Time-relevant stability of $2D$ systems

D. Napp [a], P. Rapisarda [b], P.Rocha [a]

[a] *Department of Electrical and Computer Engineering, Faculty of Engineering, University of Oporto, Portugal.*

[b] *ISIS group, School of Electronics and Computer Science, University of Southampton, Southampton, United Kingdom.*

Abstract

For many $2D$ systems, one of the independent variables plays a distinct role in the evolution of the trajectories; since often this special independent variable is time, we call such systems 'time-relevant'. In this paper, we introduce a stability notion for time-relevant systems described by higher-order difference equations. We give algebraic tests in terms of the location of the zeros of the determinant of a polynomial matrix describing the system. We also give an LMI characterization of time-relevant stability involving only constant matrices.

1 Introduction

In this paper we study the stability of *time-relevant* $2D$ *systems*, that is $2D$ systems where one independent variable - often identified with time in many applications - plays a distinguished role. Previous work on $2D$ systems has almost exclusively focused on systems where both independent variables are on the same footing; this made necessary the definition of the concept of "past" and "future", so self-evident in the $1D$ framework, to the case of more than one independent variable, where there is no obvious such splitting. An eminently reasonable position is to let the laws describing the physical phenomenon themselves dictate what the "direction" is of the evolution of the system; this is the approach pioneered in the $2D$ case in [20]. Although this approach agrees with many applications of $2D$ systems (image processing, for example), it can be argued that in many other situations (for example in the modeling of physical systems, or in iterative learning control), time plays a more distinguished role than the spatial variables.

The notion of stability, because of its important consequences in the analysis and design of control systems and of filters, has attracted considerable interest also in the case of $2D$ systems (see for example [2,5,24]). The case of non time-relevant $2D$ systems has been studied in detail by Valcher in [24], where stability is defined with respect to "past" and "future" cones. For time-relevant systems, the natural direction of the independent variable "time" determines what the past and the future are, and a reasonable definition of stability must formalize the intuition that the trajectories of the system die out as time goes to infinity. A sound definition of time-relevant stability must also take into account the important role boundary conditions play in the behavior of $2D$ systems.

In section 2 of this paper we address these issues and we propose a definition of time-relevant stability corresponding to the situation in which system trajectories with "finite-energy" boundary conditions have zero energy in the limit as time goes to infinity. In section 3 we provide an algebraic characterization of square, autonomous time-stable systems in terms of the location of the zeroes of the determinant of any square polynomial matrix describing the system. This condition is rather difficult to check, since it involves determining the location of the roots of a parameter-dependent polynomial. In order to find efficient conditions we put to good use the concepts and formalism of 1-D quadratic difference forms and of dissipativity theory, summarized in section 4 of this paper. Equipped with these theoretical tools, in section 5 we give an LMI characterization of time-relevant stability for $2D$ systems, which involves only constant matrices and which can be tested using standard linear algebra computations.

The present work is greatly indebted to the results illustrated in [25] in the case of continuous independent variables. It is also important to mention here other works regarding time-relevant systems; those more in the spirit of the approach followed in this paper are [21,22], while in [27] a more algebraic approach to the issue of time-relevant stability is illustrated.

Email addresses: diego@ua.pt (D. Napp), pr3@ecs.soton.ac.uk (P. Rapisarda), mprocha@fe.up.pt (P.Rocha).

Notation and background material

We denote by $\mathbb{W}^{\mathbb{T}}$ the set consisting of all maps from a set $\mathbb{T}$ to a set $\mathbb{W}$. We call $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ a $2D$ *linear shift-invariant partial difference behavior* if $\mathfrak{B}$ is the set of solutions of a finite system of constant-coefficient partial difference equations. We denote by $\mathfrak{L}^{\mathtt{w}}(\mathbb{Z}^2, \mathbb{R}^{\mathtt{w}})$ the set of all $2D$ linear shift-invariant partial difference behaviors with $\mathtt{w}$ variables, often denoted simply with $\mathfrak{L}^{\mathtt{w}}$.

The system of constant-coefficient partial difference equations describing $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ can be efficiently represented using polynomial matrices in two variables as follows. denote by σ_i the *i-th shift operator*, defined for $i = 1$ as

$$\begin{aligned}&\sigma_1 : (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \to (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \\ &(\sigma_1 w)(k_1, k_2) := w(k_1 + 1, k_2) \ ,\end{aligned}$$

and analogously for σ_2; the inverse shift operators σ_1^{-1} and σ_2^{-1} are defined in the obvious way. Then $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ if and only if there exist nonnegative integers M and L and matrices $R_{ij} \in \mathbb{R}^{p \times \mathtt{w}}$, $i, j = -L, \ldots, M$, such that

$$[w \in \mathfrak{B}] \Longleftrightarrow \left[\sum_{i,j=-L}^{M} R_{ij} \sigma_1^i \sigma_2^j w = 0 \right] \ .$$

Define the two-variable Laurent polynomial matrix $R(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1})$ in the indeterminates ξ_1 and ξ_2 as $R(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}) := \sum_{i,j=-L}^{M} R_{ij} \xi_1^i \xi_2^j$; then we can write

$$\mathfrak{B} = \ker\, R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}) \ , \quad (1)$$

expressing $\mathfrak{B}$ as the kernel of a polynomial operator in the shifts. We call (1) a *kernel representation* of $\mathfrak{B}$.

Associating behaviors with Laurent polynomial matrices allows the development of a calculus of representations in which properties of a behavior are reflected in algebraic properties of the polynomial matrices representing it. A thorough introduction to this calculus is given in the literature; we now briefly review only those notions of this calculus necessary for the results presented in this paper.

First, we introduce some notation. We denote by $\mathbb{R}^{\mathtt{r} \times \mathtt{w}}[\xi_1, \xi_2]$ (respectively, with $\mathbb{R}^{\mathtt{r} \times \mathtt{w}}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$) the set of all $\mathtt{r} \times \mathtt{w}$ matrices with entries in the ring $\mathbb{R}[\xi_1, \xi_2]$ of polynomials in 2 indeterminates, with real coefficients (respectively in the ring $\mathbb{R}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$ of Laurent polynomials in 2 indeterminates with real coefficients). For simplicity in the following we often omit an explicit indication of the indeterminates when referring to (Laurent) polynomial matrices. When one of the dimensions of a matrix is not specified (but finite), we denote it with a bullet; for example, $\mathbb{R}^{\bullet \times \mathtt{w}}$ is the set of matrices with real entries and $\mathtt{w}$ columns. The set of real matrices with an infinite number of columns (respectively rows) and $\mathtt{w}$ rows (respectively columns) is denoted by $\mathbb{R}^{\mathtt{w} \times \infty}$ (respectively $\mathbb{R}^{\infty \times \mathtt{w}}$). The set of real matrices with an infinite number of rows and columns is denoted by $\mathbb{R}^{\infty \times \infty}$.

Inclusion and equality of behaviors are reflected in properties of the Laurent polynomial matrices associated with their kernel representations as follows. If two behaviors are represented as $\mathfrak{B}_i := \ker\, R_i(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$, with $R_i \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$, $i = 1, 2$, then $\mathfrak{B}_1 \subseteq \mathfrak{B}_2$ if and only if there exists $L \in \mathbb{R}^{\bullet \times \bullet}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$ such that $R_2 = LR_1$. Also, $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exist $L_1, L_2 \in \mathbb{R}^{\bullet \times \bullet}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$ such that $R_2 = L_1 R_1$ and $R_1 = L_2 R_2$. If the polynomial matrices R_1 and R_2 have full row rank, then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exists a unimodular matrix $L \in \mathbb{R}^{\bullet \times \bullet}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$, i.e. a matrix whose determinant is a unit in $\mathbb{R}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$, such that $R_1 = LR_2$. Note that since the determinant of a unimodular matrix is a unit, a unimodular matrix is invertible in the ring it belongs to.

A set $\mathcal{K} \subset \mathbb{R} \times \mathbb{R}$ is a *cone* if $\alpha\mathcal{K} \subset \mathcal{K}$ for all $\alpha \geq 0$; a cone is *convex* if it contains, with any two points, also the line segment between them; a convex cone is *solid* if it contains an open ball of $\mathbb{R} \times \mathbb{R}$.

We denote by $\ell_2(\mathbb{Z}, \mathbb{Z}^{\mathtt{w}})$ (often abbreviated with ℓ_2 when the trajectory dimension is evident from the context) the set of square summable trajectories:

$$\ell_2(\mathbb{Z}, \mathbb{Z}^{\mathtt{w}}) := \Big\{ w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}} \mid \sum_{k=-\infty}^{+\infty} w(k)^\top w(k) = \sum_{k=-\infty}^{+\infty} \|w(k)\|_2^2 < \infty \Big\} \ .$$

2 Time-relevant $2D$ systems

When is it reasonable to call a $2D$ system 'time-relevant'? The requirement that one of the independent variables plays a distinguished role is of course necessary; however, we believe that it is legitimate to do so only when this special independent variable has another desirable property, namely the fact that it imposes a unequivocal partition of the independent variable space in a 'past' and a 'future'. In order to articulate this point of view in a mathematically sound way, we need to recall several notions introduced in [3,20,24].

Definition 1 *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$*. A subset* $\mathcal{S} \subseteq \mathbb{Z}^2$ *is* characteristic *for* $\mathfrak{B}$ *if*

$$\{[w_1, w_2 \in \mathfrak{B}] \ \text{and} \ [w_{1|\mathcal{S}} = w_{2|\mathcal{S}}]\} \Longrightarrow [w_1 = w_2] \ .$$

The following result is a straightforward consequence of this definition and of the linearity of $\mathfrak{B}$.

