International Journal of Control
Vol. 00, No. 00, Month 200x, 1–20

RESEARCH ARTICLE

A 2D Systems Approach to Iterative Learning Control for Discrete Linear Processes with Zero Markov Parameters

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(*January 2010*)

In this paper a new approach to iterative learning control for the practically relevant case of deterministic discrete linear plants with uniform rank greater than unity is developed. The analysis is undertaken in a 2D systems setting that, by using a strong form of stability for linear repetitive processes, allows simultaneous consideration of both trial-to-trial error convergence and along the trial performance, resulting in design algorithms that can be computed using Linear Matrix Inequalities (LMIs). Finally, the control laws are experimentally verified on a gantry robot that replicates a pick and place operation commonly found in a number of applications to which iterative learning control is applicable.

Keywords: iterative learning control, repetitive process, experimental verification.

1 Introduction

Iterative learning control (ILC) is a technique for controlling systems operating in a repetitive, or trial-to-trial, mode with the requirement that a reference trajectory $y_{ref}(p)$ defined over a finite interval $p = 0, 1, \ldots, \alpha - 1$, where α denotes the trial duration, is followed to a high precision. Examples of such systems include robotic manipulators that are required to repeat a given task to high precision, chemical batch processes or, more generally, the class of tracking systems.

Since the original work Arimoto et al. (1984), the general area of ILC has been the subject of intense research effort. Initial sources for the literature here are the survey papers Bristow et al. (2006), Ahn et al. (2007). One approach to the analysis and design of ILC schemes is to use 2D systems theory where one direction of information propagation is from trial-to-trial and the other is along the trial, and it is in this setting that the results reported in this paper are developed.

In ILC, the control law must ensure convergence of the trial-to-trial error, where the error on any trial is the difference between the reference signal and the output and is defined over a finite duration. Hence it is possible for trial-to trial error convergence to occur even if the along the trial dynamics are unstable. In the case of linear dynamics, therefore, ILC can be applied to a system with an unstable state matrix or, if unstable along the trial dynamics are not allowed, a pre-stabilizing control law can be first implemented and ILC applied to the resulting stabilized dynamics. For discrete linear systems, a common way to approach ILC design is to use lifting

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This work has been partially supported by the Ministry of Science and Higher Education in Poland under the project N N514 293235.

ISSN: 0020-7179 print/ISSN 1366-5820 online

DOI: 10.1080/0020717YYxxxxxxxx
http://www.informaworld.com

to first write the dynamics in terms of a standard difference equation and for unstable examples the route is to first design a stabilizing feedback control law and then applying lifting to the resulting controlled process dynamics. The result is a two stage design process.

An alternative to lifting is to use a 2D systems approach, where previous work Hladowski et al. (2008, 2010) has been shown that ILC schemes can be designed for a class of discrete linear systems by, in effect, extending techniques developed for 2D linear systems using the framework of linear repetitive processes. Also these designs have been experimentally verified on a gantry robot executing a pick and place operation that is typical of many industrial applications to which ILC is well suited. The basis of the results in Hladowski et al. (2008, 2010) is stability along the pass, or trial, for linear repetitive processes that, in the case of the along the trial dynamics, demands bounded-input-bounded output (BIBO) stability uniformly with respect to the trial duration. Hence it is possible to simultaneously design for trial-to-trial error convergence and along the trial response. This allows the design of a single control law and adds to the options to the designer which in some cases may be more attractive than pre-stabilization followed by ILC design.

Some ILC algorithms for linear dynamics require that the first Markov parameter of the plant state-space model is not the zero matrix, that is, for the state-space triple $\{A, B, C\}$, $CB \neq 0$. A specific example is, in the single-input single-output (SISO) case for simplicity, P-type ILC of the form

$$u_{k+1}(p) = u_k + \Gamma e_k(p+1), p = 0, 1, \cdots, \alpha - 1$$

where the integer $k \geq 0$ denotes the trial number, $u_k(p)$ is the trial input, $e_k(p)$ is the difference on trial k between the reference signal $y_{ref}(p)$ and the output $y_k(p)$, and Γ is a gain to be selected. Trial-to-trial error convergence holds in this case provided

$$||1 - CB\Gamma|| < 1$$

where $||\cdot||$ is a suitably chosen operator norm. Hence if $CB = 0$, or in an implementation rounding errors enforce this condition, then this simple structure algorithm cannot be used.

This problem has been the subject of considerable research, again see the survey papers Bristow et al. (2006), Ahn et al. (2007) for a starting point on the literature, but the algorithms currently available are for trial-to-trial error convergence with pre-stabilization of the along the trial dynamics if required. This paper extends the results in Hladowski et al. (2008, 2010) to discrete linear time-invariant processes of uniform rank greater than unity, that is for some integer $h > 1$ the first non-zero Markov parameter is CA^hB. Experimental verification results are also given to allow, for example, comparison with alternatives.

In this paper, the null and identity matrices with the required dimensions are denoted by 0 and I respectively. Also $\Gamma \succ 0$ and $\Gamma \prec 0$ respectively are used to denote symmetric matrices which are positive definite and negative definite, respectively, and $\otimes$ denotes the Kronecker matrix product. The symbol $r(\cdot)$ is used to denote the spectral radius of a given matrix, that is, if H is an $n \times n$ matrix then $r(H) = max_{1 \leq i \leq n}|\lambda_i|$ where λ_i is an eigenvalue of H.

2 Background and Initial Analysis

The processes considered in this paper are assumed to be adequately represented by discrete linear time-invariant systems described by the state-space triple $\{A, B, C\}$. In an ILC setting, process state-space model is written as

$$\begin{aligned} x_k(p+1) &= Ax_k(p) + Bu_k(p),\ p = 0, 1, \ldots, \alpha - 1 \\ y_k(p) &= Cx_k(p) \end{aligned} \tag{1}$$

where on trial k, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the output vector, $u_k(p) \in \mathbb{R}^r$ is the vector of control inputs, and the trial length $\alpha < \infty$. If the signal to be tracked is denoted by $y_{ref}(p)$ then $e_k(p) = y_{ref}(p) - y_k(p)$ is the error on trial k. In this paper the analysis is for SISO processes as the multi-input multi-output (MIMO) case follows with only routine changes to replace the row and column vectors C and B respectively by matrices.

As discussed in the introduction to this paper, ILC can be treated in a 2D systems setting where information propagation in one direction is from trial-to-trial and in the other it is along the trial. In the case of discrete linear dynamics, Kurek and Zaremba (1993), and others, have used the Roesser state-space model Roesser (1975) to design a control law to ensure trial-trial error convergence, but applications will arise where it is also necessary to control the along the trial dynamics. For example, consider a gantry robot whose task is to collect an object from a location, place it on a moving conveyor, and then return for the next one and so on. If, for example, the object has an open top and is filled with liquid, and/or is fragile in nature, then unwanted vibrations during the transfer time could have very detrimental effects.

For discrete dynamics, one approach in cases such as that outlined above is to design a feedback control scheme to control the process output and then proceed to ILC design for trial-to-trial error convergence by, for example, a lifted standard linear systems state-space model of the controlled dynamics. In implementation terms, this is a two loop control scheme and this paper considers the alternative where a 2D systems setting is used to design a single controller for both tasks. Linear repetitive processes are a distinct class of 2D linear systems where the duration of information propagation in one of the two directions is finite, and next the relevant background on these processes is given.

