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Brief paper

# Identification and data-driven model reduction of state-space representations of lossless and dissipative systems from noise-free data ☆

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ABSTRACT

We illustrate procedures to identify a state-space representation of a lossless or dissipative system from a given noise-free trajectory; important special cases are passive systems and bounded-real systems. Computing a rank-revealing factorization of a Gramian-like matrix constructed from the data, a state sequence can be obtained; the state-space equations are then computed by solving a system of linear equations. This idea is also applied to perform model reduction by obtaining a balanced realization directly from data and truncating it to obtain a reduced-order model.



## 1. Introduction

We are given a noise-free discrete-time $\mathtt{w}$-dimensional trajectory $w$, and a real $\mathtt{w} \times \mathtt{w}$ symmetric matrix $\Sigma$. We assume that $w$ is produced by a linear finite-dimensional time-invariant system which is half-line lossless or dissipative with respect to the supply rate induced by $\Sigma$ (passive systems and bounded-real systems are special cases of such a situation, depending on the specific choice of $\Sigma$). We also assume certain identifiability conditions, described in detail later. The problem is to find a state-space description of the system from $w$.

To solve this problem, we could use deterministic subspace identification methods (see Moonen, De Moor, Vandenberghe, and Vandewalle (1989) and Van Overschee and De Moor (1996)) to compute a state sequence from $w$. One of the fundamental contributions of this paper is that a state sequence $x$ can be alternatively computed from any rank-revealing factorization of a Gramian-like matrix computed from the data $w$ and the supply rate $\Sigma$; the matrices corresponding to a state-space representation of the system can then be computed by solving for $(E, F, G)$ in the equations

$$[E\ F\ G]\begin{bmatrix}\sigma x\\ x\\ w\end{bmatrix} = 0, \tag{1}$$

where $\sigma$ is the forward shift defined by $(\sigma x)(k) := x(k+1)$; or, if a partition $w = (u, y)$ of the variable $w$ in inputs $u$ and outputs $y$ is known, by solving for $(A, B, C, D)$ in

$$\begin{bmatrix}\sigma x\\ y\end{bmatrix} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}x\\ u\end{bmatrix}. \tag{2}$$

We also show that special rank-revealing factorizations of the Gramian-like matrix can be used to obtain balanced state-space representations from data. In a balanced state representation the matrices corresponding to the maximal and the minimal storage function are diagonal and the inverse of each other; for passive systems and for bounded-real systems, this definition coincides with the classical one (see Desai and Pal (1984) and Opdenacker and Jonckheere (1988)). The possibility of obtaining balanced state-space representations directly from data makes our approach interesting for the *data-driven model order reduction* problem, that of obtaining a reduced-order model directly from measurements of

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a system, which we consider in the last part of this work. Structure-preserving model reduction is usually considered starting from a given state representation; recently, some authors (see Antoulas (2005), Antoulas (2004), Gugerçin and Antoulas (2004), Polyuga and van der Schaft (2010), Sorensen (2005), and Trentelman, Ha, and Rapisarda (2009)) have investigated the computation of reduced-order models from data. However, the data considered in these works are often of a special type, e.g. exponential trajectories associated with the spectral zeros of the system; a novel aspect of our work is that we use *general* measurements of the system.

The data available for identification are assumed to be noise free; we believe that, before trying to solve problems involving stochastics such as those arising in identification from noisy measurements, it makes sense to solve the simpler problem of identification from noise-free data. The results presented are consequently preliminary to an exhaustive investigation on the identification of lossless and dissipative systems in a realistic setting; in the conclusions section of this paper we discuss some of the open issues (finite data, noise, and consistency).

In this paper, we use the behavioral approach and quadratic difference forms. The former distinguishes between the external properties of a system and its representation; it is consequently appropriate for finding a state-space representation of a system through the intermediate *construction* of a state sequence from external measurements. Of course it is possible to postulate the existence of a state representation actually producing the data, but this assumption is unnecessary in the context of our problem. Moreover, choosing not to commit ourselves earlier on to a specific state representation allows us to obtain from the computed state sequence any required representation. The formalism of quadratic difference forms offers the possibility of a representation-free approach to the study of lossless and dissipative systems.

The reader unfamiliar with the behavioral approach is referred to Willems (1991) and Kaneko and Fujii (2000) for a thorough exposition; in Section 2, we only briefly review the necessary material. In Section 3.1, we consider lossless identification, and the version with dissipativity in Section 3.2. Our data-driven model reduction procedure is illustrated in Section 4. The paper ends with some concluding remarks in Section 5.

*Notation*. We denote the ring of integers with $\mathbb{Z}$, and the set $\{z \in \mathbb{Z} \mid z \geq 0\}$ with $\mathbb{Z}_+$. The space of $\mathtt{n}$-dimensional real vectors is denoted by $\mathbb{R}^{\mathtt{n}}$, and the space of $\mathtt{m} \times \mathtt{n}$ real matrices by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$. $\mathbb{R}^{\mathtt{m}\times\bullet}$ denotes the space of real matrices with $\mathtt{m}$ rows and an unspecified finite number of columns, and $\mathbb{R}^{\bullet\times\bullet}$ the space of real matrices with a finite but unspecified number of rows and columns. The symbol $\mathbb{R}^{\mathtt{m}\times\infty}$ denotes the set of real matrices with $\mathtt{m}$ rows and an infinite number of columns. The linear space of all sequences from $\mathbb{Z}$ to $\mathbb{R}^{\mathtt{w}}$ is denoted with $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$. $\ell_2^{\mathtt{w}}(\mathbb{Z})$ denotes the linear subspace of all square-summable sequences on $\mathbb{Z}$. The ring of polynomials with real coefficients in the indeterminate $\xi$ is denoted by $\mathbb{R}[\xi]$; the ring of two-variable polynomials with real coefficients in the indeterminates $\zeta$ and $\eta$ is denoted by $\mathbb{R}[\zeta, \eta]$. $\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi]$ denotes the space of all $\mathtt{r} \times \mathtt{w}$ matrices with entries in $\mathbb{R}[\xi]$. We denote with $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta, \eta]$ the space of $\mathtt{n} \times \mathtt{m}$ polynomial matrices in the indeterminates $\zeta$ and $\eta$.

## 2. Background material

In this paper, we consider linear, shift-invariant and 'complete' (see Definition II.4, p. 262 of Willems (1991)) subspaces $\mathfrak{B}$ of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$. We call them *behaviors* and denote the set consisting of all behaviors with $\mathtt{w}$ variables with $\mathfrak{L}^{\mathtt{w}}$. $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ admits several types of representation; particularly important in this paper are *hybrid* ones, in which besides the *external variable* $w$ a *latent variable* $\ell$ is also present:

$$R(\sigma)w = M(\sigma)\ell. \tag{3}$$

This has associated a *full behavior* $\mathfrak{B}_f := \{(w, \ell) \mid (w, \ell) \text{ satisfies } (3)\}$ and an *external behavior* $\mathfrak{B} := \{w \mid \exists \ell \text{ s.t. } (w, \ell) \text{satisfies } (3)\}$.

