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Strong practical stability based robust stabilization of uncertain discrete linear repetitive processes

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SUMMARY

Repetitive processes are a distinct class of $2D$ systems of both theoretical and practical interest whose dynamics evolve over a subset of the positive quadrant in the 2D plane. The stability theory for these processes originally consisted of two distinct concepts termed asymptotic stability and stability along the pass respectively where the former is a necessary condition for the latter. Stability along the pass demands a bounded-input bounded-output property over the complete positive quadrant of the 2D plane and this is a very strong requirement, especially in terms of control law design. A more feasible alternative for some cases is strong practical stability, where previous work has formulated this property and obtained necessary and sufficient conditions for its existence together with Linear Matrix Inequality (LMI) based tests, which then extend to allow control law design. This paper develops considerably simpler, and hence computationally more efficient, stability tests that extend to allow control law design in the presence of uncertainty in process model. Copyright © 2011 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Repetitive, or multipass [1], processes make a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique stabilization problem in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction. These processes operate over a subset of the positive quadrant in the 2D plane.

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Physical examples of these processes include long-wall coal cutting and metal rolling operations [1]. Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives, including classes of iterative learning control (ILC) laws [2] and iterative algorithms for solving nonlinear dynamic optimal stabilization problems based on the maximum principle [3]. In this last case, for example, use of the repetitive process setting provides the basis for the development of highly reliable and efficient iterative solution algorithms and in the former provides control law design algorithms to complement those already available. Recently experimental verification of control laws designed using this approach on a gantry robot system, executing a pick and place operation under synchronization that emulates many industrial applications, has been reported [2].

Attempts to stabilize and control these processes using standard, as termed 1D in the multidimensional systems literature, systems theory/algorithms fail, except in a few very special cases, precisely because such an approach ignores their inherent 2D systems structure, that is, information propagation occurs from pass-to-pass and along a given pass and also the initial conditions are reset before the start of each new pass. To remove these deficiencies, a rigorous stability theory has been developed [1] based on an abstract model of the dynamics in a Banach space setting that includes a very large number of processes with linear dynamics and a constant pass length as special cases. Also the results of applying this theory to a range of sub-classes, including the discrete linear repetitive processes considered here, have been reported [1]. This stability theory consists of the distinct concepts of asymptotic stability and stability along the pass respectively where the former is a necessary condition for the latter.

Recognizing the unique control problem, this stability theory is of the bounded input bounded output (BIBO) form, that is, a bounded initial pass profile is required to produce a bounded sequence of pass profiles, where boundedness is defined in terms of the norm on the underlying Banach space. Asymptotic stability guarantees this property over the finite and fixed pass length whereas stability along the pass is stronger in that it requires this property uniformly, that is, for all possible values of the pass length, and asymptotic stability is a necessary condition for stability along the pass.

Imposing stability along the pass effectively extends the operating domain of the process from a subset to the complete positive quadrant of the 2D plane where one axis of this plane represents the along the pass direction of information propagation and the other pass-to-pass. This is a very strong requirement and for linear dynamics imposes the requirement that the complete frequency content of the initial, or starting, pass profile is attenuated from pass-to-pass [1] and any control law applied to an example would have to ensure this property. In applications, it will only ever be the case that the number of passes completed and the pass length are finite and this has led to strong practical stability as an alternative to stability along the pass where the BIBO property is relaxed as both the along the pass and pass-to-pass variables tend to infinity [4].

In more recent work [5] it was shown that necessary and sufficient conditions for strong practical stability property could be formulated in LMI terms that immediately give algorithms for the design of a stabilizing control law. In this paper we develop much simpler LMI based results for these problems, which are more computationally effective and extend the analysis to the design of control laws in the presence of uncertainty in the process model.

Throughout this paper, the null and identity matrices of the required dimensions are denoted by 0 and I respectively. Moreover, $M > 0$ (< 0) denotes a real symmetric positive (negative) definite matrix, and $\mathrm{sym}(x)$ is used to denote $X + X^T$ for a square matrix X.

2. BACKGROUND

The state-space model of a discrete linear repetitive process [1] has the following form over $0 \leq p \leq \alpha - 1,\ k \geq 0$,

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p) \end{aligned} \tag{1}$$

where $\alpha < \infty$ is the pass length and on pass k $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the pass profile vector, and $u_k(p) \in \mathbb{R}^r$ is the vector of control inputs. The boundary conditions, that is, the pass state initial vector sequence and the initial pass profile) are

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \geq 0 \\ y_0(p) &= f(p), \quad 0 \leq p \leq \alpha - 1 \end{aligned} \tag{2}$$

where the $n \times 1$ vector d_{k+1} has known constant entries and $f(p)$ is an $m \times 1$ vector whose entries are known functions of p.

Applying the stability theory of [1] to (1) and (2) gives the necessary and sufficient condition for asymptotic stability as $r(D_0) < 1$, where $r(\cdot)$ denotes the spectral radius of its matrix argument. At first sight, this result is somewhat surprising since it is independent of the plant state dynamics and, in particular, places no constraints the location of the eigenvalues of the matrix A and hence the dynamics produced along any pass. This condition is a result of the finite pass length and its consequences are discussed next.

