# Considerate Equilibrium*

**Martin Hoefer**[1] **Michal Penn**[2] **Maria Polukarov**[3] **Alexander Skopalik**[4] **Berthold Vöcking**[1]

[1]Dept. of Computer Science, RWTH Aachen University, Germany, {mhoefer,voecking}@cs.rwth-aachen.de

[2] Faculty of Industrial Engineering and Management, Technion, Israel, mpenn@ie.technion.ac.il

[3]School of Electronics and Computer Science, University of Southampton, UK, mp3@ecs.soton.ac.uk

[4]School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore, ASkopalik@ntu.edu.sg

## Abstract

We study the existence and computational complexity of coalitional stability concepts based on social networks. Our concepts represent a natural and rich combinatorial generalization of a recent notion termed *partition equilibrium* [5]. We assume that players in a strategic game are embedded in a social (or, communication) network, and there are coordination constraints defining the set of coalitions that can jointly deviate in the game. A main feature of our approach is that players act in a "considerate" fashion to ignore potentially profitable (group) deviations if the change in their strategy may cause a decrease of utility to their neighbors in the network.

We explore the properties of such *considerate equilibria* in application to the celebrated class of *resource selection games (RSGs)*. Our main result proves existence of a super-strong considerate equilibrium in all symmetric RSGs with strictly increasing delays, for *any* social network among the players and feasible coalitions represented by the set of cliques. The existence proof is constructive and yields an efficient algorithm. In fact, the computed considerate equilibrium is a Nash equilibrium for a standard RSG, thus showing that there exists a state that is stable against selfish and considerate behavior simultaneously. Furthermore, we provide results on convergence of considerate dynamics.

## 1 Introduction

Multi-agent scenarios, in which self-motivated, rational actors share resources, allocate tasks or compete for production or communication lines, are central to AI. Natural tools for the analysis of such interactions include the well-studied solution concepts developed for strategic games. Rationality is usually captured in a way that agents are acting autonomously in order to maximize their own utility function. This leads to much interest in the study of stable outcomes, making it the central topic in game theory. In strategic games, the standard concept of stability is the *Nash equilibrium (NE)*—a state resilient to unilateral changes of players' strategies. While every finite game possesses a mixed Nash equilibrium, a pure strategy Nash equilibrium is not guaranteed in general, though has been proven to exist in several interesting classes such as congestion (or, potential) games [12, 15]. A drawback of Nash equilibrium is that it neglects coalitional deviations by groups of players; these are captured most prominently by the notion of *strong equilibrium (SE)* [3], where no coalition can strictly improve the utility of *all* participants. A slightly stronger variant, termed *super-strong equilibrium (SSE)* [5,16], guarantees that no coalition can strictly improve *any* participant without deteriorating at least one other member. SSE postulates the natural and widely considered condition of (strong) Pareto efficiency [13] for every coalition. However, while stability against deviations by coalitions of players is a most natural desideratum, it is well-known that there are only very few strategic games with SE, and SSE are even harder to guarantee.

However, in contrast to the assumptions underlying SE and SSE, many real-life scenarios allow only certain subsets of players to cooperate and apply joint deviations. Indeed, to deviate collectively, a group of players has to find a deviation, agree on it, and coordinate individual actions. This is impossible for a subset of players that are completely unrelated to each other. To this end, a promising recent approach for limited coalitional deviations was proposed in [5], where there is a given partition of the set of players such that only sets of the partition can implement joint actions. A *partition equilibrium* is a SSE subject to feasible coalitions being restricted to player sets in the partition only. In contrast to standard SSE, partition equilibrium was shown to always exist in resource selection games (RSGs) [1]; moreover, corresponding strategy profiles are also NE—that is, coalitional and unilateral stability are obtained simultaneously.

In this paper, we significantly strengthen the partition equilibrium concept by considering coalitional deviations and stable outcomes based on a rich combinatorial structure derived from a social network among the players rather than just partitions. We assume that players in a strategic game correspond to the set of nodes in a graph, where edges represent social relations (or, communication links) among the players. In addition, there are coordination constraints that prescribe what coalitions can potentially emerge and jointly deviate in the

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game. In particular, we focus on the natural case where potential coalitions of players are fully connected—that is, the set of feasible coalitions corresponds to the set of all possible cliques in the graph.