The unique characteristic of a repetitive process (Rogers et al. (2007)), is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

To introduce a formal definition, let $\alpha < +\infty$ denote the pass length. Then in a repetitive process the pass profile $y_k(p)$ generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(p)$, $p = 0, 1, \ldots, \alpha - 1$, $k \geq 0$.

Attempts to control these processes using standard, or 1D, systems theory and algorithms fail in general precisely because such an approach ignores their inherent 2D systems structure, that is, information propagation occurs from pass-to-pass (k direction) and along a given pass (p direction) and also the initial conditions are reset before the start of each new pass. To remove these deficiencies, a rigorous stability theory has been developed Rogers et al. (2007) based on an abstract model of the dynamics in a Banach space setting which includes a very large class of processes with linear dynamics and a constant pass length as special cases. In terms of their dynamics, it is the pass-to-pass coupling, noting again their unique feature, which is critical. This is of the form

$$y_{k+1} = L_\alpha y_k \tag{2}$$

where $y_k \in E_\alpha$, E_α is a Banach space with norm $||\cdot||$, and L_α is a bounded linear operator mapping E_α into itself.

The most basic discrete linear repetitive process state-space model has the following form over $p = 0, 1, \ldots, \alpha - 1$, $k \geq 0$,

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p) \end{aligned} \tag{3}$$

where on pass k,$x_k(p) \in \mathbb{R}^n$, $u_k(p) \in \mathbb{R}^r$, $y_k(p) \in \mathbb{R}^m$ are the state, input and pass profile vectors respectively. The boundary conditions are $x_k(0) = d_{k+1}$, $k \geq 0$, and $y_0(p) = f(p)$, $p = 0, 1, \ldots, \alpha - 1$, where the entries in d_{k+1} are known constants and those in $f(p)$ known functions of p.

In the repetitive process model (3), the contribution to the current pass dynamics at p only arises from the same point on the previous pass profile. The analysis in this paper will make use of, in particular, stability theory for a wave discrete linear repetitive process Galkowski et al. (2006) described by the following state-space model over $p = 0, 1, \ldots, \alpha - 1$, $k \geq 0$,

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + \sum_{i=-\underline{\gamma}}^{\overline{\gamma}} B_i y_k(p+i) \\ y_{k+1}(p) &= Cx_{k+1}(p) + \sum_{i=-\underline{\gamma}}^{\overline{\gamma}} D_i y_k(p+i) \end{aligned} \tag{4}$$

where $\underline{\gamma}$ and $\overline{\gamma}$ are positive integers and the rest of notation is the same as that for (3). Here the dynamics are more general since the previous pass contribution to the current pass dynamics at any point along the pass comes from a 'window' of points on the previous pass profile.

Recognizing the unique control problem, the stability theory Rogers et al. (2007) for linear repetitive processes is of the BIBO form and, in abstract model terms, requires that a bounded initial profile y_0 produces a bounded sequence of pass profiles $\{y_k\}_{k\geq 0}$ either over the finite pass length or uniformly, that is, for all possible values of this parameter. The former property is termed asymptotic stability and for the abstract model (2) requires the existence of finite real scalars $M_\alpha > 0$ and $\lambda_\alpha \in (0, 1)$ such that $||L_\alpha^k|| \leq M_\alpha \lambda_\alpha^k$, $k \geq 0$, where $|| \cdot ||$ also denotes the induced operator norm. For processes described by (3) it has been shown elsewhere Rogers et al. (2007)) that this property holds if, and only if, $r(D_0) < 1$.

The second form of stability for linear repetitive processes demands that there exists finite real scalars $M_\infty > 0$ and $\lambda_\infty \in (0, 1)$, which are independent of the pass length α, such that $||L_\alpha^k|| \leq M_\infty \lambda_\infty^k$, $k \geq 0$, holds for the abstract model (2). This is known as stability along the pass and the uniform boundedness property in both directions of information propagation, that is, pass-to-pass and along the pass respectively. In the case of processes described by (3), it is shown in Rogers et al. (2007) that stability along the pass holds if and only if i) $r(D_0) < 1$, ii) $r(A) < 1$, and iii) all eigenvalues of $G(z) = C(zI - A)^{-1}B_0 + D_0$ have modulus strictly less than unity for all $|z| = 1$. Hence to ensure stability along the pass in general requires more than stabilization of the state matrix A_d. A simple example is $A = -0.5$, $B = 1, B_0 = 0.5 + \beta$, $C = 1, D = D_0 = 0$, with the real scalar β selected such that $|\beta| \geq 1$.

It is known Galkowski et al. (2006)) that the abstract model based stability theory for linear repetitive processes of the form (3) can also be applied to processes described by (4) using a Lyapunov function approach. This analysis forms the basis for the new design algorithms in this paper and is introduced in context in the next section.

The basic premise in ILC is to improve performance by directly adjusting the input used on each new trial, and often this is expressed in the form

$$u_{k+1}(p) = u_k(p) + \Delta u_{k+1}(p),\ k \geq 0 \tag{5}$$

Hence the problem is to develop an algorithm to select the adjustment $\Delta u_{k+1}(p)$ to be added to the input $u_k(p)$ used on the previous trial

In this paper, the approach used for the forms of $\Delta u_{k+1}(p)$ considered is to first show that the resulting controlled dynamics can be described by a discrete linear repetitive process state-space model of the form (3) or (4) and then apply the stability theory to derive the corresponding control law design algorithm. The current pass state-vector of the resulting repetitive process model in both cases is

$$\eta_{k+1}(p) = x_{k+1}(p-1) - x_k(p-1) \tag{6}$$

Hence

$$\eta_{k+1}(p+1) = A\eta_{k+1}(p) + B\Delta u_{k+1}(p-1) \tag{7}$$

and, since $e_{k+1}(p) - e_k(p) = y_k(p) - y_{k+1}(p)$ and using (6),

$$e_{k+1}(p) - e_k(p) = -CA\eta_{k+1}(p) - CB\Delta u_{k+1}(p-1) \tag{8}$$

Consider also a control law formed by setting

$$\Delta u_{k+1}(p) = K_1\eta_{k+1}(p+1) + K_2 e_k(p+1) \tag{9}$$

in (5), which is a combination of current pass state feedback plus a feedforward term from the previous pass profile. Then (7) and (8) can be written as

$$\begin{aligned} \eta_{k+1}(p+1) &= \hat{A}\eta_{k+1}(p) + \hat{B}_0 e_k(p) \\ e_{k+1}(p) &= \hat{C}\eta_{k+1}(p) + \hat{D}_0 e_k(p) \end{aligned} \tag{10}$$

where

$$\hat{A} = A + BK_1,\ \ \hat{B}_0 = BK_2,\ \ \hat{C} = -C(A + BK_1),\ \ \hat{D}_0 = I - CBK_2 \tag{11}$$

which is of the form (3) and hence the repetitive process stability theory can be applied to this ILC control scheme.

The ILC scheme of (10) is asymptotically stable if, and only if, $r(\hat{D}_0) = r(I - CBK_2) < 1$, which is also the condition obtained by applying 2D discrete linear systems theory to this state-space model Kurek and Zaremba (1993) to ensure trial-to-trial error convergence only. One problem for some ILC control laws is immediate from this result. In particular, this condition cannot hold when the first Markov parameter in the process state-space model is zero, which can be stated as $CB = 0$ or that the system has relative degree greater than unity. As discussed in the introduction to this paper, this problem arises for P-type ILC which is of simple structure and hence very appealing from an applications standpoint. Moreover, it also applies to systems where for some integer $h > 1$ the first non-zero Markov parameter is CA^hB and for processes which are SISO or MIMO.