Suppose that asymptotic stability holds and the input sequence applied $\{u_{k+1}\}$ converges strongly as $k \to \infty$ to u_∞. Then the strong limit $y_\infty := lim_{k\to\infty} y_k$ is termed the limit profile corresponding to this input sequence and its state-space model is

$$\begin{aligned} x_\infty(p+1) &= (A + B_0(I - D_0)^{-1}C)x_\infty(p) + (B + B_0(I - D_0)^{-1}D)u_\infty(p) \\ y_\infty(p) &= (I - D_0)^{-1}Cx_\infty(p) + (I - D_0)^{-1}Du_\infty(p) \\ x_\infty(0) &= d_\infty \end{aligned} \tag{3}$$

where d_∞ is the strong limit of the sequence $\{d_k\}$. In physical terms, this result states that under asymptotic stability the repetitive dynamics can, after a sufficiently large number of passes have elapsed, be replaced by those of a 1D discrete linear system.

As an example, consider the case when $A = -0.5,\ B = 1,\ B_0 = 0.5 + \beta,\ C = 1,\ D = 0,$ $D_0 = 0$, where β is a real scalar. Asymptotic stability holds in this case with resulting limit profile

$$y_\infty(p+1) = \beta y_\infty(p) + u_\infty(p) \tag{4}$$

Hence if $|\beta| \geq 1$, the sequence of pass profiles converges in the pass-to-pass direction (k) to an unstable 1D discrete linear system. Note also that this occurs even though the state matrix A is stable in the 1D discrete linear systems sense.

The problem illustrated by (4) is the finite pass length over which duration even an unstable 1D discrete linear system can only produce a bounded output. If the limit profile is unstable, as a 1D discrete linear system, then this is unacceptable in applications where tracking a reference signal is required.

Stability along the pass prevents this problem from arising by demanding the BIBO property uniformly with respect to the pass length and can be analyzed mathematically by letting $\alpha \to \infty$. This leads to several sets of necessary and sufficient conditions [1] for this property, such as the following.

Theorem 1
Suppose that the pair $\{A, B_0\}$ is controllable and the pair $\{C, A\}$ is observable. Then a discrete linear repetitive process described by (1) and (2) is stable along the pass if, and only if, $r(D_0) < 1, \quad r(A) < 1$, and all eigenvalues of

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$$G(z) = C(zI - A)^{-1}B_0 + D_0 \tag{5}$$

have modulus strictly less than unity $\forall\, |z| = 1$

These conditions can be tested by direct application of well known 1D linear systems tests. Application of them to the example given above shows that stability along the pass also places a constraint on the state dynamics of both the current pass ($r(A) < 1$) and, in the single-input single-output (SISO) case for simplicity, the complete frequency response of the transfer-function describing the contribution of the previous pass profile or equivalently, the initial pass profile and not just on D_0. Also it is easy to see that stability along the pass ensures that the resulting limit profile is stable as a 1D discrete linear system, that is, $r(A + B_0(I - D_0)^{-1}C) < 1$.

Stability along the pass for linear repetitive processes demands that the signals involved are uniformly bounded when both independent variables k and p are of unbounded duration. Equivalently this property must hold for any k and p in the positive quadrant of the 2D plane, that is, $(k, p) \in P := \{(k, p) : k \geq 0, p \geq 0\}$. In terms of design to track a given reference vector, such as in the ILC application where the basic idea is to use information from previous passes to update the control signal on the current pass and thereby improve performance from pass-to-pass in terms of reducing the error defined on each pass as the difference between a given reference vector and the process output, imposing the requirement for stability along the pass means that the control law must achieve the required level of attenuation over the complete frequency range and this, by comparison with the 1D linear systems case, is most likely to result in a very difficult design problem. In such cases, strong practical stability may lead to acceptable design, especially for applications where an unstable limit profile is not acceptable and/or some control is required over the along the pass dynamics.

Strong practical stability relaxes the BIBO stability requirement over P by removing the uniform boundedness requirement as both $k \to \infty$ and $\alpha \to \infty$ but still demands this property for the cases when the pass number $k \to \infty$ and the pass length α finite, and also when the pass index k is finite and the pass length $\alpha \to \infty$. These requirements have strong practical motivation in that the number of passes completed in an application will always be finite and the pass length may be very long but finite. The case when α is finite and $k \to \infty$ is a mathematical formulation of the desire to operate the plant a very large number of times without the need to stop and hence, in a manufacturing example, lose throughput. The case k is finite and the pass length $\alpha \to \infty$ is the mathematical formulation of the case where the process completes a finite number of passes but the pass length is very long, and there is a requirement to control the along the pass dynamics. In the next section necessary and sufficient conditions for strong practical stability of the discrete linear repetitive processes considered in this paper are developed and previous developed tests for this property reviewed as motivation for the new results in this paper.

2.1. Strong Practical Stability

From the analysis of asymptotic stability summarized in the previous section, it follows that when α is finite and $k \to \infty$ strong practical stability requires $r(D_0) < 1$ and $A + B_0(I - D_0)^{-1}C) < 1$. The case when k is finite and $\alpha \to \infty$ results in

$$y_{k+1}(\infty) = \left(C(I - A)^{-1}B_0 + D_0\right) y_k(\infty) \tag{6}$$

and hence we require $r(C(I - A)^{-1}B_0 + D_0) < 1$.

In summary, therefore, the following result gives necessary and sufficient conditions for strong practical stability.

