Doubles of Klein surfaces

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1 Introduction

The following is a well-known limerick.

A mathematician called Klein
thought that the Möbius band was divine,
said he "if you glue
the edges of two,
you'll get a weird bottle like mine".

This is saying that a Klein bottle is an example of a *double* of a Möbius band.

Historical note. A non-orientable surface of genus 2 (meaning 2 cross-caps) is popularly known as the Klein bottle. However, the term Klein surface comes from Felix Klein's book "On Riemann's Theory of Algebraic Functions and their Integrals" (1882) where he introduced such surfaces in the final chapter.

A Klein surface is a surface with a dianalytic structure and we are mainly concerned here with compact surfaces. (For dianalytic structure see [AG] or [BEGG]). Topologically compact Klein surfaces are surfaces which might be non-orientable and might have boundary. We are interested in this paper in studying the doubles of Klein surfaces. For example, a Klein bottle turns out to be a double of a Möbius band. By a double here we mean a smooth double although we do allow folding. The folding map $\phi : \mathbb{C} \longrightarrow \mathcal{U}$ where $\mathcal{U}$ is the upper-half complex plane, is defined by $\phi(x + iy) = x + i|y|$, and a smooth morphism of Klein surfaces is a map which is locally smooth or locally the folding map, the latter occurring over the boundary of the image. For the precise definition see [AG]. Three types of doubles turn out to be particularly interesting; the complex double, the Schottky double and the orienting double. These doubles are defined in [AG] in terms of equivalence classes of dianalytic atlases. One of the aims of this paper is to describe these simply using index two subgroups of uniformization crystallographic groups and also with topological descriptions.

2 Klein surfaces and NEC groups

Every Klein surface can be represented as $\mathcal{U}/\Gamma$ where $\mathcal{U}$ is a simply-connected Riemann surface and Γ is a crystallographic group without elliptic elements. (It might have reflections though). If the algebraic genus of the surface is greater than 1, then $\mathcal{U} = \mathcal{H}$, the upper half-plane and Γ is a non-Euclidean crystallographic (NEC) group. If the algebraic genus is equal to 1, (for example the Möbius band) then $\mathcal{U} = \mathbb{C}$ and Γ is a Euclidean group. These groups can be assigned a signature of the form

$$(g; \pm; [\,]; \{(\,)^k\}).$$

Here, $(\,)^k$ means k empty period cycles. If this occurs $\mathcal{U}/\Gamma$ is a compact surface of genus g with k boundary components; it is orientable when the $+$ sign occurs and non-orientable when the $-$ sign occurs. If the $+$ sign occurs then the fundamental region for the group is a hyperbolic polygon with surface symbol

$$\alpha_1\beta_1\alpha_1'\beta_1' \ldots,\ \alpha_g\beta_g\alpha_g'\beta_g'\epsilon_1\gamma_1\epsilon_1' \ldots,\ \epsilon_k\gamma_k\epsilon_k' \tag{1}$$

If the $-$ sign occurs then the fundamental polygon has surface symbol

$$\alpha_1\alpha_1^* \ldots \alpha_g\alpha_g^*\epsilon_1\gamma_1\epsilon_1' \ldots \epsilon_k\gamma_k\epsilon_k' \tag{2}$$

The group has two possible presentations; if the $+$ sign occurs the presentation is

$$\langle a_1, b_1, \ldots, a_g, b_g, e_1, \ldots, e_k, c_1, \ldots, c_k \mid \Pi_{i=1}^{g}[a_1, b_i]e_1 \cdots e_k = 1, c_i^2 = 1, e_ic_ie_i^{-1} = c_i \quad (i = 1, \ldots, k)\rangle$$

Here a_i, b_i are hyperbolic, c_i are reflections and e_i are orientation-preserving though usually hyperbolic. Here $a_i(\alpha_i') = \alpha_i, b_i(\beta_i') = \beta_i, e_i(\epsilon_i') = \epsilon_i$ and c_i fixes the edge γ_i.

If the $-$ sign occurs the presentation is

$$\langle a_1 \ldots, a_g, e_1, \ldots, e_k, c_1, \ldots, c_k \mid a_1^2 \cdots a_g^2 e_1 \cdots e_k = 1, c_i^2 = 1, e_ic_ie_i^{-1} = c_i \quad (i = 1, \ldots, k)\rangle$$

Now the a_i are glide-reflections and $a_i(\alpha_i^*) = \alpha_i$

In both presentations the first relation is called the *long* relation.