This paper develops ILC laws to overcome this problem which are of relatively simple structure and can be designed in one step for both trial-to-trial error convergence and along the trial performance. The design algorithms are generalizations of those in Hladowski et al. (2008, 2010) and they are experimentally implemented on the gantry robot used in this previous work (reconfigured to result in a model with zero first Markov parameter). It turns out that results obtained for the SISO case generalize immediately to MIMO and hence for ease of presentation only the former case is treated in detail. For similar reasons, the analysis is first developed for the case when $CB = 0$ and $CAB \neq 0$ is nonzero, and then generalized to $h > 1$.

3 Analysis

3.1 $CB = 0$ *and* $CAB \neq 0$

Consider an example of (1) with $CB = 0$. Then (7) and (8) become

$$\begin{aligned} \eta_{k+1}(p+1) &= A\eta_{k+1}(p) + B\Delta u_{k+1}(p-1) \\ e_{k+1}(p) &= -CA\eta_{k+1}(p) + e_k(p) \end{aligned} \tag{12}$$

or, on substituting from the first equation into the term $-CA\eta_{k+1}(p)$ in the second and applying some routine manipulations,

$$\begin{aligned} \eta_{k+1}(p+1) &= A^2\eta_{k+1}(p-1) + B\Delta u_{k+1}(p-1) + AB\Delta u_{k+1}(p-2) \\ e_{k+1}(p) &= -CA^2\eta_{k+1}(p-1) - CAB\Delta u_{k+1}(p-2) + e_k(p) \end{aligned} \tag{13}$$

The problem now is that this state-space model is not of the form (3) and hence the stability analysis in a repetitive process setting cannot proceed.

It turns out that if the control law (9) is changed then the controlled system state-space model is in the form of a wave linear repetitive process with state-space model (4). Hence the stability theory for this model can be applied and moreover the control law has a relatively simple structure. The starting point is to replace (9) by

$$\Delta u_{k+1}(p) = K_1\eta_{k+1}(p+1) + K_2 e_k(p+2) \tag{14}$$

Then, on introducing,

$$\widehat{\eta}_{k+1}(p) = \begin{bmatrix} \eta_{k+1}(p) \\ \eta_{k+1}(p-1) \end{bmatrix} \tag{15}$$

the controlled dynamics can be written in the form

$$\begin{aligned} \widehat{\eta}_{k+1}(p+1) &= \tilde{A}\widehat{\eta}_{k+1}(p) + \tilde{B}_1 e_k(p+1) + \tilde{B}_0 e_k(p) \\ e_{k+1}(p) &= \tilde{C}\widehat{\eta}_{k+1}(p) + \tilde{D}_1 e_k(p+1) + \tilde{D}_0 e_k(p) \end{aligned} \tag{16}$$

where

$$\begin{aligned} \tilde{A} &= \begin{bmatrix} BK_1 & (A^2 + ABK_1) \\ I & 0 \end{bmatrix} \\ \tilde{B}_1 &= \begin{bmatrix} BK_2 \\ 0 \end{bmatrix} \\ \tilde{B}_0 &= \begin{bmatrix} ABK_2 \\ 0 \end{bmatrix} \\ \tilde{C} &= -\begin{bmatrix} 0 & (CA^2 + CABK_1) \end{bmatrix} \\ \tilde{D}_1 &= 0 \\ \tilde{D}_0 &= (I - CABK_2) \end{aligned}$$

Also this state-space model is a special case of (4) with $\underline{\gamma} = 0$ and $\overline{\gamma} = 1$.

It is known Galkowski et al. (2006)) that the abstract model based stability theory for linear repetitive processes of the form (3) can also be applied to processes described by (4) using a Lyapunov function approach. Consider, therefore, the following candidate Lyapunov function for processes described by (16)

$$V(k,p) = V_1(k,p) + V_2(k,p) \tag{17}$$

where

$$\begin{aligned} V_1(k,p) &= \eta_{k+1}^T(p)Q_0\eta_{k+1}(p) \\ V_2(k,p) &= e_k^T(p+1)P_1 e_k(p+1) + e_k^T(p)P_0 e_k(p) \end{aligned}$$

$Q_0 \succ 0$, $P_1 \succ 0$, $P_0 \succ 0$, and take the associated increment as

$$\Delta V(k,p) = \Delta V_1(k,p) + \Delta V_2(k,p) \tag{18}$$

where

$$\Delta V_1(k,p) = \eta_{k+1}^T(p+1)\ Q_0\ \eta_{k+1}(p+1) - \eta_{k+1}^T(p)\ Q_0\ \eta_{k+1}(p)$$
$$\Delta V_2(k,p) = e_{k+1}^T(p)\ [P_0 + P_1]\ e_{k+1}(p) - \left[e_k^T(p{+}1)\ \ e_k^T(p)\right] \operatorname{diag}(P_1,\ P_0) \begin{bmatrix} e_k(p+1) \\ e_k(p) \end{bmatrix}$$

Then (4) is stable along the trial if

$$\Delta V(k,p) < 0$$

for all non-zero $\eta_{k+1}(p)$ and $e_k(p)$.

Introduce the following notation for the controlled system state-space model (16)

$$\Phi = \begin{bmatrix} \tilde{A} & \tilde{B}_0 & \tilde{B}_1 \\ \tilde{C} & \tilde{D}_0 & \tilde{D}_1 \\ \tilde{C} & \tilde{D}_0 & \tilde{D}_1 \end{bmatrix} \tag{19}$$

and

$$\theta = \begin{bmatrix} Q_0 & 0 & 0 \\ 0 & P_0 & 0 \\ 0 & 0 & P_1 \end{bmatrix} \tag{20}$$

Then the following result is proved by obvious extension of the proof of the main stability result in Galkowski et al. (2006) and hence it is omitted here.

Theorem 3.1 : *The wave repetitive process of (16) is stable along the trial if there exist matrices $Q_0 \succ 0$, $P_1 \succ 0$, and $P_0 \succ 0$*

$$\Phi^T \theta \Phi - \theta \prec 0 \tag{21}$$

where Φ and θ are given by (19) and (20) respectively.

The following result solves the ILC design problem considered in this section.

Theorem 3.2 : *An ILC scheme which can be written in the form (16) is stable along the trial if there exist matrices $X_1 \succ 0$ and $X_2 \succ 0$, N_1 and N_2 such that the LMI*

$$\begin{bmatrix} -X_1 & * & * & * & * & * & * & * \\ 0 & -X_1 & * & * & * & * & * & * \\ 0 & 0 & -X_2 & * & * & * & * & * \\ 0 & 0 & 0 & -X_2 & * & * & * & * \\ BN_1 & A^2X_1 + ABN_1 & ABN_2 & BN_2 & -X_1 & * & * & * \\ X_1 & 0 & 0 & 0 & 0 & -X_1 & * & * \\ 0 & -CA^2X_1 - CABN_1 & X_2 - CABN_2 & 0 & 0 & 0 & -X_2 & * \\ 0 & -CA^2X_1 - CABN_1 & X_2 - CABN_2 & 0 & 0 & 0 & 0 & -X_2 \end{bmatrix} \preceq 0 \tag{22}$$

is feasible.