Crucially, besides the ability of cooperation, the presence of social links may also affect the strategic interests of players in the game. In this spirit, *social context games* [2] were proposed to model scenarios where a player's utility can depend on the payoffs of other players. For example, a player may be interested in ranking his payoff as high as possible comparing to the others' payoffs [4], or a player may care about the total payoff of a subset of his "friends", as in *coalitional congestion games* [8, 11]. A social context game is then defined by some underlying game, the social context given by some topological or graph-theoretic structure of neighborhood, and aggregation functions capturing the effects of utility changes in the underlying game on player incentives. In [2], RSGs are considered as the underlying games, and four natural social contexts are studied. However, unlike for partition equilibrium, this work deals only with unilateral deviations.

This paper studies the interplay between social structure and the outcome of multi-agent interaction in yet another way. Instead of relating a player's utility to the *payoffs* of other participants, we consider the effects their *actions* may have on it. In presence of social connections, these effects introduce additional incentives for the players and may have crucial influence on the decisions they make in the game. For example, in social networks such as FaceBook or LinkedIn, links among the agents represent friendships, professional partnerships, or even family relations. In other contexts of interest, agents may be tied by business contracts, technological dependencies or communication lines. In such scenarios, it is natural to expect that an agent will behave in a "considerate" manner and avoid taking actions that may harm his neighbors in the network. This motivates the study of *consideration* in strategic games, which is in the main focus of our work. As far as we are aware, this paper is the first to address this issue.

The solution concepts naturally corresponding to considerate behavior extend the notions of NE, SE and SSE to consider decisions (either group or individual) that do not deteriorate any neighboring players. Focusing on the natural case where coalitions of players that execute a strategy change must be fully connected, we define the *considerate equilibrium* to be a state in which (1) no coaliton formed by a clique in the social network can deviate so that the utility of at least one member of the coalition strictly improves and (2) none of the players neighboring the clique gets worse.

We observe that partition equilibrium evolves as a special case of considerate equilibrium when the social network is composed of a set of disjoint cliques. Indeed, one may find that the restriction of coalitional deviations in partition equilibrium essentially postulates two structural properties: (1) coalitions of players that execute a strategy change have to be "close" to each other, and (2) their decision must strictly benefit at least one of them but not strictly deteriorate any other player close to them. The notion of closeness is defined in both cases simply as being in the same partition set. However, while [2, 5] are initial steps in relating the social structure to the outcome of a game, they are quite restrictive in that only particular social contexts and fixed coalitional structures (partitions) are considered. In addition, they generally ignore the phenomenon of considerate behavior which is present in our work. Similar arguments apply w.r.t. [7], where fixed coalition structures in load balancing and congestion games are studied. Here coalitions act as single "splittable" players that strive to minimize the makespan or the sum of costs of the agents in the coalition.

We explore the concept of considerate behavior in the prominent class of resource selection games. In an RSG, each player chooses one of a finite set of resources, and its cost is given by a delay function depending on the number of players choosing the same resource. RSGs are a fundamental setting in computer science, operations research and economics, due to their practical applicability (e.g., in electronic commerce and communication networks) and plausible analytical properties. In particular, for strictly increasing delay functions, SE always exist [9, 10], but SSE is not guaranteed [5]. The latter fact has been prominently utilized to demonstrate the power of limited coalitional deviations [1, 5].

### 1.1 Our Results

We show that regardless of the social network topology, all RSGs with strictly increasing delay functions possess a considerate equilibrium. Our proof in Section 3 is constructive and yields an efficient algorithm for computing such an equilibrium. Importantly, the computed considerate equilibrium is also a standard NE for a given RSG, thus showing that there exists a state that is stable against selfish and considerate behavior simultaneously. Observe that the number of cliques might be exponential in the number of players, which makes non-trivial even the computation of a single improving move. We solve this problem by showing that, in an NE, every profitable deviation of a clique is witnessed by a move of a single player that decreases a suitably defined potential function. In addition, our proof is fundamentally different and significantly simpler than the existence proof for the special case of partition equilibrium in [1].

In Section 4, we study convergence properties of considerate dynamics. Let us remark that the potential function argument used in our existence proof does not imply that the sequential dynamics defined by deviations of cliques is acyclic, since the single player moves considered in the existence proof do not necessarily correspond to allowed improving moves. Indeed, we show that even for identical, strictly increasing delays there are infinite sequences of improving moves of cliques. This is in contrast to the dynamics corresponding to partition equilibrium, for which we show the finite improvement property in this setting.

