Technical Report: Bessel Filter Analysis

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In this technical report, we prove two fundamental theorems for an edge detection algorithm based on a Bessel filter.

Theorem 1: *The gradient magnitude of the convolved image* $u: R^2 \to R^+$ *calculated by* $u(x, y) = h * g$ *is unbounded on the discontinuities of a given piecewise-constant image* $g: \Omega \to R^+$ *where* Ω *and* R^+ *are the image domain and the set of positive real numbers respectively.*

Proof:

Before presenting the proof for this theorem, we briefly explain the structure of the proof. The convolution (in Cartesian coordinates) between a modified Bessel function of the second kind and zero degree and an input image (a piece-wise constant function) depicted in figure (A) (see the end of section A of this document) at the center of coordinate system (point *O*) is evaluated in polar coordinate system. Next, we calculate the directional derivative of this convolution (at point *O*) along a line passing through point *O* with any given slope between $\tan(\vartheta)$ and $\tan(\vartheta + \phi)$ (see figure (A) in this document). By exploiting the asymptotic properties of the Bessel function, we then prove that this directional derivate is infinity (unbounded). We immediately conclude that the gradient magnitude of the convolved function at point *O* is also unbounded. To explain the proof in details, let us now consider the piece-wise constant image shown in figure (A) described in polar coordinates as:

$$g(r,\theta) = aH(r)\big(H(\theta - \vartheta) - H(\theta - \vartheta - \varphi)\big) \tag{A-1}$$

where *H*(.), *a*, *r*, and θ are the Heaviside step function, a constant representing luminance, radial and angular coordinates respectively and $\vartheta, \phi \in [0, 2\pi)$ are constants

In figure (A), the origin of the coordinate system, point O is placed on the image discontinuity forming a sharp corner. We aim to show that the gradient magnitude of u at O (i.e. $\left|\nabla u\right|_O$) is infinite. If u is the convolved image with the Bessel filter, we evaluate $u_n = \left.\frac{\partial u}{\partial \mathbf{n}}\right|_{(0,0)}$ where $\mathbf{n} = \cos(\theta_n)\,\mathbf{i} + \sin(\theta_n)\,\mathbf{j}$, $\theta_n \in (\vartheta, \vartheta+\phi)$, and $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors along x and y axes respectively:

$$\frac{\partial u}{\partial \mathbf{n}} = \frac{\partial h}{\partial \mathbf{n}} * g = \underset{\varepsilon\to 0}{Lim}\int_{-\infty}^{-\varepsilon}\int_{-\infty}^{-\varepsilon} g(x-x', y-y')\frac{\partial h(x',y')}{\partial \mathbf{n}}dx'dy' + \underset{\varepsilon\to 0}{Lim}\int_{-\infty}^{-\varepsilon}\int_{\varepsilon}^{+\infty} g(x-x', y-y')\frac{\partial h(x',y')}{\partial \mathbf{n}}dx'dy'$$

$$+\underset{\varepsilon\to 0}{Lim}\int_{\varepsilon}^{+\infty}\int_{-\infty}^{-\varepsilon} g(x-x', y-y')\frac{\partial h(x',y')}{\partial \mathbf{n}}dx'dy' + \underset{\varepsilon\to 0}{Lim}\int_{\varepsilon}^{+\infty}\int_{\varepsilon}^{+\infty} g(x-x', y-y')\frac{\partial h(x',y')}{\partial \mathbf{n}}dx'dy' \quad \text{(A-2)}$$

where h is a Bessel filter defined as:

$$h(x) = \begin{cases} K_0(\frac{|x|}{\sqrt{\mu}}) & |x| > \dfrac{\varepsilon}{2} \\ K_0(\frac{\varepsilon}{2\sqrt{\mu}}) & |x| \le \dfrac{\varepsilon}{2} \end{cases}$$

In the above equations, K_0 is the modified Bessel function of the second type and zero degree.

