sum_{i=1}^{n} \binom{n}{i} \left(\frac{x}{1-x}\right)^i. \tag{75}$$

Based on the binomial formula

$$\sum_{i=1}^{n} \binom{n}{i} y^i = (1+y)^n - 1 \tag{76}$$

and by deriving both members of (76) in y, we obtain

$$\sum_{i=1}^{n} \binom{n}{i} i y^i = ny(1+y)^{n-1}. \tag{77}$$

By considering $y = (x/1-x)$, (76) and (77) lead to

$$\sum_{i=1}^{n} \binom{n}{i} \left(\frac{x}{1-x}\right)^i = \left[1 + \left(\frac{x}{1-x}\right)\right]^n - 1 = \left(\frac{1}{1-x}\right)^n - 1 \tag{78}$$

and

$$\sum_{i=1}^{n} \binom{n}{i} i \left(\frac{x}{1-x}\right)^i = n \left(\frac{x}{1-x}\right) \left[1 + \left(\frac{x}{1-x}\right)\right]^{n-1} = nx \left(\frac{1}{1-x}\right)^n \tag{79}$$

respectively. By substituting (78) and (79) in (75), we obtain

$$\sum_{i=1}^{n} \binom{n}{i} \left(\frac{i}{nx} - 1\right) \left(\frac{x}{1-x}\right)^i = 1 \tag{80}$$

which ends the proof. ■

REFERENCES

[1] S. Hara and R. Prasad, "Overview of multicarrier CDMA," *IEEE Commun. Mag.*, vol. 35, no. 12, pp. 126–133, Dec. 1997.

[2] N. Yee, J.-P. Linnartz, and G. Fettweis, "BER of multicarrier CDMA in an indoor Rician fading channel," in *Conf. Rec. 27th Asilomar Conf. Signals, Syst. Comput.*, Pacific Grove, CA, Nov. 1993, vol. 1, pp. 426–430.

[3] L. Hanzo, M. Müster, B. Choi, and T. Keller, *OFDM and MC–CDMA*. Hoboken, NJ: Wiley, 2003.

[4] I. Cosovic and S. Kaiser, "A unified analysis of diversity exploitation in multicarrier CDMA," *IEEE Trans. Veh. Technol.*, vol. 56, no. 4, pp. 2051–2062, Jul. 2007.

[5] S. Kaiser, "OFDM code-division multiplexing in fading channels," *IEEE Trans. Commun.*, vol. 50, no. 8, pp. 1266–1273, Aug. 2002.

[6] L. Hanzo and B. J. Choi, "Near-instantaneously adaptive HSDPA-style OFDM versus MC–CDMA transceivers for WiFi, WiMAX, and next-generation cellular systems," *Proc. IEEE*, vol. 95, no. 12, pp. 2368–2392, Dec. 2007.

[7] H. Wei and L. Hanzo, "Semiblind and group-blind multiuser detection for the MC–CDMA uplink," in *Proc. IEEE Semiannual Veh. Technol. Conf.*, Milan, Italy, May 2004, vol. 3, pp. 1727–1731.

[8] L.-L. Yang and L. Hanzo, "Performance of fractionally spread multicarrier CDMA in AWGN as well as slow and fast Nakagami-m fading channels," *IEEE Trans. Veh. Technol.*, vol. 54, no. 5, pp. 1817–1827, Sep. 2005.

[9] K. Fazel and S. Kaiser, *Multicarrier and Spread Spectrum Systems*. Hoboken, NJ: Wiley, 2003.

[10] J.-P. Linnartz, "Performance analysis of synchronous MC–CDMA in mobile Rayleigh channel with both delay and Doppler spreads," *IEEE Trans. Veh. Technol.*, vol. 50, no. 6, pp. 1375–1387, Nov. 2001.

[11] M. Simon and M. Alouini, *Digital Communication Over Fading Channels: A Unified Approach to Performance Analysis*, 1st ed. New York: Wiley, 2000.

