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Semidefinite Programming Relaxation Based Virtually Antipodal Detection for Gray Coded 16-QAM MIMO Signalling

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***Abstract*—An efficient semidefinite programming relaxation (SDPR) based virtually antipodal (VA) detection approach is proposed for Gray coded 16-QAM signalling over multiple-input–multiple-output (MIMO) channels. The existing index-bit-based VA-SDPR (IVA-SDPR) method is incapable of making direct binary decisions concerning the individual information bits without making symbol decisions first, except for the linear natural-mapping aided rectangular QAM constellations. By contrast, our new method is capable of directly deciding on the information bits of the ubiquitous Gray-mapping aided 16-QAM by employing a strikingly simple linear matrix representation (LMR) of 4-QAM. As an appealing benefit, the conventional "signal-to-symbol-to-bits" decision process is substituted by a simpler "signal-to-bits" decision process for the classic Gray-mapping aided rectangular 16-QAM. Furthermore, when combined with low-complexity bit-flipping based "hill climbing", the proposed direct-bit-based VA-SDPR (DVA-SDPR) detector achieves the best bit-error-ratio (BER) performance among the known SDPR-based MIMO detectors in the context considered, while still maintaining a worst-case complexity order as low as $O\left[(4N_T+1)^{3.5}\right]$.**

***Index Terms*—Binary constrained quadratic programming, Gray mapping, primal-dual interior-point algorithm, QAM, semidefinite programming relaxation (SDPR), virtually-antipodal detection.**

I. Introduction

The tree-search based sphere decoder (SD) [1], [2] derived for multiple-input–multiple-output (MIMO) channels is probably the best-known computationally efficient algorithm capable of achieving the exact maximum-likelihood (ML) performance. However, the SD is only efficient for relatively high signal-to-noise ratios (SNR) and a low number of transmit antennas N_T. Furthermore, it has an exponentially increasing *expected complexity order* of $O(M^{\beta N_T})$ in both the worst-case and the average-case, where M is the constellation size, and $\beta \in (0, 1]$ is a small factor depending on the value of SNR [3].

In contrast to the classic tree-search philosophy, the semidefinite programming [4] relaxation (SDPR) approach is based on convex optimization theory [5] and has recently received much research attention [6]–[12]. The most attractive characteristic of the SDPR-aided detectors is that they guarantee a so-called polynomial-time[1] worst-case computational complexity, while achieving a high performance. The numerical and analytical results of [6] confirmed that the SDPR detector achieves the maximum possible diversity order, when using binary phase shift keying (BPSK) for transmission over a real-valued fading MIMO channel. The SDPR approach was also further developed for high-order modulation schemes, such as for M-ary phase shift keying (M-PSK) scenario in [7], and for high-order quadrature amplitude modulation (QAM) in [8]–[11]. As for the 16-QAM scenario, it was recently shown in [12] that the so-called polynomial-inspired SDPR (PI-SDPR) [8], the bound-constrained SDPR (BC-SDPR) [9] and the virtually antipodal SDPR (VA-SDPR) [11] are actually equivalent in the sense that they attain the same optimal objective values and exhibit an identical symbol error ratio (SER) performance[2].

The VA-SDPR detector is of particular interest to us, since it may be shown to have a strong connection to the SDPR detector used in BPSK, where the SDPR shows near-optimal performance. The VA-SDPR converts the M-ary integer programming problem into a binary integer programming problem. However, in the VA-SDPR detector of [11] the binary decisions are made on the "index bits" rather than on the "information bits". These two types of bits are in general different [11] from each other, except for the linear natural-mapping[3] aided rectangular M-QAM [13]. Consequently, when the ubiquitous Gray-mapping aided rectangular M-QAM is used, in order to make correct decisions on the information bits, the VA-SDPR detector of [11] has to obtain its symbol decisions based on the decided "index bits". As shown in [13], the Gray-mapping of high-order rectangular QAM is nonlinear for $M > 4$ and the relationship between the transmitted symbol vector and the information bits *cannot* be characterized by a compact linear matrix transformation of the form $\mathbf{s} = \mathbf{Wb}$ as for the linear natural-mapping aided rectangular QAM, where $\mathbf{s}$ is the transmitted symbol vector, $\mathbf{b}$ is the associated antipodal information bit vector, and $\mathbf{W}$ is the constellation-specific modulation matrix known to both the transmitter and receiver.