An important special type of hybrid representation is when the latent variable $\ell$ has the state property (see Definition VII.1, p. 268 of Willems (1991)); then the latent variable is denoted by $x$, and (3) is called a *state representation* of $\mathfrak{B}$. In this case it can be shown that (3) allows an alternative hybrid representation (1) of first order in $x$ and zeroth order in $w$. If, in addition, an input–output (i/o) partition $w = (u, y)$ of the external variable $w$ is given, then $\mathfrak{B}$ allows an input–state–output representation (i/s/o) (2) with state variable $x$. A state representation of $\mathfrak{B}$ is called *minimal* if the number of components of the state variable $x$ is minimal over all state representations of $\mathfrak{B}$. This number is called the *McMillan degree* of $\mathfrak{B}$, denoted by $\mathtt{n}(\mathfrak{B})$. If for a given $\mathfrak{B}$ an i/o partition $w = (u, y)$ is given, then we define $\mathtt{m}(\mathfrak{B}) := \mathtt{m}$, where $\mathtt{m}$ denotes the number of components of the input variable $u$, and $\mathtt{m}(\mathfrak{B})$ is called the *input cardinality* of $\mathfrak{B}$. $\mathtt{m}(\mathfrak{B})$ is an invariant for $\mathfrak{B}$, since every i/o partition of the external variable $w$ yields the same number of input variables.

We do not define explicitly what a controllable behavior is; for more details, see section V of Willems (1991). For our purposes, it is important to mention that a controllable behavior always contains nonzero $\ell_2^{\mathtt{w}}(\mathbb{Z})$-trajectories, in particular finite support ones. In the following, we denote the subset of $\mathfrak{L}^{\mathtt{w}}$ consisting of all controllable behaviors with $\mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$.

We now review the basic concepts regarding *bilinear* and *quadratic difference forms*. Let $\Phi \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta, \eta]$; then $\Phi(\zeta, \eta) = \sum_{h,k=0}^{N} \Phi_{h,k}\zeta^h\eta^k$, where $\Phi_{h,k} \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}$ and $N$ is a nonnegative integer. $\Phi(\zeta, \eta)$ induces the *bilinear difference form* (*BdF*)

$$L_\Phi : (\mathbb{R}^{\mathtt{w}_1})^{\mathbb{Z}} \times (\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}} \longrightarrow \mathbb{R}^{\mathbb{Z}}$$

$$L_\Phi(w_1, w_2)(t) := \sum_{h,k=0}^{N} w_1(t+h)^\top \Phi_{h,k} w_2(t+k).$$

If $\mathtt{w}_1 = \mathtt{w}_2$, then $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ also induces a *quadratic difference form* (*QdF*)

$$Q_\Phi : (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}} \longrightarrow \mathbb{R}^{\mathbb{Z}}$$

$$Q_\Phi(w)(t) := \sum_{h,k=0}^{N} w(t+h)^\top \Phi_{h,k} w(t+k).$$

When considering QdFs, without loss of generality, we assume the two-variable polynomial matrix $\Phi(\zeta, \eta)$ to be *symmetric*, i.e. $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top$.

The *rate of change* of a QdF $Q_\Phi$ is the QdF $\nabla Q_\Phi$ defined by $\nabla Q_\Phi(w)(k) := Q_\Phi(w)(k+1) - Q_\Phi(w)(k)$; note that the two-variable polynomial matrix associated with this QdF is given by $\nabla\Phi(\zeta, \eta) = (\zeta\eta - 1)\Phi(\zeta, \eta)$.

A controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is *dissipative* with respect to the *supply rate* $Q_\Phi$ if there exists a QdF $Q_\Psi$, called a *storage function*, such that

$$\nabla Q_\Psi(w) \leq Q_\Phi(w) \quad \text{for all } w \in \mathfrak{B}. \tag{4}$$

Inequality (4) is equivalent (see Proposition 3.3 of Kaneko and Fujii (2000)) to the existence of a *dissipation function*, i.e. a QdF $Q_\Delta \geq 0$ such that

$$\sum_{k=-\infty}^{+\infty} Q_\Delta(w)(k) = \sum_{k=-\infty}^{+\infty} Q_\Phi(w)(k) \quad \text{for all } w \in \mathfrak{B} \cap \ell_2^{\mathtt{w}}(\mathbb{Z}). \tag{5}$$

Moreover, there is a one-to-one correspondence between storage and dissipation functions, in the sense that for every dissipation function $Q_\Delta$ there exists a unique storage function $Q_\Psi$, and for every storage function $Q_\Psi$ there exists a unique dissipation function $Q_\Delta$, such that, for all $w \in \mathfrak{B}$,

$$\nabla Q_\Psi(w) + Q_\Delta(w) = Q_\Phi(w). \tag{6}$$

This translates in two-variable polynomial terms as

$$(\zeta\eta-1)\Psi(\zeta,\eta)+\Delta(\zeta,\eta)=\Phi(\zeta,\eta). \tag{7}$$

If (4) is an equality, or equivalently if $\Delta(\zeta,\eta)=0$ in (7), then $\mathfrak{B}$ is called *lossless with respect to* $Q_\Phi$. If $\sum_{k=-\infty}^{0} Q_\Phi(w)(k)\geq 0$ for all $w\in\mathfrak{B}_{|(-\infty,0]}\cap\ell_2^{\mathtt{w}}(\mathbb{Z}_-)$, then $\mathfrak{B}$ is called *half-line dissipative*; and if $\sum_{k=-\infty}^{0} Q_\Phi(w)(k)=0$ for all $w\in\mathfrak{B}_{|(-\infty,0]}\cap\ell_2^{\mathtt{w}}(\mathbb{Z}_-)$, then $\mathfrak{B}$ is called *half-line lossless*.

In this paper, we will restrict ourselves to supply rates $Q_\Phi$ with $\Phi$ constant, i.e. $\Phi(\zeta,\eta)=\Sigma$, for some real symmetric $\mathtt{w}\times\mathtt{w}$ matrix $\Sigma$. Obviously, in that case we have $Q_\Phi(w)=w^\top\Sigma w$. Two prominent special cases of this situation are passive systems and bounded-real systems. In both cases, the external variable is partitioned as $w=(u,y)$, with $u$ input and $y$ output. In the passive case, we have $\mathtt{m}=\mathtt{p}$, i.e. the number of inputs and outputs are equal, and

$$\Sigma=\begin{bmatrix}0 & I_{\mathtt{m}}\\ I_{\mathtt{m}} & 0\end{bmatrix}, \tag{8}$$

and for *bounded-real systems*, we have

$$\Sigma=\begin{bmatrix}I_{\mathtt{m}} & 0\\ 0 & -I_{\mathtt{p}}\end{bmatrix}. \tag{9}$$

Generalizing Proposition 2 in Trentelman and Willems (2002) to the discrete-time case, it can be shown that a behavior $\mathfrak{B}$ with i/o partition $w=(u,y)$ and $\mathtt{m}=\mathtt{p}$ is passive if and only if $\mathfrak{B}$ is $\Sigma$-half-line dissipative, with $\Sigma$ given by (8). The same holds for bounded-real systems and $\Sigma$ given by (9); this can be shown generalizing Proposition 1 in Trentelman and Willems (2002) to the discrete-time case.