3 Double covers of Klein surfaces

A double cover of a Klein surface $\mathcal{U}/\Gamma$ has the form $\mathcal{U}/\Lambda$ where Λ is a subgroup of index 2 in Γ. There is then a natural epimorphism $\theta : \Gamma \longrightarrow C_2 = \{1, t\}$, with $\ker\theta = \Lambda$, called the monodromy epimorphism.

Theorem 1 *Let X be a compact Klein surface of genus g with $k > 0$ boundary components. Then there are $2^\lambda - 1$ double covers of X where*

$$\lambda = \begin{cases} 2g + 2k - 1 \text{ if } X \text{ is orientable} \\ g + 2k - 1 \text{ if } X \text{ is non-orientable} \end{cases}$$

Proof. We just need to find the number of index 2 subgroups of NEC surface groups. Let $X = \mathcal{U}/\Gamma$, where Γ is an NEC group as above. Thus when there is a + sign in the signature Γ has generators

$$a_i, b_i, (i = 1, \cdots, g),\ e_i(i = 1, \cdots, k),\ c_i(i = 1, \cdots, k)$$

and relations as listed above. We wish to find all epimorphisms $\theta : \Gamma \longrightarrow C_2$. All the relations for Γ that we listed above will hold in C_2 as long as $\theta(e_1)\theta(e_2)\cdots\theta(e_k) = 1$. Thus $\theta(a_i), \theta(b_i)$ can be chosen in 2 ways and the same is true of the $\theta(c_i)$. We can choose $\theta(e_1), \cdots, \theta(e_{k-1})$ in each of two ways and then $\theta(e_k)$ is uniquely determined by the long relation. As we cannot have every generator mapping to the identity we find $2^{2g+2k-1} - 1$ epimorphisms $\theta : \Gamma \longrightarrow C_2$ as required. The proof for groups with a minus sign in the signature is exactly the same, except we only have g generators $a_1, \cdots, a_g$. ■

The same result appears in [AG]. We wish to study double covers of surfaces by studying all the epimorphisms $\theta : \Gamma \longrightarrow C_2$. Let Λ be the kernel of θ. The question we are interested in is to determine the topological nature of $\mathcal{U}/\Lambda$.

Theorem 2 *[H] Define a map $\tau_\theta\{c_1, ..., c_k\} \longrightarrow \{0, 1, 2\}$ by*

$$\tau_\theta = \begin{cases} 2 \text{ for } \theta(c_i) = \theta(e_i) = 1 \\ 1 \text{ for } \theta(c_i) = 1,\ \theta(e_i) = t \\ 0 \text{ for } \theta(c_i) \neq 1 \end{cases}$$

then the number of boundary components of $\mathcal{U}/\Lambda$ is

$$s = \sum_{i=1}^{k} \tau_\theta(c_i).$$

Proof. Let Λ be the kernel of θ of We need to show that s is the number of conjugacy classes of reflections in Λ. Write Γ as a disjoint union of two cosets Λ and $h\Lambda$, $h \in \Gamma \setminus \Lambda$.

Now it is known that the centralizer of c in Γ is in the group $\langle e, c\rangle$, the group generated by c and e [S]. If $\theta(c) = \theta(e) = 1$, then $e, c \in \Lambda$, so that if $hch^{-1} = kck^{-1}$ then $h^{-1}k$ centralizes c and so $h^{-1}k \in \Lambda$ or $\Lambda h = \Lambda k$. Thus each coset corresponds to a different conjugacy class of Λ in Γ which is two in this case.

If $\theta(e) = t$, then $e \in \Gamma \setminus \Lambda$. If $h \in \Gamma \setminus \Lambda$, then $he \in \Lambda$ and $hch^{-1} = hece^{-1}h^{-1}$ which is conjugate to c in Λ, and so there is only one conjugacy class of reflections in Λ.