Lemma 1
A discrete linear repetitive process described by (1) and (2) is strongly practically stable if, and only if,

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[a] $r(D_0) < 1$

[b] $r(A) < 1$

[c] $r(A + B_0(I - D_0)^{-1}C) < 1$, and

[d] $r(C(I - A)^{-1}B_0 + D_0) < 1$

The conditions of Lemma 1 can, assuming no numerical problems associated with computing the eigenvalues of the matrices involved, be easily checked for a given example. Previous research [5] used results from 1D singular discrete linear systems theory for the state-space model to obtain the following result.

Theorem 2
[5] A discrete linear repetitive process described by (1) and (2) is strongly practically stable if, and only if, there exist compatibly dimensioned matrices

$$W_1 > 0, \quad W_2 > 0, \quad X_{21}^1, \quad X_{21}^2, \quad X_{11}^1 = (X_{11}^1)^T, \quad X_{22}^1 = (X_{22}^1)^T, \quad Y_{11}^2, \quad Y_{22}^1,$$
$$X_{11}^2 = (X_{11}^2)^T, \quad X_{22}^2 = (X_{22}^2)^T, \quad \widetilde{G_1}, \quad \widetilde{G_2}$$

such that the following LMIs are feasible for scalars $\beta_1 > 1$, $\beta_2 > 1$

$$\begin{bmatrix} -W_1 & W_1^T D_0^T \\ D_0 W_1 & -W_1 \end{bmatrix} < 0 \tag{7}$$

$$\begin{bmatrix} -W_2 & W_2^T A^T \\ A W_2 & -W_2 \end{bmatrix} < 0 \tag{8}$$

$$\begin{bmatrix} -X_{11}^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & (Y_{22}^1)^T \\ 0 & 0 & X_{11}^1 & (X_{21}^1)^T \\ 0 & Y_{22}^1 & X_{21}^1 & X_{22}^1 \end{bmatrix} + Sym\left\{ \begin{bmatrix} \tilde{A}_1 \widetilde{G_1} \\ -\widetilde{G_1} \end{bmatrix} \begin{bmatrix} I & \beta_1 I \end{bmatrix} \right\} < 0 \tag{9}$$

$$\begin{bmatrix} 0 & 0 & (Y_{11}^2)^T & 0 \\ 0 & -X_{22}^2 & 0 & 0 \\ Y_{11}^2 & 0 & X_{11}^2 & (X_{21}^2)^T \\ 0 & 0 & X_{21}^2 & X_{22}^2 \end{bmatrix} + Sym\left\{ \begin{bmatrix} \tilde{A}_2 \widetilde{G_2} \\ -\widetilde{G_2} \end{bmatrix} \begin{bmatrix} U_2 & \beta_2 U_2 \end{bmatrix} \right\} < 0 \tag{10}$$

where

$$U_2 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

As detailed in [5], Theorem 2 leads to the computationally tractable stability tests that, in turn, extend to enable control laws to be designed to ensure strong practical stability of the controlled process and also can be extended to the robust control law design. This, however, comes at the cost of large dimensioned LMIs of complicated structure in terms of how the block entries are constructed from the plant model and control law matrices. The new results in this paper substantially remove this problem.

3. NEW STRONG PRACTICAL STABILITY TESTS

The route to developing simpler computationally efficient tests to those of Theorem 2 again uses 1D descriptor nonsingular linear systems theory and starts from from conditions [c] and [d] of Lemma 1. To begin, first note that (3) can be rewritten in the form

$$\begin{aligned} x_\infty(h+1) - B_0 y_\infty(h) &= A x_\infty(h) \\ (I - D_0) y_\infty(h) &= C x_\infty(h) \end{aligned} \tag{11}$$

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where, given [a] of Lemma 1 $I-D_0$ is a nonsingular matrix. In particular, the condition [c] of Lemma 1 is equivalent to stability of the 1D descriptor linear system with the state-space model

$$E_1 z(h+1) = A_1 z(h) + \Pi u(h) \tag{12}$$

where

$$z(h) = \begin{bmatrix} x_\infty(h) \\ y_\infty(h-1) \end{bmatrix}, \qquad A_1 = \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, \qquad E_1 = \begin{bmatrix} I & -B_0 \\ 0 & I-D_0 \end{bmatrix}$$

Similarly, (6) can be rewritten as

$$\begin{aligned} x_{k+1}(\infty) &= Ax_{k+1}(\infty) + B_0 y_k(\infty) \\ y_{k+1}(\infty) &= Cx_{k+1}(\infty) + D_0 y_k(\infty) \end{aligned} \tag{13}$$

or

$$E_2 z(h+1) = A_2 z(h) + \Pi u(h) \tag{14}$$

where

$$z(h) = \begin{bmatrix} x_k(\infty) \\ y_k(\infty) \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & B_0 \\ 0 & D_0 \end{bmatrix}, \qquad E_2 = \begin{bmatrix} I-A & 0 \\ -C & I \end{bmatrix}$$

Hence the condition [d] of Lemma 1 is equivalent to stability of the 1D descriptor linear system (14). The following result is essential to the proof of the first new result of this paper stated as Theorem 3 below.

Lemma 2
[6] A 1D discrete linear system with state matrix of the form $E^{-1}\hat{A}$ is stable if, and only if $\exists$ a matrix $\hat{Q} > 0$ and a nonsingular matrix $\hat{G}$ such that the following LMI is feasible

$$\begin{bmatrix} -\hat{Q} & \hat{A}^T E^{-T} \hat{G}^T \\ \hat{G} E^{-1} \hat{A} & \hat{Q} - \hat{G} - \hat{G}^T \end{bmatrix} < 0 \tag{15}$$

The following is the first new result of this paper.

