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$\mathscr{H}_2/\mathscr{H}_\infty$ output information-based disturbance attenuation for differential linear repetitive processes

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SUMMARY

Repetitive processes propagate information in two independent directions where the duration of one is finite. They pose control problems that cannot be solved by application of results for other classes of 2D systems. This paper develops controller design algorithms for differential linear processes, where information in one direction is governed by a matrix differential equation and in the other by a matrix discrete equation, in an $\mathscr{H}_2/\mathscr{H}_\infty$ setting. The objectives are stabilization and disturbance attenuation, and the controller used is actuated by the process output and hence the use of a state observer is avoided.
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1. INTRODUCTION

The unique feature of a repetitive process is a series of sweeps, termed passes, through a set of dynamics defined over a finite duration known as the pass length. The exact sequence is that a pass is completed, the process is reset and the output, or pass profile, produced on the previous pass acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile and so on. Hence the updating structure in the pass-to-pass direction is spatial. The result can be oscillations that increase in amplitude in the pass-to-pass direction where this instability cannot be analyzed using standard, or 1D, theory. Repetitive processes have their origins in the coal mining and metal rolling industries, see [1] and the relevant cited references for details.

Applications have also arisen where using a repetitive process setting for analysis has distinct advantages, including classes of iterative learning control (ILC) schemes (see, for example, [2]) and iterative algorithms for solving nonlinear dynamic optimal control/optimization problems based on the maximum principle [3]. ILC can be treated as a 2D system where one direction of information is from trial-to-trial and the other along a trial. In particular, since the trial length is finite, ILC can therefore be treated as a repetitive process. Recently, it has been shown [4], with experimental verification on a gantry robot system, how the repetitive process setting can be used to solve problems in ILC that cannot be treated by alternatives.

Attempts to stabilize and/or meet performance specifications, such as the level of disturbance attenuation, for these processes using 1D systems theory/algorithms fail (except in a few very restrictive special cases), precisely because such an approach ignores their inherent 2D systems

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structure, that is, information propagation occurs from pass-to-pass and along a given pass. In addition the initial conditions are reset before the start of each new pass and the structure of these can be somewhat complex. For example, if they are an explicit function of points on the previous pass profile then this alone can destroy stability [1]. In seeking a rigorous foundation on which to analyze such features, it is natural to attempt to exploit structural links that exist between these processes and other classes of 2D linear systems.

The case of 2D discrete linear systems recursive in the positive quadrant $(i, j): i \geqslant 0,\ j \geqslant 0$ (where i and j denote the directions of information propagation) has been the subject of much research effort over the years using, in the main, the well-known Roesser [5] and Fornasini–Marchesini [6] state–space models. Productive research has been reported on $\mathscr{H}_\infty$ and $\mathscr{H}_2$ approaches to analysis and design, see, for example, [7, 8] for such systems. In this paper we consider differential linear repetitive processes where information propagation along the pass is governed by a matrix differential equation. The systems theory for 2D discrete linear systems is, therefore, not applicable.

In this paper we develop results in an $\mathscr{H}_2$ setting that are then combined with others obtained in an $\mathscr{H}_\infty$ setting to produce potentially very powerful mixed $\mathscr{H}_2/\mathscr{H}_\infty$ results. These are developed through the use of an output dynamic controller, where the major advance over previously reported results [9] is that access to the current pass state vector is not required. The computations are in the form of linear matrix inequalities (LMIs).

Throughout this paper, the null matrix and the identity matrix with appropriate dimensions are denoted by 0 and I, respectively. Moreover, $M \succ 0$ (respectively, $\succcurlyeq 0$) denotes a real symmetric positive-definite (respectively, semi-definite) matrix. Similarly, $M \prec 0$ (respectively, $\preccurlyeq 0$) denotes a real symmetric negative-definite (respectively, semi-definite) matrix. We also require the following signal space definition.

Definition 1
Consider a $q \times 1$ vector sequence $\{w_j(t)\}$ defined over the real interval $0 \leqslant t \leqslant \infty$ and the nonnegative integers $0 \leqslant j \leqslant \infty$, which is written as $\{[0, \infty), [0, \infty)\}$. Then the L_2 norm of this vector sequence is given by

$$\|w\|_2 = \sqrt{\sum_{j=0}^{\infty} \int_0^{\infty} w_j^{\mathrm{T}}(t) w_j(t) \mathrm{d}t}$$

and this sequence is said to be a member of $L_2^q\{[0, \infty), [0, \infty)\}$, or L_2^q for short, if $\|w\|_2 < \infty$.

2. BACKGROUND

This paper considers differential linear repetitive processes described by the following state–space model over $0 \leqslant t \leqslant \alpha$, $k \geqslant 0$:

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + B_0 y_k(t) + Bu_{k+1}(t) + B_{11} w_{k+1}(t) + B_{21} v_{k+1}(t) \\ y_{k+1}(t) &= Cx_{k+1}(t) + D_0 y_k(t) + Du_{k+1}(t) + B_{12} w_{k+1}(t) + B_{22} v_{k+1}(t) \end{aligned} \tag{1}$$

Hereon pass k, $x_k(t)$ is the $n \times 1$ state vector, $y_k(t)$ is the $m \times 1$ pass profile vector and $u_k(t)$ is the $l \times 1$ vector of control inputs, $w_{k+1}(t)$ and $v_{k+1}(t)$ are $r \times 1$ disturbance vectors acting on the current pass state and pass profile vectors, respectively. Note that these disturbances are allowed to vary from pass-to-pass.