## 2 Preliminaries and Initial Results

A *strategic game* is a tuple $(N, (S_i)_{i\in N}, (u_i)_{i\in N})$, where $N$ is the set of $n$ *players*, ans $S_i$ is a *strategy space* of player $i$. A *state* $s$ of the game is a vector of strategies $(s_1, \ldots, s_n)$, where $s_i \in S_i$. For convenience, we use $s_{-i}$ to denote $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$, i.e., $s$ reduced by the single entry of player $i$. Similarly, for a state $s$ we use $s_C$ to denote the strategy choices of a coalition $C \subseteq N$ and $s_{-C}$ for the complement, and we write $s = (s_C, s_{-C})$. The *utility* of player $i$ in state $s$ is $u_i(s) \in \mathbb{R}$. For a state $s$ a coalition $C \subseteq N$ is said to have an *improving move* if there is $s'_C$ such that $u_i(s'_C, s_{-C}) > u_i(s)$ for every player $i \in C$. In particular, the improving move is *unilateral* if $|C| = 1$. A state has a *weak improving move* if there is $C \subseteq N$ and $s'_C$ such that $u_i(s'_C, s_{-C}) \geq u_i(s)$ for every $i \in C$ and $u_i(s'_C, s_{-C}) > u_i(s)$ for *at least one* $i \in C$. A *(pure strategy) Nash equilibrium (NE)* [14] is a state that has no unilateral improving moves, a *strong equilibrium (SE)* [3] is a state that has no improving moves, and a *super-strong equilibrium (SSE)* [5] is a state that has no weak improving moves.

To model considerate behavior, we adjust the definition of improving moves. In particular, there is an undirected, unweighted graph $G = (N, E)$ over the set of players. For a subset $C \subseteq N$, consider the *neighborhood* of $C$ as $\mathcal{N}(C) = \{j \in N \mid \exists i \in C, \{i, j\} \in E\}$.

**Definition 1 (Considerate Improving Moves)** *A state $s$ has a* considerate improving move *for a coalition $C$ if there is $s'_C$ such that $u_i(s'_C, s_{-C}) > u_i(s)$ for all $i \in C$ and $u_j(s'_C, s_{-C}) \geq u_j(s)$ for all $j \in \mathcal{N}(C)$. For a* unilateral considerate improving move *we have $|C| = 1$. A state $s$ has a* weak considerate improving move *for a coalition $C$ if there is $s'_C$ such that $u_i(s'_C, s_{-C}) \geq u_i(s)$ for all $i \in C \cup \mathcal{N}(C)$ and $u_i(s'_C, s_{-C}) > u_i(s)$ for at least one $i \in C$.*

Note that every (weak/unilateral) considerate improving move is also a (weak/unilateral) improving move but not vice versa. To define coalitional equilibria, let us, for the time being, also assume that there is a set system of *feasible coalitions $\mathcal{C} \subseteq 2^N$*. A *considerate Nash equilibrium (CNE)* is a state $s$ that has no unilateral considerate improving moves. A *(super) strong considerate equilibrium ((S)SCE)* is a state $s$ that has no (weak) considerate improving move for a coalition $C \in \mathcal{C}$. Note that for CNE we implicitly assume $\mathcal{C}$ is the set of all singleton sets $\{i\}$ for all $i \in N$. Every NE is a CNE, and every (S)SE is a (S)SCE. The converse only holds for CNE and NE if $E = \emptyset$. In general, (S)SCE are (S)SE only if $E = \emptyset$ and $\mathcal{C} = 2^N$. In this way, the presence of social ties and a non-trivial set of feasible coalitions weaken the structural requirements for the existence of equilibrium.

In the rest of the paper, we make the natural assumption that the set of feasible coalitions corresponds to the set of cliques in $G$. In our analysis, we focus on weak improving moves and study super strong considerate equilibria as we believe that this solution concept is most interesting not only from a technical point of view but also a natural and convincing model for the interaction of coalitional structures in the presence of a social network.