Theorem 3
A discrete linear repetitive process described by (1) and (2) is strongly practically stable if and only if $\exists$ compatibly dimensioned matrices $W_1 > 0$, $W_2 > 0$, $Q_1 > 0$, $Q_2 > 0$, and nonsingular matrices S_1 and S_2 such that the following LMIs are feasible

$$\begin{bmatrix} -W_1 & W_1 D_0^T \\ D_0 W_1 & -W_1 \end{bmatrix} < 0 \tag{16}$$

$$\begin{bmatrix} -W_2 & W_2 A^T \\ A W_2 & -W_2 \end{bmatrix} < 0 \tag{17}$$

$$\begin{bmatrix} -Q_1 & S_1^T A_1^T \\ A_1 S_1 & Q_1 - E_1 S_1 - S_1^T E_1^T \end{bmatrix} < 0 \tag{18}$$

$$\begin{bmatrix} -Q_2 & S_2^T A_2^T \\ A_2 S_2 & Q_2 - E_2 S_2 - S_2^T E_2^T \end{bmatrix} < 0 \tag{19}$$

Proof
The LMIs (16) and (17) are easily seen to be equivalent to $r(D_0) < 1$ and $r(A) < 1$ respectively and hence to conditions [a] and [b] of Lemma 1for strong practical stability. The proof that the LMI (18) is equivalent to condition [c] for strong practical stability proceeds by applying Lemma 2 to (12) and introducing the new variables $\hat{S}_1^T = G_1 E_1^{-1}$ to obtain

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the LMI

$$\begin{bmatrix} -\hat{Q}_1 & A_1^T\hat{S}_1 \\ \hat{S}_1^T A_1 & \hat{Q}_1 - \hat{S}_1^T E_1 - E_1^T \hat{S}_1 \end{bmatrix} < 0 \tag{20}$$

Left- and right multiplying this last condition by $\begin{bmatrix} \hat{S}_1^{-T} & 0 \\ 0 & \hat{S}_1^{-T} \end{bmatrix}$ and its transpose respectively and then introducing $S_1 = \hat{S}_1^{-1}$ and $Q_1 = \hat{S}_1^{-T}\hat{Q}_1\hat{S}_1^{-1}$ completes this part of the proof.

The equivalence of the LMI (19) to condition [d] of Lemma 1 follows identical steps to that above and hence the details are omitted here. □

4. STABILIZATION IN THE STRONG PRACTICAL SENSE

This section considers the design of a control law of the form

$$u_{k+1}(p) = K_1 x_{k+1}(p) + K_2 y_k(p) \tag{21}$$

where K_1 and K_2 are matrices to be selected. The previous pass profile is a measured output and here we assume that it not significantly corrupted by noise etc. Moreover, the current pass state vector in this control law could be replaced by the current pass profile or estimated using an observer if not all entries are available for measurement.

Application of (21) to (1) gives the controlled process state-space model

$$\begin{aligned} x_{k+1}(p+1) &= (A + BK_1)x_{k+1}(p) + (B_0 + BK_2)y_k(p) \\ y_{k+1}(p) &= (C + DK_1)x_{k+1}(p) + (D_0 + DK_2)y_k(p) \end{aligned} \tag{22}$$

and the following result gives necessary and sufficient for strong practical stability of the controlled process.

Lemma 3
A discrete linear repetitive process described by (22) is strongly practically stable if, and only if,

[e] $r(D_0 + DK_2) < 1$

[f] $r[A + BK_1] < 1$

[g] $r[(B_0 + BK_2)(I - D_0 - DK_2)^{-1}(C + DK_1) + (A + BK_1)] < 1$, and

[h] $r[(C + DK_1)(I - A - BK_1)^{-1}(B_0 + BK_2) + (D_0 + DK_2)] < 1$

A simpler control law structure would result if $K_2 = 0$ which is equivalent to stabilization using only current pass state feedback but examples are easily constructed where strong practical stability can never be achieved.

The problem of developing a computationally efficient method to design the control such that the conditions of Lemma 3 hold is more complicated relative to the stability only case because we only have two matrices K_1 and K_2 available for selection.

Theorem 4
A controlled discrete linear repetitive process described by (22) is strongly practically stable if ∃ compatibly dimensioned matrices $Q_1 > 0$, $Q_2 > 0$, a nonsingular matrix $S = diag(S_1, S_2)$, and rectangular matrices $\widetilde{N}_1 = \begin{bmatrix} N_1 & 0 \end{bmatrix}$, $\widetilde{N}_2 = \begin{bmatrix} 0 & N_2 \end{bmatrix}$ such that the following LMIs are feasible

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$$\begin{bmatrix} -Q_1 & S^T A_1^T + \widetilde{N}_1^T \Pi^T \\ A_1 S + \Pi \widetilde{N}_1 & Q_1 - (E_1 S - \Pi \widetilde{N}_2) - (E_1 S - \Pi \widetilde{N}_2)^T \end{bmatrix} < 0 \tag{23}$$

$$\begin{bmatrix} -Q_2 & S^T A_2^T + \widetilde{N}_2^T \Pi^T \\ A_2 S + \Pi \widetilde{N}_2 & Q_2 - (E_2 S - \Pi \widetilde{N}_1) - (E_2 S - \Pi \widetilde{N}_1)^T \end{bmatrix} < 0 \tag{24}$$