Proposition 2 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. A subset $\mathcal{S} \subseteq \mathbb{Z}^2$ is characteristic for $\mathfrak{B}$ if and only if*

$$\left\{[w \in \mathfrak{B}] \text{ and } \left[w_{|\mathcal{S}} = 0\right]\right\} \Longrightarrow [w = 0] \ .$$

Of course the trivial set $\mathcal{S} = \mathbb{Z}^2$ is characteristic for every behavior $\mathfrak{B}$; however, in the following we are only interested in those behaviors with nontrivial characteristic sets. We call these systems autonomous (see Def. 1 p. 1503 of [3] and also Def. 2.2 p. 292 of [24]).

Definition 3 *A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called* autonomous *if it admits a characteristic set $\mathcal{S} \subset \mathbb{Z} \times \mathbb{Z}$ whose complementary set $(\mathbb{Z} \times \mathbb{Z}) \setminus \mathcal{S}$ includes the intersection $\mathcal{K} \cap (\mathbb{Z} \times \mathbb{Z})$ of a closed solid convex cone $\mathcal{K}$ of $\mathbb{R} \times \mathbb{R}$ with $\mathbb{Z} \times \mathbb{Z}$.*

It has been shown in [3] that every $2D$ autonomous behavior admits a kernel representation (1) with $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$ of full column rank. The following result states that every autonomous behavior can be decomposed (non-uniquely) as the sum of a finite-dimensional part and of an infinite-dimensional, "square" part, the latter being unique.

Proposition 4 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be autonomous. There exists a finite dimensional autonomous behavior $\mathfrak{B}^{\mathrm{fd}}$ and a square nonsingular polynomial matrix $S \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$ such that $\mathfrak{B} = \mathfrak{B}^{\mathrm{fd}} + \ker S(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$, where*

$$\mathfrak{B}^{\mathrm{sq}} := \ker S(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$$

is uniquely determined by $\mathfrak{B}$.

Proof. The result follows from Proposition 2.3 of [23]. ■

In the sequel we follow [24] and we call behaviors $\mathfrak{B}^{\mathrm{sq}}$ as in Proposition 4 *square autonomous behaviors.*

We now introduce time-relevant behaviors; of special importance in this case are the sets

$$\mathcal{S}_{t_1} := \{(k_1, k_2) \in \mathbb{Z}^2 \mid k_1 \leq t_1\} \ , \tag{2}$$

and their subsets

$$\mathcal{S}_{t_0,t_1} := \{(k_1, k_2) \in \mathbb{Z}^2 \mid t_0 \leq k_1 \leq t_1\} \ . \tag{3}$$

These are illustrated in Fig.s 1 and 2, respectively. Often in the following we call a set $\mathcal{L}_t = \mathcal{S}_{t,t}$ a *vertical line.*

The definition of time-relevant behavior is the following.

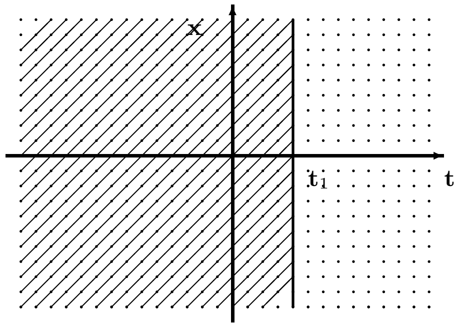

Fig. 1. A set $\mathcal{S}_{t_1}$, see formula (2).

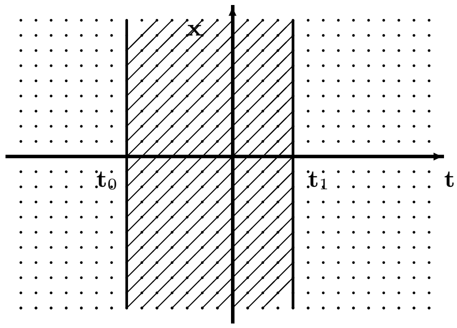

Fig. 2. A set $\mathcal{S}_{t_0,t_1}$, see formula (3).

Definition 5 $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ *is* time-relevant *if for all $t \in \mathbb{Z}$ the sets S_t of the form (2) are characteristic.*

Observe that linear shift-invariant finite-dimensional $2D$ behaviors $\mathfrak{B}^{\mathrm{fd}}$ are time-relevant: indeed, it can be shown that any sufficiently large (finite) rectangle in $\mathbb{Z}^2$ is characteristic for $\mathfrak{B}^{\mathrm{fd}}$, and consequently all the sets S_t are characteristic for $\mathfrak{B}^{\mathrm{fd}}$. From Proposition 4 it follows then that a behavior $\mathfrak{B}$ is time-relevant if and only if its square part $\mathfrak{B}^{\mathrm{sq}}$ is time-relevant. Therefore, in the rest of this section we concentrate on square autonomous behaviors.

We next give a characterization of time-relevance for square autonomous systems; namely, we show that a time-relevant (square, autonomous) behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ has a special kernel representation; this will be useful in proving several important results later on in the paper.

Proposition 6 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be a square autonomous behavior. Then $\mathfrak{B}$ is time-relevant if and only if there exists $R \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi_1^{-1}, \xi_2, \xi_2^{-1}]$ such that $\mathfrak{B} = \ker R(\sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$ and*

$$\begin{aligned} R(\xi_1^{-1}, \xi_2, \xi_2^{-1}) = I_{\mathtt{w}} + R_1(\xi_2, \xi_2^{-1})\xi_1^{-1} + \ldots \\ + R_L(\xi_2, \xi_2^{-1})\xi_1^{-L} \ , \end{aligned} \tag{4}$$

where $L \in \mathbb{N}$, and $R_i \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi_2, \xi_2^{-1}]$, $i = 1, \ldots, L$.

Proof. We begin with the following preliminary result.

Lemma 7 *Let $Q \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[s,\xi_2,\xi_2^{-1}]$, given by*

$$Q(s,\xi_2,\xi_2^{-1}) = Q_0(\xi_2,\xi_2^{-1}) + \ldots + Q_L(\xi_2,\xi_2^{-1})s^L$$

with $Q_L \neq 0$, be nonsingular. Then there exists a unimodular matrix $U \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[s^{-1},\xi_2,\xi_2^{-1}]$ such that

$$U(s^{-1},\xi_2,\xi_2^{-1})Q(s,\xi_2,\xi_2^{-1})$$

has the following properties:

(1) $U(s^{-1},\xi_2,\xi_2^{-1})Q(s,\xi_2,\xi_2^{-1}) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[s,\xi_2,\xi_2^{-1}]$;
(2) $\tilde{Q}(s,\xi_2,\xi_2^{-1}) := U(s^{-1},\xi_2,\xi_2^{-1})Q(s,\xi_2,\xi_2^{-1})$ is such that $\tilde{Q}(0,\xi_2,\xi_2^{-1})$ is nonsingular.

Proof. Assume that $Q(0,\xi_2,\xi_2^{-1})$ has not full rank; then there exists a unimodular matrix $V(\xi_2,\xi_2^{-1})$ such that

$$V(\xi_2,\xi_2^{-1})Q(0,\xi_2,\xi_2^{-1}) = \begin{bmatrix} F(\xi_2,\xi_2^{-1}) \\ 0 \end{bmatrix},$$

with F of full row rank, and consequently there exist $F_e, H \in \mathbb{R}^{\bullet\times\bullet}[s,\xi_2,\xi_2^{-1}]$ such that $F_e(0,\xi_2,\xi_2^{-1}) = F(\xi_2,\xi_2^{-1})$ and

$$V(\xi_2,\xi_2^{-1})Q(s,\xi_2,\xi_2^{-1}) = \begin{bmatrix} F_e(s,\xi_2,\xi_2^{-1}) \\ sH(s,\xi_2,\xi_2^{-1}) \end{bmatrix},$$

or equivalently,

$$\begin{bmatrix} I & 0 \\ 0 & s^{-1}I \end{bmatrix} V(\xi_2,\xi_2^{-1})Q(s,\xi_2,\xi_2^{-1}) = \begin{bmatrix} F_e(s,\xi_2,\xi_2^{-1}) \\ H(s,\xi_2,\xi_2^{-1}) \end{bmatrix} =: Q^1(s,\xi_2,\xi_2^{-1}) . \quad (5)$$

For future reference, denote by N_0 the highest power of s in det $Q(s,\xi_2,\xi_2^{-1})$, and by N_1 the highest power of s in det $Q^1(s,\xi_2,\xi_2^{-1})$. It follows from (5) that $N_0 > N_1$.

Now check the rank of $Q^1(0,\xi_2,\xi_2^{-1})$; if it is full, then the claim of the Lemma is proved with $U(s^{-1},\xi_2,\xi_2^{-1}) = \begin{bmatrix} I & 0 \\ 0 & s^{-1}I \end{bmatrix} V(\xi_2,\xi_2^{-1})$ and $\tilde{Q}(s,\xi_2,\xi_2^{-1}) = Q^1(s,\xi_2,\xi_2^{-1})$. If not, we can apply a transformation to $Q^1(s,\xi_2,\xi_2^{-1})$ analogous to the one performed in the previous step. Since at each step k the highest power of s in the determinant of $Q^k(s,\xi_2,\xi_2^{-1})$ decreases, the procedure terminates, yielding $\tilde{Q}$ with the desired properties. ■

We now prove the claim of the Proposition.

(*If*) Assume that a representation (4) of $\mathfrak{B}$ exists, and observe that $w \in \mathfrak{B}$ is equivalent to

$$w = -R_1(\sigma_2,\sigma_2^{-1})\sigma_1^{-1}w - \cdots - R_L(\sigma_2,\sigma_2^{-1})\sigma_1^{-L}w .$$

Consequently if w is zero in $\mathcal{S}_0$, then its restriction to the vertical line $\{(1,k) \mid k \in \mathbb{Z}\}$ is zero. Extending this solution to the next vertical line, the one after that, and so on, shows that w is zero everywhere. Therefore from Proposition 2 it follows that $\mathfrak{B}$ is time-relevant.