**Definition 2 (Considerate Equilibria)** *A* considerate equilibrium (CE) *is a state $s$ that has no weak considerate improving move for a coalition corresponding to a clique in $G$.*

The notion of CE nicely generalizes partition equilibrium. In particular, a partition equilibrium is a CE if the social network $G$ is partitioned into isolated cliques. Note that we do not explicitly assume that the set of feasible coalitions is restricted to *maximal* cliques. If the graph is partitioned into isolated cliques, however, this rather technical assumption made in the definition of partition equilibrium is a natural consequence of the assumption that the coalitions behave considerately towards their neighbors. In this way, since weak improving moves do not decrease the utility of neighboring players, one can assume w.l.o.g. that all members of a partition set participate in a coalition.

We apply the concept of consideration to *resource selection games (RSG)*[1]—a basic class of potential games [12, 15]. In an RSG, there is a set of resources $R$, and $S_i = R$ for every player $i \in N$. For a state $s$ we denote by $\ell_r(s)$ the number of players that pick $r \in R$ in $s$. For each resource $r \in R$, there is a delay function $d_r(x) \in \mathbb{R}$. Throughout the paper we assume that all delay functions are non-negative and strictly increasing. In a state with $s_i = r$, player $i$ has *cost* $c_i(s) = -u_i(s) = d_r(\ell_r(s))$.

In this paper, we focus on RSGs with strictly increasing delays. In this case, it is known that NE exist [15], can be computed in polynomial time [6], and are equivalent to SE [9]. Moreover, the games possess a (strong) *potential function* [9, 12], i.e., every sequence of unilateral improving moves has finite length and ends in a NE/SE. Trivially, by restriction of improving moves, the same holds also for CNE and SCE. Interestingly, however, SSE are not guaranteed to exist even in simplest games.[2] In contrast, we prove below that all RSGs with strictly increasing delays possess considerate equilibria. However, even for identical resources, we show that there are infinite sequences of weak considerate improving moves of coalitions being cliques in $G$. This is in contrast to a special case where $G$ is a disjoint set of cliques and CE reduces to partition equilibrium; in this case, there exists a potential function for weak (considerate) improving moves in games with identical resources.

# 3 Existence and Computation

This section contains our main theorem showing the existence of CE in RSGs with strictly increasing delay functions. The existence proof is constructive and yields a polynomial time algorithm computing a state that is both a CE and a standard NE, thus showing that the two equilibrium concepts intersect.

**Theorem 1** *For any RSG with strictly increasing delay functions and any associated social network $G$, there exists at least one state that is an NE and a CE. Moreover, there is a polynomial time algorithm computing such a state.*

**Proof :** We describe a process that starts in a Nash equilibrium and converges to a CE. This process consists of movements of single players. Every strategy profile in this sequence is a standard Nash equilibrium.

Consider a state $s$. Let $d_{\max}$ denote the maximal delay of a player in $s$. Note that in a Nash equilibrium, each used resource $r$ has either delay $d_r(\ell_r) = d_{\max}$ or $d_r(\ell_r) < d_{\max}$ and $d_r(\ell_r + 1) \geq d_{\max}$. In the former case, we call resource $r$ a *high* resource, in the latter case, we call it a *low* resource if additionally $d_r(\ell_r + 1) = d_{\max}$. Let $\mathcal{N}_{i,r}(s)$ denote the set of neighbors of player $i$ in $G$ that are on resource $r$ in $s$.

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[1]Also being referred to as *simple congestion games* , *singleton congestion games* or *parallel link games*

[2]Consider a game with $N = \{1, 2, 3\}$, $R = \{r_1, r_2\}$, and $d_{r_1}(x) = d_{r_2}(x) = x$.

We are now ready to describe the process:

1. Compute a Nash equilibrium $s$.
2. If there is a player $i$ placed on a high resource $r$ and there is a low resource $r'$ with $|\mathcal{N}_{i,r}(s)| > |\mathcal{N}_{i,r'}(s)|$ then set $s = (s_{-i}, r')$, and repeat this step.
3. If there is a player $i$ placed on a high resource $r$ and there is a low resource $r'$ with $|\mathcal{N}_{i,r}(s)| = |\mathcal{N}_{i,r'}(s)|$ and $d_r(\ell_r(s) - 1) < d_{r'}(\ell_{r'}(s))$ then set $s = (s_{-i}, r')$, and continue with step 2.
4. Output $s$.