[12] M. Z. Win and J. H. Winters, "Virtual branch analysis of symbol error probability for hybrid selection/maximal-ratio combining in Rayleigh fading," *IEEE Trans. Commun.*, vol. 49, no. 11, pp. 1926–1934, Nov. 2001.

[13] M. Guenach and H. Steendam, "Performance evaluation and parameter optimization of MC–CDMA," *IEEE Trans. Veh. Technol.*, vol. 56, no. 3, pp. 1165–1175, May 2007.

[14] K. Zhang, Y. L. Guan, and Q. Shi, "Complexity reduction for MC–CDMA with MMSEC," *IEEE Trans. Veh. Technol.*, vol. 57, no. 3, pp. 1989–1993, May 2008.

[15] S. Chen, A. Livingstone, and L. Hanzo, "Minimum bit-error rate design for space–time equalization-based multiuser detection," *IEEE Trans. Commun.*, vol. 54, no. 5, pp. 824–832, May 2006.

[16] L. Hanzo, L.-L. Yang, E.-L. Kuan, and K. Yen, *Single and Multicarrier DS–CDMA: Multiuser Detection, Space–Time Spreading, Synchronization and Standards*. Hoboken, NJ: Wiley, 2003.

[17] S. Slimane, "Partial equalization of multicarrier CDMA in frequency-selective fading channels," in *Proc. IEEE Int. Conf. Commun.*, New Orleans, LA, Jun. 2000, pp. 26–30.

[18] A. Conti, "MC–CDMA bit error probability and outage minimization through partial combining," *IEEE Commun. Lett.*, vol. 9, no. 12, pp. 1055–1057, Dec. 2005.

[19] A. Conti, B. Masini, F. Zabini, and O. Andrisano, "On the downlink performance of multicarrier CDMA systems with partial equalization," *IEEE Trans. Wireless Commun.*, vol. 6, no. 1, pp. 230–239, Jan. 2007.

[20] R. J. McEliece and W. E. Stark, "Channels with block interference," *IEEE Trans. Inf. Theory*, vol. IT-30, no. 1, pp. 44–53, Jan. 1984.

[21] M. Chiani, A. Conti, and O. Andrisano, "Outage evaluation for slow frequency-hopping mobile radio systems," *IEEE Trans. Commun.*, vol. 47, no. 12, pp. 1865–1874, Dec. 1999.

[22] A. Conti, M. Z. Win, M. Chiani, and J. H. Winters, "Bit error outage for diversity reception in shadowing environment," *IEEE Commun. Lett.*, vol. 7, no. 1, pp. 15–17, Jan. 2003.

[23] P. Mary, M. Dohler, J. M. Gorce, G. Villemaud, and M. Arndt, "BPSK bit error outage over Nakagami-m fading channels in lognormal shadowing environments," *IEEE Commun. Lett.*, vol. 11, no. 7, pp. 565–567, Jul. 2007.

[24] M. Chiani, "Error probability for block codes over channels with block interference," *IEEE Trans. Inf. Theory*, vol. 44, no. 7, pp. 2998–3008, Nov. 1998.

[25] A. Lodhi, F. Said, M. Dohler, and H. Aghvami, "Closed-form symbol error probabilities of STBC and CDD MC–CDMA with frequency-correlated subcarriers over Nakagami-m fading channels," *IEEE Trans. Veh. Technol.*, vol. 57, no. 2, pp. 962–973, Mar. 2008.

[26] W. Gifford, M. Win, and M. Chiani, "Antenna subset diversity with non-ideal channel estimation," *IEEE Trans. Wireless Commun.*, vol. 7, no. 5, pp. 1527–1539, May 2008.

[27] P. Tan and N. Beaulieu, "Effect of channel estimation error on bit error probability in OFDM systems over Rayleigh and Ricean fading channels," *IEEE Trans. Commun.*, vol. 56, no. 4, pp. 1527–1539, Apr. 2008.