At point O in figure (A) in polar coordinates, we can write:

$$\frac{\partial u}{\partial \mathbf{n}}(0,0) = \underset{\varepsilon\to 0}{Lim}\int_{\varepsilon}^{+\infty}\int_{0}^{2\pi} g(-r,-\theta)\frac{\partial h}{\partial \mathbf{n}}\, r\, drd\theta \quad \text{(A-3)}$$

Or

$$\frac{\partial u}{\partial \mathbf{n}}(0,0) = a\,\underset{\varepsilon\to 0}{Lim}\int_{\varepsilon}^{+\infty}\int_{\vartheta+\pi}^{\vartheta+\varphi+\pi} \frac{\partial h}{\partial \mathbf{n}}\, r\, drd\theta \quad \text{(A-4)}$$

On the other hand, $\frac{\partial h}{\partial \mathbf{n}}$ can be calculated as:

$$\frac{\partial h}{\partial \mathbf{n}} = \vec{\nabla} h \cdot \mathbf{n} = \frac{\partial h}{\partial x}\cos(\theta_n) + \frac{\partial h}{\partial y}\sin(\theta_n) \quad \text{(A-5)}$$

Equation (A-4) can then be written as:

$$\frac{\partial\, u}{\partial\, \mathbf{n}}(0,0) = \frac{a}{\sqrt{\mu}}\underset{\varepsilon\to 0}{Lim}\int_{\varepsilon}^{+\infty}\int_{\vartheta+\pi}^{\vartheta+\varphi+\pi} r\,\frac{\partial K_0(\frac{r}{\sqrt{\mu}})}{\partial\, r}\cos\left(\theta-\theta_n\right)drd\theta \quad \text{(A-6)}$$

Therefore:

$$\frac{\partial u}{\partial \mathbf{n}}(0,0)=\frac{a}{\sqrt{\mu}}\left(\sin(\vartheta-\theta_n)-\sin(\vartheta+\varphi-\theta_n)\right)\underset{\varepsilon\to 0}{Lim}\int_{\varepsilon}^{+\infty} r\ \frac{\partial K_0(\frac{r}{\sqrt{\mu}})}{\partial\ r}dr \qquad \text{(A-7)}$$

Equation (A-7) can be integrated by using integration by part:

$$\frac{\partial u}{\partial \mathbf{n}}(0,0)=\frac{a}{\sqrt{\mu}}\left(\sin(\vartheta-\theta_n)-\sin(\vartheta+\varphi-\theta_n)\right)\underset{\varepsilon\to 0}{Lim}\left[\left(rK_0(\tfrac{r}{\sqrt{\mu}})\right)\Big|_{\varepsilon}^{+\infty}-\int_{\varepsilon}^{+\infty} K_0(\tfrac{r}{\sqrt{\mu}})dr\right] \quad \text{(A-8)}$$

Using asymptotic behaviors of $K_0(r)$, as $r=\varepsilon\to 0$ and $r\to+\infty$ and l'Hopital's rule for indeterminate terms, we can determine the asymptotic behavior of term $rK_0(\frac{r}{\sqrt{\mu}})$ in equation (A-8), when $r=\varepsilon\to 0$ and $r\to+\infty$, i.e.:

$$\underset{r\to 0}{Lim}\left[rK_0(\tfrac{r}{\sqrt{\mu}})\right]=\underset{r\to 0}{Lim}\left[-r\,Log(\tfrac{r}{\sqrt{\mu}})\right]=0$$

and

$$\underset{r\to+\infty}{Lim}\left[rK_0(\tfrac{r}{\sqrt{\mu}})\right]=\underset{r\to+\infty}{Lim}\left[r\sqrt{\frac{\pi}{2\,r}}\exp(\tfrac{-r}{\sqrt{\mu}})\right]=0$$

Equation (A-8) can therefore be rewritten as:

$$\frac{\partial u}{\partial \mathbf{n}}(0,0)=\frac{a}{\sqrt{\mu}}\left(\sin(\vartheta+\varphi-\theta_n)-\sin(\vartheta-\theta_n)\right)\underset{\varepsilon\to 0}{Lim}\left(\int_{\varepsilon}^{+\infty} K_0(\tfrac{r}{\sqrt{\mu}})dr\right) \qquad \text{(A-9)}$$

as $\varepsilon\to 0$, the integration term in equation (A-9) approaches to infinity.