Remark 1
Here the disturbance terms are taken as white noise with unit variance and bounded energy ($\|\cdot\|_2 < \infty$) but are denoted by different symbols, that is, $w_{k+1}(t)$ and $v_{k+1}(t)$, respectively. Without

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loss of generality, these can be written as one disturbance in the form

$$\eta_{k+1}(t)=\begin{bmatrix} w_{k+1}(t) \\ v_{k+1}(t) \end{bmatrix}, \quad \text{and} \quad B_1=[B_{11} \;\; B_{21}], B_2=[B_{12} \;\; B_{22}] \tag{2}$$

To complete the process description, it is necessary to specify the boundary conditions, that is, the state initial vector on each pass and the initial pass profile (pass 0). For the purposes of this paper, it is assumed that the state initial vector at the start of each new pass has zero entries, and that the initial pass profile, $y_0(t)$, is equal to an $m\times 1$ vector, $f(t)$, with known entries over $0\leqslant t\leqslant \alpha$.

The stability theory [1] for linear repetitive processes is based on an abstract model in a Banach space setting which includes a large number of such processes as special cases. In this setting, let the pass profile be $y_k \in E_\alpha$, where E_α is an appropriately chosen Banach space. Then the process dynamics, or pass profile updating, can be written in the form

$$y_{k+1}=L_\alpha y_k + b_{k+1}, \quad k\geqslant 0 \tag{3}$$

where L_α is a bounded linear operator mapping E_α into itself and $b_{k+1}\in W_\alpha$, a linear subspace of E_α, represents contributions that enter on pass $k+1$, that is, from the control input and disturbances. For the state–space model (1), L_α is the linear convolution operator defined by the state–space matrices $\{A, B_0, C, D_0\}$.

Noting again the unique feature of these processes, that is, oscillations that increase in amplitude from pass-to-pass (the k direction in the notation for variables used here), this theory is based on ensuring that such a response cannot occur by demanding that the output sequence of pass profiles generated $\{y_k\}$ has a bounded-input bounded-output stability property defined in terms of the norm on the underlying Banach space. This property can be demanded over the finite pass length α or independent of this parameter in the form of stability along the pass which, in terms of the abstract model (3), requires the existence of finite real scalars $M_\infty>0$ and $\lambda_\infty\in(0,1)$ which are independent of α such that $\|L_\alpha^k\|\leqslant M_\infty \lambda_\infty^k$, $k\geqslant 0$, where $\|\cdot\|$ denotes the induced norm. Stability along the pass includes all other forms of stability for these processes as special cases and hence in this paper considers controller design for this property plus disturbance rejection.

Several equivalent sets of conditions for stability along the pass are known but here the starting point is the 2D transfer-function matrix description of the process dynamics, and hence the 2D characteristic polynomial. Since the state on pass 0 plays no role, it is convenient to re-label the state vector as $x_{k+1}(t)\mapsto x_k(t)$ (keeping of course the same interpretation). Also define the pass-to-pass shift operator as z_2 applied, for example, to $y_k(t)$ as follows:

$$y_k(t) := z_2 y_{k+1}(t)$$

and for the along the pass dynamics we use the Laplace transform variable s, where it is routine to argue that the finite pass length does not cause a problem provided that the variables considered are suitably extended from $[0,\alpha]$ to $[0,\infty)$, and here we assume that this has been done (stability along the pass is independent of the pass length, that is, holds for $0\leqslant\alpha\leqslant\infty$).

Let $Y(s,z_2)$ and $U(s,z_2)$ denote the results of applying these transforms to the sequences $\{y_k\}$ and $\{u_k\}$, respectively. Then the process dynamics, in the absence of disturbances, can be written as

$$Y(s,z_2)=G_{yu}(s,z_2)U(s,z_2)$$

where the 2D transfer-function matrix, $G_{yu}(s,z_2)$, is given by

$$G_{yu}(s,z_2)=[0 \quad I]\left(\begin{bmatrix} sI-A & -B_0 \\ -z_2 C & I-z_2 D_0 \end{bmatrix}\right)^{-1}\begin{bmatrix} B \\ D \end{bmatrix}$$

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and the 2D characteristic polynomial by

$$\mathscr{C}(s,z_2) := \det\left(\begin{bmatrix} sI-A & -B_0 \\ -z_2C & I-z_2D_0 \end{bmatrix}\right)$$

It has been shown elsewhere [1] that stability along the pass holds if, and only if,

$$\mathscr{C}(s,z_2) \neq 0 \tag{4}$$

in $\mathscr{U}(s,z_2) := \{(s,z_2): Re(s) \geqslant 0,\ |z_2| \leqslant 1\}$.

3. DISTURBANCE REJECTION

One key aspect of control law design beyond stabilization for the repetitive processes considered is, as for other classes of systems, disturbance rejection or, more physically realistic, attenuation. Here we consider two cases, the first of which is when the disturbance is impulsive and the second when it is of finite energy (taken as belonging to the space L_2). To analyze the first case we use the $\mathscr{H}_2$ norm for impulsive inputs and the $\mathscr{H}_\infty$ norm to measure the performance objective which is the attenuation of the effects of the disturbances in the model of (1). These norms are introduced next.