In this paper, the fact that under suitable conditions *storage functions for discrete-time systems are quadratic functions of the state* plays an essential role. We say that a storage function $Q_\Psi$ is a quadratic function of the state if, given a state representation for $\mathfrak{B}$ with state variable $x$, there exists $K=K^\top\in\mathbb{R}^{\bullet\times\bullet}$ such that for every trajectory $(x,w)\in\mathfrak{B}_f$ it holds that $Q_\Psi(w)=x^\top Kx$. Every storage function is a quadratic function of the state for continuous-time systems (see Theorem 5.5 of Willems and Trentelman (1998)); however, this is not true in discrete-time cases; see Kaneko and Fujii (2003): additional assumptions are needed. It can be shown that this is the case when the system is lossless (see Theorem 5.3 of Kaneko and Fujii (2003)) or when the storage function is nonnegative (see Theorem 5.1 of Kaneko and Fujii (2003)). Another sufficient condition is given in Theorem 5.2 of Kaneko and Fujii (2003), and a necessary and sufficient condition is given in Proposition 2 of Kojima, Takaba, Kaneko, and Rapisarda (2006).

## 3. Noise-free identification

In the rest of the paper, we deal with trajectories $w$ defined on $\mathbb{Z}_+$. Given $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}}$, we denote with $\mathfrak{B}_+:=\{w_{|\mathbb{Z}_+}\mid w\in\mathfrak{B}\}$, and with $\ell_2^{\mathtt{w}}(\mathbb{Z}_+)$ the set of $\mathtt{w}$-dimensional square-summable trajectories on $\mathbb{Z}_+$.

We first introduce the notion of *persistency of excitation*. A sequence $f:\mathbb{Z}_+\to\mathbb{R}^{\mathtt{f}}$ is said to be *persistently exciting of order* $L$ (abbreviated as *p.e. of order* $L$) if

$$\operatorname{rank}\begin{bmatrix} f(0) & f(1) & f(2) & \cdots\\ f(1) & f(2) & f(3) & \cdots\\ \vdots & \vdots & \cdots & \\ f(L-1) & f(L) & f(L+1) & \cdots\end{bmatrix}=L\mathtt{f}.$$

In Corollary 2 of Willems, Rapisarda, Markovsky, and De Moor (2005), it has been shown that, for every trajectory $(u,x)$ of a state-space system $\sigma x=Ax+Bu$ with $\mathtt{n}$-dimensional state vector $x$ and $\mathtt{m}$-dimensional input $u$, it holds that

$$u \text{ p.e. of order } \mathtt{n}\Longrightarrow \operatorname{rank}\begin{bmatrix}u(0) & \cdots & u(T)\\ x(0) & \cdots & x(T)\end{bmatrix}=\mathtt{n}+\mathtt{m} \tag{10}$$

for $T$ is "sufficiently large", i.e. $T\geq\mathtt{nm}$. It follows that, for such $T$,

$[u \text{ p.e. of order } \mathtt{n}]\Longrightarrow[\operatorname{rank}[x(0)\ \cdots\ x(T)]=\mathtt{n}]$.

The result expressed in (10), referred to in Katayama (2005) as the 'fundamental lemma', is the only identifiability condition required by our algorithms.

### 3.1. The lossless case

We first define the $S$-matrix associated with a trajectory and a BdF.

**Definition 1.** Let $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, $w\in\mathfrak{B}_+\cap\ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, and let $\Phi\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be symmetric. The *S-matrix* is the infinite matrix $S(w)$ whose $(i,j)$-th entry is

$$[S(w)]_{i,j=0,\ldots}:=\sum_{k=0}^{\infty}L_\Phi(\sigma^i w,\sigma^j w)(k). \tag{11}$$

Since $w\in\ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, $S(w)$ is a well-defined real matrix. Note that if $\Phi(\zeta,\eta)=\Sigma$, with $\Sigma\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ a symmetric matrix, then $S(w)_{i,j=0,\ldots}=\sum_{k=0}^{\infty}w(i+k)^\top\Sigma w(k+j)$.

The most important result of this section is the following.

**Proposition 2.** *Let $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, and let $\mathfrak{B}_f$ be a minimal state representation of $\mathfrak{B}$ with state variable $x$. Let $\Sigma=\Sigma^\top\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and assume that $\mathfrak{B}$ is $\Sigma$-half-line lossless. Then there exists $K=K^\top\in\mathbb{R}^{\bullet\times\bullet}$ such that, for every $w\in\mathfrak{B}_+\cap\ell_2^{\mathtt{w}}(\mathbb{Z}_+)$ with associated state trajectory $x$, i.e. $(x,w)\in\mathfrak{B}_f$, the following equality holds:*

$$S(w)=\begin{bmatrix}x(0)^\top\\ x(1)^\top\\ \vdots\end{bmatrix}K[x(0)\ x(1)\ \cdots]. \tag{12}$$

**Proof.** Since $w\in\ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, $\lim_{k\to\infty}w(k)=0$. From the losslessness of $\mathfrak{B}$ and Theorem 5.3 of Kaneko and Fujii (2003), it follows that every storage function is a quadratic function of the state. Consequently, there exists $K'=K'^\top\in\mathbb{R}^{\bullet\times\bullet}$ such that, for every $(x_i,w_i)$, $i=1,2$, in the full behavior $\mathfrak{B}_f$, $\sum_{k=0}^{+\infty}w_1(k)^\top\Sigma w_2(k)$ equals

$$\begin{aligned}&-x_1(0)^\top K'x_2(0)+x_1(1)^\top K'x_2(1)\\ &\quad -x_1(1)^\top K'x_2(1)+x_1(2)^\top K'x_2(2)-\cdots\\ &=-x_1(0)^\top K'x_2(0)+(\lim_{k\to\infty}x_1(k))^\top K'(\lim_{k\to\infty}x_2(k))\\ &=-x_1(0)^\top K'x_2(0),\end{aligned}$$

where we have used the fact that, by minimality of the state representation, since $\lim_{k\to\infty}w(k)=0$, also $\lim_{k\to\infty}x(k)=0$. This implies that $\sum_{k=0}^{+\infty}w(k+i)^\top\Sigma w(k+j)=-x(i)^\top K'x(j)$ for $i,j=0,\ldots$, and proves Eq. (12), with $K:=-K'$. □

To use Proposition 2 for identifying a state representation of $\mathfrak{B}$, we need the $S$-matrix appearing on the left-hand side of Eq. (12) to contain all the information needed to construct a (minimal) state sequence. This amounts to requiring that $\operatorname{rank}(S(w))=\mathtt{n}(\mathfrak{B})$, the McMillan degree of $\mathfrak{B}$. There are many sufficient conditions ensuring this; a realistic one in an identification context, since it requires a minimal amount of a priori knowledge about the system, is to assume that the system is half-line lossless, and that the number $\mathtt{m}(\mathfrak{B})$ of input variables equals the number $\sigma_+(\Sigma)$ of positive eigenvalues of the supply rate matrix $\Sigma$. Indeed, in this case, using Theorem 5.3 of Kaneko and Fujii (2003) and the same argument of the proof of Theorem 6.4 of Willems and Trentelman (1998) for the continuous-time case, it can be shown that all storage functions are positive, in the sense that, if a storage function is given by $x^\top Kx$, with $x$ a minimal state, then $K > 0$. Note that the condition $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$ holds both for passive systems and for bounded-real systems.

The following result holds.