[28] Y. Chen and N. Beaulieu, "Optimum pilot-symbol-assisted modulation," *IEEE Trans. Commun.*, vol. 55, no. 8, pp. 1536–1546, Aug. 2007.

[29] M. Chiani, A. Conti, and C. Fontana, "Improved performance in TD–CDMA mobile radio system by optimizing energy partition in channel estimation," *IEEE Trans. Commun.*, vol. 51, no. 3, pp. 352–355, Mar. 2003.

[30] A. Conti, W. M. Gifford, M. Z. Win, and M. Chiani, "Optimized simple bounds for diversity systems," *IEEE Trans. Commun.*, vol. 57, no. 9, pp. 2674–2685, Sep. 2009.

[31] *Standard for Local and Metropolitan Area Networks Part 16: Air Interface for Fixed and Mobile Broadband Wireless Access Systems Amendment 2*, IEEE Std. 802.16e, Feb. 2006.

[32] *Digital Video Broadcasting (DVB): Framing Structure, Channel Coding and Modulation for Digital Terrestrial Television*, ETSI Std. DVB-T v.1.4.1, Jan. 2001.

[33] I. S. Gradshteyn and I. Ryzhik, *Table of Integrals, Series and Products*, 6th ed. San Diego, CA: Academic, 2000.

[34] V. Erceg, L. J. Greenstein, S. Y. Tjandra, S. R. Parkoff, A. Gupta, B. Kulic, A. A. Julius, and R. Bianchi, "An empirically based path loss model for wireless channels in suburban environments," *IEEE J. Sel. Areas Commun.*, vol. 17, no. 7, pp. 1205–1211, Jul. 1999.

[35] M. Chiani, A. Conti, and R. Verdone, "Partial compensation signal-level-based uplink power control to extend terminal battery duration," *IEEE Trans. Veh. Technol.*, vol. 50, no. 4, pp. 1125–1131, Jul. 2001.

[36] C.-K. Jao, Y.-C. Cheng, and S.-W. Wei, "Code selection method for downlink MC–CDMA over frequency-selective fading channel," *IEEE Trans. Veh. Technol.*, vol. 57, no. 6, pp. 3875–3879, Nov. 2008.

[37] M. El-Hajjar, O. Alamri, R. Maunder, and L. Hanzo, "Layered steered space–time spreading-aided generalized MC–DS–CDMA," *IEEE Trans. Veh. Technol.*, vol. 59, no. 2, pp. 999–1005, Feb. 2010.

[38] B. Hu, L.-L. Yang, and L. Hanzo, "Time- and frequency-domain-spread generalized multicarrier DS–CDMA using subspace-based blind and group-blind space–time multiuser detection," *IEEE Trans. Veh. Technol.*, vol. 57, no. 5, pp. 3235–3241, Sep. 2008.

Flavio Zabini was born in San Giovanni, Persiceto, Italy, on June 30, 1979. He received the Dr.Ing. degree (with honors) and the Ph.D. degree in telecommunications engineering from the University of Bologna, Bologna, Italy, in 2004 and 2010, respectively. In 2004, he did his Master's thesis with the Department of Electrical and Computer Engineering (ECE), University of California, San Diego, working on beam-forming techniques in multiple-antenna ultrawideband systems in the presence of interference.

Since 2005, he has been with the Wireless Communications Laboratory, Institute for Electronics, Information, and Telecommunications Engineering, Research Unit of Bologna, National Research Council, University of Bologna, working on wireless transmission techniques. From 2006 to 2009, he worked for the project DVB 2006, with ALMAMATER s.rl.l., for the realization of an on-channel repeater with echo canceller for digital video broadcasting. In 2008, he was a Visiting Student with the DOCOMO Communications Laboratories Europe (DoCoMo EuroLabs), Munich, Germany. His research interests include novel combining techniques in multicarrier code-division multiple access systems, multiple antennas, orthogonal frequency-division multiplexing with nonlinearities, echo cancellation for on-channel repeaters, and channel estimation.

