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RESEARCH ARTICLE

Lyapunov stability of 2D *finite-dimensional behaviors*

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In this paper we investigate a Lyapunov approach to the stability of finite-dimensional $2D$ systems. We use the behavioral framework and consider a notion of stability following the ideas in Pillai and Shankar (1998), Rocha (2008), Valcher (2000). We characterize stability in terms of the existence of a (quadratic) Lyapunov function and provide a constructive algorithm for the computation of all such Lyapunov functions.

Keywords: 2-D system; Lyapunov function; quadratic difference form; behavioral approach;

1 Introduction

The stability of two dimensional ($2D$) systems has been the subject of extensive investigation in the past decades; among these research efforts, some have also been focused on the computation of Lyapunov functions. Past research has predominantly been concerned with systems whose set of trajectories is infinite-dimensional, and almost exclusively has concerned specific class of models, for example Fornasini-Marchesini or Roesser models (see Fornasini and Marchesini (1980), Lu and Lee (1985)). Moreover, in those investigations a specific (usually nonnegative quarter-plane) notion of causality has been assumed. In this paper we follow the behavioral approach: we study the stability of $2D$ systems described by higher-order difference equations without reference to special representations; the central object of interest in our investigation is the set of all admissible trajectories of the system, the *behavior*, rather than any of its specific representations. Following the pioneering approach of Valcher (2000), stability is accordingly defined at the level of trajectories, although we will be using a different but ultimately equivalent definition to that proposed in Valcher (2000) for finite dimensional case. We also adopt the eminently reasonable position proposed in Valcher (2000) to let the system dynamics themselves dictate what notion of causality is most appropriate for the case at hand.

A Lyapunov analysis of stability of *infinite-dimensional square* $2D$ behaviors has been presented in Kojima et al. (2010); in this paper we concentrate our attention on the case of finite-dimensional $2D$ discrete behaviors, i.e. finite-dimensional subspaces of the set of trajectories from $\mathbb{Z}^2$ to $\mathbb{R}^{\mathrm{w}}$. We define a finite-dimensional $2D$ system stable if all trajectories of the behavior go to zero along every "discrete line" in a cone, which without loss of generality we take to be the first orthant of the lattice $\mathbb{Z}^2$; this is a different but equivalent definition from that adopted in section 3, Def. 3.1 of Valcher (2000), and follows the approach of Napp Avelli and Rocha (2010), Pillai and Shankar (1998), Rocha (2008). The main result of the paper is a characterization of

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stability for finite-dimensional $2D$ behaviors in term of the existence of a *Lyapunov function*, defined as a quadratic function of the system variables and their $2D$ shifts which is positive along each discrete line in the first orthant, and whose increment along each such line is negative. In this paper we also give necessary and sufficient conditions for a quadratic function of the system variables and their $2D$ shifts to be a Lyapunov function; and we illustrate an algorithm to compute a Lyapunov function for a given finite-dimensional $2D$ behavior.

In the following we make extensive use of the concepts and calculus of $2D$ *quadratic difference forms* (see Kojima and Takaba (2006)), and their association with four-variable polynomial matrices. We will also use extensively the concepts and terminology of the behavioral approach to $2D$ systems. In order to make the paper self-contained we have included some background material in section 2; the reader interested in a more thorough introduction to $2D$ behavioral system theory is referred to Pillai and Willems (2002), Rocha (1990), Valcher (2000). The main result of the paper is illustrated in section 3. In section 4 we outline an algorithm for the construction of Lyapunov functions.

2 Preliminaries

We consider sets $\mathcal{B}$ of trajectories defined over $\mathbb{Z}^2$ that can be described by a set of linear constant coefficient partial difference equations, i.e.,

$$\mathcal{B} = \ker R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}) \subseteq (\mathbb{R}^{\mathrm{w}})^{\mathbb{Z}^2} , \tag{1}$$

where σ_i's are the 2D shift operators defined by

$$\sigma_i w(k_1, k_2) = w((k_1, k_2) + e_i) ,$$

for $(k_1, k_2) \in \mathbb{Z}^2$ and e_i the ith element of the canonical basis of $\mathbb{R}^2$, $i = 1, 2$; and $R(s_1, s_1^{-1}, s_2, s_2^{-1})$ is a $2D$ $(\mathrm{p} \times \mathrm{w})$-dimensional Laurent-polynomial matrix. We call (1) a *kernel representation* of the *behavior* $\mathcal{B}$.

We now introduce finite-dimensional behaviors, and briefly discuss their representation by means of state equations.

Given a full column rank Laurent-polynomial matrix $R \in \mathbb{R}^{\mathrm{p}\times\mathrm{w}}[s_1, s_1^{-1}, s_2, s_2^{-1}]$, we define its *Laurent variety* (or simply variety) as

$$\mathcal{V}(R) := \{(\lambda_1, \lambda_2) \in (\mathbb{C} \setminus 0)^2 \mid \mathrm{rank}(R(\lambda_1, \lambda_2)) < \mathrm{rank}(R) = \mathrm{w}\},$$

where $\mathrm{rank}(R(\lambda_1, \lambda_2))$ is the rank of the complex matrix $R(\lambda_1, \lambda_2)$, while $\mathrm{rank}(R)$ is the rank of the Laurent-polynomial matrix R. It can be shown Zerz (1996) that any two different representations of $\mathcal{B}$ share the same Laurent variety; consequently in the following we refer to the *variety of* $\mathcal{B}$, denoted by $\mathcal{V}(\mathcal{B})$, as the Laurent variety of any of its kernel representations. It is well known (see Bisiacco and Valcher (2005), Rocha (1990)) that a behavior $\mathcal{B}$ is finite dimensional (when considered as a subspace of the vector space over $\mathbb{R}$ consisting of all trajectories from $\mathbb{Z}^2$ to $\mathbb{R}^{\mathrm{w}}$) if and only if $\mathcal{V}(\mathcal{B})$ consists of a finite number of points, or equivalently if $\mathcal{B}$ admits a right factor prime representation (see Fornasini and Valcher (1997) for a definition).