A differential linear repetitive process described by (1) is said to have $\mathscr{H}_\infty$ disturbance attenuation (or $\mathscr{H}_\infty$ norm bound) γ_∞ if it is stable along the pass and

$$\sup_{0 \neq v \in L_2^r} \frac{\|y\|_2}{\|v\|_2} < \gamma_\infty \tag{5}$$

In effect, this is a worst case bound as it corresponds to a bound on the maximum peak gain of the 2D frequency response between v and y and is given, with $\overline{\sigma}(\cdot)$ denoting the maximum singular value of its matrix argument, by

$$\|G_{yv}(s,z_2)\|_\infty = \sup_{\omega_1 \in \mathbb{R}, \omega_2 \in [0,2\pi]} \overline{\sigma}[G(j\omega_1, \mathrm{e}^{j\omega_2})]$$

where

$$G_{yv}(s,z_2) = [0 \quad I]\left(\begin{bmatrix} sI-A & -B_0 \\ -z_2C & I-z_2D_0 \end{bmatrix}\right)^{-1}\begin{bmatrix} B_{21} \\ B_{22} \end{bmatrix} \tag{6}$$

that is, the 2D transfer-function matrix describing the coupling between v_{k+1} and the current pass profile. This result invokes Parseval's theorem for 2D signals. The proof of this for the 2D discrete linear systems case can be found in, for example, [10] and in [9] for the continuous-discrete signal case which is needed here.

Another commonly used performance measure for analysis and synthesis in the 1D systems case is the $\mathscr{H}_2$ norm. It is widely recognized that this norm is a useful tool to optimize the transient behavior of a system and the same is also true for repetitive processes. In this paper, we will minimize $\mathscr{H}_2$ norm of the 2D transfer-function matrix between w and y, that is,

$$G_{yw}(s,z_2) = [0 \quad I]\left(\begin{bmatrix} sI-A & -B_0 \\ -z_2C & I-z_2D_0 \end{bmatrix}\right)^{-1}\begin{bmatrix} B_{11} \\ 0 \end{bmatrix} \tag{7}$$

to reduce the output energy in response to impulse inputs or the output variance in response to white noise inputs.

It is important to note that the $\mathscr{H}_2$ norm of the 2D transfer-function matrix can be only defined if there is no direct coupling between the impulsive disturbance signal and the pass profile on any pass. Hence, in common with the 1D linear systems case where the $\mathscr{H}_2$ norm is only defined for

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strictly proper systems, we set $D=0$ when computing this norm of the 2D transfer-function matrix linking u and y, and set $B_{12}=0$ for the same computation applied to the 2D transfer-function matrix linking w and y. This ensures that the $\mathcal{H}_2$ norm is always defined and never infinite.

The following result is fundamental to the analysis and the computations involved are expressed in terms of LMIs.

Theorem 1 ([9])
For some prescribed $\gamma_\infty>0$, suppose that there exist matrices $R_1\succ 0$ and $R_2\succ 0$ such that the following LMI holds for a differential linear repetitive process described by (1):

$$\begin{bmatrix} -R_2 & R_2 C & R_2 D_0 & R_2 B_{22} \\ C^{\mathrm{T}} R_2 & A^{\mathrm{T}} R_1 + R_1 A & R_1 B_0 & R_1 B_{21} \\ D_0^{\mathrm{T}} R_2 & B_0^{\mathrm{T}} R_1 & -R_2+I & 0 \\ B_{22}^{\mathrm{T}} R_2 & B_{21}^{\mathrm{T}} R_1 & 0 & -\gamma_\infty^2 I \end{bmatrix} \prec 0 \tag{8}$$

Then this process is stable along the pass and $\|G_{yv}(s,z_2)\|_\infty<\gamma_\infty$.

In 1D systems theory the $\mathcal{H}_\infty$ norm is used as a measure of system robustness. Suppose also that the $\mathcal{H}_\infty$ norm of the controlled process 2D transfer-function matrix from v to y is below the level γ_∞. Then the above result can be viewed as the differential repetitive process generalization in the sense that the process under consideration is robust to unstructured perturbations of the form

$$v=\mathbf{\Delta} y, \quad \|\mathbf{\Delta}\|_\infty \leqslant \gamma_\infty$$

This means that choosing a lower value of γ_∞ reduces the robustness to un-modeled dynamics (as measured in this way) and vice versa.

For the $\mathcal{H}_2$ norm, we follow the commonly used 1D approach and consider first a SISO stable along the pass process, which can be analyzed mathematically by letting the pass length $\alpha\to\infty$, and let the $m\times 1$ vector $g(k,t)$ denote the response to an impulse, denoted by $\delta(k,t)$, applied at $t=0$ on pass k. Then, by invoking Parseval's theorem in the along pass direction on each pass and summing over the pass index, the $\mathcal{H}_2$ norm is given by

$$\|G\|_2=\sqrt{\|g(k,t)\|_2^2}=\sqrt{\sum_{k=0}^{\infty}\int_0^\infty g^{\mathrm{T}}(k,t)g(k,t)\mathrm{d}t} \tag{9}$$

This last result is easily extended to the multiple-input multiple-output case and leads to the following result which gives a sufficient condition for stability along the pass with an upper bound on the $\mathcal{H}_2$ norm of the 2D transfer-function matrix.