In this paper, we propose a novel direct-bit-based VA-SDPR (DVA-SDPR) detector for the ubiquitous Gray-mapping aided rectangular 16-QAM, which is capable of directly deciding on the information bits for transmission over fading MIMO channels. By exploiting the specific structure of the Gray-mapping aided 16-QAM constellation, our approach transforms the original 16-QAM aided $(N_T \times N_R)$-element MIMO system to a virtual 4-QAM aided $(2N_T \times N_R)$-element MIMO system. Since the modulation matrix of 4-QAM is identical for both the natural-mapping and the Gray-mapping [13], the proposed DVA-SDPR detector finally converts the classic nonlinear Gray-mapping aided 16-QAM symbol detection problem to a Boolean quadratic programming (BQP) problem [5]. When relying on this technique, the conventional "signal-to-symbol-to-bits" decision process is substituted by a simpler "signal-to-bits" decision process for the classic nonlinear Gray-mapping aided rectangular 16-QAM. In other words, we can directly carry out the information-bit decisions without invoking first conventional symbol decisions for the nonlinear Gray-mapping aided rectangular 16-QAM scheme. Furthermore, when combined with low-complexity bit-flipping based "hill climbing", the DVA-SDPR detector achieves the best bit-error-ratio (BER) performance among the known SDPR-based MIMO detectors in the context considered, while still maintaining a polynomial-time worst-case complexity order as low as $O\left[(4N_T+1)^{3.5}\right]$.

II. System Model And Problem Statement

Consider a perfectly symbol-synchronized memoryless spatial multiplexing MIMO system having N_T transmit and N_R receive

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[1]The computational complexity increases as a polynomial function of N_T.

[2]The SDPR QAM detector of [10] exhibits a better performance than that of [8], [9], [11], but has a much higher complexity.

[3]The linear natural mapping is defined as the mapping which satisfies eq.(3) of [13].

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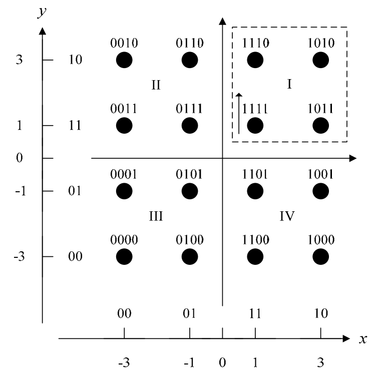

Fig. 1. Signal space diagram of the Gray-mapping aided 16-QAM.

antennas. The baseband equivalent system model is written as

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n}, \tag{1}$$

where $\mathbf{y}$ is the N_R-element received signal vector, $\mathbf{s}$ is the N_T-element transmitted symbol vector, whose elements are from the Gray-coded rectangular 16-QAM constellation shown in Fig. 1, $\mathbf{H}$ is the $(N_R \times N_T)$-element complex-valued channel matrix, and $\mathbf{n}$ is the N_R-element complex Gaussian noise vector with a zero mean and covariance matrix of $2\sigma^2\mathbf{I}$.

The ML detection conceived for the MIMO system of (1) can be formulated as the following constrained discrete least-squares optimization problem

$$\hat{\mathbf{s}}_{\mathrm{ML}} = \arg\min_{\mathbf{s}\in\mathbb{D}} \|\mathbf{y} - \mathbf{H}\mathbf{s}\|_2^2, \tag{2}$$

where the alphabet set $\mathbb{D}$ represents the Gray-mapping aided rectangular 16-QAM constellation of Fig. 1.