If $\theta(c) \neq 1$ then $c \notin \Lambda$ and so there are no conjugacy classes of reflections in Λ. ■

Theorem 3 *(i) If Γ has orientable quotient space then Λ has non-orientable quotient space if and only if $\Gamma \setminus \Lambda$contains both orientation-preserving and orientation-reversing transformations of Γ.*

(ii) If Γ has non-orientable quotient space then Λ has non-orientable quotient space if and only if $\Gamma \setminus \Lambda$ contains both orientation preserving and orientation -reversing generators of Γ or Λ contains any of the glide reflections of Λ.

Proof. The proof follows from the techniques in [HS]. (Alternatively, we could use Theorem 2.1.3 of [BEGG].) Basically $\mathcal{U}/\Lambda$ is non-orientable if and only if the coset graph $C(\Gamma, \Lambda)$ with reflection loops deleted, contains orientation-reversing loops. In (i) all generators are orientation-preserving (hyperbolic) or orientation-reversing (reflections). The coset graph only has two points, corresponding to the two cosets. If $\Gamma \setminus \Lambda$ has hyperbolic and reflection generators then one followed by the other gives an orientation - reversing loop in $C(\Gamma, \Lambda)$, and all orientation-reversing loops have this form.

(ii) follows in the same way. Now Γ contains glide-refections and if any of these lie in Λ then they give an orientation-reversing loop in $C(\Gamma, \Lambda)$. ■

4 Standard homomorphisms and doubles

In Theorem 1 we saw that a Klein surface could have a large number of doubles. For this reason we highlight doubles which are easier to study, and we find these include the most important doubles mentioned in the introduction.

First, we assume that Γ has a negative sign in its signature so that $X = \mathcal{U}/\Gamma$ is non-orientable. We consider only the epimorphisms $\theta : \Gamma \longrightarrow C_2$

for which all the e_i generators have the same image, all the reflection generators have the same image and all the glide-reflection generators have the same image. We let E denote the set $\{e_1, \ldots, e_k\}$, $C = \{c_1, \ldots, c_k\}$, $A = \{a_1, \ldots, a_g\}$. Let $C_2 = \langle t \mid t^2 = 1 \rangle$. If we write $\theta(E) = t$ we mean $\theta(e_i) = t$ for $i = 1, \cdots k$ etc. For groups with orientable quotient space we follow a similar idea except that now the set $A = \{a_1, b_1, \ldots, a_g, b_g\}$.

Theorem 4 *If k is even then there are 7 standard epimorphisms $\theta : \Gamma \longrightarrow C_2$, while if k is odd there are only 3 standard homomorphisms.*

Proof. We must have $\theta(A) = 1$ or t, $\theta(E) = 1$ or t and $\theta(C) = 1$ or t. As we have an epimorphism they cannot all map to 1 so we have $2^3 - 1$ standard epimorphisms. If k is odd then because of the long relation we cannot have $\theta(e_i) = t$ for $i = 1, \ldots, k$ so that $\theta(E) = 1$ and so we only have $2^2 - 1 = 3$ standard homomorphisms. ■

If Λ is the kernel of a standard epimorphism then we call $\mathcal{U}/\Lambda$ a *standard double* of $\mathcal{U}/\Gamma$.

In Table 1 we list the standard epimorphisms and for each one we give the topological type of the standard double, which we have obtained by Theorems 2 and 3. We distinguish between the cases where Γ has orientable or non-orientable quotient space. Here, k is the number of boundary components of $\mathcal{U}/\Gamma$, B is the number of boundary components of the standard double and the orientability of the doubles are denoted by + or −.

			Orientability of double	
	Standard epimorphism	B	$\mathcal{U}/\Gamma$ non-orientable	$\mathcal{U}/\Gamma$ orientable
1.	$E \to 1$ $C \to t$ $A \to t$	0	+	−
2.	$E \to 1$ $C \to 1$ $A \to t$	$2k$	+	+
3.	$E \to 1$ $C \to t$ $A \to 1$	0	−	+
4.	$E \to t$ $C \to 1$ $A \to 1$	k	−	+
5.	$E \to t$ $C \to 1$ $A \to t$	k	−	+
6.	$E \to t$ $C \to t$ $A \to 1$	0	−	−
7.	$E \to t$ $C \to t$ $A \to t$	0	−	−

(Table 1)