Theorem 2 ([9])
A differential linear repetitive process described by (1) is stable along the pass and has $\mathcal{H}_2$ norm bound $\gamma_2>0$, that is, $\|G\|_2<\gamma_2$, if there exist matrices $P_1\succ 0$, $P_2\succ 0$ and $P_3\succ 0$ such that the following LMIs hold:

$$\mathrm{trace}(\widetilde{B}_1^{\mathrm{T}} P_1 \widetilde{B}_1)-\gamma_2^2<0 \tag{10}$$

and

$$\begin{bmatrix} -Y & Y\widehat{A}_2 & 0 \\ \widehat{A}_2^{\mathrm{T}} Y & \widehat{A}_1^{\mathrm{T}} Z + Z\widehat{A}_1 - V & M^{\mathrm{T}} \\ 0 & M & -I \end{bmatrix} \prec 0 \tag{11}$$

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where $Z = \text{diag}(R_1, 0)$, $Y = \text{diag}(P_3, P_2)$ and $V = \text{diag}(0, P_2)$. The matrices $\widehat{A}_1$ and $\widehat{A}_2$ are given by

$$\widehat{A}_1 = \begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{A}_2 = \begin{bmatrix} 0 & 0 \\ C & D_0 \end{bmatrix} \tag{12}$$

and

$$M = [C \quad D_0], \quad \widetilde{B} = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}$$

4. $\mathscr{H}_2$ AND MIXED $\mathscr{H}_2/\mathscr{H}_\infty$ CONTROL WITH A DYNAMIC PASS PROFILE CONTROLLER

Some application areas will clearly require the design of compensators that guarantee stability along the pass and also have the maximum possible disturbance attenuation. The relevance of rejecting the effects of disturbances on measurements (and subsequent computations) of variables is well founded physically by noting the conditions in which physical examples have to operate, for example, in ILC applications such as using a gantry robot to synchronously place objects on a chain conveyor [11].

In previous work [9] it was assumed that all entries in the current pass state vector are available for measurement and noting again the critical role of the previous pass profile vector and hence the weakness of current pass action alone, the use of a control law of the form

$$u_{k+1}(t) = [K_1 \ \ K_2] \begin{bmatrix} x_{k+1}(t) \\ y_k(t) \end{bmatrix} \tag{13}$$

was considered, where K_1 and K_2 are appropriately dimensioned matrices to be designed. This control law is termed static as it has no internal dynamics.

If the current pass state vector is not available for measurement, (13) could still be used but a state observer would be required to estimate the entries in this vector that cannot be measured. An alternative is to replace this term in (13) by the current pass profile vector. This is feasible since, by definition, the pass profile is the process output vector and hence available for measurement but is weak since, for example, it may not be possible to choose the matrix K_1 (this is an 1D systems pole placement problem with output as opposed to state feedback) such that all eigenvalues of $A + BK_1C$ have strictly negative real parts, which is a necessary condition for stability along the pass of the controlled process. For such cases, the next stage is to allow internal dynamics (at the possible cost of increased design complexity) in the form of the following dynamic pass profile controller:

$$\begin{aligned} \begin{bmatrix} \dot{x}^c_{k+1}(t) \\ y^c_{k+1}(t) \end{bmatrix} &= \begin{bmatrix} A_{c11} & A_{c12} \\ A_{c21} & A_{c22} \end{bmatrix} \begin{bmatrix} x^c_{k+1}(t) \\ y^c_k(t) \end{bmatrix} + \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} y_{k+1}(t) \\ u_{k+1}(t) &= [C_{c1} \ \ C_{c2}] \begin{bmatrix} x^c_{k+1}(t) \\ y^c_k(t) \end{bmatrix} + D_c y_{k+1}(t) \end{aligned} \tag{14}$$

where $x^c_{k+1}(t)$ and $y^c_k(t)$ are internal vectors for the controller.

In the analysis that follows use will be made of the following well-known result.

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Lemma 1 ([*12*])
Suppose that the $n\times n$ matrices $\Sigma\succ 0$ and $\Gamma\succ 0$ are given and n_c is a positive integer. Then there exists $n\times n_c$ matrices Σ_2, Γ_2 and $n_c\times n_c$ symmetric matrices Σ_3, and Γ_3, such that

$$\begin{bmatrix}\Sigma & \Sigma_2\\ \Sigma_2^{\mathrm{T}} & \Sigma_3\end{bmatrix}\succ 0 \quad\text{and}\quad \begin{bmatrix}\Sigma & \Sigma_2\\ \Sigma_2^{\mathrm{T}} & \Sigma_3\end{bmatrix}^{-1}=\begin{bmatrix}\Gamma & \Gamma_2\\ \Gamma_2^{\mathrm{T}} & \Gamma_3\end{bmatrix}$$

if, and only if,

$$\begin{bmatrix}\Sigma & I\\ I & \Gamma\end{bmatrix}\succcurlyeq 0$$

Introduce

$$\Phi=\begin{bmatrix}A & B_0\\ C & D_0\end{bmatrix},\quad B_2=\begin{bmatrix}B\\ D\end{bmatrix},\quad A_c=\begin{bmatrix}A_{c11} & A_{c12}\\ A_{c21} & A_{c22}\end{bmatrix},\quad B_c=\begin{bmatrix}B_{c1}\\ B_{c2}\end{bmatrix},\quad C_2=[0\ \ I],$$
$$C_c=[C_{c1}\quad C_{c2}] \tag{15}$$