It was shown in Fornasini et al. (1993) that for every finite dimensional behavior $\mathcal{B}$ there exist a *hybrid representation* of first order, i.e., there exist matrices $A_1 \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$, $A_2 \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ and $C \in \mathbb{R}^{\mathrm{w}\times\mathrm{n}}$ such that $\mathcal{B}$ consists of all trajectories w for which there exists a trajectory

$x : \mathbb{Z}^2 \to \mathbb{R}^{\mathrm{n}}$ such that

$$\begin{aligned} \sigma_1 x &= A_1 x \\ \sigma_2 x &= A_2 x \\ w &= C x, \end{aligned} \tag{2}$$

holds, where also the matrices A_1 and A_2 commute: $A_1 A_2 = A_2 A_1$. In particular, A_1, A_2 and C can be chosen so that the *state variable* x is *observable* from w, i.e.,

$$[(w, x) \text{ satisfy (2) and } w = 0] \Longrightarrow [x = 0] \ .$$

There are several different characterization of observability in terms of the algebraic properties of the representation; the first one is

$$\ker \begin{bmatrix} \sigma_1 I - A_1 \\ \sigma_2 I - A_2 \\ C \end{bmatrix} = \{0\}.$$

This condition is equivalent with the *extended observability matrix*, defined as the column block matrix

$$\mathcal{O}(A_1, A_2, C) := \begin{bmatrix} C \\ CA_1 \\ CA_2 \\ CA_1^2 \\ CA_1A_2 \\ CA_2^2 \\ \vdots \\ CA_2^{\mathrm{n}-1} \end{bmatrix}, \tag{3}$$

having rank equal to n. This implies that there exists a matrix E such that $E \cdot \mathcal{O}(A_1, A_2, C) = I_{\mathrm{n}}$, where I_{n} is the $\mathrm{n} \times \mathrm{n}$ identity matrix. Thus we obtain that $x = X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w$ where

$$X(s_1, s_1^{-1}, s_2, s_2^{-1}) := E \begin{bmatrix} C \\ Cs_1 \\ Cs_2 \\ Cs_1s_2 \\ Cs_1^2 \\ Cs_2^2 \\ \vdots \\ Cs_2^{\mathrm{n}-1} \end{bmatrix} \in \mathbb{R}[s_1, s_1^{-1}, s_2, s_2^{-1}]^{\mathrm{n} \times \mathrm{w}}. \tag{4}$$

In this case the *state map* $X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$ is minimal, i.e., the dimension of the state variable x is minimal among all possible representations (2). Note that here X is a polynomial matrix, i.e., its entries contain only nonnegative powers of s_1 and s_2.

In the following, discrete lines in the lattice $\mathbb{Z}^2$ will play an important role; we now introduce the basic notation and discuss behaviors restricted to lines.

The set of lines in the first orthant of $\mathbb{Z}^2$ is defined by

$$L := \{\ell \subset \mathbb{N}^2 \mid \ell = \{\alpha(a, b) \in \mathbb{N}^2 \mid \alpha \in \mathbb{N}\},\ a, b \in \mathbb{N} \text{ are coprime}\}.$$

The lines in the first orthant corresponding to the vertical, respectively horizontal, axes, will be denoted in the following with ℓ_1 and ℓ_2 respectively. Given a $2D$ behavior $\mathcal{B}$ and a line $\ell \in L$, we define the *restriction of* $\mathcal{B}$ *to* ℓ as

$$\mathcal{B}|_\ell := \{w_\ell : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} \mid \text{ there exists } w \in \mathcal{B} \text{ such that } w|_\ell = w_\ell\} ,$$

where $w\,|_\ell$ denotes the restriction of the trajectory w to the domain ℓ. Note that w_ℓ is a $1D$ trajectory, while $w\,|_\ell$ is a trajectory depending on two indices. It has been shown in (Napp Avelli and Rocha 2010, Th.6 and Th.7) that a $2D$ behavior restricted to a line is a $1D$ behavior; that is, $\mathcal{B}$ is the kernel of some polynomial operator in the $1D$ shift defined by a Laurent-polynomial matrix $R(s) \in \mathbb{R}[s, s^{-1}]$. It is easy to see that if $\mathcal{B}$ is described by (2) and if $\mathcal{B}_1$ and $\mathcal{B}_2$ denote the restrictions of $\mathcal{B}$ to the axes, then $\mathcal{B}_i$ is described in the state-space form as

$$\begin{aligned} \sigma_i x &= A_i x \\ w &= Cx, \end{aligned} \tag{5}$$

$i = 1, 2$. Note that these state representations may be non-minimal even if (2) is minimal.

In many modeling and control problems it is necessary to study quadratic functionals of the system variables and their derivatives; for example, in linear quadratic optimal control, H_∞-control, dissipativity theory, etc. Following the seminal work of Willems and Trentelman (1998), successively extended to the $2D$ case in Kojima et al. (2007, 2010), Pillai and Willems (2002), we will use polynomial matrices in 4 variables as a tool to express quadratic functionals of functions of 2 independent variables and their shifts. We next review the definitions regarding these functionals which are most relevant to the problems treated in this paper.

In the following, we use the multi-indices $\mathbf{k} = (k_1, k_2)$ and $\mathbf{l} = (l_1, l_2)$, and multi-indeterminates denoted by $\zeta = (\zeta_1, \zeta_2)$ and $\eta = (\eta_1, \eta_2)$. We also denote $\zeta^{\mathbf{k}} = \zeta_1^{k_1}\zeta_2^{k_2}$ and $\eta^{\mathbf{k}} = \eta_1^{k_1}\eta_2^{k_2}$. We denote with $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ the set of real polynomial $\mathtt{w} \times \mathtt{w}$ matrices in the 4 indeterminates ζ and η; that is, an element of $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$ is of the form

$$\Phi(\zeta, \eta) = \sum_{\mathbf{k},\mathbf{l}} \Phi_{\mathbf{k},\mathbf{l}} \zeta^{\mathbf{k}} \eta^{\mathbf{l}} \tag{6}$$