Example. We consider the doubles of the Möbius band. The Möbius band is a non-orientable surface of genus 1 with one boundary component and is represented by a group Γ_1 of signature $(1; -; \{()\})$, and presentation

$$\langle a, e, c \mid a^2 e = c^2 = ece^{-1}c = 1 \rangle. \qquad (3)$$

For any epimorphism $\theta : \Gamma_1 \longrightarrow C_2$, because a^2e=1 we must have $\theta(e) = 1$, and so there are only 3 such epimorphisms and they are all standard epimorphisms, namely 1,2,3 above. If, for example we consider the epimorphism 3, we see from Table 1 that the double is non-orientable without boundary. The Riemann-Hurwitz formula tells us that this double has non-orientable genus 2 and so is a Klein bottle; compare the limerick at the beginning of this paper. We shall find all the doubles of the Möbius band (note that for the Möbius band all the doubles are standard doubles). As we shall see, these three doubles are part of general families of doubles.

The doubles have the form $\mathbb{C}/\Lambda$ where Λ has index two in Γ. Write $\Gamma = \Lambda + \Lambda q$ where Λ has index two in Γ and $q \in \Gamma \setminus \Lambda$ If F_Γ is a fundamental region for Γ then $F_\Lambda = F_\Gamma \cup qF_\Gamma$ is a fundamental region for Λ. As shown in [HS] the sides of F_Λ are paired by the Schreier generators of Λ in Γ. These Schreier generators have the form gdh^{-1} where g, h are coset representatives (in this case only 1 and q) and d is one of the standard generators of Γ which pair sides of F_Γ. If $d(s') = s$, where s, s' are sides of $\mathcal{F}_\Gamma$ then gdh^{-1} pairs the sides $h(s')$ and $g(s)$ of F_Λ.

It is instructive to see these epimorphisms geometrically. The sides of F_Λ are $\epsilon', \epsilon, c\epsilon, c\alpha^*, c\alpha, c\epsilon'$ In epimorphism 1 of table 1 the Schreier generators and the side pairings are $1e1^{-1} : \epsilon' \longrightarrow \epsilon$, $cec^{-1} : c\epsilon' \longrightarrow c\epsilon$, $1ac^{-1} : c(\alpha^*) \longrightarrow \alpha$, $ca1^{-1} : \alpha^* \longrightarrow c(\alpha)$. This is pictured in Figure 1. This epimorphism maps every orientation preserving element of Γ_1 to 1, and every orientation reversing element to t. If Λ_1 is the kernel then $\mathbb{C}/\Lambda_1$ is known as the *complex double* X_C of $X = \mathbb{C}/\Gamma_1$. More about complex doubles later.

Figure 1 gives a picture of F_Λ with the side identifications. We find that the surface obtained carries a map with three vertices A, B, C. There are 4 edges, (ϵ, ϵ'), $(c\epsilon, c\epsilon')$, (α, α^*) and $(c\alpha, c\alpha^*)$ and 1 face. Thus the Euler characteristic is 0, and as the sides are all paired orientably we get a torus as expected.

Now consider the epimorphism 2. Now a coset representative is a. We take $F_\Gamma \cup aF_\Gamma$ as fundamental region of Λ. The Schreier generators and side pairings are $1e1^{-1} : \epsilon' \longrightarrow \epsilon$, $aea^{-1} : a(\epsilon') \longrightarrow a(\epsilon)$, $1c1^{-1}$ fixes γ, aca^{-1} fixes $a(\gamma)$, $aa1^{-1} : \alpha \longrightarrow a(\alpha^*)(= a(a(\alpha)))$. The diagram is as in Figure 2.

We see that the quotient space has two boundary components corresponding to γ and $a(\gamma)$. All side pairings are orientable and again the Euler characteristic is 0 and so and so we get a cylinder. This is an example of an orienting double; again more of this later.