(*Only if*) Assume that $\mathfrak{B}$ is time-relevant, and let $\widetilde{R} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1}]$ induce a kernel representation of $\mathfrak{B}$, i.e. $\mathfrak{B} = \ker \widetilde{R}(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})$. If necessary, premultiply $\widetilde{R}$ on the left by a unimodular matrix of the form $\xi_1^{-K}I_{\mathtt{w}}$, $K \in \mathbb{N}$, to obtain an equivalent representation of the form

$$R'(\xi_1^{-1},\xi_2,\xi_2^{-1}) = R'_0(\xi_2,\xi_2^{-1}) + R'_1(\xi_2,\xi_2^{-1})\xi_1^{-1} + \ldots + R'_L(\xi_2,\xi_2^{-1})\xi_1^{-L} .$$

Consider

$$Q(s,\xi_2,\xi_2^{-1}) := R'_0(\xi_2,\xi_2^{-1}) + R'_1(\xi_2,\xi_2^{-1})s + \ldots + R'_L(\xi_2,\xi_2^{-1})s^L ;$$

in view of the result of Lemma 7, we can assume without loss of generality that $R'_0(\xi_2,\xi_2^{-1})$ is nonsingular.

Now recall that the trajectories in $\mathfrak{B}$ satisfy

$$R'_0(\sigma_2,\sigma_2^{-1})w = -R'_1(\sigma_2,\sigma_2^{-1})\sigma_1^{-1}w - \cdots - R'_L(\sigma_2,\sigma_2^{-1})\sigma_1^{-L}w .$$

Consequently, $w \in \mathfrak{B}$ and $w_{|\mathcal{S}_0} = 0$ imply

$$R'_0(\sigma_2,\sigma_2^{-1})w(1,\cdot) = 0 . \quad (6)$$

Now define

$$\mathcal{Z} := \{v \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}} \ : \ \exists w \in \mathfrak{B} \text{ such that } w|_{\mathcal{S}_0} = 0 \text{ and } w|_{\mathcal{L}_1} = v\} ;$$

this is the set of the restrictions to the vertical line $\mathcal{L}_1$ of $\mathfrak{B}$-trajectories which are zero in the past. Since $\mathfrak{B}$ is time-relevant by assumption, $\mathcal{Z}$ only consists of the zero trajectory. We now proceed to show that $\mathcal{Z} = \ker R'_0(\sigma_2,\sigma_2^{-1})$.

It follows from (6) that $\mathcal{Z} \subset \ker R'_0(\sigma_2,\sigma_2^{-1})$; we next prove the converse inclusion. Let $v \in \ker R'_0(\sigma_2,\sigma_2^{-1})$ and consider $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ such that: $w(k,\cdot) = 0,\ k \leq 0$, $w(1,\cdot) = v(\cdot)$ and

$$R'_0(\sigma_2,\sigma_2^{-1})w(k,\cdot) = = -R'_1(\sigma_2,\sigma_2^{-1})w(k-1,\cdot) \ldots - R'_L(\sigma_2,\sigma_2^{-1})w(k-L,\cdot),$$

for $k \geq 2$. Note that such a trajectory exists since $R_0'(\sigma_2, \sigma_2^{-1})$ is a surjective operator (as $R_0'(\xi_2, \xi_2^{-1})$ is nonsingular). Moreover, by construction $w|_{\mathcal{S}_0} = 0$, $w \in \mathfrak{B}$ and $w|_{\mathcal{L}_1} = v$, and therefore $v \in \mathcal{Z}$. This means that ker $R_0'(\sigma_2, \sigma_2^{-1}) = \mathcal{Z}$.

It follows from this argument that $\mathfrak{B}$ is time-relevant if and only if ker $R_0'(\sigma_2, \sigma_2^{-1}) = \{0\}$ i.e. if and only if the matrix R_0' is unimodular (invertible in $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_2, \xi_2^{-1}]$). The unimodular transformation $R_0'(\xi_2, \xi_2^{-1})^{-1} R'(\xi_1^{-1}, \xi_2, \xi_2^{-1})$ yields the desired form of the kernel representation. This concludes the proof of the claim. ∎

The result of Proposition 6 shows that $\mathfrak{B}$ is time-relevant if and only if the restriction of $w \in \mathfrak{B}$ to a vertical line $\mathcal{L}_{t_1}$ where $t_1 \in \mathbb{Z}$, is a linear combination of the restrictions of w and its shifts $\sigma_2^k w$ to a finite number of similar lines $\mathcal{L}_{t_0}$ with $t_0 < t_1$. The minimal number of such lines will be called the *time-lag* of $\mathfrak{B}$.

3 Time-relevant stability, and its algebraic characterization

In the seminal paper [24], it has been discussed that a sound definition of stability for $2D$ systems needs to take into account the "initial conditions" of trajectories. Indeed, as pointed out in [24], when dealing with square autonomous behaviors, which have infinite characteristic sets, certain choices of the free initial conditions may produce situations such that the corresponding trajectory does not asymptotically decay to zero. Therefore, some requirements on the initial conditions must be imposed.

For time-relevant systems, we require that the restrictions of a trajectory $w \in \mathfrak{B}$ to a finite family of vertical lines are square summable, i.e. have finite energy along those lines. As we shall see, this is enough to ensure that then the restriction of w to any vertical line in the "future" is an ℓ_2-trajectory.

Proposition 8 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be a time-relevant square autonomous behavior with time-lag L. Assume that there exists $t_0 \in \mathbb{Z}$ and $w \in \mathfrak{B}$ such that, for all $k \in \mathbb{Z} \cap [t_0, t_0 + L)$, $w_{|\mathcal{L}_k} := w(k, \cdot) \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ is square-summable. Then $w_{|\mathcal{L}_k} \in \ell_2(\mathbb{Z}, \mathbb{R}^{\mathtt{w}})$ for all $k \geq t_0 + L$.*

Proof. This follows from the fact that the restriction of $w \in \mathfrak{B}$ to any set $\mathcal{L}_k$ is a linear combination of a finite number of restrictions of w and its shifts to sets $\mathcal{L}_{k'}$ with $k' < k$. Therefore, if the latter are ℓ_2-trajectories, so is w restricted to $\mathcal{L}_k$. ∎

This result supports the definition of time-relevant stability that follows: a system is time-relevant stable if whenever $w \in \mathfrak{B}$ has 'initial conditions' of finite energy in a set $\mathcal{S}_{t_0, t_0+N-1} = \cup_{k=t_0,\ldots,t_0+N-1} \mathcal{L}_k$, then the energy of the restrictions of w along vertical lines goes to zero with time.

Definition 9 *A time-relevant behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is* time-relevant stable *if there exists $N \in \mathbb{N}$ such that*

$$\{[w \in \mathfrak{B}] \text{ and } [w(k, \cdot) \in \ell_2(\mathbb{Z}, \mathbb{R}^{\mathtt{w}}) \text{ for all } 0 \leq k \leq N-1]\} \Longrightarrow \left[\lim_{k\to\infty} \|w(k,\cdot)\|_{\ell_2} = 0\right].$$

Note that, if it exists, the integer N of the previous definition must be greater than or equal to the time lag of the behavior. For the sake of simplicity when considering time-relevant systems we sometimes simply refer to time-relevant stability as stability.

It is easy to see that if $\mathfrak{B}$ is finite-dimensional, the only trajectory which is square summable along a vertical line is the zero trajectory. Thus, it follows from Definition 9 that a finite-dimensional behavior is always time-relevant stable. From the decomposition result of Proposition 4 it then also follows that an autonomous behavior $\mathfrak{B}$ is time-relevant stable if and only if its square part is time-relevant stable. Consequently, in the rest of this paper we will be focusing on square autonomous behaviors $\mathfrak{B}$.

We now give an algebraic test for the stability of a time-relevant system.

Theorem 10 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be a time-relevant square autonomous behavior, and let $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$ induce a kernel representation of $\mathfrak{B}$. The behavior $\mathfrak{B}$ is time-relevant stable if and only if for all $\omega \in \mathbb{R}$ the Laurent polynomial* det $R(\xi_1, \xi_1^{-1}, e^{i\omega}, e^{-i\omega})$ *has all its roots in the open unit disk.*

Proof. We first assume that the kernel representation is induced by a matrix R of the form (4). The claim for the general case follows easily, as we show further on.

Note that if R is of the form (4), namely

$$R(\xi_1^{-1}, \xi_2, \xi_2^{-1}) = I_{\mathtt{w}} + R_1(\xi_2, \xi_2^{-1})\xi_1^{-1} + \ldots + R_N(\xi_2, \xi_2^{-1})\xi_1^{-L},$$

then it is straightforward to see that its determinant is of the form

$$d(\xi_1^{-1}, \xi_2, \xi_2^{-1}) := \det R(\xi_1^{-1}, \xi_2, \xi_2^{-1}) = 1 + d_1(\xi_2, \xi_2^{-1})\xi_1^{-1} + \ldots + d_N(\xi_2, \xi_2^{-1})\xi_1^{-N},$$

for some $N \in \mathbb{N}$.

(If) We reduce ourselves to the scalar case in the following way, analogously to [24]. Denote by $\mathrm{Adj}(R)(\xi_1^{-1},\xi_2,\xi_2^{-1})$ the adjoint matrix of $R(\xi_1^{-1},\xi_2,\xi_2^{-1})$, i.e.

$$\mathrm{Adj}(R)(\xi_1^{-1},\xi_2,\xi_2^{-1})\, R(\xi_1^{-1},\xi_2,\xi_2^{-1}) = d(\xi_1^{-1},\xi_2,\xi_2^{-1})\, I_{\mathtt{w}} \ . \tag{7}$$

Now define $\mathfrak{B}^{\sup} := \ker\ d(\sigma_1^{-1},\sigma_2,\sigma_2^{-1})\ I_{\mathtt{w}}$, and note that, because of (7), $\mathfrak{B}^{\sup} \supset \mathfrak{B}$. Note also that $w \in \mathfrak{B}^{\sup}$ if and only if each of the components of w satisfies the scalar difference equation $d(\sigma_1^{-1},\sigma_2,\sigma_2^{-1})w' = 0$.