the augmented state and pass profile vectors

$$\dot{\overline{x}}_{k+1}(t)=\begin{bmatrix}\dot{x}_{k+1}(t)\\ \dot{x}^c_{k+1}(t)\end{bmatrix},\quad \overline{y}_k(t)=\begin{bmatrix}y_k(t)\\ y^c_k(t)\end{bmatrix}$$

and also the matrices

$$\Pi=\begin{bmatrix}I & 0 & 0 & 0\\ 0 & 0 & I & 0\\ 0 & I & 0 & 0\\ 0 & 0 & 0 & I\end{bmatrix},\quad \Pi_1=\begin{bmatrix}I & 0 & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\quad \Pi_2=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & 0 & I\end{bmatrix},\quad \Pi_3^{\mathrm{T}}=\begin{bmatrix}0\\ I\\ 0\\ 0\end{bmatrix}$$

Then the state–space model of the controlled process resulting from application of (14) can be written in the form

$$\begin{aligned}\begin{bmatrix}\dot{x}_{k+1}(t)\\ \overline{y}_{k+1}(t)\end{bmatrix}&=(\overline{A}_1+\overline{A}_2)\begin{bmatrix}\overline{x}_{k+1}(t)\\ \overline{y}_k(t)\end{bmatrix}+(\overline{B}_{11}+\overline{B}_{12})w_{k+1}(t)+(\overline{B}_{21}+\overline{B}_{22})v_{k+1}(t)\\ z_{k+1}(t)&=\overline{C}\begin{bmatrix}\overline{x}_{k+1}(t)\\ \overline{y}_{k+1}(t)\end{bmatrix}\end{aligned} \tag{16}$$

where

$$\begin{aligned}\overline{A}_1+\overline{A}_2&=\Pi_1\begin{bmatrix}\Phi+B_2D_cC_2 & B_2C_c\\ B_cC_2 & A_c\end{bmatrix}\Pi^{\mathrm{T}}+\Pi_2\begin{bmatrix}\Phi+B_2D_cC_2 & B_2C_c\\ B_cC_2 & A_c\end{bmatrix}\Pi^{\mathrm{T}}\\ &=\begin{bmatrix}A & BC_{c1} & B_0+BD_c & BC_{c2}\\ 0 & A_{c11} & B_{c1} & A_{c12}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ C & DC_{c1} & D_0+DD_c & DC_{c2}\\ 0 & A_{c21} & B_{c2} & A_{c22}\end{bmatrix}\end{aligned}$$

$$\overline{B}_{11}+\overline{B}_{12}=\Pi_1\begin{bmatrix}B_{11}\\ 0\\ 0\\ 0\end{bmatrix}+\Pi_2\begin{bmatrix}B_{11}\\ 0\\ 0\\ 0\end{bmatrix}=\begin{bmatrix}B_{11}\\ 0\\ 0\\ 0\end{bmatrix}+\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

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$$\overline{B}_{21}+\overline{B}_{22}=\Pi_1\begin{bmatrix}B_{21}\\B_{22}\\0\\0\end{bmatrix}+\Pi_2\begin{bmatrix}B_{21}\\B_{22}\\0\\0\end{bmatrix}=\begin{bmatrix}B_{21}\\0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\B_{22}\\0\end{bmatrix}$$

$$\overline{C}=[C_2 \quad 0]\Pi^{\mathrm{T}}$$

Additionally define

$$\overline{A}_3=\Pi_3\begin{bmatrix}\Phi+B_2D_cC_2 & B_2C_c\\ B_cC_2 & A_c\end{bmatrix}\Pi^{\mathrm{T}}=[C \quad DC_{c1} \quad D_0+DD_c \quad DC_{c2}]$$

Then, based on Theorems 1 and 2, we have the following result.

Theorem 3
Suppose that a controller of the form (14) is applied to a differential linear repetitive process described by (1). Then the resulting controlled process is stable along the pass and has prescribed $\mathscr{H}_2$ and $\mathscr{H}_\infty$ norm bounds $\gamma_2>0$ and $\gamma_\infty>0$, respectively, if there exist matrices $P_1\succ 0$, $P_2\succ 0$ such that the following inequalities hold:

$$\mathrm{trace}(\overline{B}_{11}^{\mathrm{T}}P\overline{B}_{11})-\gamma_2^2<0 \qquad (17)$$

$$\begin{bmatrix}-W & W\overline{A}_2 & 0\\ \overline{A}_2^{\mathrm{T}}W & \overline{A}_1^{\mathrm{T}}P+P\overline{A}_1-R & \overline{A}_3^{\mathrm{T}}\\ 0 & \overline{A}_3 & -I\end{bmatrix}\prec 0 \qquad (18)$$

$$\begin{bmatrix}-W & W\overline{A}_2 & W\overline{B}_{22} & 0\\ \overline{A}_2^{\mathrm{T}}W & \overline{A}_1^{\mathrm{T}}P+P\overline{A}_1-R & P\overline{B}_{21} & \overline{C}^{\mathrm{T}}\\ \overline{B}_{22}^{\mathrm{T}}W & \overline{B}_{21}^{\mathrm{T}}P & -\gamma_\infty^2 I & 0\\ 0 & \overline{C} & 0 & -I\end{bmatrix}\prec 0 \qquad (19)$$

where $W=\mathrm{diag}(I, P_2)$, $P=\mathrm{diag}(P_1, I)$ and $R=\mathrm{diag}(0, P_2)$.