In [11], (2) was further formulated as[4]

$$\hat{\mathbf{d}}_{\mathrm{ML}} = \arg\min_{\mathbf{d}\in\{+1,-1\}^{4N_T}} \left\|\tilde{\mathbf{y}} - \tilde{\mathbf{H}}\mathbf{T}\mathbf{d}\right\|_2^2, \tag{3}$$

where $\mathbf{d}$ represents the vector of "index bits" [11][5], which are different from the (antipodal) information-bit vector $\mathbf{b}$. $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{y}}$ are the real-valued versions of $\mathbf{H}$ and $\mathbf{y}$ in (2) respectively, while $\mathbf{T}$ is the real-valued transformation matrix, which is fixed for a specific constellation, similar to the complex-valued modulation matrix $\mathbf{W}$ of [13]. After obtaining $\hat{\mathbf{d}}_{\mathrm{ML}}$, the original real-valued symbol vector corresponding to the real-valued system model is estimated as

$$\hat{\tilde{\mathbf{s}}}_{\mathrm{ML}} = \mathbf{T}\hat{\mathbf{d}}_{\mathrm{ML}}. \tag{4}$$

In contrast to this solution, the problem of interest to us is — how can we develop a VA-SDPR detector that directly estimates the (antipodal) information bit vector $\mathbf{b}$ without estimating the symbol vector $\mathbf{s}$?

III. Structure of Gray-Mapping Aided 16-QAM

Assume that the jth component of the transmitted 16-QAM symbol vector $\mathbf{s}$ is obtained using the bit-to-symbol mapping function $s_j = \mathrm{map}(\mathbf{u}_j)$, $j = 1, 2, \cdots, N_T$, where $\mathbf{u}_j = [u_{j,1}, u_{j,2}, u_{j,3}, u_{j,4}]^T$ is the vector of information bits with each element being 1 or 0. The vector of information bits corresponding to $\mathbf{s}$ is denoted as $\mathbf{u}$, which satisfies $\mathbf{s} = \mathrm{map}(\mathbf{u})$ and is formed by concatenating the N_T antennas' information bits $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{N_T}$, yielding $\mathbf{u} = [u_1, u_2, \cdots, u_k, \cdots, u_{4N_T}]^T = [\mathbf{u}_1^T, \mathbf{u}_2^T, \cdots, \mathbf{u}_{N_T}^T]^T \in \{1, 0\}^{4N_T}$. The antipodal information bits are obtained from the original information bits of logical 1 or 0 using $b_k = 2u_k - 1$, where $b_k \in \{+1, -1\}$.

As shown in [13], the nonlinear Gray-mapping aided 16-QAM scheme may be formulated as $\mathbf{s} = \mathbf{W}(\mathbf{b})\mathbf{b}$, where the structure of the modulation matrix $\mathbf{W}(\mathbf{b})$ exhibits multiple forms, depending on the antipodal information bit vector $\mathbf{b}$. Hence $\mathbf{W}(\mathbf{b})$ is not readily available at the receiver side. Although it may be possible to estimate the modulation matrix $\mathbf{W}(\mathbf{b})$ at the receiver, the estimation error will inevitably degrade the achievable performance.

Let us revisit the "generating units" of the Gray-mapping aided 16-QAM scheme shown in Table I [13]. Since the four constellation points in the same quadrant share the same generating units, without loss of generality, we will consider the constellation points in Quadrant IV of Fig. 1 as an example.

The legitimate original information-bit-sequences $[u_1 u_2 u_3 u_4]$ are:

$$[1\ 1\ 0\ 0] \qquad [1\ 0\ 0\ 0] \qquad [1\ 0\ 0\ 1] \qquad [1\ 1\ 0\ 1]. \tag{5}$$

The above-mentioned generating units $[g_1 g_2 g_3 g_4]$ corresponding to $[u_1 u_2 u_3 u_4]$ are:

$$[2\ -1\ 2i\ i] \quad [2\ -1\ 2i\ i] \quad [2\ -1\ 2i\ i] \quad [2\ -1\ 2i\ i]. \tag{6}$$