Assuming that for all $\omega \in \mathbb{R}$ $\det R(\xi_1^{-1}, e^{i\omega}, e^{-i\omega})$ has all its roots in the open unit disk, we will show that $\ker\ d(\sigma_1^{-1},\sigma_2,\sigma_2^{-1})$ is time-relevant stable; this implies that $\mathfrak{B} \subset \mathfrak{B}^{\sup}$ is also time-relevant stable.

Denote by w' any trajectory in $\ker\ d(\sigma_1^{-1},\sigma_2,\sigma_2^{-1})$, and define

$$x(k,\cdot) := \begin{bmatrix} w'(k-N,\cdot) \\ w'(k-N+1,\cdot) \\ \vdots \\ w'(k-1,\cdot) \end{bmatrix} .$$

Now it can be shown that $w' \in \ker\ d(\sigma_1^{-1},\sigma_2,\sigma_2^{-1})$ if and only if there exists x such that

$$\begin{aligned} \sigma_1 x(k_1,k_2) &= A(\sigma_2,\sigma_2^{-1})\ x(k_1,k_2) \\ w'(k_1,k_2) &= C(\sigma_2,\sigma_2^{-1}) x(k_1,k_2)\ , \end{aligned} \tag{8}$$

$(k_1,k_2) \in \mathbb{Z}^2$, where $A(\sigma_2,\sigma_2^{-1})$ is a companion matrix of polynomial shift operators:

$$A(\sigma_2,\sigma_2^{-1}) := \tag{9}$$
$$\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -d_N(\sigma_2,\sigma_2^{-1}) & -d_{N-1}(\sigma_2,\sigma_2^{-1}) & \cdots & -d_1(\sigma_2,\sigma_2^{-1}) \end{bmatrix},$$

and

$$C(\sigma_2,\sigma_2^{-1}) := \begin{bmatrix} -d_N(\sigma_2,\sigma_2^{-1}) & \cdots & -d_1(\sigma_2,\sigma_2^{-1}) \end{bmatrix} .$$

Let now $w' \in \ker\ d(\sigma_1^{-1},\sigma_2,\sigma_2^{-1})$ be such that $w'(k,\cdot) \in \ell_2(\mathbb{Z},\mathbb{R})$, $0 \le k \le N$; then the corresponding $x(0,\cdot)$ belongs to $\ell_2(\mathbb{Z},\mathbb{R}^N)$. Observe that for every $x(k,\cdot) \in \ell_2(\mathbb{Z},\mathbb{R}^N)$ it holds that $x(k+1,\cdot) \in \ell_2(\mathbb{Z},\mathbb{R}^N)$, $k \ge 0$, i.e. $A(\sigma_2,\sigma_2^{-1})$ is an operator from $\ell_2(\mathbb{Z},\mathbb{R}^N)$ to $\ell_2(\mathbb{Z},\mathbb{R}^N)$. Therefore, $w'(k,\cdot) \in \ell_2(\mathbb{Z},\mathbb{R})$ for every $k \ge 0$. Further,

$$\left[\lim_{k\to\infty} x(k,\cdot) = 0\right] \Longrightarrow \left[\lim_{k\to\infty} w'(k,\cdot) = 0\right] ,$$

and hence

$$\left[\lim_{k\to\infty} \|x(k,\cdot)\|_{\ell_2} = 0\right] \Longrightarrow \left[\lim_{k\to\infty} \|w'(k,\cdot)\|_{\ell_2} = 0\right] .$$

Thus, the stability of $\mathfrak{B}^{\sup} = \ker\ d(\sigma_1^{-1},\sigma_2,\sigma_2^{-1})\ I_{\mathtt{w}}$ is proved if we show that the x-trajectories of (8) satisfy $\lim_{k\to\infty} x(k,\cdot) = 0$, for every $x(0,\cdot) \in \ell_2(\mathbb{Z},\mathbb{R}^N)$.

In order to do this, since $x(k,\cdot) = A(\sigma_2,\sigma_2^{-1})^k x(0,\cdot)$, $\lim_{k\to\infty} x(k,\cdot) = 0$ for all $x(0,\cdot) \in \ell_2(\mathbb{Z},\mathbb{R}^N)$ if and only if

$$\begin{aligned} &\lim_{k\to\infty} \|A(\sigma_2,\sigma_2^{-1})^k\|_\infty \\ &= \lim_{k\to\infty} \sup\nolimits_{\omega\in\mathbb{R}} \sqrt{\rho(A(e^{i\omega},e^{-i\omega})^{k\top} A(e^{i\omega},e^{-i\omega})^k)} = 0 \ , \end{aligned}$$

or, equivalently, for all $\omega \in \mathbb{R}$,

$$\lim_{k\to\infty}\ \rho(A(e^{i\omega},e^{-i\omega})^{k\top} A(e^{i\omega},e^{-i\omega})^k) = 0 \ .$$

Consequently, $\lim_{k\to\infty} x(k,\cdot) = 0$ if and only if for each fixed but otherwise arbitrary $\omega \in \mathbb{R}$

$$\begin{aligned} &\lim_{k\to\infty} \rho(A(e^{i\omega},e^{-i\omega})^{k\top} A(e^{i\omega},e^{-i\omega})^k) = 0 \\ &\Longleftrightarrow \lim_{k\to\infty} \|A(e^{i\omega},e^{-i\omega})^k\|_2 = 0 \\ &\Longleftrightarrow \lim_{k\to\infty} \|A(e^{i\omega},e^{-i\omega})^k v^*\|_2 = 0 \text{ for all } v^* \in \mathbb{C}^N \ . \end{aligned}$$

The last statement is equivalent to saying that the complex system

$$v(k+1) = A(e^{i\omega},e^{-i\omega}) v(k)$$

is stable, which in turn is equivalent to

$$\rho(A(e^{i\omega},e^{-i\omega})) < 1 \ , \tag{10}$$

which again is equivalent to $d'(s,e^{i\omega},e^{-i\omega}) := \det(Is - A(e^{i\omega},e^{-i\omega}))$ having all its roots in the unit circle. Since $d(\xi_1^{-1},e^{i\omega},e^{-i\omega}) = \xi_1^{-N} d'(\xi_1,e^{i\omega},e^{-i\omega})$, it follows that the assumption that all the roots of $d(\xi_1^{-1},e^{i\omega},e^{-i\omega})$ are in the unit circle implies that the condition (10) holds.

(Only if): Assume that $\mathfrak{B}$ is time-relevant stable and let N be as in Definition 9. Further, let $w \in \mathfrak{B}$ be such that

$$w(k,\cdot) \in \ell_2(\mathbb{Z},\mathbb{R}^N) \text{ for all } 0 \le k \le N-1.$$

Then, by Proposition 8, $w(k,\cdot) \in \ell_2(\mathbb{Z},\mathbb{R}^N)$ also for $k \ge N$ and, for all $k \ge 0$, $w(k,\cdot)$ admits a discrete-time Fourier transform $\widehat{w}(k,e^{i\omega}) := \sum_{k_2=-\infty}^{+\infty} w(k,k_2)e^{-i\omega k_2}$. Since w satisfies the equation

$$\left(R(\sigma_1^{-1},\sigma_2,\sigma_2^{-1})w\right)(k_1,k_2) = 0$$

for all $k_1, k_2 \in \mathbb{Z}$, then $\widehat{w}$ satisfies the equation

$$R(\sigma_1^{-1}, e^{i\omega}, e^{-i\omega})\widehat{w}(k, e^{i\omega}) = 0 \tag{11}$$

for $k \geq N$.

Due to the assumption of time-relevant stability, it holds that $\lim_{k\to\infty} \|w(k,\cdot)\|_{\ell_2} = 0$. Now note that

$$\begin{aligned}
\|w(k,\cdot)\|_{\ell_2} &= \sum_{k_2=-\infty}^{+\infty} \|w(k,k_2)\|_2^2 \\
&= \sum_{k_2=-\infty}^{+\infty} \|w(k,k_2)e^{-ik_2\omega}\|_2^2 \\
&\geq \| \sum_{k_2=-\infty}^{+\infty} w(k,k_2)e^{-ik_2\omega}\|_2^2 \\
&= \|\hat{w}(k,e^{i\omega})\|_2^2 \ ,
\end{aligned}$$

for every fixed but otherwise arbitrary $\omega \in \mathbb{R}$. Therefore, for all $\omega \in \mathbb{R}$ it holds that $\lim_{k\to\infty} \|\hat{w}(k,e^{i\omega})\|_2^2 = 0$. This implies that for all $\omega \in \mathbb{R}$, $\lim_{k\to\infty} \widehat{w}(k,e^{i\omega}) = 0$ and consequently that the complex 1D system (11) is stable. This in turn implies that $\det R(\xi^{-1}, e^{i\omega}, e^{-i\omega})$ has all its roots in the open unit circle.