We now consider the epimorphism 3. As $c \longrightarrow t$ we take c as the coset representative. The Schreier coset representatives and the side pairings are $1e1^{-1} : \epsilon' \longrightarrow \epsilon, cec^{-1} : c(\epsilon') \longrightarrow c(\epsilon)$, $1a1^{-1} : \alpha \longrightarrow \alpha^*$, $cac^{-1} : c(\alpha) \longrightarrow c(\alpha^*)$. As a reverses orientation the quotient surface is non-orientable and

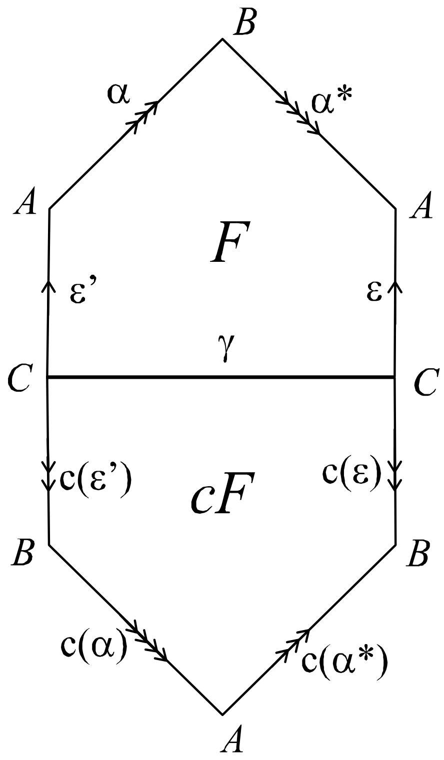

Figure 1:

so we get a Klein bottle as a double of a Möbius band. See the limerick at the beginning. The diagram is in figure 3. Note that γ projects to a simple closed curve on X which separates X into two Möbius bands. This is an example of a Schottky double, and again we discuss these later.

5 The most natural doubles

In the previous section we found three doubles of the Möbius band. We can study these three doubles for any Klein surface. These three doubles have been considered in [AG].

1. The complex double

If X is a Klein surface then its complex double is the unique double which is a Riemann surface without boundary. If $X = \mathcal{U}/\Gamma$, where Γ is an NEC

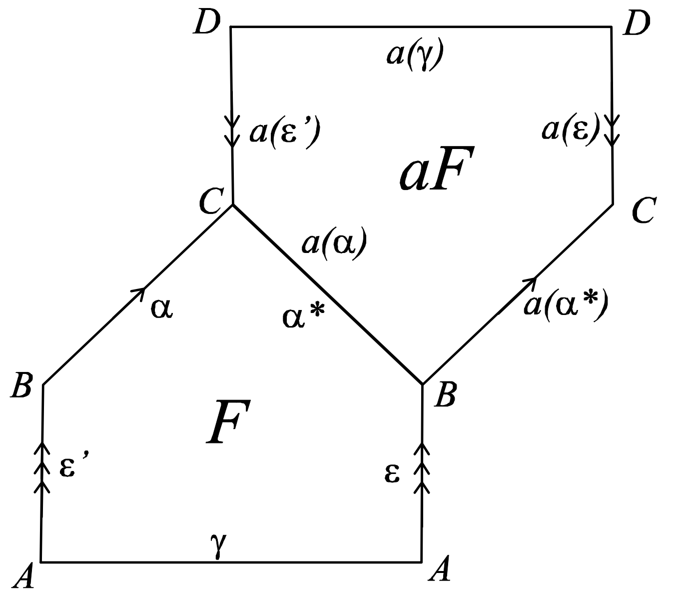

Figure 2:

surface group then its complex double is $\mathcal{U}/\Gamma^+$ where Γ^+ is the canonical fuchsian group of Γ that is the index two subgroup of Γ consisting of those transformations that preserve orientation. If $\mathcal{U}/\Gamma$ is non-orientable then the generators of A are glide reflections and so the epimorphism 1 gives the complex double. If $\mathcal{U}/\Gamma$ is orientable then the generators in A are hyperbolic and so the epimorphism 3 gives the complex double. Finally if X is an orientable surface without boundary the complex double consist of two connected components X_1, X_2 each one homeomorphic to X, the epimorphism is the trivial one. The important point is that, in general, each connected component has a different analytic structure and there is an anticonformal isomorphism from X_1 to X_2.