where $\Phi_{k,l} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$; the sum ranges over a finite set of multi-indices $\mathbf{k}, \mathbf{l} \in \mathbb{N}^2$. The 4-variable polynomial matrix $\Phi(\zeta, \eta)$ is called *symmetric* if $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top$, equivalently if $\Phi_{\mathbf{k},\mathbf{l}} = \Phi_{\mathbf{l},\mathbf{k}}^\top$ for all $\mathbf{l}$ and $\mathbf{k}$. In this paper we restrict our attention to the symmetric elements in $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$, and denote this subset by $\mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]$. Any symmetric Φ induces a quadratic functional

$$\begin{aligned} Q_\Phi &: (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}} \times (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}\times\mathbb{Z}} \longrightarrow (\mathbb{R})^{\mathbb{Z}\times\mathbb{Z}} \\ Q_\Phi(w) &= \sum_{\mathbf{k},\mathbf{l}} (\sigma^{\mathbf{k}} w)^\top \Phi_{\mathbf{k},\mathbf{l}} \sigma^{\mathbf{l}}(w) \end{aligned}$$

where the k-th shift operator $\sigma^{\mathbf{k}}$ is defined as $\sigma^{\mathbf{k}} = \sigma^{k_1}\sigma^{k_2}$ (similarly for $\sigma^{\mathbf{l}}$). We call Q_Φ the *quadratic difference form* (in the following abbreviated with QDF) associated with Φ. Given two QDFs Q_{Φ_1}, Q_{Φ_2} we say that Q_{Φ_1} *is equivalent to* Q_{Φ_2} *on* $\mathcal{B}$, denoted by $Q_{\Phi_1} \overset{\mathcal{B}}{=} Q_{\Phi_2}$, if

$$Q_{\Phi_1}(w) = Q_{\Phi_2}(w) \text{ for all } w \in \mathcal{B} .$$

We call a QDF Q_Φ *nonnegative along* $\mathcal{B}$, denoted $Q_\Phi \overset{\mathcal{B}}{\geq} 0$, if $Q_\Phi(w) \geq 0$ for all $w \in \mathcal{B}$. We call Q_Φ *positive along* $\mathcal{B}$, denoted $Q_\Phi \overset{\mathcal{B}}{>} 0$, if $Q_\Phi \overset{\mathcal{B}}{\geq} 0$, and moreover $\forall w \in \mathcal{B}\ [Q_\Phi(w) = 0] \Longrightarrow [w = 0]$.

In the following we will be also using operations on QDFs. Given a QDF Q_Φ and a line $\ell = \{\alpha(a,b) \mid \alpha \in \mathbb{N}\} \in L$, we define the *increment of* Q_Φ *along the line* ℓ, denoted $\nabla_\ell(Q_\Phi)$, as

$$\nabla_\ell Q_\Phi(w)(\alpha(a,b)) := Q_\Phi(w)((\alpha+1)(a,b)) - Q_\Phi(w)(\alpha(a,b)).$$

The increment along the vertical, respectively horizontal, line will be denoted by ∇_{ℓ_1}, ∇_{ℓ_2} respectively.

3 2D stability and Lyapunov functions

In this paper we will consider stability as defined in Rocha (2008), i.e. with respect to a cone S. A set $S \subset \mathbb{R} \times \mathbb{R}$ is called a *cone* if $\alpha S \subset S$ for all $\alpha \geq 0$. A cone S is *solid* if it contains an open ball in $\mathbb{R} \times \mathbb{R}$, and *pointed* if $S \cap (-S) = \{(0,0)\}$. A cone is *proper* if it is closed, pointed, solid, and convex. Since an appropriate change of independent variables transforms any proper cone S into the first orthant, in the following we assume, without loss of generality, that S is the first orthant in $\mathbb{Z}^2$. For the sake of brevity, in the following we will use the expression "stable" instead of "stable with respect to the first orthant".

The definition of asymptotic stability that we shall use in the rest of this paper is the following; note that it is the discrete counterpart of that considered in the continuous-time case in Pillai and Shankar (1998) for nD behaviors.

Definition 3.1: Let $\mathcal{B}$ be a 2D behavior. $\mathcal{B}$ is *asymptotically stable* if

$$[w \in \mathcal{B}] \Longrightarrow [\forall\ (a,b) \in \mathbb{N}^2 \ \lim_{\alpha \to \infty} w(\alpha(a,b)) = 0]\ .$$

It is straightforward to see that $\mathcal{B}$ is asymptotically stable if and only if $\forall\ \ell \in L$ it holds that w_ℓ, the 1D trajectory associated with the restriction $w_{|_\ell}$, goes to zero as the independent variable goes to infinity. This definition of stability is equivalent to the definition considered in Valcher (2000) for (finite dimensional) 2D behaviors. It was shown in Rocha (2008) that in the discrete case all stable behaviors according to Definition 3.1 are finite dimensional; note that in the continuous case this is not necessarily true.

Having defined stability as in Definition 3.1, we now define Lyapunov functions as follows.

Definition 3.2: A functional $F : (\mathbb{R}^{\mathrm{w}})^{\mathbb{Z}^2} \to (\mathbb{R})^{\mathbb{Z}^2}$ is a *Lyapunov function* for a 2D behavior $\mathcal{B}$ if for all $\ell \in L$ it holds that for all $w \in \mathcal{B}$,

$$F(w)|_\ell > 0 \qquad \text{and} \qquad \nabla_\ell F(w) < 0.$$

If F is a quadratic functional of $w \in \mathcal{B}$ and its shifts, we call it a *quadratic Lyapunov function* (QLF) for $\mathcal{B}$.

In order to state the main result of this section, a characterization of asymptotic stability in terms of Lyapunov functions, we need some preliminary concepts and results. The first one is the notion of quadratic functionals of the state. Let $\mathcal{B}$ be a finite-dimensional 2D behavior, and let (2) be a minimal state representation of $\mathcal{B}$. We say that a quadratic functional $Q_\Phi(w)$ of $w \in \mathcal{B}$ and its shifts is a *quadratic function of the state of* $\mathcal{B}$ if there exists a symmetric constant matrix P such that for all trajectories (w,x) satisfying (2) it holds that $Q_\Phi(w) = x^\top P x$. As the following result shows, in the finite-dimensional case *any* quadratic functional of the system variables and their shifts is a quadratic functional of the state.