**Proposition 3.** *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$, and assume that $w = (u, y)$ is an i/o partition of the external variable $w$. Let $\mathfrak{B}_f$ be a minimal state representation of $\mathfrak{B}$ with state variable $x$. Assume that $\mathfrak{B}$ is $\Sigma$-half-line lossless, where $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Assume that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Let $w = (u, y) \in \mathfrak{B}_+ \cap l_2^{\mathtt{w}}(\mathbb{Z}_+)$ be a given sequence, and assume that $u$ is p.e. of order $\mathtt{n}(\mathfrak{B})$. Then the S-matrix has rank $\mathtt{n}(\mathfrak{B})$.*

**Proof.** Recall that, in the lossless case, every storage function is a quadratic function of the state. Now, apply the same argument used to prove the equivalence of (1) and (6) in Theorem 6.4 of Willems and Trentelman (1998) to conclude that, under the assumptions of $\Sigma$-losslessness of $\mathfrak{B}$ and $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, the matrix $K' \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$ used in the proof of Proposition 2 is positive definite. Then, use the persistency of excitation of $u$ to conclude that rank $[x(0)\ x(1)\ \cdots] = \mathtt{n}(\mathfrak{B})$. Finally, use (12). □

It follows from Propositions 2 and 3 that, to compute a minimal state sequence corresponding to the data $w$, one can proceed as follows. Define a *rank-revealing factorization* of $S(w)$ to be any factorization $S(w) = U\Lambda U^\top$ with $U \in \mathbb{R}^{\infty\times\mathtt{n}(\mathfrak{B})}$ and $\Lambda \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$ both having full rank, equal to $\mathtt{n}(\mathfrak{B}) = \text{rank } S(w)$. It follows from (12) that a state sequence $x(0), x(1), \ldots$ can be obtained from such a rank-revealing factorization as

$$[x(0)\ x(1)\ \ldots] := U^\top.$$

Once a state sequence is known, one can readily compute the matrices $E$, $F$, and $G$ in a state representation (1) or, if an i/o partition of $w$ is known, the matrices $A$,$B$, $C$, and $D$ corresponding to a minimal i/s/o representation (2) of $\mathfrak{B}$ by solving systems of linear equations. Note that solutions $(A, B, C, D)$ and $(E, F, G)$ to such systems of equations always exist, since $x$ is a state variable; see, for example, Proposition VII.3 of Willems (1991).

We now illustrate our identification procedure with an example.

**Example 4.** Consider the controllable system $\mathfrak{B} \in \mathfrak{L}^2$ with i/o-partitioned external variable $w = (u, y)$, described by the transfer function $G(z) = \frac{\frac{1}{8}z^2-\frac{3}{4}z+1}{z^2-\frac{3}{4}z+\frac{1}{8}}$. $\mathfrak{B}$ is bounded-real lossless. We generate a system trajectory using the Matlab© `lsim` command; we choose a zero initial state and an input sequence $u$ whose first 100 samples are a pseudo-random sequence, and whose last 200 are zero. The corresponding $y$ is for practical purposes zero after a finite number of instants; in this way, we can treat the finite data at our disposal as if they were the truncation of a (half-line) infinite trajectory $w$ vanishing at infinity.

It is easy to see that, since $w$ has finite support, the $S$-matrix $S(w)$ corresponding to this data is zero apart from its $100 \times 100$ principal submatrix:

$$S(w) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & -0.5462 & -0.6997 & -0.1983 & \cdots \\ 0 & -0.6997 & -1.2000 & -0.8451 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that the first row and column of $S(w)$ are zero since from Proposition 2 and Eq. (12) it follows that $S_{1,j} = -x(0)^\top K'x(j) = -x(j)^\top K'x(0)$, with $K'$ the matrix associated with the storage function, and since $x(0) = 0$.

Define $S$ to be the $100 \times 100$ submatrix of $S(w)$; factorize $S = U\Lambda U^\top$ with $\Lambda = \text{diag}(-2.2947, -0.6747)$ and

$$U^\top = \begin{bmatrix} 0 & -0.3513 & -0.7024 & -0.6190 & \cdots \\ 0 & 0.6243 & 0.3170 & -0.7140 & \cdots \end{bmatrix};$$

note that $S(w)$ has rank 2, as expected. From this, we obtain the state trajectory

$$[x(0)\ x(1)\ x(2)\ \cdots] := U^\top.$$

Solving Eq. (2) in the least-squares sense yields

$$A = \begin{bmatrix} 0.6901 & -0.0627 \\ 1.3330 & 0.0599 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.4716 \\ 0.8381 \end{bmatrix}$$
$$C = [-0.0558\ -0.8144], \qquad D = [0.125].$$

It can be verified that $C(zI - A)^{-1}B + D = \frac{0.125z^2-0.75z+1}{z^2-0.75z+0.125}$.

**Remark 5.** The data of Example 4 is of finite support; consequently, for computational purposes, we were able to use a finite submatrix of $S(w)$ without loss of information about the system. However, in real applications only a finite number of measurements of $w$ are available. In such cases, only an *approximation* of the entries of the $S$-matrix can be computed; consequently, a rank-revealing factorization of this approximate $S$-matrix only corresponds to an *approximation* of an actual state sequence of the data-producing system. An easy way out of this problem is to assume that a “sufficiently large” time window of the data is given, so that these approximation issues are negligible; given that $w \in \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, for “large enough” $T$, $w(T)$ is approximately zero. Note that this is exactly the approach taken in Moonen et al. (1989). This expedient solution, however, cannot be considered satisfactory, and a thorough investigation on the generalization of the above procedure to the finite-time case is required; we will pursue this elsewhere.

Different rank-revealing factorizations of the Gramian-like matrix produce different state sequences, in their turn corresponding to different state representations; this can be exploited to obtain balanced state-space representations, defined as follows. Assume $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ given, and $\sigma_+(\Sigma) = \mathtt{m}(\mathfrak{B})$; then, if $x$ is a minimal state variable, the matrix $K$ associated to a storage function $x^\top Kx$ is positive definite. A minimal state-space representation of $\mathfrak{B}$ is *balanced* if the matrices $K_-$ and $K_+$ corresponding to the minimal and the maximal storage functions $x^\top K_- x$ and $x^\top K_+ x$ are diagonal and the inverse of each other. In the case of $\Sigma$ given by (8), respectively (9), this definition of balanced state representation coincides with the classical one. Note that, in the lossless case, the maximal and minimal storage functions coincide, and a realization is balanced if the matrix $K$ corresponding to the unique storage function is the identity.

If an i/o partition $w = (u, y)$ of $w$ is known, then, by choosing appropriately the rank-revealing factorization (12) and solving (2), a balanced i/s/o representation can be obtained.

**Proposition 6.** *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with external variable $w$ i/o-partitioned as $w = (u, y)$. Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, and partition accordingly*

$$\Sigma = \begin{bmatrix} \Sigma_{uu} & \Sigma_{uy} \\ \Sigma_{uy}^\top & \Sigma_{yy} \end{bmatrix} \in \mathbb{R}^{(\mathtt{m}+\mathtt{p})\times(\mathtt{m}+\mathtt{p})}. \tag{13}$$

*Assume that $\mathfrak{B}$ is $\Sigma$-half-line lossless, and assume that $\sigma_+(\Sigma) = \mathtt{m}(\mathfrak{B})$. Let $\mathfrak{B}_f$ be a minimal i/s/o representation of $\mathfrak{B}$ with state* *variable $x$, associated with the matrices $A$, $B$, $C$, and $D$. Let $w = (u, y) \in \mathfrak{B}_+ \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$ be a given sequence. Then the matrix $K = K^\top \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$ satisfying Eq. (12) is equal to the unique real symmetric solution of the equations*

$$\begin{aligned}
&B(-K)B^\top - \Sigma_{uu} - D^\top \Sigma_{uy}^\top - \Sigma_{uy} D - D^\top \Sigma_{yy} D = 0\\
&A^\top(-K)B - \Sigma_{uy} C - D^\top \Sigma_{yy} C = 0\\
&A^\top(-K)A - (-K) - C^\top \Sigma_{yy} C = 0.
\end{aligned} \tag{14}$$

**Proof.** Use the $\Sigma$-losslessness of $\mathfrak{B}$ in order to conclude that the system with external variable $(u, x)$ described by $x(k+1) = Ax(k) + Bu(k)$ is $\Sigma'$-lossless, where

$$Q_{\Sigma'}(u, x) := [u^\top\ x^\top] \begin{bmatrix} \Sigma'_{uu} & \Sigma'_{ux} \\ \Sigma'^\top_{ux} & \Sigma'_{xx} \end{bmatrix} \begin{bmatrix} u \\ x \end{bmatrix}, \tag{15}$$

with $\Sigma'_{uu} := \Sigma_{uu} + D^\top \Sigma_{uy}^\top + \Sigma_{uy} D + D^\top \Sigma_{yy} D$, $\Sigma'_{ux} := \Sigma_{uy} C + D^\top \Sigma_{yy} C$, $\Sigma'_{xx} := C^\top \Sigma_{yy} C$.