The above inequalities cannot be directly solved with efficient computational methods. This difficulty is removed by the following result expressed in terms of LMIs. Note, however, that these results can introduce a significant degree of conservativeness due to the need to use the same matrix variable W in transforming the matrix inequalities (18) and (19) into LMIs.

Theorem 4
Suppose there exist matrices $W_{11}\succ 0$, $P_{11}\succ 0$, $S_{11}\succ 0$, $R_{11}\succ 0$, D_c, $X_1\div X_8$ such that the LMIs defined by (20)–(24) hold. Then there exists a controller of the form (14) such that a differential linear repetitive process described by (16) is stable along the pass with prescribed $\mathscr{H}_2$ and $\mathscr{H}_\infty$

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norm bounds $\gamma_2>0$ and $\gamma_\infty>0$, respectively.

$$\mathrm{trace}(B_{11}^{\mathrm{T}}P_{11}B_{11})-\gamma_2^2<0 \tag{20}$$

$$\begin{bmatrix} -S_{11} & -I & CR_{11}+DX_1 & C & D_0+DD_c & D_0S_{11}+DX_2 & 0 \\ (\bullet)^{\mathrm{T}} & -W_{11} & X_5 & W_{11}C & W_{11}D_0+X_3 & X_7 & 0 \\ (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & AR_{11}+BX_1+R_{11}A^{\mathrm{T}}+X_1^{\mathrm{T}}B^{\mathrm{T}} & X_6^{\mathrm{T}}+A & B_0+BD_c & B_0S_{11}+BX_2 & R_{11}C^{\mathrm{T}}+X_1^{\mathrm{T}}D^{\mathrm{T}} \\ (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & P_{11}A+A^{\mathrm{T}}P_{11} & P_{11}B_0+X_4 & X_8 & C^{\mathrm{T}} \\ (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & -W_{11} & -I & D_0^{\mathrm{T}}+D_c^{\mathrm{T}}D^{\mathrm{T}} \\ (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & -S_{11} & S_{11}D_0^{\mathrm{T}}+X_2D^{\mathrm{T}} \\ (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & -I \end{bmatrix} \prec 0 \tag{21}$$

(where the notation $(\bullet)^{\mathrm{T}}$ denotes block entries in symmetric LMIs.

$$\begin{bmatrix} -S_{11} & -I & CR_{11}+DX_1 & C & D_0+DD_c & D_0S_{11}+DX_2 & B_{22} & 0 \\ (\bullet)^{\mathrm{T}} & -W_{11} & X_5 & W_{11}C & W_{11}D_0+X_3 & X_7 & W_{11}B_{22} & 0 \\ (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & AR_{11}+BX_1+R_{11}A^{\mathrm{T}}+X_1^{\mathrm{T}}B^{\mathrm{T}} & X_6^{\mathrm{T}}+A & B_0+BD_c & B_0S_{11}+BX_2 & B_{21} & 0 \\ (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & P_{11}A+A^{\mathrm{T}}P_{11} & P_{11}B_0+X_4 & X_8 & P_{11}B_{21} & 0 \\ (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & -W_{11} & -I & 0 & I \\ (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & -S_{11} & 0 & S_{11} \\ (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & -\gamma_\infty^2 I & 0 \\ (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & -I \end{bmatrix} \prec 0 \tag{22}$$

and

$$\begin{bmatrix} W_{11} & I \\ I & S_{11} \end{bmatrix} \succcurlyeq 0 \tag{23}$$

$$\begin{bmatrix} P_{11} & I \\ I & R_{11} \end{bmatrix} \succcurlyeq 0 \tag{24}$$

Suppose now that the LMIs (21), (22), (23) and (24) are feasible. Then the following is a systematic procedure for obtaining the corresponding controller matrices.

Step 1. Compute the singular value decomposition (SVD) of $I-W_{11}S_{11}$ and $I-P_{11}R_{11}$ to obtain the matrices U_1, U_2, V_1 and V_2 such that

$$I-W_{11}S_{11}=U_1\Sigma_1V_1^{\mathrm{T}}, \quad I-P_{11}R_{11}=U_2\Sigma_2V_2^{\mathrm{T}}$$

Step 2. Choose the matrices W_{12}, S_{12}, P_{12} and R_{12} as

$$W_{12}=U_1\Sigma_1^{\frac{1}{2}}, \quad S_{12}=V_1\Sigma_1^{\frac{1}{2}}, \quad P_{12}=U_2\Sigma_2^{\frac{1}{2}}, \quad R_{12}=V_2\Sigma_2^{\frac{1}{2}}$$