In order to conclude the proof of the Theorem, we need to show that the result also holds for general kernel representations of $\mathfrak{B}$. Consider an arbitrary $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$ such that $\mathfrak{B} = \ker R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$, and observe that it is unimodularly equivalent in $\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$ to a matrix R' of the form (4). It follows from this that $\det(R)$ is also unimodularly equivalent to $\det(R')$. Consequently they share the same nonzero finite roots, and hence $\det\ R(\xi_1, \xi_1^{-1}, e^{i\omega}, e^{-i\omega})$ has all its roots in the open unit circle if and only if $\det\ R'(\xi_1^{-1}, e^{i\omega}, e^{-i\omega})$ has, which concludes the proof of the theorem. ■

The result of Theorem 10 suggests to verify the time-relevant stability of a behavior $\ker R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$ by checking the location of the roots of the ω-dependent Laurent polynomial $\det\ R(\xi_1, \xi_1^{-1}, e^{i\omega}, e^{-i\omega})$ as ω varies in $\mathbb{R}$. We illustrate efficient ways to perform this check in section 5 of this paper, where we develop tests for time-relevant stability which do not involve dependency on the variable ω. To obtain those results we will put to strenuous use the calculus of *1-D quadratic difference forms* (*1-D QDFs*), briefly reviewed in the next section.

4 1-D quadratic difference forms

In [26] it has been shown that by using two-variable polynomial matrices a calculus of functionals can be developed which seamlessly integrates with the calculus of continuous-time representations based on one-variable polynomial algebra. This framework was extended to the discrete-time case in [8], which we take as a basis for the following summary.

Quadratic difference forms (QDF) are mappings from $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ to $\mathbb{R}^{\mathbb{Z}}$ defined in the following way.

Let

$$\mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] := \{\Phi(\zeta,\eta) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta] \ : \ \Phi(\zeta,\eta) = \Phi(\eta,\zeta)^\top\}$$

denote the set of symmetric real two-variable (or 2D) $\mathtt{w} \times \mathtt{w}$ polynomial matrices. Given $\Phi(\zeta,\eta) \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ the QDF Q_Φ associated with Φ is defined as

$$\begin{aligned}
Q_\Phi : (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}} &\longrightarrow \mathbb{R}^{\mathbb{Z}} \\
w \ &\mapsto \ Q_\Phi(w) = \sum_{k,\ell} (\sigma^k w)^\top \Phi_{k,\ell} \sigma^\ell w \ ,
\end{aligned}$$

where $\sigma : (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}} \to (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ is the 1D shift, defined as $(\sigma w)(k) := w(k+1)$.

Note that $\Phi(\zeta,\eta)$ can be identified with its *coefficient matrix*, defined as

$$\tilde{\Phi} := \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} ;$$

which is an infinite matrix, but only has a finite number of nonzero rows and columns. Moreover,

$$\Phi(\zeta,\eta) = \begin{bmatrix} I_{\mathtt{w}} & \zeta I_{\mathtt{w}} & \cdots \end{bmatrix} \tilde{\Phi} \begin{bmatrix} I_{\mathtt{w}} \\ \eta I_{\mathtt{w}} \\ \vdots \end{bmatrix} . \tag{12}$$

Note that factorizations of the coefficient matrix of a QDF give rise to factorizations of the corresponding two-variable polynomial matrix. This will be used later on in the proof of Theorem 17.

We call a QDF Q_Φ *nonnegative* , denoted $Q_\Phi \geq 0$, if

$$Q_\Phi(\ell)(k) \geq 0 \text{ for all } \ell \in (\mathbb{R}^\bullet)^{\mathbb{Z}} \text{ and for all } k \in \mathbb{Z} \ .$$

We call Q_Φ *positive* , denoted $Q_\Phi > 0$, if

$$Q_\Phi \geq 0 \text{ and } [Q_\Phi(\ell) = 0] \Longrightarrow [\ell = 0] \ .$$

A QDF Q_Φ is called *average nonnegative* if

$$\sum_{k=-\infty}^{\infty} Q_\Phi(\ell)(k) \geq 0$$

for all finite support $\ell \in (\mathbb{R}^\bullet)^\mathbb{Z}$. Q_Ψ is a *storage function for Q_Φ* if the following *dissipation inequality* holds:

$$\nabla Q_\Psi \leq Q_\Phi \ , \tag{13}$$

where ∇Q_Ψ denotes the increment of Q_Ψ along the (single) independent variable, i.e. $\nabla Q_\Psi(\ell) := Q_\Psi(\sigma(\ell)) - Q_\Psi(\ell)$ and where $\nabla Q_\Psi \leq Q_\Phi$ denotes $Q_\Phi - \nabla Q_\Psi \geq 0$ (as a QDF).

A QDF Q_Δ is said to be a *dissipation function for Q_Φ* if

$$Q_\Delta \geq 0 \quad \text{and} \quad \sum_{k=-\infty}^{+\infty} Q_\Phi(\ell)(k) = \sum_{k=-\infty}^{+\infty} Q_\Delta(\ell)(k) \tag{14}$$

for all finite support trajectories $\ell \in (\mathbb{R}^\bullet)^\mathbb{Z}$.

The following equivalences are well known, see Proposition 3.3 of [8].

Proposition 11 *Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. The following conditions are equivalent:*

(1) Q_Φ is average nonnegative;
(2) $\Phi(e^{-i\omega}, e^{i\omega}) \geq 0$ for all $\omega \in \mathbb{R}$;
(3) Q_Φ admits a storage function Q_Ψ;
(4) Q_Φ admits a dissipation function Q_Δ.

Moreover, the following holds:

$$Q_\Phi = \nabla Q_\Psi + Q_\Delta \ ; \tag{15}$$

or equivalently in two-variable polynomial terms

$$\Phi(\zeta,\eta) = (\zeta\eta - 1)\Psi(\zeta,\eta) + \Delta(\zeta,\eta) \ . \tag{16}$$

The equality (15) is usually referred to as the *dissipation equality.*

5 LMI conditions for time-relevant stability

LMI tests for checking various properties of $2D$ systems, among which stability, have been proposed in [1,6], building on the pioneering work done in [17] for the study of parameter-dependent inequalities. Other relevant results in this area aimed at eliminating the parameter-dependency in stability tests for $2D$ systems have been reported in [5]; applications to discrete linear repetitive processes have been reported in [9]. In these approaches, however, the focus is on specific types of $2D$ systems representations, typically 'state-space' ones; in this section instead we consider the general case of higher-order systems, using heavily the calculus of quadratic difference forms developed in [8] and briefly recalled in the previous section.

The main line of exposition in this section is the following. We first introduce the ω-dependent Bézoutian associated with a polynomial $d(\xi_1, e^{i\omega})$ obtained from a polynomial $d \in \mathbb{R}[\xi_1,\xi_2]$ by letting $\xi_2 = e^{i\omega}$. The relationship between the positivity of the Bézoutian and the Schurness of a polynomial (meaning all its roots are in the open unit circle) is well known (see for example [7,14]), and this provides a first test for stability. However, this test is unsatisfactory since it requires to check the positivity of a parameter-dependent matrix; using dissipativity theory we will develop alternative tests based on LMIs involving finite constant matrices.

In the sequel it will be convenient to consider behaviors $\mathfrak{B}$ whose trajectories take values in $\mathbb{C}^{\mathtt{w}}$. Moreover, since the calculus of QDFs has been developed only for polynomial representations, in the following without loss of generality (i.e. possibly multiplying a given kernel representation by a unimodular matrix) we will consider polynomial kernel representations $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1,\xi_2]$.

In general, complex behaviors correspond to kernel representations induced by polynomials with complex coefficients; consequently, we now introduce some new notation. Given a polynomial $p \in \mathbb{C}[s]$

$$p(s) = p_0 + \cdots + p_L s^L \ , \tag{17}$$

with $p_L \neq 0$, the *reciprocal* of p is defined as

$$p_r(s) := p_0 s^L + \cdots + p_L = s^L p(s^{-1}) \ . \tag{18}$$

The *conjugate* of the polynomial p is defined as

$$\overline{p}(s) := \overline{p_0} + \cdots + \overline{p_L} s^L \ ,$$

and the *Bézoutian* of p is

$$B_p(\mu,\nu) := \frac{\overline{p}(\mu)p(\nu) - p_r(\mu)\overline{p_r}(\nu)}{\mu\nu - 1} \ .$$

Now consider

$$d(\xi_1,\xi_2) = d_0(\xi_2) + \cdots + d_L(\xi_2)\xi_1^L \in \mathbb{R}[\xi_1,\xi_2] \ , \tag{19}$$

with $d_L(\xi_2) \neq 0$ and the j-th coefficient $d_j \in \mathbb{R}[\xi_2]$. Note that $d(\xi_1, e^{i\omega})$ is a polynomial in ξ_1 with complex coefficients, and that, since the polynomials $d_j \in \mathbb{R}[\xi_2]$, $\overline{d}(\xi_1, e^{i\omega}) = d(\xi_1, e^{-i\omega})$. We denote the Bézoutian $B_{d(\xi_1,e^{i\omega})}(\mu,\nu)$ by $B_\omega(\mu,\nu)$, that is

$$B_\omega(\mu,\nu) := B_{d(\xi_1,e^{i\omega})}(\mu,\nu) \ . \tag{20}$$

It is worth mentioning here the important role played by the ω-dependent Bézoutian (20) in the study of stability of $2D$ filters in the approach of [10,11].

Observe that $B_\omega(\mu,\nu)$ can be written as

$$B_\omega(\mu,\nu) =: S(\mu)^\top \tilde{B}(e^{-i\omega}, e^{i\omega}) S(\nu) \qquad (21)$$

where $S(\xi) := \begin{bmatrix} 1 & \xi & \cdots & \xi^{L-1} \end{bmatrix}^\top$ and the $L \times L$ matrix $\tilde{B}(e^{-i\omega}, e^{i\omega})$ is the *coefficient matrix of* $B_\omega(\mu,\nu)$, defined as:

$$\tilde{B}(e^{-i\omega}, e^{i\omega}) =: \begin{bmatrix} \tilde{B}_{00}(e^{-i\omega}, e^{i\omega}) & \cdots & \tilde{B}_{0,L-1}(e^{-i\omega}, e^{i\omega}) \\ \tilde{B}_{01}(e^{i\omega}, e^{-i\omega}) & \cdots & \tilde{B}_{1,L-1}(e^{-i\omega}, e^{i\omega}) \\ \vdots & \cdots & \vdots \\ \tilde{B}_{0,L-1}(e^{i\omega}, e^{-i\omega}) & \cdots & \tilde{B}_{L-1,L-1}(e^{-i\omega}, e^{i\omega}) \end{bmatrix}.$$

The following result can be proved using standard polynomial algebra arguments, see for example Theorem 1 p. 172 of [14] and section VI.C of [7].