If $x^\top K' x$ is the storage function, then (15) equals

$$\sigma x^\top K' \sigma x - x^\top K' x = (Ax + Bu)^\top K'(Ax + Bu) - x^\top K' x.$$

To conclude the proof, use the fact that $K = -K'$. □

Now, assume that $u$ is an input sequence which is p.e. of order $\mathtt{n}(\mathfrak{B})$, and that $\sigma_+(\Sigma) = \mathtt{m}(\mathfrak{B})$; then Proposition 6 implies that, if the matrix $K$ satisfying (12) is $-I_{\mathtt{n}(\mathfrak{B})}$, i.e. if the factorization of the $S$-matrix is of the form $S(w) = -UU^\top$, then the i/s/o representation obtained by solving (2) is balanced.

**Example 7.** We consider the system of Example 4. If we choose a factorization $S = -UI_2U^\top$, with

$$U^\top = \begin{bmatrix} 0 & -0.5322 & -1.0640 & -0.9377 & \cdots \\ 0 & 0.5128 & 0.2604 & -0.5865 & \cdots \end{bmatrix}$$

we obtain the (classically) balanced realization

$$A' = \begin{bmatrix} 0.6901 & -0.1157 \\ 0.7228 & 0.0599 \end{bmatrix}, \qquad B' = \begin{bmatrix} -0.7144 \\ 0.6884 \end{bmatrix}$$

$$C' = [-0.0369\ \ -0.9915], \qquad D' = [0.1250].$$

**Remark 8.** Approaches to the identification of balanced state models in the deterministic case similar to the one presented in this paper are studied in Moonen and Ramos (1993) and in Chapter 5 of Van Overschee and De Moor (1996); and in a combined deterministic–stochastic setting in Van Overschee and De Moor (1995). On the related topic of stochastic balancing of autoregressive systems, see Dahlén and Scherrer (2004). See also the discussion in Remark 14 of this paper.

We conclude with the statement of an algorithm for the identification of a state representation of a lossless system from noise-free data. If $U$ is a matrix, we denote with $U(1 : j, :)$ its submatrix consisting of the first $j$ rows.

**Algorithm 1.** Input: $w \in \mathfrak{B}_+ \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, with $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ $\Sigma$-half-line lossless, with $\sigma_+(\Sigma) = \mathtt{m}(\mathfrak{B})$.
Output: A minimal state representation of $\mathfrak{B}$.
Step 1: Compute the $S$-matrix (11).
Step 2: Compute $\mathtt{n} := \operatorname{rank} S(w)$.
Step 3: Factor $S(w) = U\Lambda U^\top$, $U \in \mathbb{R}^{\infty\times\mathtt{n}}$, $\Lambda \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$.
Comment: For balancing, do Step 3 with $\Lambda = -I_{\mathtt{n}}$.
Step 4: Let $x := U(1 : \infty, :)^\top$, $\sigma x := U(2 : \infty, :)^\top$.
Step 5: Solve (1) or (2) if an i/o partition is known.

### 3.2. The dissipative case

If a dissipation function is known, or if the data $w$ is a trajectory of zero dissipation, i.e. the dissipation function is identically zero along it, it makes sense to consider the extension of the approach illustrated in Section 3 to the case of dissipative systems. This is straightforward: if a system is dissipative with respect to a supply rate $Q_\Sigma := w^\top \Sigma w$, then it is lossless with respect to the supply rate $Q_\Sigma - Q_\Delta$, with $Q_\Delta$ a dissipation function. We now formalize this intuition.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ be $\Sigma$-half-line dissipative, and assume that a dissipation function induced by $\Delta \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ is known. Moreover, assume that the storage function $Q_\Psi$ associated with $Q_\Delta$ is a quadratic function of the state; see Section 2 for several sufficient conditions.

From (7), it follows that for every $w_1, w_2 \in \mathfrak{B}$ we have $L_\Sigma(w_1, w_2) = L_\Delta(w_1, w_2) + \nabla L_\Psi(w_1, w_2)$, where $L_\Sigma(w_1, w_2) = w_1^\top \Sigma w_2$, and where $L_\Delta$ and $L_\Psi$ are the *bilinear* difference forms induced by $\Psi(\zeta, \eta)$, and $\Delta(\zeta, \eta)$, respectively. Now, let $\mathfrak{B}_f$ be a minimal state representation of $\mathfrak{B}$ with state variable $x$; if $w_1, w_2 \in \mathfrak{B}_+ \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, with associated full trajectories $(w_i, x_i) \in \mathfrak{B}_f$, $i = 1, 2$, then $\sum_{k=0}^\infty w_1(k)^\top \Sigma w_2(k) = \sum_{k=0}^\infty L_\Delta(w_1, w_2)(k) - x_1(0)^\top K' x_2(0)$, where $K' = K'^\top \in \mathbb{R}^{\bullet\times\bullet}$ is the matrix corresponding to the storage function $Q_\Psi$ and the state variable $x$. Define the *generalized $S$-matrix* $S(w)$ as

$$\begin{aligned}
S(w)_{i,j=0,1,\ldots} := &\sum_{k=0}^{\infty} L_\Sigma(\sigma^i w, \sigma^j w)(k)\\
&- \sum_{k=0}^{\infty} L_\Delta(\sigma^i w, \sigma^j w)(k),
\end{aligned} \tag{16}$$

and note that

$$S(w) = \begin{bmatrix} x(0)^\top \\ x(1)^\top \\ \vdots \end{bmatrix} \underbrace{(-K')}_{=:K} [x(0)\ x(1)\ \cdots]. \tag{17}$$

This argument proves the following result, analogous to Proposition 2 of Section 3.1.

**Proposition 9.** *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$, and let $\mathfrak{B}_f$ be a minimal state representation of $\mathfrak{B}$ with state variable $x$. Assume that $\mathfrak{B}$ is $\Sigma$-half-line dissipative, and let $\Delta \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ induce a dissipation function for $\mathfrak{B}$. Assume that the storage function associated with $Q_\Delta$ is a quadratic function of the state. Then there exists $K = K^\top \in \mathbb{R}^{\bullet\times\bullet}$ such that, for all $w \in \mathfrak{B}_+ \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, with associated state trajectory $x$, i.e. $(x, w) \in \mathfrak{B}_f$, the generalized $S$-matrix (16) satisfies (17).*

If, in addition, an i/o partition $w = (u, y)$ of $w$ is known, then, under the assumption of persistency of excitation of the input sequence $u$, from the factorization (17) of the generalized $S$-matrix (16) it follows that, if $K$ is non-singular, then $S(w)$ has rank $\mathtt{n}(\mathfrak{B})$. These considerations lead us to the following result, proven analogously to Proposition 3.