Observing (5) and (6), two remarks can be made. Firstly, the bits u_1 and u_3 of the four information-bit-sequences that are mapped to the specific constellation points dwelling in the same quadrant are identical. Secondly, the first and the third components of the generating units are $[g_1 g_3] = [2\ 2i]$, which indicates that the bits u_1 and u_3 of each information-bit-sequence are mapped to a 4-QAM constellation, whose amplitude is doubled[6]. Therefore, the Gray-mapping aided 16-QAM symbols of Quadrant IV can be formulated as

$$\begin{aligned} s &= \mathrm{map}_{\text{16-QAM}}(u_1 u_2 u_3 u_4) \\ &= 2 \times \mathrm{map}_{\text{4-QAM}}(u_1 u_3) + \mathrm{map}_x(u_2 u_4) \\ &= 2s^{(1)} + s^{(2)}. \end{aligned} \tag{7}$$

To elucidate the notation of $\mathrm{map}_x(u_2 u_4)$ further, let us observe

$$[u_2 u_4]: \quad [1\ 0] \qquad [0\ 0] \qquad [0\ 1] \qquad [1\ 1], \tag{8}$$

$$[g_2 g_4]: \quad [-1\ i] \qquad [-1\ i] \qquad [-1\ i] \qquad [-1\ i], \tag{9}$$

$$s^{(2)}: \quad -1-i \qquad 1-i \qquad 1+i \qquad -1+i, \tag{10}$$

where we have $s^{(2)} = (2[u_2 u_4] - 1) \times [g_2 g_4]^T$. Note that $s^{(2)}$ may also be obtained by mapping the bits

$$[\tilde{u_2}\tilde{u_4}]: \quad [0\ 0] \qquad [1\ 0] \qquad [1\ 1] \qquad [0\ 1] \tag{11}$$

to 4-QAM, where $s^{(2)} = (2[\tilde{u_2}\tilde{u_4}] - 1) \times [1\ i]^T$.

Therefore, (7) may be reformulated as

$$s = 2 \times \mathrm{map}_{\text{4-QAM}}(u_1 u_3) + \mathrm{map}_{\text{4-QAM}}(\tilde{u}_2 \tilde{u}_4). \tag{12}$$

On the other hand, we have

$$\begin{aligned} 00 &= (\underline{1} \oplus 1)(\underline{0} \oplus 0), \\ 10 &= (\underline{0} \oplus 1)(\underline{0} \oplus 0), \\ 11 &= (\underline{0} \oplus 1)(\underline{1} \oplus 0), \\ 01 &= (\underline{1} \oplus 1)(\underline{1} \oplus 0), \end{aligned} \tag{13}$$

where $\oplus$ represents the XOR operation. Eq. (13) may be written in a more compact manner as

$$[\tilde{u_2}\tilde{u_4}] = [u_2 u_4] \boxplus [u_1 u_3], \tag{14}$$

hence we have

$$[u_2 u_4] = [u_1 u_3] \boxplus [\tilde{u_2}\tilde{u_4}], \tag{15}$$

[4] The real-valued model is used in [11], whereas we use the more general complex-valued model here.

[5] In general, the (real-valued) Gray-coded QAM symbol vector $\tilde{\mathbf{s}}$ *cannot* be represented as a linear transformation of $\tilde{\mathbf{s}} = \mathbf{T}\mathbf{b}$, as shown in [13]. However, it was formulated as $\tilde{\mathbf{s}} = \mathbf{T}\mathbf{d}$ in [11], where $\mathbf{d}$ was termed as "index bits".

[6] As shown in [13], the natural-mapping and the Gray-mapping are identical for 4-QAM.