Proposition 12 *Let d be defined as in (19). Then $d(\xi_1, e^{i\omega})$ is Schur for all $\omega \in \mathbb{R}$ if and only if the coefficient matrix $\tilde{B}(e^{-i\omega}, e^{i\omega})$ of the Bézoutian $B_\omega(\mu,\nu)$ defined in (20) is positive definite for all $\omega \in \mathbb{R}$.*

The result of Proposition 12 allows in principle to check the time-relevant stability of a $2D$ system by applying standard tests for the positive-definiteness of matrices to the coefficient matrix $\tilde{B}(e^{-i\omega}, e^{i\omega})$.

Example 13 Consider the system with $\mathtt{w} = 2$ external variables

$$\mathfrak{B} = \ker\, R(\sigma_1,\sigma_2) := \ker \begin{bmatrix} 1+\sigma_1 & -\frac{1}{4} + 2\sigma_1 - \frac{3}{8}\sigma_2 \\ 1 & \sigma_1 - \frac{1}{4}\sigma_2 \end{bmatrix};$$

premultiplying $R(\xi_1,\xi_2)$ by $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \xi_1^{-1}$ yields a representation as in Proposition 6 from which it follows that $\mathfrak{B}$ is time-relevant. The determinant of $R(\xi_1,\xi_2)$ is

$$\begin{aligned} d(\xi_1,\xi_2) &:= \det\; R(\xi_1,\xi_2) \\ &= \underbrace{\left(\frac{1}{4} + \frac{\xi_2}{8}\right)}_{r_0(\xi_2)} + \underbrace{\left(-1 - \frac{\xi_2}{4}\right)}_{r_1(\xi_2)}\xi_1 + \underbrace{1}_{r_2(\xi_2)} \cdot \xi_1^2 . \end{aligned}$$

The coefficient matrix of the Bézoutian $B_\omega(\mu,\nu)$ of $d(\xi_1, e^{i\omega})$ is

$$\begin{bmatrix} 1 - r_0(e^{-i\omega})r_0(e^{i\omega}) & r_1(e^{-i\omega}) - r_0(e^{-i\omega})r_1(e^{i\omega}) \\ r_1(e^{i\omega}) - r_0(e^{i\omega})r_1(e^{-i\omega}) & 1 \end{bmatrix}.$$

It is positive definite for all $\omega \in \mathbb{R}$ if and only if the following inequalities hold for all $\omega \in \mathbb{R}$:

$$\begin{aligned} &1 - r_0(e^{-i\omega})r_0(e^{i\omega}) > 0 \\ &1 - r_0(e^{-i\omega})r_0(e^{i\omega}) > (r_1(e^{-i\omega}) - r_0(e^{-i\omega})r_1(e^{i\omega})) \\ &\qquad\qquad \cdot (r_1(e^{i\omega}) - r_0(e^{i\omega})r_1(e^{-i\omega})) . \end{aligned}$$

It is a matter of straightforward computations to show that $1 - r_0(e^{-i\omega})r_0(e^{i\omega}) = \frac{1}{64}(59 - 4\cos\omega)$ and that $(r_1(e^{-i\omega}) - r_0(e^{-i\omega})r_1(e^{i\omega}))(r_1(e^{i\omega}) - r_0(e^{i\omega})r_1(e^{-i\omega})) = 549 + 92\cos\omega - \frac{16}{1024}\cos 2\omega$. These two functions of ω are positive for all $\omega \in \mathbb{R}$, and consequently we conclude that $d(\xi_1, e^{i\omega})$ is Schur for all $\omega \in \mathbb{R}$. ■

When the coefficient matrix $\tilde{B}(e^{-i\omega}, e^{i\omega})$ is of higher dimension than that considered in the previous example, the positive definiteness condition is obviously more difficult to check. Therefore a different approach is required to develop an efficient test. In the following we obtain an LMI condition for the stability of $\mathfrak{B} = \ker\, R(\sigma_1,\sigma_2)$; in order to do this, we need first to consider the next result.

Lemma 14 *The entries $\tilde{B}_{\ell,k}(e^{-i\omega}, e^{i\omega})$ of the coefficient matrix $\tilde{B}(e^{-i\omega}, e^{i\omega})$ of the Bézoutian defined in (20) equal*

$$\tilde{B}_{0,k}(e^{-i\omega}, e^{i\omega}) = d_L(e^{i\omega})d_{L-k}(e^{-i\omega}) - d_0(e^{-i\omega})d_k(e^{i\omega})$$

for $k = 0, \ldots, L-1$; and

$$\begin{aligned} \tilde{B}_{\ell,k}(e^{-i\omega}, e^{i\omega}) &= \tilde{B}_{\ell-1,k-1}(e^{-i\omega}, e^{i\omega}) \\ &\quad + d_{L-\ell}(e^{i\omega})d_{L-k}(e^{-i\omega}) - d_\ell(e^{-i\omega})d_k(e^{i\omega}) \end{aligned}$$

for $\ell = 1, \ldots, L-1$, $k = 1, \ldots, L-1$.

Proof. It follows from the definition of $B_\omega(\mu,\nu)$ that:

$$\begin{aligned} (\mu\nu - 1\,)\; B_\omega(\mu,\nu) &= \\ &= \overline{d}(\mu, e^{i\omega})d(\nu, e^{i\omega}) - d_r(\mu, e^{i\omega})\overline{d}_r(\nu, e^{i\omega}) \\ &= d(\mu, e^{-i\omega})d(\nu, e^{i\omega}) - d_r(\mu, e^{i\omega})d_r(\nu, e^{-i\omega}). \end{aligned}$$

This can be reformulated in terms of coefficient matrices as:

$$(\mu\nu - 1\ )\ \begin{bmatrix}1 & \mu & \cdots \mu^{L-1}\end{bmatrix} \tilde{B}(e^{-i\omega}, e^{i\omega}) \begin{bmatrix}1 \\ \nu \\ \vdots \\ \nu^{L-1}\end{bmatrix} \tag{22}$$

$$= \begin{bmatrix}1 & \mu & \cdots \mu^{L}\end{bmatrix} D(e^{-i\omega}, e^{i\omega}) \begin{bmatrix}1 \\ \nu \\ \vdots \\ \nu^{L}\end{bmatrix},$$

with

$$D(e^{-i\omega}, e^{i\omega}) := \begin{bmatrix}d_0(e^{-i\omega}) \\ \vdots \\ d_L(e^{-i\omega})\end{bmatrix} \begin{bmatrix}d_0(e^{i\omega}) & \cdots & d_L(e^{i\omega})\end{bmatrix} - \begin{bmatrix}d_L(e^{i\omega}) \\ \vdots \\ d_0(e^{i\omega})\end{bmatrix} \begin{bmatrix}d_L(e^{-i\omega}) & \cdots & d_0(e^{-i\omega})\end{bmatrix}$$

Now, the left-hand side of (22) is given by

$$\begin{bmatrix}1 & \mu & \cdots \mu^{L}\end{bmatrix} N(e^{-i\omega}, e^{i\omega}) \begin{bmatrix}1 \\ \nu \\ \vdots \\ \nu^{L}\end{bmatrix}, \tag{23}$$

with

$$N(e^{-i\omega}, e^{i\omega}) := \begin{bmatrix}0 & 0 & 0 & \cdots \\ 0 & \tilde{B}_{0,0}(e^{-i\omega}, e^{i\omega}) & \tilde{B}_{0,1}(e^{-i\omega}, e^{i\omega}) & \cdots \\ 0 & \tilde{B}_{0,1}(e^{i\omega}, e^{-i\omega}) & \tilde{B}_{1,1}(e^{-i\omega}, e^{i\omega}) & \cdots \\ \vdots & \vdots & & \ddots\end{bmatrix} - \begin{bmatrix}\tilde{B}_{0,0}(e^{-i\omega}, e^{i\omega}) & \tilde{B}_{0,1}(e^{-i\omega}, e^{i\omega}) & \cdots \\ \tilde{B}_{0,1}(e^{i\omega}, e^{-i\omega}) & \tilde{B}_{1,1}(e^{-i\omega}, e^{i\omega}) & \cdots \\ \vdots & \vdots & \ddots\end{bmatrix},$$

from which the claim is obtained in a straightforward manner. ■

In order to obtain LMI tests for $2D$ stability, we now associate to the coefficient matrix $\tilde{B}(e^{-i\omega}, e^{i\omega})$ of the ω-dependent Bézoutian (20) a two-variable polynomial matrix

$$\Phi(\zeta, \eta) = [\Phi_{\ell,k}(\zeta, \eta)]_{\ell,k=0,\ldots,L-1} \in \mathbb{R}_s^{L\times L}[\zeta, \eta]$$

such that $\Phi(e^{-i\omega}, e^{i\omega}) = \tilde{B}(e^{-i\omega}, e^{i\omega})$, as follows. Define

$$\begin{aligned}\Phi_{0,k}(\zeta, \eta) &:= d_L(\eta)d_{L-k}(\zeta) - d_0(\zeta)d_k(\eta) \\ \Phi_{k,0}(\zeta, \eta) &:= \Phi_{0,k}(\eta, \zeta)\ ,\end{aligned} \tag{24}$$

for $k = 0, \ldots, L-1$;

$$\begin{aligned}\Phi_{\ell,k}(\zeta, \eta) &:= \Phi_{\ell-1,k-1}(\zeta, \eta) + d_{L-\ell}(\eta)d_{L-k}(\zeta) \\ &\quad - d_\ell(\zeta)d_k(\eta)\end{aligned} \tag{25}$$

for $\ell = 1, \ldots, L-1$, $k = 1, \ldots, L-1$. Use the identities established in Lemma 14 to conclude that $\tilde{B}(e^{-i\omega}, e^{i\omega}) = \Phi(e^{-i\omega}, e^{i\omega})$. If $\Phi(e^{-i\omega}, e^{i\omega}) > 0$ for all $\omega \in \mathbb{R}$, the two-variable polynomial matrix Φ satisfies condition (2) of Proposition 11; consequently, the corresponding QDF admits a storage function and the dissipation equation (15) holds. This observation leads to the following Proposition, that will be instrumental in proving the main result of this section.