Proposition 3.3: *Let* $\mathcal{B}$ *be a finite dimensional* 2D *behavior, and let (2) be a state representation of* $\mathcal{B}$ *with state* x*. Let* $Q_\Phi : (\mathbb{R}^{\mathrm{w}})^{\mathbb{Z}^2} \to (\mathbb{R})^{\mathbb{Z}^2}$ *be a QDF. Then there exists a symmetric* *matrix* $P \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ *such that for all* $w \in \mathcal{B}$ *with associated state trajectory* x *it holds*

$$Q_\Phi(w) = x^\top P x \; .$$

Proof: Write

$$\begin{aligned} Q_\Phi(w) &= \sum_{i_1,i_2,k_1,k_2=0}^{N} (\sigma_1^{i_1}\sigma_2^{i_2}w)^\top \Phi_{i_1,i_2,k_1,k_2}(\sigma_1^{k_1}\sigma_2^{k_2}w) \\ &= \begin{pmatrix} w^\top & \sigma_1 w^\top & \sigma_2 w^\top & \dots \end{pmatrix} \begin{pmatrix} \Phi_{0000} & \Phi_{0010} & \Phi_{0001} & \cdots \\ \Phi_{1000} & \Phi_{1010} & \Phi_{0011} & \cdots \\ \Phi_{0100} & \Phi_{0110} & \Phi_{0101} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} w \\ \sigma_1 w \\ \sigma_2 w \\ \vdots \end{pmatrix} . \end{aligned}$$

Now observe that $\sigma_1^{i_1}\sigma_2^{i_2}w = CA_1^{i_1}A_2^{i_2}x$; consequently the last expression can be rewritten as

$$\begin{pmatrix} (Cx)^\top & (CA_1x)^\top & (CA_2x)^\top & \dots \end{pmatrix} \begin{pmatrix} \Phi_{0000} & \Phi_{0010} & \Phi_{0001} & \cdots \\ \Phi_{1000} & \Phi_{1010} & \Phi_{0011} & \cdots \\ \Phi_{0100} & \Phi_{0110} & \Phi_{0101} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Cx \\ CA_1x \\ CA_2x \\ \vdots \end{pmatrix} .$$

Now define

$$P := \mathcal{O}(A_1, A_2, C)^\top \begin{pmatrix} \Phi_{0000} & \Phi_{0010} & \Phi_{0001} & \cdots \\ \Phi_{1000} & \Phi_{1010} & \Phi_{0011} & \cdots \\ \Phi_{0100} & \Phi_{0110} & \Phi_{0101} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mathcal{O}(A_1, A_2, C) \; ;$$

the claim follows. □

It follows from the proof of Proposition 3.3 that every QDF Q_Φ can be written as $Q_\Phi(w) = (X(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})w)^\top P(X(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})w) = x^\top Px$ for $w \in \mathcal{B}$, where $X(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})$ is a polynomial operator in the shift that induces the state variable x when acting on the trajectory w; note that if the expression for X is derived from the extended observability matrix (3), then X is a polynomial matrix, and therefore Φ is a polynomial matrix in four indeterminates. Consequently, every QDF Q_Φ is equivalent on $\mathcal{B}$ to a QDF $Q_{\Phi'}$ induced by a 4-variable polynomial matrix of the form $\Phi'(\zeta_1,\zeta_2,\eta_1,\eta_2) = X(\zeta_1,\zeta_2)^\top PX(\eta_1,\eta_2)$, with P symmetric and X inducing a state variable. We call such a QDF $Q_{\Phi'}$ a *canonical representative of* Q_Φ. Note that if $\widehat{x}$ is another minimal state variable for $\mathcal{B}$, then it is easy to see that there exists a nonsingular matrix T such that $\widehat{x} = Tx$. Consequently, $Q_\Phi(w) = \widehat{x}^\top \widehat{P}\widehat{x}$ where $\widehat{P} = (T^\top)^{-1}PT^{-1}$. If a well-ordering (see Becker and Weispfenning (1993)) has been fixed in the space of polynomials $\mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\zeta_1,\zeta_2,\eta_1,\eta_2]$, then a *unique* canonical representative can be defined, see Kojima et al. (2007).

We now give an example of the computation of a canonical representative of a QDF.

Example 3.4 Let $\mathcal{B} = \ker R(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})$ be a $2D$ behavior where

$$R(s_1,s_1^{-1},s_2,s_2^{-1}) = \begin{pmatrix} (s_2-\frac{1}{2})(s_1-\frac{1}{4}) & 0 \\ (s_1-\frac{1}{2})(s_2-\frac{1}{4}) & 0 \\ 0 & s_1-\frac{1}{3} \\ 0 & s_2-\frac{1}{5} \end{pmatrix} ,$$

and $\Phi(\eta_1, \eta_2, \zeta_1, \zeta_2)$ a 4-variable polynomial matrix given by

$$\Phi(\eta_1, \eta_2, \zeta_1, \zeta_2) = \begin{pmatrix} 1 + \eta_1\zeta_1 & \zeta_1 \\ \eta_1 & \eta_2\zeta_2 \end{pmatrix} .$$

It is a matter of straightforward verification to check that the variety of $\mathcal{B}$ consists of the points

$$\mathcal{V}(\mathcal{B}) = \{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{4}, \frac{1}{4}), (\frac{1}{3}, \frac{1}{5})\} ;$$

since $\mathcal{V}(\mathcal{B})$ is finite, it follows that $\mathcal{B}$ is finite-dimensional.