Note that each state produced by this process is a Nash equilibrium. During this process, the following potential function

$$\phi(s) = \sum_{i \in N} M|\mathcal{N}_i(s)| + \sum_{r \in R} d_r(\ell_r(s))$$

decreases strictly from step to step, where we use $\mathcal{N}_i(s) = \mathcal{N}_{i,s_i}(s)$ as a shorthand for the neighbors of $i$ on the same resource and assume $M > \sum_{r \in R} d_r(n)$. One can easily modify the delay functions such that $M = n|R|^2$ without changing the players' preferences which implies that the process terminates after polynomially many steps.

To prove that this process results in a CE, we show that if a state $s$ is a NE and there exists weak considerate improving move $s'_C$, then there is also a move of a single player $i \in C$ as described above.

Let $H$ and $L$ denote the set of high and low resources in $s$, respectively. Let $R_h$ be the set of resources that are high in $s$ but no longer high in $(s'_C, s_{-C})$, and let $R_l$ be the set of resources that are low in $s$ and become high in $(s'_C, s_{-C})$. By definition, $R_h \subseteq H$ and $R_l \subseteq L$. Let $N_h$ be the set of players of $C$ on resources of $R_h$ in $s$, and let $N_l$ be the set of players of $C$ on resources of $R_l$ in $s$.

**Lemma 2** *During the move $s'_C$, all players in $N_l$ are moving from resources in $R_l$ to resources outside of $R_l$. In turn, $|N_l| + |R_l|$ players are moving from resources in $H$ to the resources in $R_l$. Finally, at least $|N_l| + |R_l|$ players are leaving $R_h$ towards resources outside of $R_h$.*

**Proof :** Since $s'_C$ is a weak considerate improving move, all players in $N_l$ must move from resources in $R_l$ to resources outside of $R_l$ as their delay would increase, otherwise. These players can only be replaced by players from $H$ as other players would have an increased delay after the move, otherwise. In turn, altogether $|N_l| + |R_l|$ players need to move from $H$ to $R_l$ so that the resources of $R_l$ become high resources after the move. Furthermore, we observe that the number of players on resources in $H \setminus R_h$ does not change during the considered move, and there are no players entering $H \setminus R_h$ from outside of $H$ as such players would have an increased delay, otherwise. As a consequence, there must be at least $|N_l| + |R_l|$ players that are leaving $R_h$ towards $H \setminus R_h$ or $R_l$ in order to have $|N_l| + |R_l|$ players that move from $H$ to $R_l$. This proves Lemma 2. □ The lemma implies

$$|N_h| \geq |N_l| + |R_l| \tag{1}$$

Let $\max_h = \max_{i \in N_h} \mathcal{N}_i(s)$ denote the maximum number of neighbors that a player of $N_h$ has on his resource. The definition $\max_h$ implies

$$|N_h| \leq (\max_h + 1) \cdot |R_h| \tag{2}$$

Note that no player of $C$ has a neighbor that has chosen a resource from $R_l$ and is not in $C$. Otherwise, this neighbor's delay would increase during the move so that $s'_C$ would not be a considerate move. Therefore, we can set $\min_l = \min_{i \in N_h, r \in R_l} \mathcal{N}_{i,r}(s)$, where the choice of $i$ is irrelevant. The definition of $\min_l$ immediately implies

$$|N_l| \geq \min_l \cdot |R_l| \tag{3}$$

Let us derive some more helpful equations regarding the different kinds of resources. For each resource that decreases its load during the improving move, there is at least one resource that increases its load by one because the number of players on each low resource can only increase by one. This gives

$$|R_h| \leq |R_l| \tag{4}$$

Combining the Equations (2), (1), and (3) gives

$$(\max_h + 1) \cdot |R_h| \geq |N_h| \geq |N_l| + |R_l| \geq (\min_l + 1) \cdot |R_l| \tag{5}$$

Now, we distinguish between the following two cases.

**Case 1:** $\max_h > \min_l$. In this case, we can set $i = \arg\max_{j \in N_h} \mathcal{N}_j(s)$ and $r' = \arg\min_{r \in R_l} \mathcal{N}_{i,r}(s)$, which satisfies the conditions of step 2 of the process.

**Case 2:** $\max_h \leq \min_l$. In this case, Equation 5 yields $|R_h| \geq |R_l|$, which, coupled with Equation 4, implies $|R_h| = |R_l|$. Substituting this equality back into the Equation 5 gives $\max_h \geq \min_l$ which implies $\max_h = \min_l$. Define $q = |R_h| = |R_l|$ and $k = \max_h = \min_l$. Now Equations 2 and 3 yield $|N_h| \leq |N_l| + q$, which in combination with Equation 1 gives $|N_h| = |N_l| + q$.