Proposition 15 *Let* $\mathfrak{B} = \ker\ R(\sigma_1, \sigma_2)$ *with* $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1, \xi_2]$ *be a square autonomous time-relevant behavior, and define* $d(\xi_1, \xi_2) := \det\ R(\xi_1, \xi_2) =: d_0(\xi_2) + \cdots + d_L(\xi_2)\xi_1^L \in \mathbb{R}[\xi_1, \xi_2]$ *with* $d_L(\xi_2) \neq 0$ *and* $\Phi(\zeta, \eta) \in \mathbb{R}_s^{L\times L}[\zeta, \eta]$ *as in equations (24)-(25). Then the following statements are equivalent:*

(1) $\mathfrak{B}$ *is time-relevant stable;*
(2) $\Phi(e^{-i\omega}, e^{i\omega}) > 0$ *for all* $\omega \in \mathbb{R}$*;*
(3) There exist polynomial matrices $\Psi \in \mathbb{R}_s^{L\times L}[\zeta, \eta]$ *and* $\Delta \in \mathbb{R}_s^{L\times L}[\zeta, \eta]$ *such that the equation*

$$\Phi(\zeta, \eta) = (\zeta\eta - 1)\Psi(\zeta, \eta) + \Delta(\zeta, \eta) \tag{26}$$

is satisfied, and moreover $\Delta(e^{-i\omega}, e^{i\omega}) > 0$ *for all* $\omega \in \mathbb{R}$*;*
(4) There exist polynomial matrices $\Psi \in \mathbb{R}_s^{L\times L}[\zeta, \eta]$ *and* $F \in \mathbb{R}^{L\times L}[\xi]$ *such that equation (26) is satisfied with* $\Delta(\zeta, \eta) = F(\zeta)^\top F(\eta)$*, and moreover* $\det(F(e^{i\omega})) \neq 0$ *for all* $\omega \in \mathbb{R}$*;*

Proof. The equivalence of (1) and (2) follows from Proposition 12 and the fact that $\Phi(e^{-i\omega}, e^{i\omega}) = \tilde{B}(e^{-i\omega}, e^{i\omega})$.

(2) $\Longrightarrow$ (3) Assume that $\Phi(e^{-i\omega}, e^{i\omega}) > 0$ for all $\omega \in \mathbb{R}$; then it follows from Proposition 11 that matrices Ψ and Δ exist such that the dissipation equation (16) holds. The second part of the claim follows by letting $\zeta = e^{-i\omega}$, $\eta = e^{i\omega}$, from which we obtain $\Delta(e^{-i\omega}, e^{i\omega}) = \Phi(e^{-i\omega}, e^{i\omega}) = \tilde{B}(e^{-i\omega}, e^{i\omega}) > 0$ for all $\omega \in \mathbb{R}$.

(3) $\Longrightarrow$ (2) It follows from (26) that $\Phi(e^{-i\omega}, e^{i\omega}) = \Delta(e^{-i\omega}, e^{i\omega}) > 0$.

The equivalence of (3) and (4) can be proved using standard results on the factorization of para-Hermitian Laurent polynomial matrices which are positive on the unit circle, see for example Proposition 4.1 of [8]. ■

Example 16 Consider the time-relevant system with $\mathtt{w} = 1$ described by the equation $r(\sigma_1, \sigma_2)w = 0$ with

$$r(\xi_1, \xi_2) = \xi_1^2 + \frac{1}{2}\xi_1 + \frac{1}{10}(1 + 3\xi_2)\xi_1^0 \ .$$

The coefficient matrix $\tilde{B}(e^{-i\omega}, e^{i\omega})$ of the Bézoutian of $r(\xi_1, e^{i\omega})$ is

$$\begin{bmatrix} 1 - \frac{1}{100}(1+3e^{-i\omega})(1+3e^{i\omega}) & \frac{1}{2} - \frac{1}{20}(1+3e^{-i\omega}) \\ \frac{1}{2} - \frac{1}{20}(1+3e^{i\omega}) & 1 \end{bmatrix} .$$

We define the two-variable polynomial matrix $\Phi(\zeta, \eta)$ from (24)-(25):

$$\Phi(\zeta, \eta) := \begin{bmatrix} 1 - \frac{1}{100}(1+3\zeta)(1+3\eta) & \frac{1}{2} - \frac{1}{20}(1+3\zeta) \\ \frac{1}{2} - \frac{1}{20}(1+3\eta) & 1 \end{bmatrix} .$$

Truncating the results of the solutions of the LMI at the fourth decimal digit, let

$$F(\xi) := \begin{bmatrix} -0.1913 + 0.1568\xi & -0.9565 \\ 0.9159 & 0.2915 \end{bmatrix} ;$$

it can be verified that $\det(F(\xi)) = 0.8203 + 0.0457\xi$, and consequently $\det(F(i\omega)) \neq 0$ for all $\omega \in \mathbb{R}$. Define $\Delta(\zeta, \eta) = F(\zeta)^\top F(\eta)$, and

$$\Psi(\zeta, \eta) := \begin{bmatrix} -0.1146 & 0 \\ 0 & 0 \end{bmatrix} .$$

It can be verified that $\Phi(\zeta, \eta) - F(\zeta)^\top F(\eta) = (\zeta\eta - 1)\Psi(\zeta, \eta)$; statement 4 of Proposition 15 is thus verified, and hence ker $r(\sigma_1, \sigma_2)$ is time-relevant stable. ■

The result of Proposition 15 leads us to the following LMI-based test for the stability of a square autonomous time-relevant behavior $\mathfrak{B}$.

Theorem 17 *Let $\mathfrak{B}$ = ker $R(\sigma_1, \sigma_2)$ with $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1, \xi_2]$ be a square autonomous time-relevant behavior, and define $\Phi(\zeta, \eta)$ as in equations (24)-(25). Denote the coefficient matrix of Φ by $\tilde{\Phi}$. Then the following statements are equivalent:*

(1) $\mathfrak{B}$ is time-relevant stable;

(2) There exist matrices $\tilde{\Psi} \in \mathbb{R}^{\infty\times\infty}$ and $\tilde{F} \in \mathbb{R}^{L\times\infty}$ with all but a finite number of entries different from zero, such that:

2.1 The following linear matrix inequality holds:

$$\tilde{\Phi} - \left(\begin{bmatrix} 0 & 0 & \cdots & \cdots \\ 0 & \tilde{\Psi}_{0,0} & \tilde{\Psi}_{0,1} & \cdots \\ 0 & \tilde{\Psi}_{0,1}^\top & \tilde{\Psi}_{1,1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} - \begin{bmatrix} \tilde{\Psi}_{0,0} & \tilde{\Psi}_{0,1} & \cdots \\ \tilde{\Psi}_{0,1}^\top & \tilde{\Psi}_{1,1} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \right) \geq 0 \ ; \tag{27}$$

2.2 The left-hand side of (27) equals $\tilde{F}^\top \tilde{F}$, and moreover the polynomial matrix

$$\tilde{F} \begin{bmatrix} I \\ I\xi \\ \vdots \end{bmatrix} \tag{28}$$

is such that $\det(F(i\omega)) \neq 0$ *for all* $\omega \in \mathbb{R}$.

Proof. If $\tilde{\Psi}$ is the coefficient matrix of a two-variable polynomial matrix $\Psi(\zeta, \eta)$, then the coefficient matrix of $(\zeta\eta - 1)\Psi(\zeta, \eta)$ is

$$\begin{bmatrix} 0 & 0 & \cdots & \cdots \\ 0 & \tilde{\Psi}_{0,0} & \tilde{\Psi}_{0,1} & \cdots \\ 0 & \tilde{\Psi}_{0,1}^\top & \tilde{\Psi}_{1,1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} - \begin{bmatrix} \tilde{\Psi}_{0,0} & \tilde{\Psi}_{0,1} & \cdots \\ \tilde{\Psi}_{0,1}^\top & \tilde{\Psi}_{1,1} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} .$$

Consequently, the left-hand side of (27) is the coefficient matrix of $\Phi(\zeta, \eta) - (\zeta\eta - 1)\Psi(\zeta, \eta) = \Delta(\zeta, \eta)$. As noticed in the discussion appearing after equation (12), there is a one-to-one correspondence between the factorization of a two-variable polynomial matrix and the factorization of its coefficient matrix. Consequently, a factorization $\tilde{F}^\top \tilde{F}$ of the left-hand side of (27) yields a factorization $F(\zeta)^\top F(\eta)$ of $\Phi(\zeta, \eta) - (\zeta\eta - 1)\Psi(\zeta, \eta) = \Delta(\zeta, \eta)$, with $F(\xi)$ defined as $F(\xi) = \tilde{F} \begin{bmatrix} I \\ \xi I \\ \vdots \end{bmatrix}$. Now let $\zeta = e^{-i\omega}$ and $\eta = e^{i\omega}$; then it follows that $\Phi(e^{-i\omega}, e^{i\omega}) = F(e^{-i\omega})^\top F(e^{i\omega})$. Since by statement 2 of Proposition 15 $\Phi(e^{-i\omega}, e^{i\omega}) > 0$, necessarily F has no singularities on the unit circle, i.e. det $F(e^{i\omega}) \neq 0$ for all $\omega \in \mathbb{R}$. Then, the claim of the Theorem is a direct consequence of Proposition 15. ■

Remark 18 Note that the result of the previous theorem involves infinite matrices, but with only a finite number of nonzero rows and columns; moreover, once a representation of $\mathfrak{B}$ is given, this number can be computed. Consequently, the checking of the conditions of this theorem can be performed on the corresponding finite truncations of the relevant matrices.