Since $\frac{\partial u}{\partial \mathbf{n}}(0,0)=\vec{\nabla}u\big|_O\cdot\mathbf{n}$, then

$$\left|\nabla u\right|_O=+\infty$$

We notice that for $\phi=\pi$ the wedge shown in figure (A) changes to an edge along a straight line on which point O can be regarded as an arbitrary point. In such a case, equation (A-9) is reduced to

$$\frac{\partial u}{\partial \mathbf{n}}(0,0)=\frac{-2a}{\sqrt{\mu}}\left(\sin(\vartheta-\theta_n)\right)\underset{\varepsilon\to 0}{Lim}\left(\int_{\varepsilon}^{+\infty} K_0(\tfrac{r}{\sqrt{\mu}})dr\right) \qquad \text{(A-10)}$$

The absolute value of the right hand side of equation (A-10) also approaches infinity as $\varepsilon \to 0$, hence $|\nabla u|_O = +\infty$. We also note that by considering a regular curve approximated by an infinite number of infinitesimal straight line segments and by assuming point O to be on one of these infinitesimal straight line segments, we exploit the above argument obtained from (A-10) to conclude that $|\nabla u|_O$ approaches infinity as $\varepsilon \to 0$ for this case as well. It is also noted that for boundaries forming straight lines and/or regular curves the unit vector **n** is conveniently considered to be normal unit vector to the boundary.

$\square$

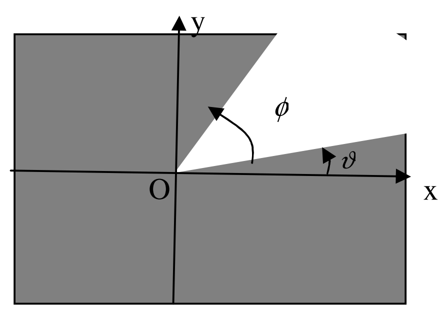

Fig (A): An original image containing discontinuities forming a sharp corner at the center of coordinates

Theorem 2: *The gradient magnitude of the convolved image* $u: R^2 \to R^+$ *calculated in* $u(x, y) = h * g$ *has local maxima on discontinuities of a given piecewise-constant image* $g: \Omega \to R^+$.

Proof:

At the beginning, let us briefly explain the structure of the proof. The directional derivative of the convolution between the isotropic Bessel function and the input image shown in figure (A) along a straight line passing through point O with a slope between $\tan(\vartheta)$ and $\tan(\vartheta+\phi)$ (see figure (A)) is considered. We initially assume that the local maximum associated to point O is located at point O_0 with polar coordinates (r_0, θ_0). By using the asymptotic properties of the Bessel function, we finally prove that point O_0 is located at the center of coordinates, i.e. point O_0 is the same as point O. To explain the proof in details, let us now consider again g in figure

(A). We now aim to prove that $|\nabla u|$ has a maximum at point *O,* as $\varepsilon \to 0$. By considering $U(0,0) = \dfrac{\partial u}{\partial \mathbf{n}}(0,0)$, we can calculate $\left.\dfrac{\partial U}{\partial \mathbf{n}}\right|_{(0,0)}$ as

$$\left.\frac{\partial U}{\partial \mathbf{n}}\right|_{(0,0)} = \int_0^{+\infty}\int_0^{2\pi}\left(r\frac{\partial^2 h}{\partial r^2}\cos^2(\theta-\theta_n) - \frac{\partial h}{\partial r}\sin^2(\theta-\theta_n)\right) g\,dr\,d\theta$$