If this condition holds, the control law matrices K_1 and K_2 of (14) can be computed from

$$K_1 = N_1 X_1^{-1},\ K_2 = N_2 X_2^{-1} \tag{23}$$

Proof Application of the Schur's complement formula to (21) yields

$$\begin{bmatrix} -Q_0 & * & * & * & * & * \\ 0 & -P_0 & * & * & * & * \\ 0 & 0 & -P_1 & * & * & * \\ \tilde{A} & \tilde{B}_0 & \tilde{B}_1 & -Q_0^{-1} & 0 & * \\ \tilde{C} & \tilde{D}_0 & \tilde{D}_1 & 0 & -P_0^{-1} & * \\ \tilde{C} & \tilde{D}_0 & \tilde{D}_1 & 0 & 0 & -P_1^{-1} \end{bmatrix} \prec 0 \tag{24}$$

Applying obvious congruence transforms and introducing

$$\begin{aligned} Q_0^{-1} &= \begin{bmatrix} X_1 & 0 \\ 0 & X_1 \end{bmatrix} \\ P_0^{-1} &= X_2,\ P_1^{-1} = X_2 \end{aligned} \tag{25}$$

now yields (after application of routine algebraic manipulations which are omitted here)

$$\begin{bmatrix} -X_1 & * & * & * & * & * & * & * \\ 0 & -X_1 & * & * & * & * & * & * \\ 0 & 0 & -X_2 & * & * & * & * & * \\ 0 & 0 & 0 & -X_2 & * & * & * & * \\ BK_1X_1 & A^2X_1 + ABK_1X_1 & ABK_2X_2 & BK_2X_2 & -X_1 & * & * & * \\ X_1 & 0 & 0 & 0 & 0 & -X_1 & * & * \\ 0 & -CA^2X_1 - CABK_1X_1 & X_2 - CABK_2X_2 & 0 & 0 & 0 & -X_2 & * \\ 0 & -CA^2X_1 - CABK_1X_1 & X_2 - CABK_2X_2 & 0 & 0 & 0 & 0 & -X_2 \end{bmatrix} \preceq 0 \tag{26}$$

Finally, introducing

$$N_1 = K_1 X_1,\ \ N_2 = K_2 X_2 \tag{27}$$

yields (22) and the proof is complete. □

To apply the control law (14), simple algebraic manipulations we obtain

$$u_k(p) = u_{k-1}(p) + K_1\left(x_k(p) - x_{k-1}(p)\right) + K_2\left(y_{ref}(p+2) - y_{k-1}(p+2)\right) \tag{28}$$

This control law does require access to state vector information but it does have a relatively simple structure which will be even more the case when design based on process output information is considered in the next section.

3.2 *Generalization to $CA^hB \neq 0$*

The analysis below shows that the result of the previous sub-section for $CB = 0$ extends to the case when $CA^{\iota}B = 0$, $\iota = 0, 1, \ldots, h-1$, $h \geq 1$, $CA^hB \neq 0$. Here the case when $CB = 0$ and $CAB = 0$ but $CA^2B \neq 0$ only is detailed since the others follow as natural generalizations.

Writing (7) and (8 for the case when $CB = 0$ and $CAB = 0$ and then substituting from the first of the resulting equations into the term $-CA\eta_{k+1}(p)$ in the second yields, after some routine manipulations,

$$\begin{aligned}\eta_{k+1}(p+1) &= A^2\eta_{k+1}(p-1) + B\Delta u_{k+1}(p-1) + AB\Delta u_{k+1}(p-2)\\ e_{k+1}(p) &= -CA^3\eta_{k+1}(p-2) - CA^2B\Delta u_{k+1}(p-3) + e_k(p)\end{aligned}$$

Consider also the following control law, which a modification of (9) to account for the additional zero Markov parameter

$$\Delta u_{k+1}(p) = K_1\eta_{k+1}(p+1) + K_2\eta_{k+1}(p) + K_3e_k(p+2) + K_4e_k(p+3) \tag{29}$$

This results, after some routine manipulations, in the controlled process state-space model

$$\begin{aligned}\eta_{k+1}(p+1) &= BK_1\eta_{k+1}(p) + (A^2 + BK_2 + ABK_1)\eta_{k+1}(p-1) + BK_4e_k(p+2)\\ &\quad +(BK_3 + ABK_4)e_k(p+1) + ABK_3e_k(p)\\ e_{k+1}(p) &= -(CA^3 + CA^2BK_1)\eta_{k+1}(p-2) - CA^2BK_2\eta_{k+1}(p-3)\\ &\quad + (I - CA^2BK_4)e_k(p) - CA^2BK_3e_k(p-1)\end{aligned} \tag{30}$$

which is a special case of the wave discrete linear repetitive process state-space model (14) with $\underline{\gamma} = 0$ and $\overline{\gamma} = 2$. However, proceeding directly from (30) will lead to very conservative results and to increase the design freedom, this state-space model is first rewritten as follows.

Introduce

$$\widehat{\eta}_{k+1}(p) = \begin{bmatrix}\eta_{k+1}(p)\\ \eta_{k+1}(p-1)\\ \eta_{k+1}(p-2)\\ \eta_{k+1}(p-3)\end{bmatrix} \tag{31}$$

and

$$\begin{aligned}\tilde{A} &= \begin{bmatrix}BK_1 & A^2 + BK_2 + ABK_1 & 0 & 0\\ \phi_1 I & (1-\phi_1)BK_1 & (1-\phi_1)(A^2 + BK_2 + ABK_1) & 0\\ 0 & \phi_2 I & (1-\phi_2)BK_1 & (1-\phi_2)(A^2 + BK_2 + ABK_1)\\ 0 & 0 & I & 0\end{bmatrix}\\ \tilde{B}_2 &= \begin{bmatrix}BK_4\\ 0\\ 0\\ 0\end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix}BK_3 + ABK_4\\ (1-\phi_1)BK_4\\ 0\\ 0\end{bmatrix}, \quad \tilde{B}_0 = \begin{bmatrix}ABK_3\\ (1-\phi_1)(BK_3 + ABK_4)\\ (1-\phi_2)BK_4\\ 0\end{bmatrix}\\ \tilde{B}_{-1} &= \begin{bmatrix}0\\ (1-\phi_1)ABK_3\\ (1-\phi_2)(BK_3 + ABK_4)\\ 0\end{bmatrix}, \quad \tilde{B}_{-2} = \begin{bmatrix}0\\ 0\\ (1-\phi_2)ABK_3\\ 0\end{bmatrix}\\ \tilde{C} &= \begin{bmatrix}0 \; 0 \; -CA^3 - CA^2BK_1 \; CA^2BK_2\end{bmatrix}, \quad \tilde{D}_2 = 0, \quad \tilde{D}_1 = 0\\ \tilde{D}_0 &= I - CA^2BK_4, \quad \tilde{D}_{-1} = -CA^2BK_3\end{aligned} \tag{32}$$

where $\phi_1 \in [0,1]$ and $\phi_2 \in [0,1]$. Then the controlled dynamics are equivalently described by

$$\begin{aligned}\widehat{\eta}_{k+1}(p+1) &= \tilde{A}\widehat{\eta}_{k+1}(p) + \tilde{B}_2 e_k(p+2) + \tilde{B}_1 e_k(p+1) + \tilde{B}_0 e_k(p)\\ &\quad + \tilde{B}_{-1} e_k(p-1) + \tilde{B}_{-2} e_k(p-2)\\ e_{k+1}(p) &= \tilde{C}\widehat{\eta}_{k+1}(p) + \tilde{D}_2 e_k(p+2) + \tilde{D}_1 e_k(p+1) + \tilde{D}_0 e_k(p) + \tilde{D}_{-1} e_k(p-1)\end{aligned} \tag{33}$$

which is again a special case of the wave discrete linear repetitive process state-space model (4) with $\underline{\gamma} = 0$ and $\overline{\gamma} = 2$. The previous theorem can now be directly applied to give the following result.