If these LMIs hold, stabilizing control law matrices are given by

$$K_1 = N_1 S_1^{-1},\ K_2 = N_2 S_2^{-1} \tag{25}$$

Proof

We first show that (23) and (24) respectively guarantee that [g] and [h] of Lemma 3 hold. In particular, we apply Theorem 3 with $A_{1new} = A_1 + \Pi \begin{bmatrix} K_1 & 0 \end{bmatrix}$ and $E_{1new} = E_1 - \Pi \begin{bmatrix} 0 & K_2 \end{bmatrix}$ in the case of (23), and $A_{2new} = A_2 + \Pi \begin{bmatrix} 0 & K_2 \end{bmatrix}$ and $E_{2new} = E_2 - \Pi \begin{bmatrix} K_1 & 0 \end{bmatrix}$ in the case of (24). Introducing the additional variables $N_1 = K_1 S_1$ and $N_2 = K_2 S_2$ completes this part of the proof.

The next task is to show that the LMIs (23) and (24) guarantee that [e] and [f] hold respectively, where by Lemma 2 these are equivalent to the following LMIs

$$\begin{bmatrix} -W_1 & S_2^T D_0^T + N_2^T D^T \\ D_0 S_2 + D N_2 & W_1 - S_2 - S_2^T \end{bmatrix} < 0 \tag{26}$$

$$\begin{bmatrix} -W_2 & S_1^T A^T + N_1^T B^T \\ A S_1 + B N_1 & W_2 - S_1 - S_1^T \end{bmatrix} < 0 \tag{27}$$

Also (23) can be rewritten in extended form as

$$\begin{bmatrix} -Q_{11} & -Q_{12} & S_1^T A^T + N_1^T B^T & S_1^T C^T + N_1^T D^T \\ -Q_{12}^T & -Q_{22} & 0 & 0 \\ A S_1 + B N_1 & 0 & Q_{11} - S_1 - S_1^T & Q_{12} + B_0 S_2 + B N_2 \\ C S_1 + D N_1 & 0 & Q_{12}^T + S_2^T B_0^T + N_2^T B^T & \Xi \end{bmatrix} < 0 \tag{28}$$

where

$$\Xi = Q_{22} - S_2 - S_2^T + D_0 S_2 + D N_2 + S_2^T D_0^T + N_2^T D^T$$

A real symmetric matrix $F(x) \in \mathcal{R}^{n \times n}$ is positive or negative definite if and only if, all of its principal minors are positive or negative definite respectively. Hence for (28) to be feasible

$$\left[\begin{array}{cc|c} -Q_{11} & -Q_{12} & S_1^T A^T + N_1^T B^T \\ -Q_{12}^T & -Q_{22} & 0 \\ \hline A S_1 + B N_1 & 0 & Q_{11} - S_1 - S_1^T \end{array}\right] < 0 \tag{29}$$

must hold (as the matrix on the left-hand side is a principal minor of the matrix of (28)). Next left- and right- multiply this last inequality by $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ to obtain

$$\left[\begin{array}{cc|c} Q_{11} - S_1 - S_1^T & A S_1 + B N_1 & 0 \\ S_1^T A^T + N_1^T B^T & -Q_{11} & -Q_{12} \\ \hline 0 & -Q_{12}^T & -Q_{22} \end{array}\right] < 0 \tag{30}$$

As all principal minors of the underlying matrix of (30) must be negative definite, the following must hold

$$\begin{bmatrix} Q_{11} - S_1 - S_1^T & A S_1 + B N_1 \\ S_1^T A^T + N_1^T B^T & -Q_{11} \end{bmatrix} < 0 \tag{31}$$

Left- and right- multiplying this last condition by $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ and the LMI (27) is obtained for $Q_{11} = W_1$.

In the case of (24) first rewrite this LMI in extended form as

$$\begin{bmatrix} -Q_{211} & -Q_{212} & 0 & 0 \\ -Q_{212}^T & -Q_{222} & S_2^T B_0^T + N_2^T B^T & S_2^T D_0^T + N_2^T D^T \\ 0 & B_0 S_2 + B N_2 & \Upsilon & Q_{212} + S_1^T C^T + N_1^T D^T \\ 0 & D_0 S_2 + D N_2 & Q_{212}^T + C S_1 + D N_1 & Q_{222} - S_2 - S_2^T \end{bmatrix} < 0 \tag{32}$$

where

$$\Upsilon = Q_{211} - S_1 - S_1^T + A S_1 + B N_1 + S_1^T A^T + B^T N_1^T$$

Also for (32) to hold

$$\left[\begin{array}{cc|c} -Q_{222} & S_2^T B_0^T + N_2^T B^T & S_2^T D_0^T + N_2^T D^T \\ B_0 S_2 + B N_2 & \Upsilon & Q_{212} + S_1^T C^T + N_1^T D^T \\ \hline D_0 S_2 + D N_2 & Q_{212}^T + C S_1 + D N_1 & Q_{222} - S_2 - S_2^T \end{array}\right] < 0 \tag{33}$$

Left- and right multiplying this last result by $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ gives

$$\left[\begin{array}{c|cc} Q_{222} - S_2 - S_2^T & D_0 S_2 + D N_2 & Q_{212}^T + C S_1 + D N_1 \\ \hline (D_0 S_2 + D N_2)^T & -Q_{222} & S_2^T B_0^T + N_2^T B^T \\ (Q_{212}^T + C S_1 + D N_1)^T & B_0 S_2 + D N_2 & \Upsilon \end{array}\right] < 0 \tag{34}$$

and for (34) to hold

$$\begin{bmatrix} Q_{222} - S_2 - S_2^T & D_0 S_2 + D N_2 \\ (D_0 S_2 + D N_2)^T & -Q_{222} \end{bmatrix} < 0 \tag{35}$$

Finally, left- and right- multiply this last inequality by $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ to obtain (26) when $Q_{222} = W_2$, and the proof is complete. □

In the proof of this last result the block diagonal structure of the matrix S is required to avoid additional strong links between the control law matrices K_1 and K_2.