On average, the resources in $R_l$ hold $|N_l|/q$ players from $C$ in state $s$ and the resources $R_h$ hold $|N_h|/q$ players from $C$. We claim that this implies that each resource in $R_l$ holds exactly $|N_l|/q$ players from $C$; and each resource in $R_h$ holds exactly $|N_h|/q$ players from $C$ and no additonal neighbor of one of them. To see this, let $r_h$ denote a resource from $R_h$ holding a maximum number of players from $C$ and let $r_l$ denote a resource from $R_l$ holding a minimum number of players from $C$. Let $i \in N_h$ be a player assigned to $r_h$. As $s'_C$ is a considerate move, $i$ does not have neighbors outside of $C$ on $r_l$. Thus, if the claim above would not hold, $i$ would have either at least $|N_h|/q$ neighbors on $r_h$ or strictly less than $|N_h|/q - 1 = |N_l|/q$ neighbors on $r_l$, which would imply $\max_h > \min_l$ and thus contradict our assumption. As a consequence, $|\mathcal{N}_{i,r}(s)| = k = \mathcal{N}_{i,r'}(s)|$, for every $i \in N_h$, $r \in R_h$, and $r' \in R_l$.

Now, Lemma 2 implies that each of the $q$ resources in $R_l$ is left by its $k$ players from $C$ and each of the $q$ resources in $R_h$ is left by its $k+1$ players from $C$.

We make a few further observations: The definition of $R_h$ implies that the number of players on a resource from $H \setminus R_h$ does not decrease during the considered move. Besides, this number cannot increase due to a weak improving move. Next consider a resource $r \notin H \cup R_l$. The definition of $R_l$ implies that the number of players on $r$ cannot increase during a weak improving move. Now suppose the number of players on $r$ would decrease. Then there is a leaving player $i$, who moves to either $R_h$ or another resource in $L \setminus R_l$, as its delay would increase, otherwise. In the latter case, a different player must make room for $i$. By following this player, we can iteratively construct a chain of moving players until finally there is a player that moves to a resource in $R_h$. Thus, together with the players leaving the resources in $R_l$ there are at least $qk + 1$ that need to migrate to a resource with a delay of less than $d_{\max}$ (after the move). However, the resources in $R_h$ have only a capacity for taking $qk$ many of such players. Hence, the number of players on any resource outside of $R_l$ or $R_h$ does not change during this move.

Now, take one of the players from $N_l$. During the considered move, this player migrates to another resource having a delay strictly less than $d_{\max}$ (after the move). If this resource does not belong to $R_h$ then another player needs to leave this resource in order to compensate for the arriving player. Following that player, we iteratively construct a chain of moving players, leading from a resource in $R_l$ to a resource in $R_h$. In this manner, we can decompose the set of moving players into a collection of $qk$ many chains each of which leads from $R_l$ to $R_h$. As we are considering a weak improving move, the delays in each of these chains does not increase and there is at least one such chain leading from a resource $r' \in R_l$ to a resource $r \in R_h$ with $d_r(\ell_r(s) - 1) < d_{r'}(\ell_{r'}(s))$. We choose an arbitrary player $i \in N_h$ assigned to resource $r$ in $s$. We have shown above that for this player $|\mathcal{N}_{i,r}(s)| = \mathcal{N}_{i,r'}(s)|$ holds. Thus, player $i$ satisfies the condition in step 3 of our process, which completes our analysis for Case 2.

This shows that, when the process terminates, there are no weak considerate improving moves. Therefore, the resulting state is a CE. □

## 4 Convergence

Next we show that the dynamics of weak considerate improving moves by general cliques does not have the *finite improvement property*, i.e., the dynamics corresponding to CE might cycle (Theorem 3). Our construction works even for resources with identical delays. This separates considerate equilibrium from partition equilibrium as, in the same setting, the dynamics corresponding to partition equilibrium admits the finite improvement property (Proposition 4).

**Theorem 3** *There are symmetric RSGs with strictly increasing and identical delays, for which there are infinite sequences of weak considerate improving moves by coalitions that are cliques in $G$.*

**Proof :** For the proof we construct a game with a modular structure. Our game consists of a number of smaller games, referred to as *blocks*. Each block consists of 14 players and 5 resources, and by itself it is acyclic. However, by creating social ties across blocks, we create larger cliques that are able to perform "resets" in one block while making improvements in other blocks. By a careful scheduling of such reset moves we construct an infinite sequence of moves.