Following the procedure illustrated in Fornasini et al. (1993), a minimal state-space realization of $\mathcal{B}$ as in (2) is given by the matrices

$$A_1 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \quad A_2 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} .$$

It is easy to compute that $x = X(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})w$ is the state variable corresponding to the matrices A_1, A_2 and C, where

$$X(s_1, s_1^{-1}, s_2, s_2^{-1}) = \begin{pmatrix} -4(s_1 - \frac{1}{2}) & 0 \\ 0 & 1 \\ 4(s_1 - \frac{1}{4}) & 0 \end{pmatrix} .$$

We now compute a canonical representative of Q_Φ. It is easy to check that $Q_\Phi(w) = x^\top P x$, where

$$P = \mathcal{O}(A_1, A_2, C)^\top \begin{pmatrix} \Phi_{0000} & \Phi_{0010} & \Phi_{0001} & \cdots \\ \Phi_{1000} & \Phi_{1010} & \Phi_{0011} & \cdots \\ \Phi_{0100} & \Phi_{0110} & \Phi_{0101} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mathcal{O}(A_1, A_2, C)$$

$$= \begin{pmatrix} 1 & 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & 0 & \frac{1}{9} & 0 & \frac{1}{9} & 0 & \frac{1}{25} & 0 & \frac{1}{15} \\ 1 & 0 & \frac{1}{4} & 0 & \frac{1}{16} & 0 & \frac{1}{4} & 0 & \frac{1}{16} & 0 & \frac{1}{16} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & \frac{1}{5} & 0 \\ \frac{1}{4} & 0 & \frac{1}{6} \\ 0 & \frac{1}{25} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{9} & 0 \\ \frac{1}{4} & 0 & \frac{1}{16} \\ 0 & \frac{1}{25} & 0 \\ \frac{1}{4} & 0 & \frac{1}{16} \\ 0 & \frac{1}{15} & 0 \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & \frac{1}{3} & \frac{9}{8} \\ \frac{1}{3} & \frac{1}{25} & \frac{1}{3} \\ \frac{9}{8} & \frac{1}{3} & \frac{9}{8} \end{pmatrix} .$$

Hence, a canonical representative for Q_Φ is

$$\Phi'(\zeta_1, \zeta_2, \eta_1, \eta_2) = X(\zeta_1, \zeta_2)^\top P X(\eta_1, \eta_2) .$$

The notion of canonical representative of a QDF is important for the proof of the following theorem, that constitutes the main result of this section. It relates the $2D$ stability of the $2D$

behavior $\mathcal{B}$ with the 1D stability of the 1D behaviors resulting from the restriction of $\mathcal{B}$ to the axes and with the existence of a quadratic Lyapunov functional.

Theorem 3.5 : *Let $\mathcal{B}$ be a finite dimensional $2D$ behavior and denote with $\mathcal{B}_1, \mathcal{B}_2$ the restrictions of $\mathcal{B}$ to the vertical, respectively horizontal, axis. The following statements are equivalent:*

(1) $\mathcal{B}$ is stable;
(2) $\mathcal{B}_1$ and $\mathcal{B}_2$ are stable 1D behaviors;
(3) There exist three 4 variable polynomial matrices Φ, Δ_1 and Δ_2 such that

$$Q_\Phi \overset{\mathcal{B}_i}{>} 0, \qquad Q_{\Delta_i} \overset{\mathcal{B}_i}{>} 0 \qquad \text{and} \qquad \nabla_{\ell_i} Q_\Phi \overset{\mathcal{B}_i}{=} -Q_{\Delta_i}, \qquad i = 1, 2.$$

(4) There exists a QLF for $\mathcal{B}$.

Proof: We begin with some general considerations about finite-dimensional behaviors which will make the proof of the result easier. Since $\mathcal{B}$ is finite dimensional, $\mathcal{V}(\mathcal{B})$ is finite and every trajectory of $\mathcal{B}$ is a linear combination of polynomial exponentials of the form

$$w_{\lambda_1,\lambda_2}(k_1, k_2) = p_{\lambda_1,\lambda_2}(k_1, k_2)\lambda_1^{k_1}\lambda_2^{k_2}, \tag{7}$$

for some suitable nonzero w-vector polynomial function p_{λ_1,λ_2}, i.e.

$$p_{\lambda_1,\lambda_2}(k_1, k_2) = \sum_{(i,j)\in I} \alpha_{ij} k_1^i k_2^j,$$

where $I \subset \mathbb{N}^2$ is a finite bi-index set and $\alpha_{ij} \in \mathbb{R}^2$. This implies that the trajectories of $\mathcal{B}_i$ are linear combinations of trajectories of the form

$$p_{\lambda_1,\lambda_2}(k_i e_i)\lambda_i^{k_i}, \quad i = 1, 2. \tag{8}$$

Furthermore, it follows from (Rocha 2008, Th.8) that $\mathcal{B}$ is stable if and only if $\mathcal{V}(\mathcal{B})$ is finite and for all $(\lambda_1, \lambda_2) \in \mathcal{V}(\mathcal{B})$, it holds that $|\lambda_i| < 1$, $i = 1, 2$.

1) $\Rightarrow$ 2): Let $\widehat{\lambda}_1 \in \mathcal{V}(\mathcal{B}_1)$. By assumption $\mathcal{B}$ is finite dimensional and the trajectories of $w \in \mathcal{B}$ are of the form described in (7). Also, the trajectories of $\mathcal{B}_1$ are of the form described in (8) with $i = 1$. This implies that there exists $\widehat{\lambda}_2$ such that $p_{\widehat{\lambda}_1,\widehat{\lambda}_2}(k_1, k_2)\widehat{\lambda}_1^{k_1}\widehat{\lambda}_2^{k_2} \in \mathcal{B}$. Since $\mathcal{B}$ is stable it follows that $|\widehat{\lambda}_1^{\alpha_1}\widehat{\lambda}_2^{\alpha_2}| < 1$, for all $(\alpha_1, \alpha_2) \in \mathbb{N}^2$ which is equivalent to saying that $|\widehat{\lambda}_i| < 1$ for $i = 1, 2$, i.e., $\mathcal{B}_i$ is stable for $i = 1, 2$.

2) $\Rightarrow$ 1): Let $(\lambda_1, \lambda_2) \in \mathcal{V}(\mathcal{B})$. Since $\mathcal{B}_i$ is stable, $i = 1, 2$ and $\lambda_1 \in \mathcal{V}(\mathcal{B}_1)$ and $\lambda_2 \in \mathcal{V}(\mathcal{B}_2)$ we have that $|\lambda_1^{\alpha_1}| < 1$ and $|\lambda_2^{\alpha_2}| < 1$ for all $(\alpha_1, \alpha_2) \in \mathbb{N}^2$. Therefore $|\lambda_i| < 1$ for all $i = 1, 2$, i.e., $\mathcal{B}$ is stable.