Theorem 4 is much simpler than the corresponding one in [5] and hence more computationally tractable and less conservative. Also, the resulting LMIs are not parameterized and hence there is no parameter which must to be tuned to achieve stabilized process dynamics. Moreover, the number of LMIs has been reduced by a factor of two.

5. UNCERTAIN PROCESS STABILITY AND ROBUST STABILIZATION

Often an exact model of the process dynamics is not available due to the presence of uncertainty. In this section, the new results of the previous section are generalized to the case when the process uncertainty is modelled in the polytopic form. In this case the process state-space model matrices are assumed to lie in the matrix polytope

$$\begin{bmatrix} A & B & B_0 \\ C & D & D_0 \end{bmatrix} \in Co\left(\begin{bmatrix} A^i & B^i & B_0^i \\ C^i & D^i & D_0^i \end{bmatrix}\right) \tag{36}$$

where $i = 1, 2, \ldots, h$ and

$$Co(\mathcal{Z}_i) := \left\{ X := \textstyle\sum_{i=1}^h \alpha_i \mathcal{Z}_i, \quad \alpha_i \geq 0, \quad \displaystyle\sum_{i=1}^h \alpha_i = 1 \right\} \tag{37}$$

The matrices E_1 and E_2 in(12) and (14) depend on matrices from the uncertainty description, but the sub-polytopes $Co(E_1^i)$ and $Co(E_2^i)$ contain no singular matrices.

Introducing the notation

$$A_1^i = \begin{bmatrix} A^i & 0 \\ C^i & 0 \end{bmatrix}, \quad E_1^i = \begin{bmatrix} I & -B_0^i \\ 0 & I - D_0^i \end{bmatrix}$$

and

$$A_2^i = \begin{bmatrix} 0 & B_0^i \\ 0 & D_0^i \end{bmatrix}, \quad E_2^i = \begin{bmatrix} I - A^i & 0 \\ -C^i & I \end{bmatrix}, \quad \Pi^i = \begin{bmatrix} B^i \\ D^i \end{bmatrix}$$

for $i = 1, 2, \ldots, h$. Then application of Theorem 3 gives the following result.

Theorem 5

A discrete linear repetitive process described by (1) and (2) with uncertainty defined by (36) is strongly practically stable if, and only if, $\exists$ appropriately dimensioned matrices $W_1 > 0$, $W_2 > 0$, $Q_1^i > 0$, $Q_2^i > 0$, for all $i = 1, \ldots, v$ and nonsingular matrices S_1 and S_2 such that the following LMIs are feasible

$$\begin{bmatrix} -W_1 & W_1 D_0^{iT} \\ D_0^i W_1 & -W_1 \end{bmatrix} < 0 \tag{38}$$

$$\begin{bmatrix} -W_2 & W_2 A^{iT} \\ A^i W_2 & -W_2 \end{bmatrix} < 0 \tag{39}$$

$$\begin{bmatrix} -Q_1^i & S_1^T A_1^{iT} \\ A_1^i S_1 & Q_1^i - E_1^i S_1 - S_1^T E_1^{iT} \end{bmatrix} < 0 \tag{40}$$

$$\begin{bmatrix} -Q_2^i & S_2^T A_2^{iT} \\ A_2^i S_2 & Q_2^i - E_2^i S_2 - S_2^T E_2^{iT} \end{bmatrix} < 0 \tag{41}$$

Proof

This follows the same steps as that for Theorem 3 for each polytope vertex, and then make a convex combination as in (37). □

The following result enables stabilization in the presence of uncertainty by design of the control law (21). However the left hand side matrices E_1 and E_2 can remain uncertain.

Theorem 6

A controlled discrete linear repetitive process described by (22) and uncertainty defined by (36) is strongly practically stable if $\exists$ the appropriately dimensioned matrices $Q_1^i > 0$, $Q_2^i > 0$, for all $i = 1, \ldots, v$, a nonsingular matrix $S = diag(S_1, S_2)$, and rectangular matrices $\widetilde{N}_1 = \begin{bmatrix} N_1 & 0 \end{bmatrix}$, $\widetilde{N}_2 = \begin{bmatrix} 0 & N_2 \end{bmatrix}$ such that the following LMIs are feasible

$$\begin{bmatrix} -Q_1^i & S^T A_1^{iT} + \widetilde{N}_1^T \Pi^{iT} \\ A_1^i S + \Pi^i \widetilde{N}_1 & Q_1^i - (E_1^i S - \Pi^i \widetilde{N}_2) - (E_1^i S - \Pi^i \widetilde{N}_2)^T \end{bmatrix} < 0 \tag{42}$$

$$\begin{bmatrix} -Q_2^i & S^T A_2^{iT} + \widetilde{N}_2^T \Pi^{iT} \\ A_2^i S + \Pi^i \widetilde{N}_2 & Q_2^i - (E_2^i S - \Pi^i \widetilde{N}_1) - (E_2^i S - \Pi^i \widetilde{N}_1)^T \end{bmatrix} < 0 \tag{43}$$