Theorem 3.3: *An ILC scheme which can be written in the form (33) is stable along the trial if for some real scalars $\delta_x > 0$, $x \in \{-2,-1,0,1,2\}$ and $\phi_1 \in [0,1]$, $\phi_2 \in [0,1]$ there exist matrices $N_{11} \succ 0$, $N_0 \succ 0$ and R_1, R_2, R_3, R_4 such that the LMI*

$$\begin{bmatrix} -I_4 \otimes N_{11} & * & * & * \\ 0 & -\Theta_{22} & * & * \\ -\Theta_{31} & -\Theta_{32} & -I_4 \otimes N_{11} & * \\ -\Theta_{41} & -\Theta_{42} & 0 & -\Theta_{22} \end{bmatrix} \prec 0 \tag{34}$$

where

$$\Theta_{22} = \Delta \otimes N_0$$

$$\Theta_{31} = \begin{bmatrix} BR_1 & A^2N_{11} + BR_2 + ABR_1 & 0 & 0 \\ \phi_1 N_{11} & (1-\phi_1)BR_1 & (1-\phi_1)(A^2N_{11} + BR_2 + ABR_1) & 0 \\ 0 & \phi_2 N_{11} & (1-\phi_2)BR_1 & (1-\phi_2)(A^2N_{11} + BR_2 + ABR_1) \\ 0 & 0 & N_{11} & 0 \end{bmatrix}$$

$$\Theta_{32} = \begin{bmatrix} 0 & 0 & ABR_3 & \delta_1(BR_3 + ABR_4) & \delta_2 BR_4 \\ 0 & (1-\phi_1)\delta_{-2}ABR_3 & (1-\phi_1)(BR_3 + ABR_4) & (1-\phi_1)\delta_1 BR_4 & 0 \\ (1-\phi_2)\delta_{-2}ABR_3 & (1-\phi_2)\delta_{-1}(BR_3 + ABR_4) & (1-\phi_2)BR_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Theta_{4i} = 1_5 \otimes \chi_i, \quad i = 1,2$$

where

$$\Delta = \begin{bmatrix} \delta_{-2} & 0 & 0 & 0 & 0 \\ 0 & \delta_{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \delta_1 & 0 \\ 0 & 0 & 0 & 0 & \delta_2 \end{bmatrix}$$

$$\chi_1 = \begin{bmatrix} 0 \; 0 \; -CA^3N_{11} - CABR_1 \; CA^2BR_2 \end{bmatrix}$$

$$\chi_2 = \begin{bmatrix} 0 \; -\delta_{-1}CA^2BR_3 \; N_0 - CA^2BR_4 \; 0 \; 0 \end{bmatrix}$$

and $1_5 = [1, 1, 1, 1, 1]^T$ *is feasible. If this condition holds, the control law matrices* K_1*,* K_2*,* K_3*,* K_4 *of (29) can be computed using*

$$K_1 = R_1N_{11}^{-1},\; K_2 = R_2N_{11}^{-1},\; K_3 = R_3N_0^{-1},\; K_4 = R_4N_0^{-1} \tag{35}$$

Remark 1: The scalars $\delta_x > 0$, $x \in \{-2, -1, 0, 1, 2\}$ and $\phi_1 \in [0, 1]$, $\phi_2 \in [0, 1]$ in this last result are free parameters available to give extra freedom when solving (34) and need to be selected before solving the LMI.

The analysis given above extends in a natural manner to the case when the first non-zero Markov parameter is $CA^hB \neq 0$, $h \geq 2$, and hence the details are omitted. In such cases, results which extend Theorem 3.3 to the considered case may be necessary to achieve the required computational efficiency. An obvious area for further research is to enhance/extend the results given here to and arbitrary value of h, but for low values of this parameter the new results in this paper can be highly effective both computationally and in experimental application.

3.3 *Output Signal only Design*

In many practically-relevant applications some, or all, entries in the state vector may not be available for measurement. Feasible ways of proceeding in such cases is to use a suitably designed observer or to consider design using the process output only. In this section the latter option is developed, focusing on the case when $CB = 0$ and $CAB \neq 0$ as the others again follow as natural generalizations. The control law is

$$\Delta u_{k+1}(p) = \hat{K}_1\mu_{k+1}(p + 1) + K_2e_k(p + 2) \tag{36}$$

with

$$\mu_k(p) = y_k(p - 1) - y_{k-1}(p - 1) = C\eta_k(p) \tag{37}$$

Substituting (37) into (36) yields

$$\Delta u_{k+1}(p) = \hat{K}_1C\eta_{k+1}(p + 1) + K_2e_k(p + 2) \tag{38}$$

and let

$$K_1 = \hat{K}_1C \tag{39}$$

Then (38) has exactly the same form as (14) with the only difference being the value of K_1. Hence the following result.

Theorem 3.4 : *An ILC scheme which can be written in the form (16) with the control law (38) is stable along the trial if for some scalar $\phi \in (0,1)$ there exist matrices $X_1 \succ 0$, $X_2 \succ 0$, $Y_1 \succ 0$ N_1, and N_2 such that the following LMI with constraints*

$$\begin{bmatrix} -X_1 & * & * & * & * & * & * & * \\ 0 & -X_1 & * & * & * & * & * & * \\ 0 & 0 & -X_2 & * & * & * & * & * \\ 0 & 0 & 0 & -X_2 & * & * & * & * \\ BN_1C & A^2X_1 + ABN_1C & ABN_2 & BN_2 & -X_1 & * & * & * \\ \phi X_1 & (1-\phi)AX_1 + (1-\phi)BN_1C & (1-\phi)BN_2 & 0 & 0 & -X_1 & * & * \\ 0 & -CA^2X_1 - CABN_1C & X_2 - CABN_2 & 0 & 0 & 0 & -X_2 & * \\ 0 & -CA^2X_1 - CABN_1C & X_2 - CABN_2 & 0 & 0 & 0 & 0 & -X_2 \end{bmatrix} \prec 0 \tag{40}$$

$$Y_1C = CX_1$$

is feasible.

If this condition holds, the control law matrices $\hat{K}_1$ and K_2 of (38) can be computed from

$$\hat{K}_1 = N_1Y_1^{-1},\ K_2 = N_2X_2^{-1} \tag{41}$$

Proof First, use the result of Theorem 3.2 with K_1 given by (39) to obtain (26) for this case, where the similar approach as in Theorem 3.3 has been applied. Then substitute (39) into this last formula and let $Y_1C = CX_1$, where Y_1 is an unknown to be determined, to obtain the following matrix inequality with equality constraints

$$\begin{bmatrix} -X_1 & * & * & * & * & * & * & * \\ 0 & -X_1 & * & * & * & * & * & * \\ 0 & 0 & -X_2 & * & * & * & * & * \\ 0 & 0 & 0 & -X_2 & * & * & * & * \\ B\hat{K}_1Y_1C & A^2X_1 + AB\hat{K}_1Y_1C & ABK_2X_2 & BK_2X_2 & -X_1 & * & * & * \\ \phi X_1 & (1-\phi)AX_1 + (1-\phi)B\hat{K}_1Y_1C & (1-\phi)BK_2X_2 & 0 & 0 & -X_1 & * & * \\ 0 & -CA^2X_1 - CAB\hat{K}_1Y_1C & X_2 - CABK_2X_2 & 0 & 0 & 0 & -X_2 & * \\ 0 & -CA^2X_1 - CAB\hat{K}_1Y_1C & X_2 - CABK_2X_2 & 0 & 0 & 0 & 0 & -X_2 \end{bmatrix} \prec 0$$
$$Y_1C = CX_1 \tag{42}$$

Finally, introduce

$$N_1 = \hat{K}_1Y_1,\ \ N_2 = K_2X_2$$

to obtain (40) and the proof is complete. □

The scalar $\phi \in (0,1)$ in this last result gives extra design freedom. Also this result extends to the case when $CA^{\iota}B = 0$, $\iota = 0, 1, \ldots, h-1$, $h \geq 1$, follows in a routine manner and hence the details are omitted here.