2) $\Rightarrow$ 3): Consider a representation of $\mathcal{B}$ as in (2). Then $\mathcal{B}_i$ is described by (5); by assumption A_1, A_2 are Schur matrices, i.e. all their eigenvalues have modulus less than one. Since these matrices commute, there exists (see Narendra and Balakrishnan (1994)) a matrix $P > 0, \widetilde{\Delta}_1 > 0$ and $\widetilde{\Delta}_2 > 0$ of suitable sizes such that

$$A_i^\top P A_i - P = -\widetilde{\Delta}_i, \quad \text{for } i = 1, 2.$$

Define

$$Q_\Phi(w) := (X(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})w)^\top P X(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})w = x^\top P x,$$
$$Q_{\Delta_i}(w) := (X(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})w)^\top \widetilde{\Delta}_i X(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})w = x^\top \widetilde{\Delta}_i x\ ,$$

$i = 1, 2$, where X is the polynomial operator in the shift inducing the state variable x. Now since $\nabla_{\ell_i} Q_\Phi(w) = x^\top(A_i^\top P A_i - P)x,\ \forall\ w \in \mathcal{B},\ \ i = 1, 2$, it is easy to verify that Q_Φ, Q_{Δ_1} and Q_{Δ_2} satisfy the conditions of statement 3).

3) $\Rightarrow$ 4): We show that Q_Φ is a QLF for $\mathcal{B}$. It follows from the Proposition 3.3 that there exists a symmetric polynomial matrix P such that $Q_\Phi(w) = x^\top P x$ with $x = X(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})w$. Observe that the 1D dynamics along a line $\ell = \{\alpha(i,j) \mid \alpha \in \mathbb{N}\} \in L$ are described by $\sigma_\ell x = A_1^i A_2^j x$, $w = Cx$, for some i, j fixed but otherwise arbitrary. Thus, it is enough to prove that P satisfies a matrix Lyapunov equation for $A_1^i A_2^j$, i.e. $(A_1^i A_2^j)^\top P (A_1^i A_2^j) - P < 0$. Observe first that

$$\begin{aligned}(A_1^i A_2^j)^\top P (A_1^i A_2^j) - P &= (A_2^j)^\top (A_1^i)^\top P (A_1^i)(A_2^j) - P \\ &< (A_2^j)^\top P (A_2^j) - P,\end{aligned}$$

where we have used the fact that

$$[A_1^\top P A_1 - P < 0] \Rightarrow [(A_1^i)^\top P (A_1^i) - P < 0]$$

as can be readily proved by induction. From the same argument it follows that $(A_2^j)^\top P (A_2^j) - P < 0$ and consequently $(A_1^i A_2^j)^\top P (A_1^i A_2^j) - P < 0$. Now define $Q_\Phi(w) := x^\top P x$, and conclude that along the line ℓ it holds that $\nabla_\ell Q_\Phi < 0$, as was to be proved.

4) $\Rightarrow$ 2) It is a matter of straightforward verification to check that $x|_{\ell_i}$ is a state trajectory for the 1D behavior $\mathcal{B}_i$, $i = 1, 2$. Moreover, if P is a matrix corresponding to a canonical representative of the Lyapunov function Q_Φ, then $(x|_{\ell_i})^\top P x|_{\ell_i}$ is a Lyapunov function for $\mathcal{B}_i$. This implies that for $i = 1, 2$ w_i goes to zero along the line ℓ_i, $i = 1, 2$, i.e., $\mathcal{B}_i$ is stable. □

We can now characterize Lyapunov functions in terms of canonical representatives.

Definition 3.6: Let $A_1, A_2 \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$. A matrix $P > 0$ is said to be a *common Lyapunov solution* (CLS) for A_1 and A_2 if

$$\begin{aligned}A_1^\top P A_1 - P < 0 \\ A_2^\top P A_2 - P < 0.\end{aligned} \tag{9}$$

Using this definition, we can give yet another equivalent statement to those of Theorem 3.5.

Proposition 3.7: *Let $\mathcal{B}$ be a finite-dimensional 2D behavior and Φ be a QDF. Φ is a QLF for $\mathcal{B}$ if and only if given any minimal state-space representation (A_1, A_2, C) of $\mathcal{B}$ together with a state map $X \in \mathbb{R}^{\mathrm{n}\times\mathrm{w}}[\xi_1, \xi_2]$ inducing the state variable x for the representation, the following two conditions hold:*

(1) There exists a symmetric matrix $P > 0$ which is a CLS for A_1 and A_2;
(2) the canonical representative of Q_Φ is equal to $X^\top P X$.

Proof: The claim follows easily using the same arguments of the proof 4) $\Rightarrow$ 2), 2) $\Rightarrow$ 3) and 3) $\Rightarrow$ 4) of Theorem 3.5. □

An interesting question is the construction of all Lyapunov functions for a given $2D$ behavior $\mathcal{B}$. The relevance of the computation of Lyapunov functions goes beyond stability analysis. Besides its theoretical interest, the computation of Lyapunov functions is crucial, for instance, for the construction of positive storage functions in the context of non-controllable 1D dissipative systems, see Pal and Belur (2008), Çamlibel et al. (2003). We believe that this will also holds true in the 2D case. We address this issue in the next section.

4 Construction of Lyapunov functions

The result of Proposition 3.7 shows that the problem of finding $2D$ Lyapunov functions can be reduced to that of finding common solutions to a pair of $1D$ Lyapunov equations. Based on this fact, in this section we provide an algorithm for the construction of such functions. We remark that it is crucial in our approach to assume that both A_1 and A_2 are diagonalizable. Note that, although not general, this is the generic case.

We present some preliminary results which will be instrumental to this end. The first one introduces two useful maps. In the following we denote with $\mathbb{R}_s^{\mathrm{n}\times\mathrm{n}}$ the set of $\mathrm{n}\times\mathrm{n}$ symmetric matrices.

Definition 4.1: Let $A_1, A_2 \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$. The *Lyapunov map* associated with A_i, $i = 1,2$ is defined as

$$\begin{aligned} \mathcal{L}_i : \mathbb{R}_s^{\mathrm{n}\times\mathrm{n}} &\to \mathbb{R}_s^{\mathrm{n}\times\mathrm{n}} \\ P &\mapsto A_i^\top P A_i - P. \end{aligned} \tag{10}$$

The name Lyapunov map is adopted following Peeters and Rapisarda (2001).