Step 3. Perform the following computations to obtain the matrices that define the controller state–space model (14)

$$C_{c1}=X_1R_{12}^{-\mathrm{T}}$$

$$C_{c2}=(X_2-D_cS_{11})S_{12}^{-\mathrm{T}}$$

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$$B_{2c} = W_{12}^{-1}(X_3 - W_{11}DD_c)$$

$$B_{1c} = P_{12}^{-1}(X_4 - P_{11}BD_c)$$

$$A_{c21} = W_{12}^{-1}(X_5 - W_{11}CR_{11} - W_{11}DC_{c1}R_{12}^{\mathrm{T}})R_{12}^{-\mathrm{T}}$$

$$A_{c11} = P_{12}^{-1}(X_6 - P_{11}BC_{c1}R_{12} - P_{11}AR_{11})R_{12}^{-1}$$

$$A_{c22} = W_{12}^{-1}(X_7 - W_{11}D_0S_{11} - W_{11}DD_cS_{11} - W_{12}B_{c2}S_{11} - W_{11}DC_{c2}S_{12}^{\mathrm{T}})S_{12}^{-\mathrm{T}}$$

$$A_{c12} = P_{12}^{-1}(X_8 - P_{11}B_0S_{11} - P_{11}BD_cS_{11} - P_{12}B_{c1}S_{11} - P_{11}BC_{c2}S_{12}^{\mathrm{T}})S_{12}^{-\mathrm{T}}$$

Proof
First, suppose that the matrices P_1, P_2, P_1^{-1}, P_2^{-1} are partitioned as

$$P_1 = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{bmatrix}, \quad P_2 = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^{\mathrm{T}} & W_{22} \end{bmatrix}$$

$$P_1^{-1} = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^{\mathrm{T}} & R_{22} \end{bmatrix}, \quad P_2^{-1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^{\mathrm{T}} & S_{22} \end{bmatrix}$$

Hence $P_1P_1^{-1} = I$ and $P_2P_2^{-1} = I$ imply that $W_{11}S_{11} + W_{12}S_{12}^{\mathrm{T}} = I$ and $P_{11}R_{11} + P_{12}R_{12}^{\mathrm{T}} = I$. Also it follows from Lemma 1 that there exist invertible matrices $P_1 \succ 0$ and $P_2 \succ 0$ if, and only if, the LMIs (23) and (24) are feasible. Next, pre- and post-multiply both sides of (18) by $\mathrm{diag}(S^{-1}, P^{-1}, I)$ and (19) by $\mathrm{diag}(S^{-1}, P^{-1}, I, I)$ to obtain

$$\begin{bmatrix} -W^{-1} & \overline{A}_2P^{-1} & 0 \\ P^{-1}\overline{A}_2^{\mathrm{T}} & P^{-1}\overline{A}_1^{\mathrm{T}} + \overline{A}_1P^{-1} - P^{-1}RP^{-1} & P^{-1}\overline{A}_3^{\mathrm{T}} \\ 0 & \overline{A}_3P^{-1} & -I \end{bmatrix} \prec 0 \qquad (25)$$

and

$$\begin{bmatrix} -W^{-1} & \overline{A}_2P^{-1} & \overline{B}_{22} & 0 \\ P^{-1}\overline{A}_2^{\mathrm{T}} & P^{-1}\overline{A}_1^{\mathrm{T}} + \overline{A}_1P^{-1} - P^{-1}RP^{-1} & \overline{B}_{21} & P^{-1}\overline{C}^{\mathrm{T}} \\ \overline{B}_{22}^{\mathrm{T}} & \overline{B}_{21}^{\mathrm{T}} & -\gamma_\infty^2 I & 0 \\ 0 & P^{-1}\overline{C} & 0 & -I \end{bmatrix} \prec 0 \qquad (26)$$

respectively.

Introduce

$$\Omega_1 = \begin{bmatrix} I & P_{11} & 0 & 0 \\ 0 & P_{12}^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & I & S_{11} \\ 0 & 0 & 0 & S_{12}^{\mathrm{T}} \end{bmatrix}, \quad \Omega_2 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & W_{11} \\ 0 & 0 & 0 & W_{12}^{\mathrm{T}} \end{bmatrix}$$

and pre- and post-multiply (25) by $\mathrm{diag}(\Omega_2, \Omega_1, I)$ and its transpose, respectively, to obtain

$$\begin{bmatrix} -\Omega_2^{\mathrm{T}}W^{-1}\Omega_2 & \Omega_2^{\mathrm{T}}\overline{A}_2P^{-1}\Omega_1 & 0 \\ (\bullet)^{\mathrm{T}} & \Omega_1^{\mathrm{T}}P^{-1}\overline{A}_1^{\mathrm{T}}\Omega_1 + \Omega_1^{\mathrm{T}}\overline{A}_1P^{-1}\Omega_1 - \Omega_1^{\mathrm{T}}P^{-1}RP^{-1}\Omega_1 & \Omega_1^{\mathrm{T}}P^{-1}\overline{A}_3^{\mathrm{T}} \\ (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & -I \end{bmatrix} \prec 0 \qquad (27)$$

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Also pre- and post-multiplying (26) by $\text{diag}(\Omega_2, \Omega_1, I, I)$ and its transpose, respectively, gives

$$\begin{bmatrix} -\Omega_2^{\mathrm{T}} W^{-1}\Omega_2 & \Omega_2^{\mathrm{T}}\overline{A}_2 P^{-1}\Omega_1 & \Omega_2^{\mathrm{T}}\overline{B}_{22} & 0 \\ (\bullet)^{\mathrm{T}} & \Omega_1^{\mathrm{T}} P^{-1}\overline{A}_1^{\mathrm{T}}\Omega_1 + \Omega_1^{\mathrm{T}}\overline{A}_1 P^{-1}\Omega_1 - \Omega_1^{\mathrm{T}} P^{-1} R P^{-1}\Omega_1 & \Omega_1^{\mathrm{T}}\overline{B}_{21} & \Omega_1^{\mathrm{T}} P^{-1}\overline{C}^{\mathrm{T}} \\ (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & -\gamma_\infty^2 I & 0 \\ (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & (\bullet)^{\mathrm{T}} & -I \end{bmatrix} \prec 0 \tag{28}$$