**Proposition 10.** *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$, and assume that $w = (u, y)$ is an i/o partition. Let $\mathfrak{B}_f$ be a minimal state representation of $\mathfrak{B}$ with state variable $x$. Assume that $\mathfrak{B}$ is $\Sigma$-half-line dissipative, and let $\Delta \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ induce a dissipation function for $\mathfrak{B}$. Assume that the storage function associated with $Q_\Delta$ is a quadratic function of the state, and that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then the matrix $K = K^\top \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$ corresponding to the storage function is positive definite. Moreover, let $w = (u, y) \in \mathfrak{B}_+ \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$ be a given sequence, and assume that $u$ is p.e. of order $\mathtt{n}(\mathfrak{B})$. Then $\operatorname{rank} S(w) = \mathtt{n}(\mathfrak{B})$.*

**Remark 11.** From Propositions 9 and 10, it follows that Algorithm 1 can be modified to identify also dissipative systems; the only change is in Step 1, where the generalized $S$-matrix (16) is used in place of the $S$-matrix (11).

We now discuss the computation of balanced realizations from data, assuming that an i/o partition of $w$ is known.

**Proposition 12.** *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with external variable $w$ i/o-partitioned as $w = (u, y)$. Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, and partition $\Sigma$ as in (13). Assume that $\mathfrak{B}$ is $\Sigma$-half-line dissipative, and let $\Delta \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ induce a dissipation function for $\mathfrak{B}$. Let $\mathfrak{B}_f$ be a minimal i/s/o representation of $\mathfrak{B}$ with state variable $x$ associated with the matrices $(A, B, C, D)$. Assume that the storage function associated with $Q_\Delta$ is a quadratic function of the state, and moreover that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Let $w = (u, y) \in \mathfrak{B} \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$ be a given sequence, and let $S(w)$ be defined as in (16). Define $R := \Sigma_{uu} + D^\top\Sigma_{uy}^\top + \Sigma_{uy}D + D^\top\Sigma_{yy}D$, $S^\top := \Sigma_{uy}C + D^\top\Sigma_{yy}C$, $Q := C^\top\Sigma_{yy}C$. Let $K$ be such that (17) holds, and assume that $R - B^\top KB > 0$; then, the matrix $K' := -K$ satisfies the algebraic Riccati equation (ARE) $0 = A^\top K'A - K' + Q - (A^\top K'B + S)(B^\top K'B + R)^{-1}(B^\top K'A + S^\top)$.*

**Proof.** The proof follows from the well-known relationship between storage functions and solutions of the ARE. □

Now assume that the dissipation functions $\Delta_+$ and $\Delta_-$ corresponding to the maximal and the minimal storage functions $\Psi_+$ and $\Psi_-$ (see Section 4 of Kaneko and Fujii (2000) and Section 3 of Kojima et al. (2006) for details) are known, and that the corresponding storage functions are quadratic functions of the state. Assume also that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, so that, if $x^\top Kx$ is a storage function, then $K > 0$. Assume also that $u$ is a p.e. input sequence of order $\mathtt{n}(\mathfrak{B})$. We compute two generalized $S$-matrices:

$$S_-(w)_{i,j} := [L_\Sigma(\sigma^i w, \sigma^j w) - L_{\Delta_-}(\sigma^i w, \sigma^j w)]_{i,j}$$

$$S_+(w)_{i,j} := [L_\Sigma(\sigma^i w, \sigma^j w) - L_{\Delta_+}(\sigma^i w, \sigma^j w)]_{i,j}.$$

We then compute a rank-revealing factorization of $S_-(w)$ as $S_-(w) = -V^\top V$. Since the columns of $V$ form a minimal state sequence, there exists $K_+ = K_+^\top \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\mathtt{n}(\mathfrak{B})}$, $K_+ > 0$, such that $S_+(w) = V^\top(-K_+)V$. It is immediate to verify that $K_+ = -(VV^\top)^{-1}VS_+(w)V^\top(VV^\top)^{-1}$. Make a spectral decomposition $U\Lambda U^\top$ of $-(VV^\top)^{-1}VS_+(w)V^\top(VV^\top)^{-1}$, with $U$ an orthogonal matrix. Observe that $\Lambda$ is positive definite. Now, define $T := U\Lambda^{-\frac{1}{4}}$; then $S_-(w) = -V'^\top\Lambda^{-\frac{1}{2}}V'$ and $S_+(w) = -V'^\top\Lambda^{\frac{1}{2}}V'$, where $V' := T^{-1}V = \Lambda^{\frac{1}{4}}U^\top V$. Since $T$ is non-singular and the columns of $V$ form a (minimal) state sequence, the columns of $V'$ also form a (minimal) state sequence; moreover, the corresponding state representation is such that the matrices associated with the minimal and the maximal storage functions are respectively $\Lambda^{-\frac{1}{2}}$ and $\Lambda^{\frac{1}{2}}$, and consequently is a balanced i/s/o representation of $\mathfrak{B}$.

**Remark 13.** An algorithm analogous to Algorithm 1 for the computation of a balanced state representation can be derived from the results of this section. We will not enter into the details here.

**Remark 14.** The approach described in this paper is based on the same idea as subspace identification: first, a state sequence is computed from the data, and then the state matrices are computed by solving a system of linear equations. We now briefly compare the two approaches.

Most algorithms for subspace identification use orthogonal or oblique projections for the computation of a state sequence, that require the (pseudo-)inversion of matrices derived from the data. Our procedures to compute a state sequence instead require the computation of the $S$-matrices (amounting to the multiplication of structured matrices) and of a rank-revealing factorization thereof; consequently, they offer computational advantages over the (deterministic) subspace identification of Moonen et al. (1989).

If *stochastically balanced* realizations (see Desai and Pal (1984) and Opdenacker and Jonckheere (1988)) are to be identified, further computations are required. One possibility is to first identify the system matrices via (stochastic) subspace identification, compute the extremal solutions of the ARE, and balance them through standard linear algebra methods. This approach does not assume a priori knowledge of the dissipation functions, but requires the pseudo-inversion of large matrices to obtain a state sequence; the solution of a system of linear equations to obtain $(A, B, C, D)$; the solution of two AREs to obtain the extremal solutions; and, finally, balancing. If the dissipation functions are known, our approach can be used, requiring one rank-revealing factorization (of $S_-(w)$); the inversion of matrices of size $\mathtt{n}(\mathfrak{B})$ (i.e. of $VV^\top$); the balancing of $\mathtt{n}(\mathfrak{B})$-sized matrices (i.e. computation of the matrix $T$); and, finally, the computation of $(A, B, C, D)$. The a priori knowledge of $\Delta_+$ and $\Delta_-$ thus allows one to replace the pseudo-inversion of large matrices and the solution of two AREs by the inversion of a $\mathtt{n}(\mathfrak{B})$-sized matrix and the computation of one rank-revealing factorization.

It is fair to mention that there exist techniques for stochastic balancing in a subspace identification setting alternative to the direct one considered above; see, for example, Lindquist and Picci (1996) and Tanaka and Katayama (2006). These techniques are grounded in a stochastic framework, and consequently a direct comparison between them and our method is impossible. It is a matter for future research to investigate whether these approaches suggest a way of eliminating the need for a priori knowledge of $\Delta_-$ and $\Delta_+$ in order to compute a balanced state-space representation.