Lemma 4.2: *If the matrices $A_1, A_2 \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ in (10) commute then the Lyapunov maps $\mathcal{L}_1$ and $\mathcal{L}_2$ associated with A_1, A_2 commute.*

Proof:
$$\begin{aligned} (\mathcal{L}_1\mathcal{L}_2)(P) &= A_1^\top[A_2^\top P A_2 - P]A_1 - [A_2^\top P A_2 - P] \\ &= A_1^\top A_2^\top P A_2 A_1 - A_1^\top P A_1 - A_2^\top P A_2 + P \\ &= A_2^\top A_1^\top P A_1 A_2 - A_2^\top P A_2 - A_1^\top P A_1 + P \\ &= A_2^\top[A_1^\top P A_1 - P]A_2 - [A_1^\top P A_1 - P] \\ &= (\mathcal{L}_2\mathcal{L}_1)(P). \end{aligned}$$
□

The next result states a necessary and sufficient condition for the existence of a common Lyapunov solution for A_1 and A_2, given in terms of the Lyapunov maps $\mathcal{L}_i$.

Theorem 4.3: *Let the matrices $A_1, A_2 \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$ in (10) be Schur commuting matrices. Then the associated Lyapunov maps $\mathcal{L}_i$ are invertible. Moreover, a matrix $P > 0$ is a CLS for A_1 and A_2 if and only if there exists a matrix $S > 0$ such that*

$$P = (\mathcal{L}_2^{-1}\mathcal{L}_1^{-1})(S). \tag{11}$$

Proof: The fact that the maps $\mathcal{L}_i$ are invertible follows from standard knowledge regarding stability of 1D systems. We prove the second part of the claim.

($\Leftarrow$): This part of the claim can be proved using the same argument of the proof of statement *ii*) of Theorem 1 of Narendra and Balakrishnan (1994). By assumption we have that $\mathcal{L}_2(P) = \mathcal{L}_1^{-1}(S) < 0$ because of the linearity of maps $\mathcal{L}_i$, as $\mathcal{L}_1^{-1}(-S) > 0$. Thus, P is a Lyapunov solution for A_2. In the same way we prove that P is a Lyapunov solution for A_1. Thus P is a CLS for A_1 and A_2.

($\Rightarrow$): Let P be a CLS for A_1 and A_2, and define $Q_i = \mathcal{L}_i(P)$, $i = 1, 2$. Note that $Q_i < 0$, $i = 1, 2$. Define $S = \mathcal{L}_1(Q_2) = \mathcal{L}_1(\mathcal{L}_2(P)) = \mathcal{L}_2(\mathcal{L}_1(P)) = \mathcal{L}_2(Q_1)$. Then, $(\mathcal{L}_2^{-1}\mathcal{L}_1^{-1})(S) = \mathcal{L}_2^{-1}(Q_2) = P$. Note that $S > 0$. □

The result of Theorem 4.3 characterizes the common Lyapunov solutions for a pair of Schur commuting matrices; it also constitutes a generalization of Theorem 1 of Narendra and Balakrishnan (1994), since it shows that the condition (11) is not only sufficient, but also necessary for the existence of a common Lyapunov function. Moreover, Theorem 4.3 also suggests an algorithm to compute a CLS by inversion of the maps $\mathcal{L}_i$.

We now proceed to investigate further the properties of these maps; a similar approach has been adopted for one polynomial Lyapunov equation in Peeters and Rapisarda (2001).

Definition 4.4: Let $A_1, A_2 \in \mathbb{R}^{\mathrm{n}\times\mathrm{n}}$. Then, $0 \neq v \in \mathbb{C}^{\mathrm{n}}$ is an eigenvector of the matrix pair (A_1, A_2) if there exists $(\lambda, \mu) \in \mathbb{C}^2$ such that

$$v^*[\lambda I - A_1 \quad \mu I - A_2] = 0 \ .$$

In this case we say that (λ, μ) is an eigenvalue (pair) of (A_1, A_2). Equivalently we say that $0 \neq \widehat{v} \in \mathbb{C}^{\mathrm{n}\times\mathrm{n}}$ is an eigenvector of a pair $(\mathcal{L}_1, \mathcal{L}_2)$ of linear transformations from $\mathbb{C}^{\mathrm{n}\times\mathrm{n}}$ to $\mathbb{C}^{\mathrm{n}\times\mathrm{n}}$ if there exists $(\lambda, \mu) \in \mathbb{C}^2$ such that $(\mathcal{L}_1(\widehat{v}), \mathcal{L}_2(\widehat{v})) = (\lambda\widehat{v}, \mu\widehat{v})$. In this case we say that (λ, μ) is an eigenvalue (pair) of $(\mathcal{L}_1, \mathcal{L}_2)$.

Lemma 4.5: *Let A_1 and A_2 be two $\mathrm{n} \times \mathrm{n}$ diagonalizable commuting matrices and $\mathcal{L}_1, \mathcal{L}_2$ the Lyapunov maps associated with A_1, A_2, respectively. Then, the pair (A_1, A_2) has n linearly independent eigenvectors. Moreover, given $V = \{v_1, \ldots, v_n\}$ a basis of $\mathbb{C}^{\mathrm{n}}$ formed by eigenvectors of (A_1, A_2) with corresponding eigenvalues (λ_i, μ_i), we have that*

$$\widehat{V} = \{\widehat{v}_{ij} = v_i v_j^* + v_j v_i^* \mid 1 \leq i \leq j \leq n\}$$

is a set of $\frac{\mathrm{n}(\mathrm{n}+1)}{2}$ linearly independent eigenvectors of $(\mathcal{L}_1, \mathcal{L}_2)$, each associated with the pair of eigenvalues $(\lambda_i\lambda_j - 1, \mu_i\mu_j - 1)$, for $1 \leq i \leq j \leq \mathrm{n}$.