2. The orienting double.

Let X be a Klein surface and suppose that ∂X has k components. For $i = 1, \cdots, k$ fill in each boundary component with a disc D_i with centre p_i (so that $p_i \notin \partial X$.) We get a surface $\hat{X}$ without boundary, of the same orientability as X. Now consider the complex double of $\hat{X}$. (Recall that $\hat{X}$ has two components if X is orientable.) Each p_i lifts to two points p_i^1 and p_i^2 in $\hat{X}$. Let D_i^1 and D_i^2 be small discs centred at p_i^1 and p_i^2 in $\hat{X}$. If we remove these discs from $\hat{X}$ we end up with an orientable surface Y which has $2k$ boundary components and clearly Y is a two-sheeted cover of X. We call

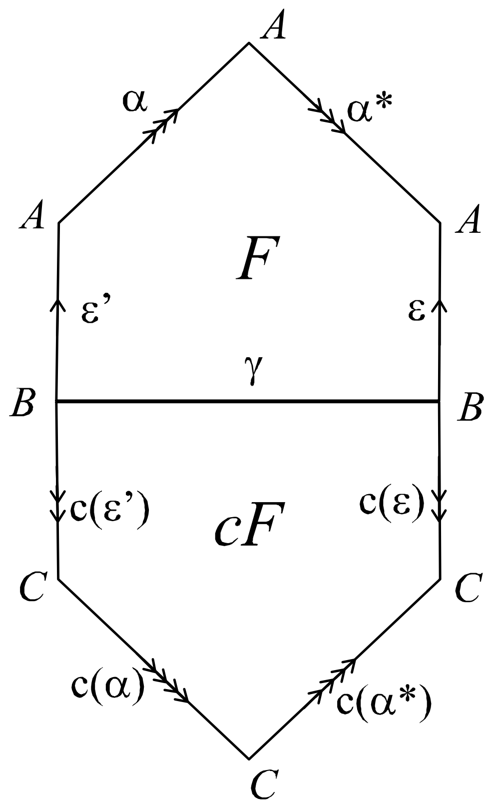

Figure 3:

Y the orienting double of X. Note that if X is orientable then Y has two connected components.

1. If we consider the epimorphisms of section 2 we see that only for epimorphism 2 do we have a covering with twice as many boundary components as the original surface so this epimorphism correspond to the orienting double of a non-orientable Klein surface. In the case of orientable Klein surfaces the epimorphism is the trivial one.

3. The Schottky double

Let $\tilde{X}$ be a double cover of the Klein surface X. Then $\tilde{X}$ admits an involution $h \in \Gamma$ such that $\tilde{X}/\langle h \rangle = X$. As we are considering unbranched but possibly folded coverings, the fixed-point set of h will include a collection of simple closed curves. (This is well-known when X is a Riemann surface and h is a symmetry, i.e. an anticonformal involution.) An analogous thing happens for Klein surfaces [BCNS]. We define the Schottky double of X to be a Klein surface $\tilde{X}$ without boundary of the same orientability as X admitting a dianalytic involution h whose fixed curves separate $\tilde{X}$ and such that $\tilde{X}/\langle h \rangle = X$.

Theorem 5 *Let $X = \mathcal{U}/\Gamma$ be a Klein surface with boundary and $\tilde{X} = \mathcal{U}/\Lambda$ its Schottky double. (Γ, Λ surface groups). Let $\theta : \Gamma \longrightarrow \Gamma/\Lambda \cong C_2$ be the natural epimorphism . Then θ is the epimorphism 3 of Table 1.*

Proof. As h has fixed curves, Γ must have some reflection generator c_1 and we can write $\Gamma = \Lambda + \Lambda c_1$. Let F be a fundamental region for Γ. Then $F \cup c_1 F$ is a fundamental region for Λ. Suppose for example that $\theta(e_i) = t$. Then a Schreier generator would be $1 e_i c_1^{-1}$. This pairs the side $c_1(\epsilon_i')$ with ϵ_i. Thus if $\pi : \mathcal{U} \longrightarrow \mathcal{U}/\Gamma$ is the natural projection and z is a point in the interior of F, then we can join z to $c_1 z$ by a path which goes from z to the edge ϵ_i', which is identified with $c\epsilon_i$ and then to $c_1 z$ without passing through $\pi(\cup \gamma_i)$. The projection of this path is then a path in $\mathcal{U}/\Lambda$ which does not pass through the fixed curves of h and we do not then have the Schottky double. Thus to have the Schottky double we must have $\theta(E) = 1$. Similarly, we must have $\theta(A) = 1$. As the Schottky double has no boundary we must have $\theta(C) = t$. We thus have the epimorphism 3. ■

Note that if $X = \mathcal{U}/\Gamma$ is orientable then $A = \{a_1, b_1, \ldots, a_g, b_g\}$ consists of hyperbolic elements as does the set $E = \{e_1, \ldots, e_k\}$ and these preserve orientation. Thus in this case, the Schottky double coincides with the complex double.