TABLE I
GENERATING UNITS OF 16QAM USING GRAY MAPPING

Index	Generating Unit	Bit Sequence	Symbol	Quadrant	Index	Generating Unit	Bit Sequence	Symbol	Quadrant
1	$2\ 1\ 2i\ i$	$-1\ -1\ -1\ -1$	$-3-3i$	III	9	$2\ -1\ 2i\ i$	$+1\ +1\ -1\ -1$	$1-3i$	IV
2	$2\ 1\ 2i\ i$	$-1\ -1\ -1\ +1$	$-3-i$		10	$2\ -1\ 2i\ i$	$+1\ +1\ -1\ +1$	$1-i$	
3	$2\ 1\ 2i\ -i$	$-1\ -1\ +1\ +1$	$-3+i$	II	11	$2\ -1\ 2i\ -i$	$+1\ +1\ +1\ +1$	$1+i$	I
4	$2\ 1\ 2i\ -i$	$-1\ -1\ +1\ -1$	$-3+3i$		12	$2\ -1\ 2i\ -i$	$+1\ +1\ +1\ -1$	$1+3i$	
5	$2\ 1\ 2i\ -i$	$-1\ +1\ +1\ -1$	$-1+3i$		13	$2\ -1\ 2i\ -i$	$+1\ -1\ +1\ -1$	$3+3i$	
6	$2\ 1\ 2i\ -i$	$-1\ +1\ +1\ +1$	$-1+i$		14	$2\ -1\ 2i\ -i$	$+1\ -1\ +1\ +1$	$3+i$	
7	$2\ 1\ 2i\ i$	$-1\ +1\ -1\ +1$	$-1-i$	III	15	$2\ -1\ 2i\ i$	$+1\ -1\ -1\ +1$	$3-i$	IV
8	$2\ 1\ 2i\ i$	$-1\ +1\ -1\ -1$	$-1-3i$		16	$2\ -1\ 2i\ i$	$+1\ -1\ -1\ -1$	$3-3i$	

where $\boxplus$ is the element-wise XOR operator. It may be readily shown that (14) and (15) also hold for the other three quadrants.

IV. THE PROPOSED DVA-SDPR DETECTOR

A. DVA-SDPR Formulation

Based on (12), the system model (1) can be rewritten as

$$\begin{aligned}\mathbf{y} &= [\mathbf{h}_1, \mathbf{h}_2, \cdots, \mathbf{h}_{N_T}] \begin{bmatrix} 2s_1^{(1)} + s_1^{(2)} \\ 2s_2^{(1)} + s_2^{(2)} \\ \vdots \\ 2s_{N_T}^{(1)} + s_{N_T}^{(2)} \end{bmatrix} + \mathbf{n} \\ &= [2\mathbf{H}\ \ \mathbf{H}]\mathbf{x} + \mathbf{n}, \end{aligned} \tag{16}$$

where $\mathbf{h}_j$ is the jth column of $\mathbf{H}$, $\mathbf{x} = [s_1^{(1)}, s_2^{(1)}, \cdots, s_{N_T}^{(1)}, s_1^{(2)}, s_2^{(2)}, \cdots, s_{N_T}^{(2)}]^T$ with each element being a standard 4-QAM symbol. At this stage, (16) may be regarded as a virtual 4-QAM aided $(2N_T \times N_R)$-element MIMO system[7].

According to the modulation matrix of 4-QAM given in [13], (16) can be further reformulated as

$$\mathbf{y} = [2\mathbf{H}\ \ \mathbf{H}]\mathbf{W}\mathbf{p} + \mathbf{n} = \mathbf{G}\mathbf{p} + \mathbf{n}, \tag{17}$$

where $\mathbf{G}$ is the "composite channel matrix", $\mathbf{p} \in \{-1, +1\}^{4N_T}$, and

$$\mathbf{W} = \begin{bmatrix} 1 & i & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & i & \cdots & 0 & 0 \\ & & & & \ddots & & \\ 0 & 0 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & i \end{bmatrix}_{2N_T \times 4N_T} \tag{18}$$

is the modulation matrix of 4-QAM for both natural-mapping and Gray-mapping. Hence the original Gray-coded 16-QAM $(N_T \times N_R)$-element MIMO channel has been converted to a virtual $(4N_T \times N_R)$-element MIMO channel relying on binary signaling.