To apply the control law of (38), simple algebraic manipulations give

$$u_k(p) = u_{k-1}(p) + \hat{K}_1\left(y_k(p) - y_{k-1}(p)\right) + K_2\left(y_{ref}(p+2) - y_{k-1}(p+2)\right) \tag{43}$$

Here the last term is phase advance on the previous trial error, where in ILC such a term is well known in simple structure algorithms. Such an advance appears in the discrete-time implementation of the derivative ILC algorithm Arimoto et al. (1985) where it was used to extend the applicability of the original ILC algorithm Arimoto et al. (1984). More recently a variable advance has been considered in Wang (1999), Wang and Longman (1996) and found to lead to accurate tracking in practice on a range of systems Freeman et al. (2007), Wallen et al. (2008). The second term is proportional in nature acting on the error between the current and previous trials at p. and $p-1$ respectively. Whilst use of current trial data has appeared in many approaches to manipulate the plant dynamics along-the-trial Chen et al. (2007), Norrlof and Gunnarsson (1999) and has been found to increase initial tracking and disturbance rejection Ratcliffe et al. (2005), the coupling of previous and current trial data is a novel addition to this class of updates. For the case of $h > 1$, the structure of the control law will be the same but the argument of the pass profile entries that form the last term will change from $p+2$ to $p+h+1$ (in this case no control will be possible over the first h steps).

The control law (43) does not require access to state vector implementation but does require access to output information which is assumed to be either noise free or that any disturbances present are of sufficiently low levels that they can be neglected. If this is not the case, then the results should easily extend to a stochastic setting. The control law here reduces to the selection of two scalars and since the LMI involved generates a family of solutions there is the opportunity for tuning the basic design. First, however, it of applications relevance to discover if the results predicted for either the state or output cases translate to experimental implementation, regarded as an essential step prior to end-user take-up.

As an example, consider the case when

$$A = \begin{bmatrix} 0.469 & 0.0013 & 0.145 \\ 0.304 & 0.063 & 0.186 \\ 0.805 & 0.344 & 0.298 \end{bmatrix}, \quad B = \begin{bmatrix} 0.567 \\ 0.328 \\ -0.450 \end{bmatrix}, \quad C = \begin{bmatrix} 0.0964 & 0.854 & 0.728 \end{bmatrix}$$

where $CB =$ and $CAB \neq 0$. Applying Theorem 3.4 gives for $\phi = 0.5$

$$\hat{K}_1 = -0.865, \;\; K_2 = 0.687$$

Figure 1 shows the reference signal and Figure 2 the sum of the squared error on each trial plotted against trial number and fast trial-to-trial error convergence occurs. Figures 3–3.3 show the evolution of the error, input and output signals and confirm that this design is capable of producing high quality performance without requiring excessive control effort. In the next section, experimental application to a gantry robot is described.

4 Experimental Verification

The gantry robot, shown in Figure 6, is a commercially available system found in a variety of industrial applications whose task is to place a sequence of objects onto a moving conveyor under synchronization. The sequence of operations is as follows: the robot collects the object from a specified location, moves until it is synchronized (in terms of both position and speed) with the conveyor, places the object on the conveyor, and then returns to the same starting location to collect the next object and so on. This is sometimes referred to as pick and place and is clearly suitable for the application of ILC. This robot system has previously been used for testing and comparing the performance of other ILC algorithms, see, for example, Ratcliffe et al. (2006).

The gantry robot can be treated as three SISO systems (one for each axis) which can operate simultaneously to locate the end effector anywhere within a cuboid work envelope. The lowest axis, X; moves in the horizontal plane, parallel to the conveyor beneath. The Y-axis is mounted on the X-axis and moves in the horizontal plane, but perpendicular to the conveyor. The Z-axis is the shorter vertical axis mounted on the Y-axis. The X and Y-axes consist of linear brushless dc motors, while the Z-axis is a linear ball-screw stage powered by a rotary brushless dc motor.

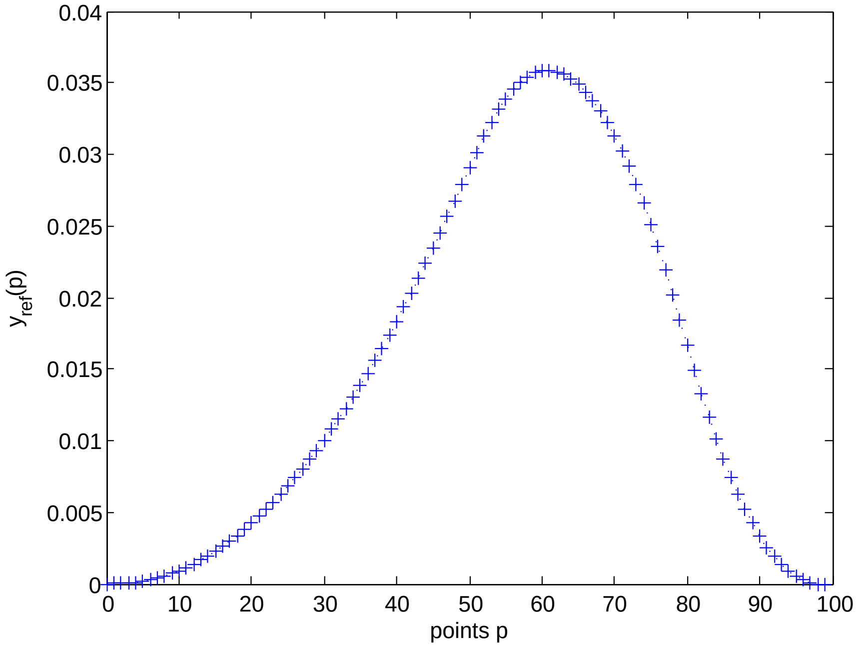

Figure 1. Reference signal.

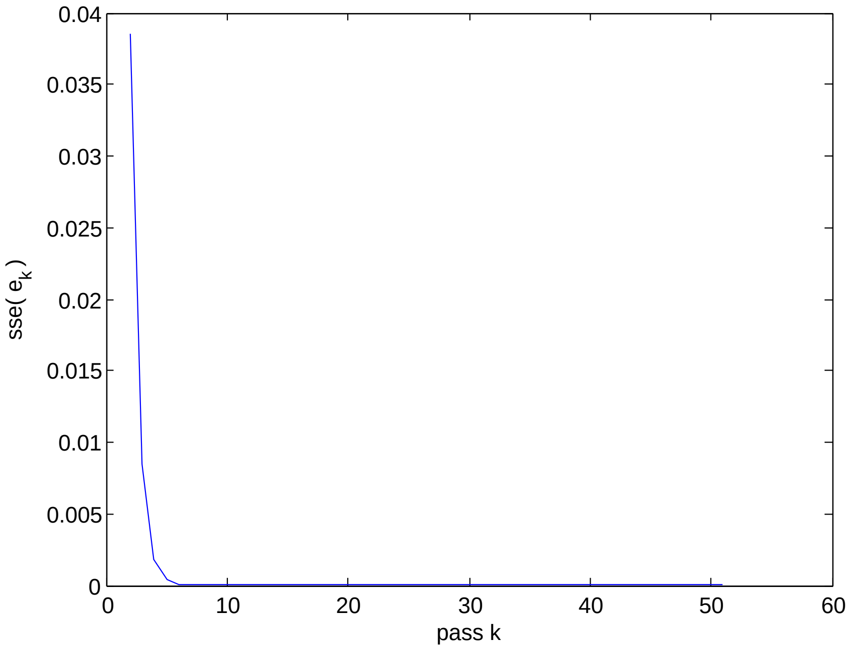

Figure 2. Sum of squared errors plotted against trial number.