To sum up, the cases 1,2,3 correspond to the complex, orienting and Schottky doubles respectively.

6 The other standard doubles

We have examined the 3 standard doubles for which $E \longrightarrow 1$ and found them to be geometrically interesting. We now briefly look at the 4 standard doubles for which $E \longrightarrow t$. We will now assume that X is non-orientable. The orientable case is similar. We first note that because of the long relation we must have s, even, where s is the number of boundary components of the double $\tilde{X}$.

4. $E \longrightarrow t, C \longrightarrow 1, A \longrightarrow 1$.

Note that by Table 1 or Theorem 2, the number of the boundary components of $\tilde{X}$ is equal to s. Also because $A \longrightarrow 1$, $\tilde{X}$ is non-orientable. We can use the Riemann-Hurwitz formula to compute the genus h of $\tilde{X}$. This gives $h = 2g - 2 + s$.

To construct the covering fill in the s boundary curves with discs $D_1, \dots D_s$ and let p_i be the centre of D_i. Let S be the resulting surface. Build a 2-sheeted cover S^* of S with simple branch points of order 2 at the points p_i. Let $\tilde{p}_i$ be the lift of p_i to S^* and let $\tilde{D}_i$ be the lifts of D_i to S^*. Finally remove the $\tilde{D}_i$ from S^* to construct $\tilde{X}$. Using the Riemann -Hurwitz formula for branched coverings of Riemann surfaces we see that the genus of $\tilde{X}$ is as above.

5. $E \longrightarrow t, C \longrightarrow 1, A \longrightarrow t$.

This is almost like 4 above. The only difference concerns the closed curves α_i on X which have neighbourhoods homeomorphic to Möbius bands, (orientation-reversing loops.) These correspond to the glide reflection generators in A. In 4 these also lie in Λ which means that these loops lift to orientation-reversing loops in $\tilde{X}$. In 5 this does not occur; now α_i^2 lifts to an orientation-preserving loop.

6. $E \longrightarrow t, C \longrightarrow t, A \longrightarrow 1$. Now, by table 1 we have that $\tilde{X}$ is non-orientable without boundary. Note that as e_i commutes with c_i e_i maps γ_i to itself. As $e_i^2 \in \Lambda$, e_i induces an automorphism of order two which rotates through 180^o degrees the closed curve in $\tilde{X}$ where γ_i projects. The map $\tilde{X} \to X$ perform the antipodal identification on the curve that is the projection of γ_i in $\tilde{X}$ and then each boundary component of X lifts to a curve in $\tilde{X}$ with a Möbius band as neighborhood.

7. $E \longrightarrow t, C \longrightarrow t, A \longrightarrow t$. This is like 6, except we have the same remark as in 4 concerning orientation-reversing loops.

References

[AG] Alling, Norman L.; Greenleaf, Newcomb Foundations of the theory of Klein surfaces. Lecture Notes in Mathematics, Vol. 219. Springer-Verlag, Berlin-New York, 1971. ix+117 pp.

[BCNS] Bujalance, E.; Costa, A. F.; Natanzon, S. M.; Singerman, D. Involutions of compact Klein surfaces. Math. Z. 211 (1992), no. 3, 461–478.

[BEGG] Bujalance, Emilio; Etayo, José J.; Gamboa, José M.; Gromadzki, Grzegorz Automorphism groups of compact bordered Klein surfaces. A combinatorial approach. Lecture Notes in Mathematics, 1439. Springer-Verlag, Berlin, 1990. xiv+201 pp.

[H] Hall, W., Automorphisms and coverings of Klein surfaces, Ph. D. Thesis, University of Southampton 1977.

[HS] Hoare, A. H. M.; Singerman, D. The orientability of subgroups of plane groups. Groups—St. Andrews 1981 (St. Andrews, 1981), pp. 221–227, London Math. Soc. Lecture Note Ser., 71, Cambridge Univ. Press, Cambridge, 1982.

[S] Singerman, D. On the structure of non-euclidean crystallographic groups, Proc. Camb. Phil. Soc. 76 (1974) 233-240.

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