where *h* is the Bessel filter and *g* is given in equation (A-1). By integrating the above equation with respect to θ and integrating by part with respect to *r* for the first term in the above equation, we can write:

$$\left.\frac{\partial U}{\partial \mathbf{n}}\right|_{(0,0)} = -a\,\underset{\varepsilon\to 0}{Lim}\Big(\sin\big(2(\varphi+\vartheta-\theta_n)\big)-\sin\big(2(\vartheta-\theta_n)\big)\Big)\int_{\varepsilon}^{+\infty}\left(\frac{dh_\varepsilon}{dr}\right)dr = a\,\Big(\sin\big(2(\varphi+\vartheta-\theta_n)\big)-\sin\big(2(\vartheta-\theta_n)\big)\Big)\underset{\varepsilon\to 0}{Lim}\,h(\varepsilon) \qquad \text{(B-1)}$$

Let us assume that *U* has a local maximum associated with point *O* in (r_0,θ_0) where r_0 representing the distance between the local maximum and the central point *O,* is a small displacement, $\theta_0 \neq \vartheta$ and $\theta_0 \neq \vartheta+\phi$. We notice that the local maxima with $r_0 \neq 0$ and angle $\theta_0 = \vartheta$ or $\theta_0 = \vartheta+\phi$ are associated with points on lower and upper discontinuity edges (along straight lines) of the wedge and therefore not associated with point *O*. A Mclaurin's series can be written in the neighbourhood of maximum point, i.e.:

$$\left.\frac{\partial U}{\partial \mathbf{n}}\right|_{(r_0,\theta_0)} = \left.\frac{\partial U}{\partial \mathbf{n}}\right|_{(0,0)} + r_0\left.\frac{\partial^2 U}{\partial \mathbf{n}^2}\right|_{(0,0)} + r_0\theta_0\left.\frac{\partial^2 U}{\partial \mathbf{n}_\perp \partial \mathbf{n}}\right|_{(0,0)} + O(r_0^2)$$

where $\mathbf{n} = \cos(\theta-\theta_n)\mathbf{e}_r - \sin(\theta-\theta_n)\mathbf{e}_\theta$, $\mathbf{n}_\perp = \sin(\theta-\theta_n)\mathbf{e}_r + \cos(\theta-\theta_n)\mathbf{e}_\theta$ ($\mathbf{n}_\perp$ is perpendicular to $\mathbf{n}$), $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit vectors along radial and angular coordinates respectively. Since r_0 is a small displacement, we ignore terms with higher degrees. On the other hand, $\left.\dfrac{\partial U}{\partial \mathbf{n}}\right|_{(r_0,\theta_0)}$ vanishes, because U has a local maximum at (r_0,θ_0), i.e.:

$$\left.\frac{\partial U}{\partial \mathbf{n}}\right|_{(r_0,\theta_0)} = 0 = \left.\frac{\partial U}{\partial \mathbf{n}}\right|_{(0,0)} + r_0\left.\frac{\partial^2 U}{\partial \mathbf{n}^2}\right|_{(0,0)} + r_0\theta_0\left.\frac{\partial^2 U}{\partial \mathbf{n}_\perp \partial \mathbf{n}}\right|_{(0,0)}$$

We can then calculate r_0 as:

$$r_0 = \frac{-\left.\dfrac{\partial U}{\partial \mathbf{n}}\right|_{(0,0)}}{\left.\dfrac{\partial^2 U}{\partial \mathbf{n}^2}\right|_{(0,0)} + \theta_0 \left.\dfrac{\partial^2 U}{\partial \mathbf{n}_\perp \partial \mathbf{n}}\right|_{(0,0)}} \tag{B-2}$$