All motors are energized by performance matched dc amplifiers. Axis position is measured by means of linear or rotary optical incremental encoders as appropriate.

Each axis of the gantry robot has been modeled based on frequency response tests where, since the axes are orthogonal, it is assumed that there is minimal interaction between them. Here only the final result for the Z-axis is given (the modeling for the other two axes can be found in Ratcliffe et al. (2005)) in the form of the 3rd order transfer-function

$$G_z(s) = \frac{15.8869(s + 850.3)}{s(s^2 + 707.6s + 3.377 \times 10^5)} \tag{44}$$

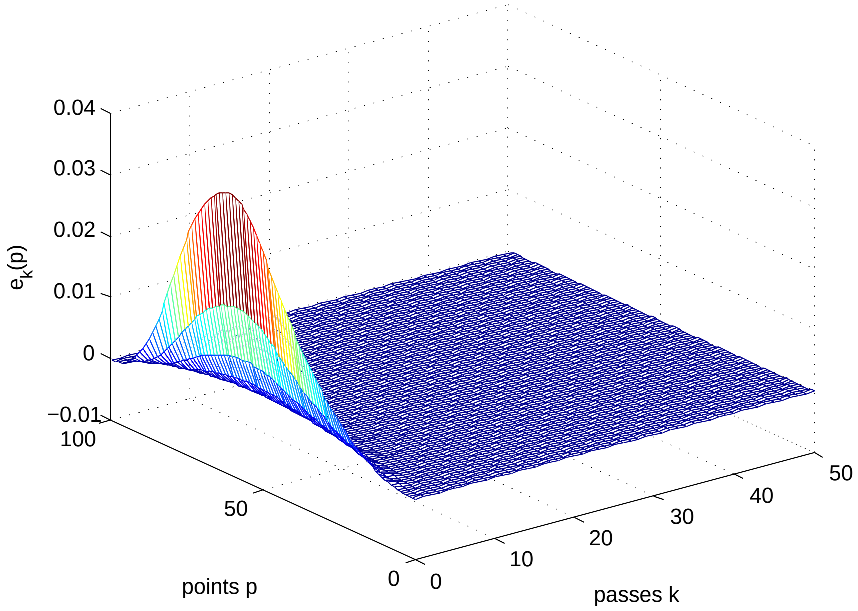

Figure 3. Evolution of the error signal.

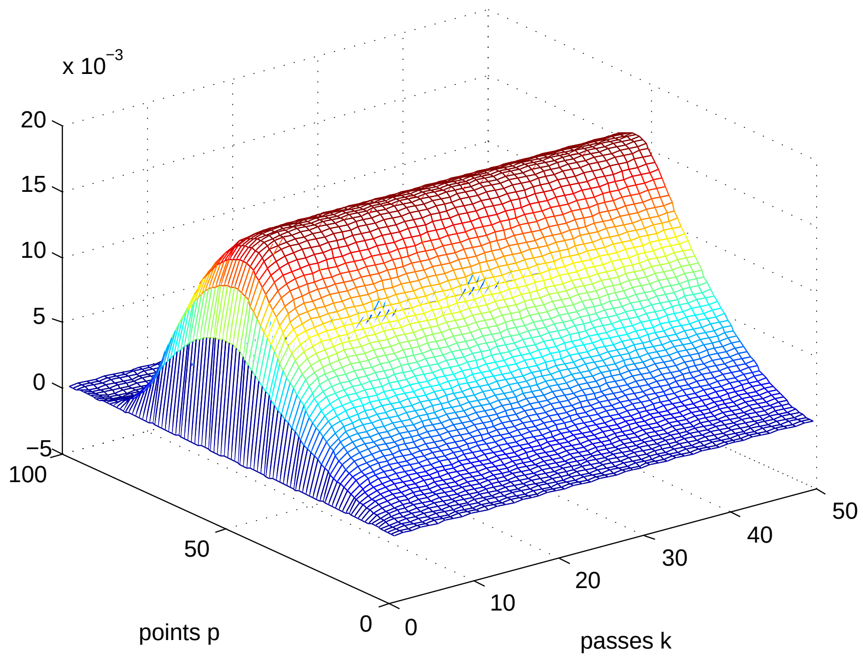

Figure 4. Evolution of the input signal.

This robot system is designed to emulate a pick and place task which arises in many applications to which ILC is applicable. In operation, the robot must undertake the following operations in synchronization with a conveyor system: i) collect an object from a fixed location, ii) transfer it over a finite duration, iii) place it on the moving conveyor, iv) return to the original location for the next object, and then v) repeat the previous four steps for as many objects as required. A desired 3D trajectory is shown in Fig. 7.

Discretization of (44) using the zero-order hold method with a sampling time of $T_s = 0.01$ sec

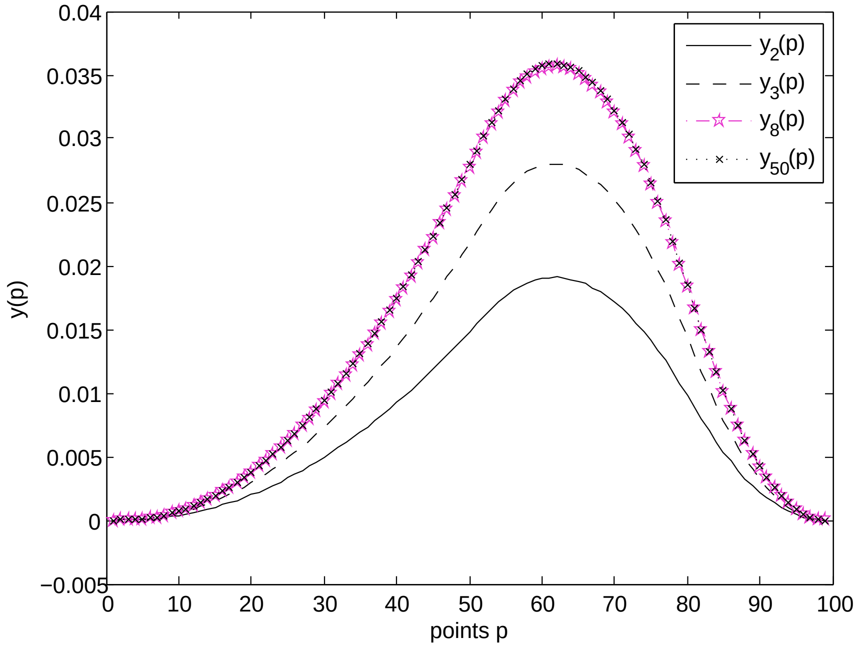

Figure 5. Evolution of the output signal.