More formally, we have 19 blocks, and in each block $i$, we have 14 players. There are 8 players $B^i, C^i, D^i, E^i, F^i, G^i, P^i, Q^i$ involved in our sequence, while 6 additional "dummy" players never move. The dummy players are singleton nodes in the social network and are only required to, in essence, simulate non-identical resources by increasing some of the delays to larger values. The social graph consists of internal links within each block and inter-block connections as follows. For each block, there are edges $\{B^i, F^i\}, \{C^i, E^i\}$ and $\{D^i, G^i\}$. In addition, for each $i = 1, ..., 19$ there are two inter-block cliques,

- $\{D^i, P^i, P^{i+1}, B^{i+1}, D^{i+2}, P^{i+2}, C^{i+6}, E^{i+6}\}$ and
- $\{D^i, Q^i, Q^{i+1}, C^{i+1}, D^{i+2}, Q^{i+2}, B^{i+9}, F^{i+9}\}$,

where the exponent is meant to cycle through the numbers 1 to 19, i.e., above $P^j$ means $P^{((j-1) \bmod 19)+1}$.

The 95 resources are denoted by $r^i_j$ with $i = 1, \ldots, 19$, $j = 1, \ldots, 5$. The delay functions are identical $d_r(x) = x$ for all $r \in R$. Note that in general, our example does not require linear delays, it suffices to ensure $d_r(3) > d_r(2)$.

Let us consider a single block $i$ and a sequence of six states within this block depicted in Fig. 1. Note that $\alpha \to \beta$ represents a weak considerate improving move for $\{D^i, G^i\}$, where $D^i$ performs the move, and $G^i$ strictly improves. Similarly, $\beta \to \gamma$ is a weak considerate improving move for $\{C^i, E^i\}$, $\delta \to \epsilon$ for $\{D^i, G^i\}$, and $\epsilon \to \zeta$ for $\{B^i, F^i\}$. The steps $\gamma \to \delta$ and $\zeta \to \alpha$ are resets, in which a cyclic switch is performed and no player within the block strictly improves. It suffices to show that these steps can be implemented with improving moves by inter-block cliques.

| | $r^i_1$ | $r^i_2$ | $r^i_3$ | $r^i_4$ | $r^i_5$ |
|---|---|---|---|---|---|
| | $C^i$ | $B^i$ | | $P^i$ | $Q^i$ |
| $\alpha$ | $E^i$ | $D^i$ | $F^i$ | x | x |
| | x | $G^i$ | x | x | x |
| | $C^i$ | | $D^i$ | $P^i$ | $Q^i$ |
| $\beta$ | $E^i$ | $B^i$ | $F^i$ | x | x |
| | x | $G^i$ | x | x | x |
| | | $C^i$ | $D^i$ | $P^i$ | $Q^i$ |
| $\gamma$ | $E^i$ | $B^i$ | $F^i$ | x | x |
| | x | $G^i$ | x | x | x |
| | | $C^i$ | $B^i$ | $P^i$ | $Q^i$ |
| $\delta$ | $E^i$ | $D^i$ | $F^i$ | x | x |
| | x | $G^i$ | x | x | x |
| | $D^i$ | | $B^i$ | $P^i$ | $Q^i$ |
| $\epsilon$ | $E^i$ | $C^i$ | $F^i$ | x | x |
| | x | $G^i$ | x | x | x |
| | $D^i$ | $B^i$ | | $P^i$ | $Q^i$ |
| $\zeta$ | $E^i$ | $C^i$ | $F^i$ | x | x |
| | x | $G^i$ | x | x | x |
| | $C^i$ | $B^i$ | | $P^i$ | $Q^i$ |
| $\alpha$ | $E^i$ | $D^i$ | $F^i$ | x | x |
| | x | $G^i$ | x | x | x |

Figure 1: Sequence of six states within a block $i$ that are attained during an infinite sequence of weak considerate improving moves.