The second directional derivatives of U along the unit vectors $\mathbf{n}$ and $\mathbf{n}_\perp$ can also be calculated as:

$$\left.\frac{\partial^2 U}{\partial \mathbf{n}^2}\right|_{(0,0)} = a\left(\sin^3(\vartheta-\theta_n) - \sin^3(\vartheta+\varphi-\theta_n)\right) \underset{\varepsilon\to 0}{Lim}\left(\int_{\varepsilon}^{+\infty} \frac{1}{r}\frac{dh}{dr}dr\right) \tag{B-3}$$

$$\left.\frac{\partial^2 U}{\partial \mathbf{n}_\perp \partial \mathbf{n}}\right|_{(0,0)} = a\left(\sin^2(\vartheta+\varphi-\theta_n)\cos\left(\vartheta+\varphi-\theta_n\right) - \sin^2(\vartheta-\theta_n)\cos\left(\vartheta-\theta_n\right)\right) \underset{\varepsilon\to 0}{Lim}\left(\int_{\varepsilon}^{+\infty} \frac{1}{r}\frac{dh}{dr}dr\right) \tag{B-4}$$

By using equations (B-1), (B-3) and (B-4), equation (B-2) can be written as:

$$r_0 = \underset{\varepsilon\to 0}{Lim} \frac{-\left(\sin\left(2\left(\varphi+\vartheta-\theta_n\right)\right) - \sin\left(2\left(\vartheta-\theta_n\right)\right)\right)h(\varepsilon)}{\left(\sin^3(\vartheta-\theta_n) - \sin^3(\vartheta+\varphi-\theta_n) + \theta_0\left(\sin^2(\vartheta+\varphi-\theta_n)\cos\left(\vartheta+\varphi-\theta_n\right) - \sin^2(\vartheta-\theta_n)\cos\left(\vartheta-\theta_n\right)\right)\right)\displaystyle\int_{\varepsilon}^{+\infty} \frac{1}{r}\frac{dh}{dr}dr}$$

The above term is indeterminate as $\varepsilon \to 0$. By using l'Hoptial's rule for indeterminate terms, we can therefore write r_0 as:

$$r_0 = \underset{\varepsilon\to 0}{Lim} \frac{\left(\sin\left(2\left(\varphi+\vartheta-\theta_n\right)\right) - \sin\left(2\left(\vartheta-\theta_n\right)\right)\right)\dfrac{dh}{d\varepsilon}}{\left(\sin^3(\vartheta-\theta_n) - \sin^3(\vartheta+\varphi-\theta_n) + \theta_0\left(\sin^2(\vartheta+\varphi-\theta_n)\cos\left(\vartheta+\varphi-\theta_n\right) - \sin^2(\vartheta-\theta_n)\cos\left(\vartheta-\theta_n\right)\right)\right)\dfrac{1}{\varepsilon}\dfrac{dh}{d\varepsilon}} = 0$$

Therefore as $\varepsilon \to 0$, the local maximum point for $U = \dfrac{\partial u}{\partial \mathbf{n}}$ approaches point *O*. Therefore $|\nabla u|$ at point *O* has a local maximum. For $\phi = \pi$, the boundary passing through point *O* is a straight line and r_0 is therefore calculated as:

$$r_0 = \underset{\varepsilon\to 0}{Lim} \frac{-\varepsilon\ \left(\sin\left(2\left(\pi+\vartheta-\theta_n\right)\right) - \sin\left(2\left(\vartheta-\theta_n\right)\right)\right)}{\left(2\sin^3(\vartheta-\theta_n) + 2\theta_0\left(\sin^2(\vartheta-\theta_n)\cos\left(\vartheta-\theta_n\right)\right)\right)} = 0 \tag{B-5}$$

Boundaries forming regular curves passing through point *O* are considered to consist of infinite numbers of infinitesimal straight line segments. For any point on such infinitesimal line segments, the above argument is applicable. The unit vector **n** is considered to be normal to boundaries forming a straight line and/or a regular curve. □

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