Figure 6. The multi-axis gantry robot with the axes marked.

gives the z transfer-function

$$G_z(z) = \frac{0.00036482(z^2 + 0.09791z + 0.005951)}{(z-1)(z^2 + 0.005922z + 0.0008451)}$$

A one sample delay is produced by the action of a zero-order hold, contained within the real-time control card, being fed to the differential equation describing the plant. Hence for design we replace $G_z(z)$ by

$$\hat{G}_z(z) = \frac{0.00036482(z^2 + 0.09791z + 0.005951)}{z(z-1)(z^2 + 0.005922z + 0.0008451)} \tag{45}$$

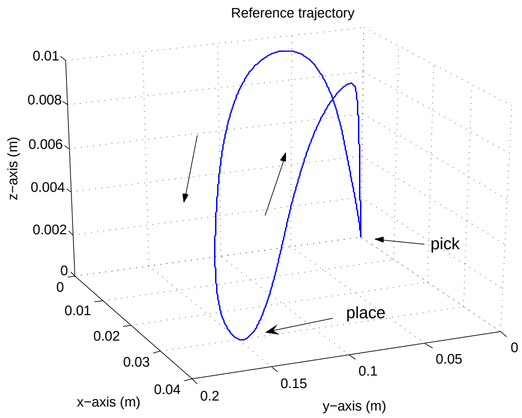

Figure 7. The 3D reference trajectory.

with corresponding state-space model matrices

$$
\begin{aligned}
A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1.0478 & 0 \\ 0 & 0 & -0.0030 & 1 \\ 0 & 0 & -0.0008 & -0.0030 \end{bmatrix} \\
B &= \begin{bmatrix} 0 & 0 & 0 & 0.0313 \end{bmatrix}^T \\
C &= \begin{bmatrix} 0.0001 & 0.0122 & 0.0117 & 0 \end{bmatrix}
\end{aligned} \tag{46}
$$

where $CB = 0$ and $CAB = 0.0003648$.

Solving the LMI of (22) and using (23) gives the control law matrices

$$
\begin{aligned}
K_1 &= \begin{bmatrix} 0.8164 & -5.028 & -6.691 & -5.315 \end{bmatrix} \\
K_2 &= 12.7
\end{aligned} \tag{47}
$$

The Z-axis component of the 3D reference trajectory with 200 samples, hence $\alpha = 200$, is shown in Fig. 8. and the experimentally measured trial-to-trial error $e_k(p) = y_{ref}(p) - y_k(p)$ over

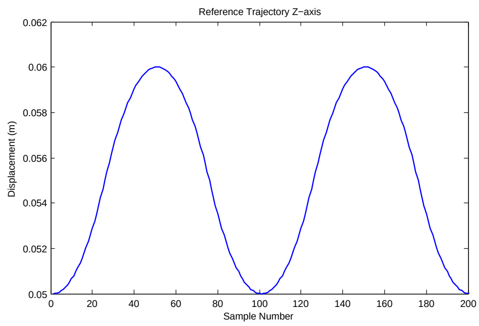

Figure 8. The Z-axis component of the reference signal.

200 trials is shown in Fig. 9. The control input sequence is acceptable for implementation and the progression of the trial outputs $y_k(p)$ is shown in Fig. 10 Finally, the mean squared error over 200 trials is shown in Fig. 11.

Analyzing these experimental results, it is clear that trial-to-trial error convergence is possible under experimental conditions with acceptable along the trial dynamics, where Fig 12 shows the input (top plot), output (middle plot) and error (bottom plot) on trial 200. For this particular case, the convergence rate is somewhat slow and how to increase this, without sacrificing along

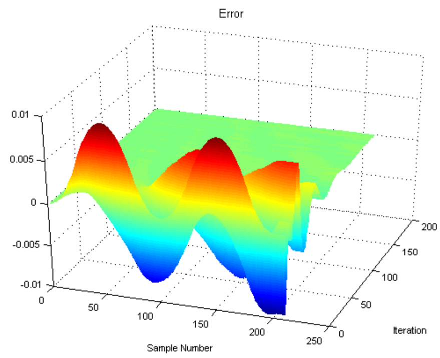

Figure 9. The trial-to-trial error.

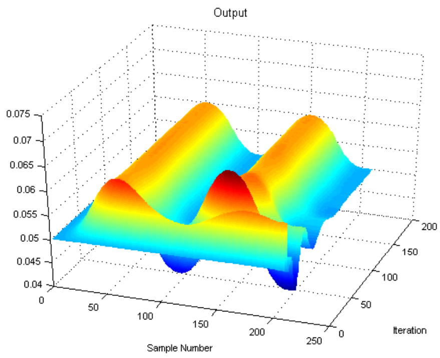

Figure 10. Trial outputs.

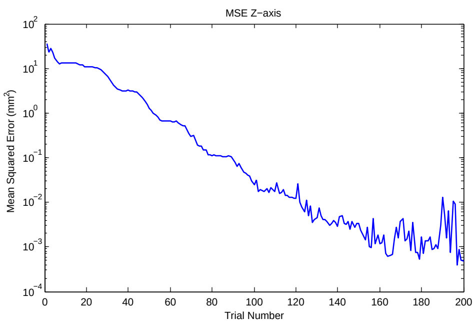

Figure 11. The mean squared error over 200 trials.

the trial performance, is the subject of on-going work. These results do, however, confirm that the ILC design algorithm here can be applied in both simulation and experiment.

5 Conclusions

This paper has considered the design of ILC schemes in a 2D linear systems setting and, in particular, the theory of discrete linear repetitive processes. This releases a stability theory for application which demands uniformly bounded along the trial dynamics (whereas previous approaches only demand bounded dynamics over the finite trial length). It has been shown that this approach leads to a stability condition expressed in terms of an LMI with immediate formulas for computing the control law matrices for the widely encountered case of a SISO discrete linear plant state-space model where the first Markov parameter (or indeed the first $h > 1$) is zero.

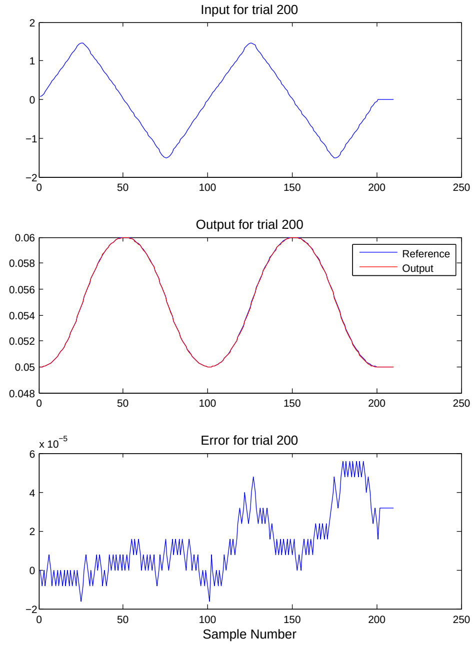

Figure 12. Along the trial performance on trial number 200, top plot: input, middle plot: output and reference signal, bottom plot: error.

Many designs have been proposed for this problem in the literature but none of these have been experimentally verified or are able to deal with any other performance specification beyond trial-to-trial error convergence. The results here establish the basic feasibility of this approach in terms of both theory and experimentation. There is a significant degree of flexibility in the resulting design algorithm and current work is undertaking a detailed investigation of how this can be fully exploited.

The analysis in this paper deals with the SISO case and it is clear that the MIMO case follows immediately on replacing row and column vectors by matrices as appropriate. Experimental verification, given reliable computations for the design phase, would require the availability of a suitable experimental facility. Finally, the question of how to guarantee monotonic trial-to-trial error convergence for these designs should be considered.

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