If these LMIs hold, stabilizing control law matrices are given by

$$K_1 = N_1 S_1^{-1}, \; K_2 = N_2 S_2^{-1} \tag{44}$$

Proof

Define the matrix Q as

$$Q = \sum_{i=1}^{h} \alpha_i Q^i$$

. Then multiplying (42) and (43), respectively, by α_i and summing from 1 to h gives the same conditions as in Theorem 4. □

6. NUMERICAL EXAMPLE - APPLICATION TO THE MATERIAL ROLLING PROCESS

As an example to illustrate the new results in this paper, consider a simplified model of a metal rolling process, see [7] for the details. The vertices for the uncertainty model in this case are taken as

Vertex 1

$$\left[\begin{array}{c|c|c} A^1 & B_0^1 & B^1 \\ \hline C^1 & D_0^1 & D^1 \end{array}\right] = \left[\begin{array}{cc|c|c} 0.9370 & 187.5 & 0.01440 & -0.08660 \\ -0.0003116 & 0.9377 & 0.00007191 & -0.0004328 \\ \hline 0.9377 & 187.5 & 0.7836 & -0.08660 \end{array}\right]$$

Vertex 2

$$\left[\begin{array}{c|c|c} A^2 & B_0^2 & B^2 \\ \hline C^2 & D_0^2 & D^2 \end{array}\right] = \left[\begin{array}{cc|c|c} 0.7676 & 153.5 & 0.05360 & -0.07090 \\ -0.001162 & 0.7676 & 0.0002682 & -0.0003543 \\ \hline 0.7676 & 153.5 & 0.8229 & -0.07090 \end{array}\right]$$

Vertex 3

$$\left[\begin{array}{c|c|c} A^3 & B_0^3 & B^3 \\ \hline C^3 & D_0^3 & D^3 \end{array}\right] = \left[\begin{array}{cc|c|c} 0.8574 & 171.5 & 0.08230 & -0.1980 \\ -0.0007130 & 0.8574 & 0.0004117 & -0.0009902 \\ \hline 0.8574 & 171.5 & 0.5049 & -0.1980 \end{array}\right]$$

Vertex 4

$$\left[\begin{array}{c|c|c} A^4 & B_0^4 & B^4 \\ \hline C^4 & D_0^4 & D^4 \end{array}\right] = \left[\begin{array}{cc|c|c} 0.9250 & 185.0 & 0.004633 & -0.02290 \\ -0.0003749 & 0.9250 & 0.00002320 & -0.0001143 \\ 0.9250 & 185.0 & 0.9428 & -0.02290 \end{array}\right]$$

Also take the pass length $\alpha = 50$ with the following boundary conditions

$$\begin{aligned} x_{k+1}(0) &= 0 \\ y_0(p) &= 6, \qquad 0 \leq p \leq 49 \end{aligned}$$

Applying Theorem 6 gives the stabilizing control law matrices

$$K_1 = \left[\begin{array}{cc} 1.920 & 1528.0 \end{array}\right], \qquad K_2 = 0.6957 \tag{45}$$

For simulations choose $\alpha_1 = 0.3$, $\alpha_2 = 0.2$, $\alpha_3 = 0.1$ and $\alpha_4 = 0.4$. Figure 1 gives the pass profile sequence generated with zero control input for the uncontrolled process and demonstrates that this process is not stable. Figure 2 shows the same simulation with the control law designed above applied.

7. CONCLUSIONS

This paper has developed new results on strong practical stability and stabilization of discrete linear repetitive processes, starting from results in descriptor, but nonsingular, 1D linear systems approach. The previous approach to strong practical stability characterization and stabilizing control law design was based on a 1D singular systems interpretation of conditions [c] and [d] of Lemma 1 and also provided parameters (β_1 and α_2) to tune the resulting control law. In the new results of this paper the LMis are reduced in number and of a much simpler structure but the ability to tune the control law is lost. Note, however,

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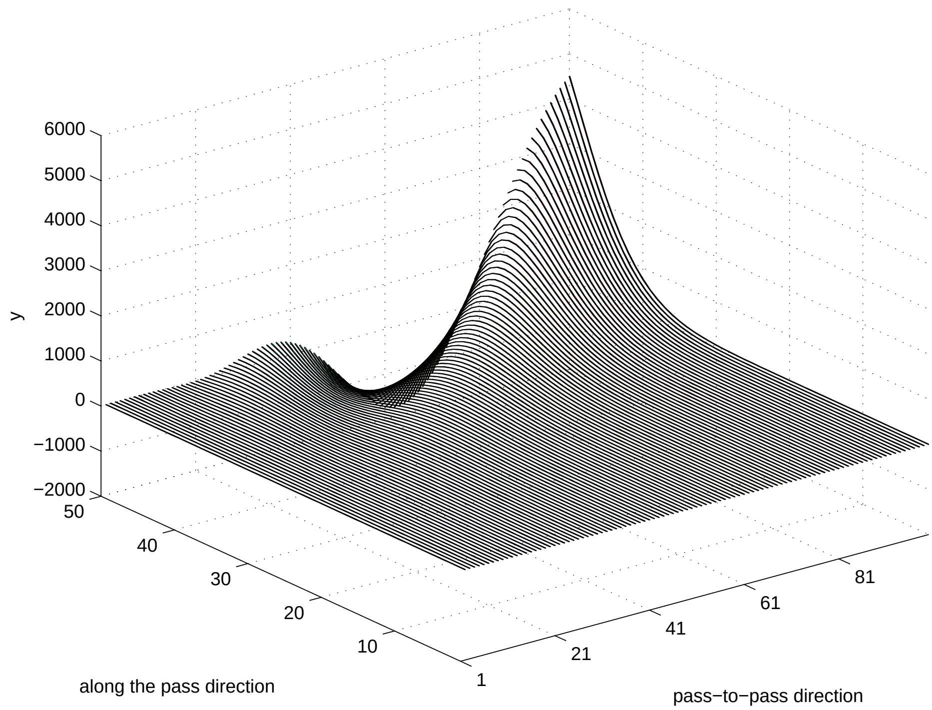