Barbara M. Masini (S'02–M'05) received the Laurea degree (with honors) and the Ph.D. degree in telecommunications engineering from the University of Bologna, Bologna, Italy, in 2001 and 2005, respectively.

In 2002, she joined the Department of Electronics, Informatics, and Systems, University of Bologna, to develop her research activity in wireless communications. Since 2005, she has been with the Wireless Communications Laboratory, Institute for Electronics, Information, and Telecommunications Engineering, Research Unit of Bologna, National Research Council, University of Bologna, working on wireless transmission techniques. Since 2007, she has been an Assistant Professor with the University of Bologna, where she teaches courses with the Telecommunication Laboratory, including experimental implementation of telecommunication systems on digital signal processing platforms. Her research interests are mainly focused on orthogonal frequency-division multiplexing and multicarrier–code-division multiple access systems, short-range wireless communications, and coexistence among different wireless systems. Her current research interests include intelligent transportation systems. She is working on research projects sponsored by industry and government ministries in vehicular networks.

Andrea Conti (S'99–M'01) received the Dr.Ing. degree in telecommunications engineering and the Ph.D. degree in electronics engineering and computer science from the University of Bologna, Bologna, Italy, in 1997 and 2001, respectively.

From 1999 to 2002, he was with the Consorzio Nazionale Interuniversitario per le Telecomunicazioni, Pisa, Italy. From 2002 to 2005, he was with the Institute for Electronics, Information and Telecommunications Engineering, Research Unit of Bologna, National Research Council, University of Bologna. Since 2005, he has been an Assistant Professor with the Engineering Department in Ferrara, University of Ferrara, Ferrara, Italy. In the summer of 2001, he joined the Wireless Section, AT&T Research Laboratories, Middletown, NJ. Since February 2003, he has been a frequent Visitor with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, where he is currently a Research Affiliate. He is a coauthor of *Wireless Sensor and Actuator Networks: Enabling Technologies, Information Processing and Protocol Design* (Elsevier, 2008). His current research interest is focused on wireless communications, including localization, adaptive transmission and multichannel reception, coding in faded multiple-input–multiple-output channels, wireless cooperative networks, and wireless sensor networks.

Dr. Conti is an Editor for the IEEE COMMUNICATIONS LETTERS. From 2003 to 2009, he was an Editor for the IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS. He is the Secretary of the IEEE Radio Communications Committee (RCC) for the period 2008–2010.

Lajos Hanzo (F'04) received the Dipl.Ing. (Master's) and Ph.D. degrees from the Technical University of Budapest, Budapest, Hungary, in 1976 and 1983, respectively, and the D.Sc. degree from the University of Southampton, Southampton, U.K., in 2004.

During his career in telecommunications, he has held various research and academic posts in Hungary, Germany, and the U.K. Since 1986, he has been with the Department of Electronics and Computer Science, University of Southampton, Southampton, U.K., where he is currently the Chair in Telecommunications. He is a coauthor of books published by John Wiley and has published about 800 research papers. Currently, he heads an academic research team, working on a range of research projects in wireless multimedia communications sponsored by industry, the Engineering and Physical Sciences Research Council, U.K., the European Information Society Technologies Program, and the Mobile Virtual Centre of Excellence, U.K. He is an enthusiastic supporter of industrial and academic liaison, and he offers a range of industrial courses.

Dr. Hanzo is an IEEE Distinguished Lecturer and is a Fellow of the Royal Academy of Engineering and the Institution of Electrical Engineers. He is an Editorial Board Member of the PROCEEDINGS OF THE IEEE, the Editor-in-Chief of the IEEE Press, and a Governor of both the IEEE Communications Society and Vehicular Technology Society. He is a coauthor of books published by the IEEE Press, totaling 10 000 pages on mobile radio communications. He has organized and chaired conferences, presented various keynote and overview lectures, and received a number of distinctions.