Consider the first reset $\gamma \rightarrow \delta$, in which $D^i$ and $B^i$ swap places, and for simplicity assume w.l.o.g. that $i = 5$. This swap is executed in three moves, where we first swap in $P^5$ for $D^5$, then swap $P^5$ and $B^5$ and finally swap out $P^5$ to bring $D^5$ back in. This cyclic switch is the result of the following sequence of weak considerate improving moves: (1) coalition $\{D^3, P^3, P^4, B^4, D^5, P^5, C^9, E^9\}$ applies a deviation where $D^5$ and $P^5$ exchange their places, and $C^9$ moves away from $E^9$ in block 9 as $\beta \rightarrow \gamma$ prescribes; (2) coalition $\{D^4, P^4, P^5, B^5, D^6, P^6, C^{10}, E^{10}\}$ improves by swapping $P^5$ and $B^5$, and moving $C^{10}$ away from $E^{10}$ in block 10; (3) finally, $D^5$ and $P^5$ swap with coalition $\{D^5, P^5, P^6, B^6, D^7, P^7, C^{11}, E^{11}\}$ where $C^{11}$ moves away from $E^{11}$ in block 11. In the final dynamics, we will use these moves also to simultaneously perform swaps in the other blocks 3, 4, 6, and 7.

The second reset swap $\zeta \rightarrow \alpha$ by $D^5$ and $C^5$ can be done in similar fashion by a circular swap involving $Q^5$ and using the $B^i$ and $F^i$ players of blocks $i = 12, 13, 14$. Note that our edges are carefully designed not to generate any undesired connections. In particular, $D^5$, $P^5$, $B^5$ rely on the movement of $C^9$, $C^{10}$ and $C^{11}$ to execute their swaps. During these swaps, $B^9$, $B^{10}$ and $B^{11}$ are deteriorated. None of the deteriorated players are attached to players in the respective improving coalitions, i.e., none of $D^3$, $P^3$, $P^4$, $B^4$, $D^5$ or $P^5$ are neighbors with $B^9$, none of $D^4$, $P^4$, $P^5$, $B^5$, $D^6$ or $P^6$ are neighbors with $B^{10}$, and none of $D^5$, $P^5$, $P^6$, $B^6$, $D^7$ or $P^7$ are neighbors with $B^{11}$. In addition, for making the switch between $D^5$, $Q^5$ and $C^5$ we use the movement of $B^{12}$, $B^{13}$ and $B^{14}$. Note that none of the players required to execute the switches are neighbors with $C^{12}$, $C^{13}$ or $C^{14}$, respectively.

An infinite sequence of weak considerate improving moves can now, for example, be obtained from a starting state as follows. We indicate for each block in which state $\alpha$ to $\zeta$ it is initialized. Here $\gamma_1$, $\gamma_2$, $\zeta_1$, and $\zeta_2$ indicate the intermediate states of the corresponding circular resetting swaps.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| | $\zeta_2$ | $\zeta_1$ | $\zeta$ | $\zeta$ | $\zeta$ | $\zeta$ | $\zeta$ | $\zeta$ | $\zeta$ |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| $\epsilon$ | $\delta$ | $\gamma_2$ | $\gamma_1$ | $\gamma$ | $\gamma$ | $\gamma$ | $\gamma$ | $\beta$ | $\alpha$ |

In the first step, we can simultaneously advance blocks 1-3 from $(\zeta_2, \zeta_1, \zeta)$ to $(\alpha, \zeta_2, \zeta_1)$ using movement of $B^{10}$, which advances block 10 to $\zeta$. In the next step we advance blocks 12-14 from $(\gamma_2, \gamma_1, \gamma)$ to $(\delta, \gamma_2, \gamma_1)$ using movement of $C^{18}$, which advances block 18 to $\gamma$. Next, we make two internal switches in blocks 11 from $\delta$ to $\epsilon$ and 19 from $\alpha$ to $\beta$. In this way, we have shifted the state sequence by one block, which implies that we can repeat this sequence endlessly. $\square$ In contrast, observe that if the graph is a set of disjoint cliques, then for games with identical and strictly increasing delay functions we can easily construct a potential function that implies acyclicity with respect to weak (considerate) improving moves.

**Proposition 4** *In every symmetric RSG with strictly increasing, identical delays functions, every sequence of weak improving moves of allowed partition sets is finite and ends in a partition equilibrium.*

Note that in this case we can assume w.l.o.g. that $d_r(x) = x$ for all $r \in R$. Also, each weak improving move decreases the sum of costs of all players in the partition set. Thereby, the results of [7] for linear delays directly imply the finite improvement property.

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