## 4. Data-driven model reduction

We describe a data-driven balanced model reduction procedure based on the factorization result of Proposition 2; we focus our attention on the lossless case. Our algorithm takes as inputs the measurements $w = (u, y) \in \mathfrak{B}_+ \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, where $u \in (\mathbb{R}^{\mathtt{m}})^{\mathbb{N}}$ is a p.e. input sequence of order $\mathtt{n}(\mathfrak{B})$, and an integer $k \leq \mathtt{n}(\mathfrak{B})$. The output is an i/s/o representation of a system of McMillan degree $\mathtt{k}$ obtained from the original by balanced truncation.

In the following, we assume that the storage function is positive, and consequently $S(w)$ is negative definite, for example because one of the conditions of Proposition 3 is satisfied. This implies that $\operatorname{rank}(S(w)) = \mathtt{n}(\mathfrak{B})$.

The following result will be useful.

**Lemma 15.** *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and assume that $w = (u, y)$ is an i/o partition of $w$. Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Assume that $\mathfrak{B}$ is $\Sigma$-half-line lossless and that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Let $w = (u, y) \in \mathfrak{B}_+ \cap l_2^{\mathtt{w}}(\mathbb{Z}_+)$, and assume that the input sequence $u$ is p.e. of order $\mathtt{n}(\mathfrak{B})$. Factorize $S(w) = -V^\top V$ for some $V \in \mathbb{R}^{\mathtt{n}(\mathfrak{B})\times\infty}$, and define $U := [u(0)\ u(1)\ \cdots]$; then $\begin{bmatrix} V \\ U \end{bmatrix}[V^\top\ U^\top]$ is invertible. Moreover, define $\Delta := UU^\top - UV^\top(VV^\top)^{-1}VU^\top$ and $R := \begin{bmatrix} I_{\mathtt{n}} & -(VV^\top)^{-1}VU^\top \\ 0_{\mathtt{m}\times n} & I_{\mathtt{m}} \end{bmatrix}$; then*

$$\left(\begin{bmatrix} V \\ U \end{bmatrix}[V^\top\ U^\top]\right)^{-1} = R\begin{bmatrix} (VV^\top)^{-1} & 0_{n\times\mathtt{m}} \\ 0_{\mathtt{m}\times n} & \Delta^{-1} \end{bmatrix}R^\top.$$

**Proof.** The first claim follows from the fundamental lemma, see (10); the second one is immediate. □

Now, define $\sigma V := V(:, 2 : \infty)$ and $Y := [y(0)\ y(1)\ \cdots]$; the following result can be verified in a straightforward way.

**Lemma 16.** *Under the same assumptions and notation of Lemma 15, the matrices A, B, C, and D solving* $\begin{bmatrix} \sigma V \\ Y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} V \\ U \end{bmatrix}$ *are unique, and* $\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma VV^\top & \sigma VU^\top \\ YV^\top & YU^\top \end{bmatrix} \begin{bmatrix} VV^\top & VU^\top \\ UV^\top & UU^\top \end{bmatrix}^{-1}}_{=:F}$.

The matrix $F$ defined in Lemma 16 is computed directly from the factorization of $S(w)$ and from the data. We are now ready to prove the main result of this section.

**Theorem 17.** *Under the same assumptions and notation of Lemmas 15 and 16, define*

$$A' := [I_{\mathtt{k}}\ 0_{\mathtt{k}\times(n+\mathtt{p}-\mathtt{k})}]F \begin{bmatrix} I_{\mathtt{k}} \\ 0_{\mathtt{k}\times(n+\mathtt{p}-\mathtt{k})} \end{bmatrix}$$
$$B' := [I_{\mathtt{k}}\ 0_{\mathtt{k}\times(n+\mathtt{p}-\mathtt{k})}]F \begin{bmatrix} 0_{n\times\mathtt{m}} \\ I_{\mathtt{m}} \end{bmatrix}$$
$$C' := [0_{\mathtt{p}\times n}\ I_{\mathtt{p}}]F \begin{bmatrix} I_{\mathtt{k}} \\ 0_{(n+\mathtt{m}-\mathtt{k})\times\mathtt{k}} \end{bmatrix}$$
$$D' := [0_{\mathtt{p}\times n}\ I_{\mathtt{p}}]F \begin{bmatrix} 0_{n\times\mathtt{m}} \\ I_{\mathtt{m}} \end{bmatrix}.$$

*Then $A', B', C', D'$ induce a* $\mathtt{k}$*-th order balanced truncation of the system represented by $A, B, C, D$.*

**Proof.** The state sequence corresponding to the columns of $V$ gives rise to a balanced i/s/o representation $A, B, C, D$, since the fact that $S(w) = -V^\top V$ implies that the storage function equals $I_{\mathtt{n}(\mathfrak{B})}$. The claim then follows immediately by observing that $A', B', C'$ are the truncations of $A, B, C$, respectively. □

We now make the formulas for $(A', B', C', D')$ more explicit; this formulation has the advantage that matrices of reduced dimensions are used, and is especially interesting in model reduction.

**Theorem 18.** *Under the assumptions and notation of Lemma 15, define $\Pi := VV^\top$, $\Pi_1$ to be the principal $\mathtt{k}\times\mathtt{k}$ submatrix of $\Pi$, and $V_1$ to be the matrix consisting of the first* $\mathtt{k}$ *rows of $V$. Then*

$$\begin{aligned} A' &:= \sigma V_1 V_1^\top \Pi_1^{-1} + \sigma V_1 V^\top \Pi^{-1} VU^\top \Delta^{-1} UV_1^\top \Pi_1^{-1} \\ &\quad - \sigma V_1 U^\top \Delta^{-1} UV_1^\top \\ B' &:= -\sigma V_1 V^\top \Pi^{-1} VU^\top \Delta^{-1} + \sigma V_1 U^\top \Delta^{-1} \\ C' &:= YV_1^\top \Pi_1^{-1} + YV^\top \Pi^{-1} VU^\top \Delta^{-1} UV_1^\top \Pi_1^{-1} \\ &\quad - YU^\top \Delta^{-1} UV_1^\top \Sigma_1^{-1} \\ D' &:= -YV^\top \Pi^{-1} VU^\top + YU^\top \Delta^{-1} \end{aligned}$$

*induce a* $\mathtt{k}$*-th order balanced truncation of the system represented in state-space form by $A, B, C, D$.*

**Proof.** The proof consists of straightforward manipulations, and is omitted. □

On the basis of the formulas obtained in Theorem 18, we can now state an algorithm for data-driven model reduction.

**Algorithm 2.** Input: $w = (u, y) \in \mathfrak{B}_+ \cap \ell_2^{\mathtt{w}}(\mathbb{Z}_+)$, with $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ $\Sigma$-half-line-lossless, and $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$; and $\mathtt{k} \in \mathbb{N}$, $\mathtt{k} \le \mathtt{n}(\mathfrak{B})$.

Output: A balanced-truncated i/s/o representation of $\mathfrak{B}$.

Steps 1–3: As in Algorithm 1.

Step 4: Factor $S(w) = -V^\top V$, $V \in \mathbb{R}^{\mathtt{n}\times\infty}$.