Example 19 Consider again the system of Example 16. In order to verify the LMI test of Theorem 17, consider the truncation of $\tilde{\Phi}$ given by

$$\tilde{\Phi}_{\text{trunc}} := \begin{bmatrix} 0.9900 & 0.4500 & -0.0300 & 0 \\ 0.4500 & 1 & -0.1500 & 0 \\ -0.0300 & -0.1500 & 0.0900 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

the corresponding truncation of the coefficient matrix of $\Psi(\zeta, \eta)$ is

$$\tilde{\Psi}_{\text{trunc}} = \begin{bmatrix} -0.1146 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of the finite matrix corresponding to the left-hand side of (27)

$$\tilde{\Phi}_{\text{trunc}} - \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1146 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \tilde{\Psi}_{\text{trunc}} \right)$$

are 1.4048, 0.4952 and 0, the latter repeated twice; moreover, this matrix can be factored as $\tilde{F}^\top \tilde{F}$, with $\tilde{F}$ such that

$$F(\xi) := \tilde{F} \begin{bmatrix} I_2 \\ I_2\xi \end{bmatrix} = \begin{bmatrix} -0.7678 + 0.1140\xi & -0.8957 \\ 0.5347 + 0.1076\xi & -0.4446 \end{bmatrix}$$

has the property $\det(F(i\omega)) \neq 0$ for all $\omega \in \mathbb{R}$. Statement 2 of Theorem 17 is consequently verified. ■

We now state an algorithm derived from the results presented in this section, to check the time-relevant stability of a $2D$ behavior using an LMI test.

Algorithm

Input: $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1, \xi_2]$ inducing a kernel representation of a time-relevant $2D$ square autonomous behavior $\mathfrak{B}$.

Output: `True` if the system is time-relevant stable, `False` if it is not.

Step 1: Compute $d(\xi_1, \xi_2) := \det\ R(\xi_1, \xi_2)$;
Step 2: Define $\Phi(\zeta, \eta)$ as in (24)-(25);
Step 3: Solve the LMI (27) considering the relevant (finite) truncated matrices;
Step 4: If a solution to the LMI does not exists, return `False` and exit: $\mathfrak{B}$ is not time-relevant stable;
Step 5: Factorize the LHS of (27) as $\tilde{F}^\top \tilde{F}$, and define

$$F(\xi) := \tilde{F} \begin{bmatrix} I_{\mathtt{w}} \\ \xi I_{\mathtt{w}} \\ \vdots \end{bmatrix};$$

Step 6: If $\det(F)$ has roots on the unit circle, then return `False` and exit: the system is not time-relevant stable;
Step 7: Return `True`.

6 Conclusions

In this paper we have considered $2D$ time-relevant systems, i.e. systems described by partial difference equations in which one of the independent variables (the "time" variable) plays a distinguished role. We have given a definition of stability for these systems, together with algebraic- and linear matrix inequality-based necessary and sufficient conditions for time-relevant stability.

Acknowledgement

The second author gratefully acknowledges the financial support of *The Royal Society* and of the *Engineering and Physical Sciences Research Council* for financially supporting his visits to the University of Oporto. The research of the other two authors has been financed by the Fundação para a Ciência e a Tecnologia, through the R&D Unit Centro de Investigação e Desenvolvimento em Matématica e Aplicações.

References

[1] Ebihara, Y., Ito, Y., and T. Hagiwara, "Exact stability analysis of 2-D systems using LMIs", *IEEE Trans. Aut. Contr.*, Vol. 51, No. 9, pp. 1509-1513, 2006.

[2] Fornasini, E. and G. Marchesini, "Stability Analysis of 2-D Systems", *IEEE Trans. Circ. Syst.*, vol. CAS-27. no. 12, pp. 1210-1217, 1980.

[3] Fornasini, E., Zampieri, S. and P. Rocha, "State space realization of $2-D$ finite-dimensional behaviours", *SIAM J. Contr. Opt.*, vol. 31, no. 6, pp. 1502-1517, 1993.

[4] Fornasini, E. and M.E. Valcher, "nD polynomial matrices with applications to multidimensional signal analysis", *Multidim. Syst. Sign. Proc.*, vol. 8, pp. 387–407, 1997.

[5] Fu, P., Chen, J. and S-I. Niculescu, "Generalized eigenvalue-based stability tests for 2-D linear systems: Necessary and sufficient conditions", *Automatica*, vol. 42, 1569–1576, 2006.

[6] Ito, Y., Hayashi, K. and H. Fujiwara, "Stability analysis and H_∞ norm computation of $2D$ discrete systems using linear matrix inequalities", *Proc. 41st IEEE CDC*, pp. 3306–3311, Las Vegas, Nevada, 2002.

[7] Jury, E. "The Roles of Sylvester and Bézoutian Matrices in the Historical Study of Stability of Linear Discrete-Time Systems", *IEEE Trans. Circ. Syst. I*, vol. 45, no. 12, pp. 1233–1251, 1988.

[8] Kaneko, O. and T. Fujii, "Discrete-time average positivity and spectral factorization in a behavioral framework", *Syst. Contr. Lett.*, vol. 39, pp. 31-44, 2000.

[9] Galkowski, K., Rogers, E., Xu, S., Lam, J. and D. H. Owens, "LMIs– A Fundamental Tool in Analysis and Controller Design for Discrete Linear Repetitive Processes", *IEEE Trans. Circ. Syst. I*, vol. 49, no. 6, pp. 768–778, 2002.

[10] Geronimo, J.S. and H.J. Woerdeman, "Positive extensions, Fejér-Riesz factorization and autoregressive filters in two-variables", *Ann. Math.*, vol. 160, pp. 839–906, 2004.

[11] Geronimo, J.S. and H.J. Woerdeman, "Two-Variable Polynomials: Intersecting Zeros and Stability", *IEEE Trans. Circ. Syst.- I: Regular Papers*, vol. 53, no. 5, pp. 1130–1139, 2006.

[12] Kojima, C., Rapisarda, P. and K.Takaba, "Canonical forms for polynomial and quadratic differential operators," *Syst. Contr. Lett.*, vol. 56, pp. 678–684, 2007.

[13] Kojima, C., Rapisarda, P. and K.Takaba, "Lyapunov stability analysis of higher-order 2-D systems", *Multidim. Syst. Sig. Proc.*, in press.

[14] Lev-Ari, H., Bistritz, Y. and T. Kailath, "Generalized Bezoutians and Families of Efficient Zero-Location Procedures", *IEEE Trans. Circ. Syst. –Part I: Fundamental Theory and Applications*, vol. 38, pp. 170–186, 1991.

[15] Lu, W.-S. and E. B. Lee, "Stability Analysis of Two-Dimensional Systems via a Lyapunov Approach", *IEEE Trans. Circ. Syst.– Part I: Fundamental Theory and Applications*, vol. CAS-32, no. 1, pp. 61–68, 1985.

[16] Napp, D., Rapisarda, P. and P. Rocha, "Lyapunov stability of $2D$ finite-dimensional behaviors", submitted to *Int. J. Contr.*, 2010.

[17] Ohara, A. and Y. Sasaki, "On solvability and numerical solutions of parameter-dependent differential matrix inequality", *Proc. 40th IEEE CDC*, pp. 3593–3594, Orlando, FL, 2001.

[18] Peeters, R. and P. Rapisarda, "A two-variable approach to solve the polynomial Lyapunov equation", *Syst. Contr. Lett.*, 42, pp. 117–126, 2001.

[19] Pillai, H. K. and J. C. Willems, "Lossless and dissipative distributed systems", *SIAM J. Contr. Opt.*, vol. 40, pp. 1406–1430, 2002.

[20] Rocha, P. *Structure and representation of 2-D systems*, Ph.D. thesis, Univ. of Groningen, The Netherlands, 1990.

[21] Sasane, A.J. "Time-autonomy and time-controllability of 2-D behaviours that are tempered in the spatial direction", *Multidimensional Systems and Signal Processing*, vol. 15, pp. 97-116, 2004.

[22] Sasane, A., Thomas, E.G.F. and J.C. Willems "Time-autonomy versus time-controllability", *Syst. Contr. Lett.* vol. 45, pp. 145–153, 2002.

[23] Valcher, M.E. "On the decomposition of two-dimensional behaviors", *Multidim. Syst. Sign. Proc.*, vol. 11, pp. 49–65, 2000.

[24] Valcher, M.E. "Characteristic Cones and Stability Properties of Two-Dimensional Autonomous Behaviors", *IEEE Trans. Circ. Syst.–Part I: Fundamental Theory and Applications*, vol. 47, no. 3, pp. 290–302, 2000.

[25] Willems, J.C., "Stability and Quadratic Lyapunov Functions for nD systems", *Proc. International Conference on Multidimensional (nD) Systems (NDs)*, Aveiro, Portugal, 2007.

[26] Willems, J.C. and H.L. Trentelman,"On quadratic differential forms", *SIAM J. Contr. Opt.*, vol. 36, no. 5, pp. 1703–1749, 1998.

[27] Wood, J., Sule, V. and E. Rogers, "Causal and stable input/output structures on multidimensional behaviors", *SIAM J. Contr. Opt.*, vol. 43, no. 4, pp. 1493–1520, 2005.