Figure 1. Pass profiles generated by the uncontrolled process with uncertainty.

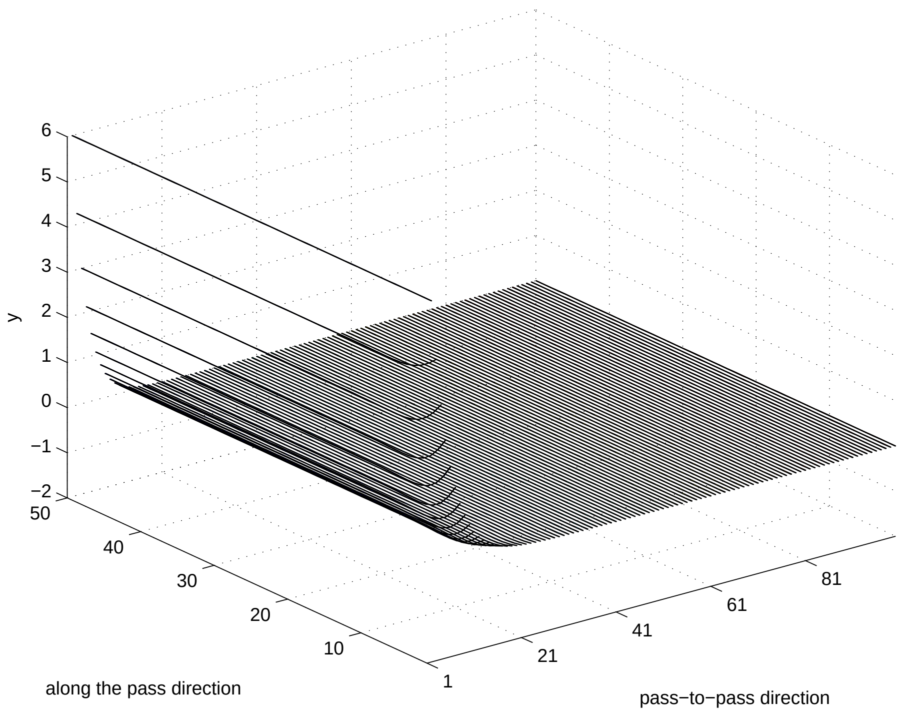

Figure 2. Pass profiles generated by the uncontrolled process with uncertainty.

that in both cases the LMIs involved are necessary and sufficient and hence equivalent. For control law design, both approaches are sufficient and the simpler structure LMIs developed in this paper limit the resulting conservativeness. The polytopic uncertainty results are also new

Future work includes extension to to processes with disturbances to, for example, the pass profile measurement that cannot be ignored at even the initial design stage. The control law considered in this paper contains a state feedback and if some its entries are not available for measurement then an observer is required for the implementation. An alternative is to extend the results here to a control law where the current pass state vector is replaced by the current pass profile. Finally, application to iterative learning control with experimental verification is a very promising applications area for these results.

REFERENCES

1. Rogers E, Gałkowski K, Owens DH. Control Systems Theory and Applications for Linear Repetitive Processes, Lecture Notes in Control and Information Sciences, vol. 349. Springer-Verlag: Berlin, Germany, 2007.
2. Hładowski L, Gałkowski K, Cai Z, Rogers E, Freeman CT, Lewin PL. Experimentally supported 2d systems based iterative learning control law design for error convergence and performance. Control Engineering Practice 2010; 18(4):339–348. URL http://eprints.ecs.soton.ac.uk/17651/.
3. Roberts PD. Numerical investigation of a stability theorem arising from 2-dimensional analysis of an iterative optimal control algorithm. Multidimensional Systems and Signal Processing 2000; 11(1-2):109–124.
4. Gałkowski K, Rogers E, Gramacki J, Gramacki A, Owens DH. Strong practical stability for a class of 2D linear systems. Proc. Int. Symposium on Circuits and Systems (ISCAS), vol. 1, Geneva, Switzerland, 2000; 403–406.
5. Dąbkowski P, Gałkowski K, Rogers E, Kummert A. Strong practical stability and stabilization of discrete linear repetitive processes. Multidimensional Systems and Signal Processing December 2009; 20(4):311–331. URL http://eprints.ecs.soton.ac.uk/17796/.
6. Oliveira MCd, Bernussou J, Geromel JC. A new discrete-time robust stability condition. Systems and Control Letters 1999; 39(4):261–265.
7. Cichy B, Gałkowski K, Rogers E, Kummert A. Robust Control of Discrete Linear Repetitive Processes with Parameter Varying Uncertainty, Lecture Notes in Electrical Engineering, vol. 80, chap. 9. Springer, 2011; 165–183, doi:10.1007/978-94-007-0602-6.