Proof: That the pair (A_1, A_2) has n linearly independent eigenvectors follows from the well-known fact that commuting matrices are diagonalizable if and only if they have a basis of common eigenvectors. In order to prove that the matrices $\widehat{v}_{ij}$ form a basis for the set of $\mathrm{n} \times \mathrm{n}$ hermitian matrices, assume by contradiction that there is a linear combination of the $\widehat{v_{ij}}$ that is zero; then there exist complex numbers k_{ij}, $i, j = 1, \ldots, n$ such that $\sum_{j=1}^{n}\sum_{i=1}^{n} k_{ij} v_i v_j^* = 0$. This equality is equivalent with $\sum_{j=1}^{n} \left(\sum_{i=1}^{n} k_{ij} v_i\right) v_j^* = 0$. Since the v_j^* are linearly independent, $j = 1, \ldots, n$, it follows that $\sum_{i=1}^{n} k_{ij} v_i = 0$ for all $j = 1, \ldots, n$. Now apply the linear independence of the v_i's, $i = 1, \ldots, n$, to conclude that all coefficients k_{ij} are zero. The claim that the matrices $\widehat{v}_{ij}$ are eigenvectors of $\mathcal{L}_i$ follows easily after verifying that $\mathcal{L}_1(\widehat{v}_{ij}) = (\lambda_i\lambda_j - 1)\widehat{v}_{ij}$ and $\mathcal{L}_2(\widehat{v}_{ij}) = (\mu_i\mu_j - 1)\widehat{v}_{ij}$; note that these formulas also show what the eigenvalues associated with each eigenvector are. □

The result of Lemma 4.5 shows that if the matrices A_i, $i = 1, 2$ are diagonalizable, a basis of common eigenvectors of $\mathcal{L}_i$, $i = 1, 2$, can be computed in a straightforward way from a basis of common eigenvectors of A_1 and A_2. Consequently, the inversion necessary to compute the matrix P as in Theorem 4.3 is a straightforward matter. These considerations lead us to stating the following algorithm to compute Lyapunov functions for a given stable behavior $\mathcal{B}$.

Algorithm

Input: A stable, finite-dimensional behavior $\mathcal{B}$;

Output: $\Phi \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ inducing a Lyapunov function for $\mathcal{B}$.

Step 1: Compute a representation of $\mathcal{B}$ as in (2), together with a state map $X \in \mathbb{R}^{\mathrm{n}\times\mathrm{w}}[\xi_1, \xi_2]$ inducing the state variable x for the representation.

Step 2: Using the matrices A_1 and A_2 from Step 1 construct V and $\widehat{V}$ as described in Lemma 4.5.

Step 3: Select $\alpha_{ij},\ 1 \leq i \leq j \leq \mathrm{n}$ such that

$$S = \sum_{1\leq i\leq j\leq \mathrm{n}} \alpha_{ij}\widehat{v}_{ij} > 0.$$

Step 4: Output

$$\Phi(\zeta_1, \zeta_2, \eta_1, \eta_2) := (X(\zeta_1, \zeta_2))^\top \sum_{1\leq i\leq j\leq \mathrm{n}} \frac{\alpha_{ij}}{(\lambda_i\lambda_j - 1)(\mu_i\mu_j - 1)} \widehat{v}_{ij} X(\eta_1, \eta_2).$$

Some remarks are in order.

Remark 1 : Note that since $\{\widehat{v}_{ij}\}_{1\leq i\leq j\leq \mathrm{n}}$ forms a basis for the space of $\mathrm{n}\times\mathrm{n}$ hermitian matrices, the construction of the matrix S in the step 3 generates all positive definite matrices; consequently the algorithm produces all possible Lyapunov functions for a given stable $2D$ behavior. In particular, a S can for instance be chosen as $S = \sum_{1\leq i\leq j\leq \mathrm{n}} v_i v_i^*$.

Remark 2 : The increments ∇_{ℓ_i}; $i = 1, 2$ of the Lyapunov function Q_Φ computed in Step 4 are easily seen to be equal respectively to

$$\Delta_1(\zeta_1, \zeta_2, \eta_1, \eta_2) = X(\zeta_1, \zeta_2)^\top \left(\sum_{1\leq i\leq j\leq n} \frac{1}{\lambda_i\lambda_j - 1} \alpha_{ij}\widehat{v}_{ij} \right) X(\eta_1, \eta_2),$$

$$\Delta_2(\zeta_1, \zeta_2, \eta_1, \eta_2) = X(\zeta_1, \zeta_2)^\top \left(\sum_{1\leq i\leq j\leq n} \frac{1}{\mu_i\mu_j - 1} \alpha_{ij}\widehat{v}_{ij} \right) X(\eta_1, \eta_2).$$

Remark 3 : It follows from Theorem 9.1.1 of Gohberg et al. (2006) that there exist non-diagonalizable matrices which do not have a basis of common *generalized* eigenvectors. The problem of finding efficient algorithms to compute a common Lyapunov solution in this non-generic case is a matter of further investigation.

Example 4.6 Consider $(A_1.A_2, X)$ as in the example 3.4. In step 3, take, for instance $S = I_3$, the 3×3 identity matrix. Step 4 produces

$$\Phi(\zeta_1, \zeta_2, \eta_1, \eta_2) = \begin{pmatrix} 9(\zeta_1 - \frac{1}{2})(\eta_1 - \frac{1}{2}) + \frac{225}{16}(\zeta_1 - \frac{1}{4})(\eta_1 - \frac{1}{4}) & 0 \\ 0 & \frac{64}{75} \end{pmatrix}.$$

5 Conclusions

In this paper we have illustrated a Lyapunov approach to the stability of finite-dimensional $2D$ systems. We have adopted as definition of stability the one given in Def. 3.1, namely the asymptotic stability along all lines in the first orthant. The main results are Theorem 3.5, which characterizes stability in terms of the existence of a Lyapunov function, defined as a quadratic functional of the system variables which is positive along all lines, and whose increments are negative along all lines; and the algorithm given in section 4 for the computation of Lyapunov functions for stable $2D$ behaviors $\mathcal{B}$, which is valid in the case $\mathcal{B}$ has a state space representation as in (2) with diagonalizable matrices $A_i,\ i = 1, 2$. The development of constructive algorithms for the general case is still a problem under current investigation.

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