Step 5: Define $\sigma V := V(:, 2:\infty)$, $V_1 := V(1:\mathtt{k}, :)$, $\sigma V_1 := V(1:\mathtt{k}, 2:\infty)$, $U := [u(0)\ \cdots]$; $Y := [y(0)\ \cdots]$; $\Pi := VV^\top$; $\Pi_1 := \Sigma(1:\mathtt{k}, 1:\mathtt{k})$; $\Delta := UU^\top - UV^\top \Pi^{-1} VU^\top$.

Step 6: Return $A', B', C', D'$ as in Theorem 18.

We conclude this section with some remarks.

**Remark 19.** An argument analogous to that used to prove Lemma 21.31 of Zhou et al. (1996) shows that the reduced-order model obtained from our procedure is $\Sigma$-half-line dissipative; in the bounded-real and positive-real cases, it is also asymptotically stable.

**Remark 20.** Using the approach outlined at the end of Section 3.2, it is possible to obtain from the data $w = (u, y)$ a balanced realization also for measurements coming from dissipative systems. Analogous results to Theorems 17 and 18 can be formulated and used in order to obtain a reduced-order model.

**Remark 21.** Algorithm 2 amounts to truncating the balanced matrices $A$, $B$, $C$, and $D$. These matrices are also obtained by solving a system of linear equations; it is worthwhile to consider then whether other possibilities exist for obtaining reduced-order models from data. We are currently investigating the possibility of approximating $S(w)$ by means of a lower-rank matrix $S'$, and of using a rank-revealing factorization of $S'$ in order to obtain a reduced-order lossless/dissipative approximation of the original system.

## 5. Conclusions

A rank-revealing factorization of a "Gramian" matrix associated with noise-free data provides a state sequence, from which a state representation is readily obtained. For lossless systems, this technique does not require any knowledge about the system except the supply rate; in the dissipative case, we must know also the dissipation function, or that the data is of zero dissipation. Our procedure yields in a straightforward manner balanced state representations; from this stems a data-driven model reduction technique.

Current research is aimed in several directions. First, we need to carry out a detailed analysis of the computational costs, and to investigate efficient and numerically accurate algorithms. Second, the generalization of our approach to the case of finite measurements must be pursued. Third, we aim to explore the research area described in Remark 21.

The most pressing issue to investigate is whether the approach illustrated in this paper can be generalized to the situation of measurements corrupted by noise. A number of preliminary remarks about this problem can be made at this early stage.

First, it is evident that a direct extension of the methods illustrated in this paper is impossible. For example, the definition of the $S$-matrix (11) requires the trajectory $w$ to be square summable, an assumption not satisfied when $w$ (or a component thereof, for example the output variable) is corrupted by noise. Consequently, the first step towards a generalization of our approach to the noisy case requires that a suitable stochastic analogue of $S(w)$ be found.

Second, considering the crucial importance of property (12) in Proposition 2, it is necessary that this stochastic version of the $S$-matrix contains enough information to extract from it (for example through a rank-revealing factorization) a state sequence to be used in order to compute the matrices $A, B, C, D$ of a state representation of the system.

Third, another issue to consider when choosing a definition of the $S$-matrix in the stochastic case is that, under suitable assumptions on the statistical properties of the noise, the passivity of the underlying system producing the data be preserved in the model identified through the stochastic analogous of our procedure. Ideally, the stochastic procedure should exhibit the same inherent robustness to noise as subspace identification methods.

Finally, the extension of the model reduction procedure of Section 4 to the case of noisy data must be investigated.

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## References

Antoulas, A. C. (2005). A new result on passivity-preserving model reduction. *Systems and Control Letters*, *54*, 361–374.
Antoulas, A. C. (2004). *Approximation of large-scale dynamical systems*. Philadelphia: SIAM Press.
Dahlén, A., & Scherrer, W. (2004). The relation of CCA subspace method to a balanced reduction of an autoregressive model. *Journal of Econometrics*, *118*(1–2), 293–312.
Desai, U. B., & Pal, D. (1984). A transformation approach to stochastic model reduction. *IEEE Transactions on Automatic Control*, *AC-29*, 1079–1100.
Gugerçin, S., & Antoulas, A. C. (2004). A survey of model reduction by balanced truncation and some new results. *International Journal of Control*, *77*(8), 748–766.
Kaneko, O., & Fujii, T. (2000). Discrete-time average positivity and spectral factorization in a behavioral framework. *Systems and Control Letters*, *39*, 31–44.
Kaneko, O., & Fujii, T. (2003). When is a storage function a state function in discrete-time? *SIAM Journal on Control and Optimization*, *42*, 1374–1394.
Katayama, T. (2005). *Subspace methods for system identification*. Berlin: Springer Verlag.
Kojima, C., Takaba, K., Kaneko, O., & Rapisarda, P. (2006). Characterization of solutions of the discrete-time algebraic Riccati equation based on quadratic difference forms. *Lin. Alg. Appl.*, *416*(2–3), 1060–1082.
Lindquist, A., & Picci, G. (1996). Canonical correlation analysis, approximate covariance extension, and identification of stationary time series. *Automatica*, *32*(5), 709–733.
Moonen, M., De Moor, B., Vandenberghe, L., & Vandewalle, J. (1989). On-and off-line identification of linear state space models. *International Journal of Control*, *49*, 219–232.
Moonen, M., & Ramos, J. (1993). A subspace algorithm for balanced state space system identification. *IEEE Transactions on Automatic Control*, *38*, 1727–1729.
Opdenacker, P. C., & Jonckheere, E. A. (1988). A contraction mapping preserving balanced reduction scheme and its infinity norm error bounds. *IEEE Transactions on Circuits and Systems*, *35*(2), 184–189.
Polyuga, R. V., & van der Schaft, A. J. (2010). Structure preserving model reduction of port-Hamiltonian systems by moment matching at infinity. *Automatica*, *46*(4), 665–672.
Sorensen, D. C. (2005). Passivity preserving model reduction via interpolation of spectral zeros. *Systems and Control Letters*, *54*(4), 347–360.
Tanaka, H., & Katayama, T. (2006). A stochastic realization algorithm via block LQ decomposition in Hilbert space. *Automatica*, *42*, 741–746.
Trentelman, H. L., Ha, B. M., & Rapisarda, P. (2009). Dissipativity preserving model reduction by retention of trajectories of minimal dissipation. *Mathematics of Control, Signals, and Systems*, *21*(3), 171–201.
Trentelman, H. L., & Willems, J. C. (2002). Synthesis of dissipative systems using quadratic differential forms — part II. *IEEE Transactions on Automatic Control*, *47*(1), 70–86.
Van Overschee, P., & De Moor, B. (1996). *Subspace identification for linear systems: theory, implementation, applications*. Kluwer Academic Publishers.
Van Overschee, P, & De Moor, B. (1995). Choice of state-space basis in combined deterministic–stochastic subspace identification. *Automatica*, *31*(12), 1877–1883.
Willems, J. C. (1991). Paradigms and puzzles in the theory of dynamical systems. *IEEE Transactions on Automatic Control*, *36*, 259–294.
Willems, J. C., Rapisarda, P., Markovsky, M., & De Moor, B. L. M. (2005). A note on persistency of excitation. *Systems and Control Letters*, *54*, 325–329.
Willems, J. C., & Trentelman, H. L. (1998). On quadratic differential forms. *SIAM Journal on Control and Optimization*, *36*, 1703–1749.
Zhou, K., et al. (1996). *Robust and optimal control*. Upper Saddle River, NJ: Prentice-Hall.

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