Finally, introduce the following change of variables:

$$\begin{aligned} X_1 &= C_{c1} R_{12}^{\mathrm{T}} \\ X_2 &= D_c S_{11} + C_{c2} S_{12}^{\mathrm{T}} \\ X_3 &= W_{11} D D_c + W_{12} B_{c2} \\ X_4 &= P_{11} B D_c + P_{12} B_{c1} \\ X_5 &= W_{11} C R_{11} + W_{11} D C_{c1} R_{12}^{\mathrm{T}} + W_{12} A_{c21} R_{12}^{\mathrm{T}} \\ X_6 &= P_{11} A R_{11} + P_{11} B C_{c1} R_{12}^{\mathrm{T}} + P_{12} A_{c11} R_{12}^{\mathrm{T}} \\ X_7 &= W_{11} D_0 S_{11} + W_{11} D D_c S_{11} + W_{12} B_{c2} S_{11} + W_{11} D C_{c2} S_{12}^{\mathrm{T}} + W_{12} A_{c22} S_{12}^{\mathrm{T}} \\ X_8 &= P_{11} B_0 S_{11} + P_{11} B D_c S_{11} + P_{12} B_{c1} S_{11} + P_{11} B C_{c2} S_{12}^{\mathrm{T}} + P_{12} A_{c12} S_{12}^{\mathrm{T}} \end{aligned} \tag{29}$$

to rewrite (27) and (28) as the LMIs (18) and (19), respectively. □

Remark 2
The controller that minimizes disturbance attenuation level γ_2 can be obtained by solving the following linear minimization problem:

$$\min_{W_{11}\succ 0, P_{11}\succ 0, S_{11}\succ 0, R_{11}\succ 0, D_c, X_1 \div X_8, \mu>0} \mu$$

$$\text{subject to (20)–(24)} \quad \text{with } \mu = \gamma_2^2$$

Remark 3
It is straightforward to see that the same controller can be applied to solve the robust $\mathscr{H}_2$ problem.

Remark 4
Theorem 4 can be viewed as the differential linear repetitive process equivalent of that in [13] for 1D differential linear systems.

5. A NUMERICAL EXAMPLE

In this section, an example is given to illustrate the application of the new results developed in this paper. This model arises from a metal rolling example [1] and the particular case considered here is when

$$A = \begin{bmatrix} 0 & 1 \\ -0.3333 & -0.0001 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 0.2778 \end{bmatrix}, \quad B = 10^{-3}\begin{bmatrix} 0 \\ -0.1667 \end{bmatrix}$$

$$C = [1 \ \ 0], \quad D_0 = 0.1667, \quad D = 0$$

and the primary goal is to design a controller such that the controlled process of Figure 1 is stable along the pass. Note that we consider a general method of formulating control problems where

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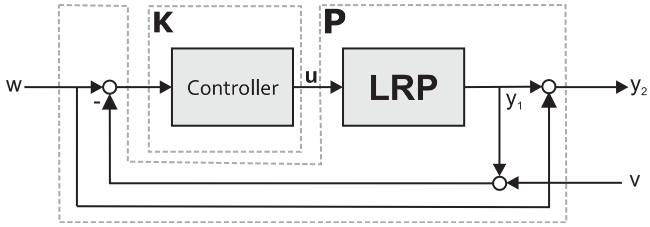

Figure 1. Control system structure.

P is the generalized plant and K is the generalized controller. In this system, w represents the tracking-reference signal and v the sensor noise. The controller also has to ensure small tracking error, that is, the difference ($y_2 = w - y_1$) between desired and actual signals must be small.

In terms of Figure 1

$$B_{11} = B, \quad B_{12} = 0, \quad B_{21} = B, \quad B_{22} = I$$

and applying Theorem 4 gives the stabilizing controller matrices

$$A_c = \begin{bmatrix} -1.1179 & -407.9422 & 0.0089 \\ 137.0345 & -112.9417 & 0.0025 \\ 0.0002 & -0.0006 & -0.0000 \end{bmatrix}, \quad B_c = \begin{bmatrix} -31.6376 \\ -8.7608 \\ 0.0106 \end{bmatrix},$$

$$C_c = 10^6 \cdot [0.0103 \;\; 3.7600 \;\; -0.0001], \quad D_c = 2.8994 \times 10^5$$

This controller ensures stability along the pass and correspondingly the $\mathscr{H}_2$ norm bound is never greater than 0.0336 for the prescribed level of $\mathscr{H}_\infty = 5.0$.

6. CONCLUSIONS

In this paper the problem of designing a dynamic output feedback controller for differential linear repetitive processes with mixed $\mathscr{H}_2/\mathscr{H}_\infty$ specifications has been addressed and a possible solution proposed. This allows controller design for stability along the pass plus disturbance attenuation. Moreover, the resulting design can be computed for a numerical example using LMIs and an illustrative example to this effect has been given. The availability of such a controller is essential for cases where static control based on the previous pass profile cannot achieve even stability along the pass. The controller itself is actuated by the previous pass profile which is a process output and hence the structure of this controller is physically well defined.

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