The original ML detection related constrained discrete least-squares optimization problem of (2) may be shown to be equivalent to the following BQP problem [5]

$$\begin{aligned} \min \quad & \mathbf{p}^T \mathbf{G}^H \mathbf{G} \mathbf{p} - 2\mathbf{y}^H \mathbf{G} \mathbf{p} \\ \text{s.t.} \quad & \mathbf{p} \in \{+1, -1\}^{4N_T}, \end{aligned} \tag{19}$$

which is difficult to solve due to the non-convex constraints of $p_i^2 = 1$.

In order to cast the objective function of (19) into a homogeneous quadratic form, we introduce a redundant scalar $t \in \{+1, -1\}$. Since $t\mathbf{p} \in \{+1, -1\}^{4N_T}$ for any $\mathbf{p} \in \{+1, -1\}^{4N_T}$, (19) may also be formulated as

$$\begin{aligned} \min \quad & \left[\mathbf{p}^T\ t\right] \Re\{\mathbf{Q}_c\} \left[\mathbf{p}^T\ t\right]^T \\ \text{s.t.} \quad & \left[\mathbf{p}^T\ t\right] \in \{+1, -1\}^{4N_T+1}, \end{aligned} \tag{20}$$

where $\Re\{\mathbf{Q}_c\}$ represents the real part of the Hermitian matrix

$$\mathbf{Q}_c \triangleq \begin{bmatrix} \mathbf{G}^H \mathbf{G} & -\mathbf{G}^H \mathbf{y} \\ -\mathbf{y}^H \mathbf{G} & 0 \end{bmatrix}. \tag{21}$$

Upon defining $\mathbf{x} \triangleq \left[\mathbf{p}^T\ t\right]^T$ and $\mathbf{Q} \triangleq \Re\{\mathbf{Q}_c\}$, (20) may be written in the following homogeneous quadratic form

$$\begin{aligned} \min \quad & \mathbf{x}^T \mathbf{Q} \mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in \{+1, -1\}^{4N_T+1}, \end{aligned} \tag{22}$$

where $\mathbf{Q}$ is a symmetric matrix. Since we have $\mathbf{x}^T\mathbf{Q}\mathbf{x} = \text{Trace}\left(\mathbf{Q}\mathbf{x}\mathbf{x}^T\right) = \text{Trace}\left(\mathbf{x}\mathbf{x}^T\mathbf{Q}\right)$, the problem of (22) may be equivalently rewritten as

$$\begin{aligned} \min \quad & \text{Trace}\,(\mathbf{X}\mathbf{Q}) \\ \text{s.t.} \quad & \mathbf{X} \succeq 0, \\ & \text{rank}(\mathbf{X}) = 1, \\ & \text{diag}(\mathbf{X}) = \mathbf{e}_{4N_T+1}, \end{aligned} \tag{23}$$

where $\mathbf{X} = \mathbf{x}\mathbf{x}^T$, $\mathbf{x} \in \{+1, -1\}^{4N_T+1}$, diag$(\mathbf{X})$ is the vector composed by the diagonal elements of $\mathbf{X}$, $\mathbf{e}_{4N_T+1}$ is the "all-ones" vector of length $4N_T+1$, and $\mathbf{X} \succeq 0$ indicates that $\mathbf{X}$ is a symmetric and positive semidefinite (PSD) matrix. Due to the constraint of rank$(\mathbf{X}) = 1$, the problem (23) is non-convex, hence it is difficult to solve. However, by dropping the constraint of rank$(\mathbf{X}) = 1$, the problem of (23) may be *relaxed* to

$$\begin{aligned} \min \quad & \text{Trace}\,(\mathbf{X}\mathbf{Q}) \\ \text{s.t.} \quad & \mathbf{X} \succeq 0, \\ & \text{diag}(\mathbf{X}) = \mathbf{e}_{4N_T+1}. \end{aligned} \tag{24}$$

The problem of (24) is known as an instance of semidefinite programming (SDP) [4], which constitutes a more general class of optimization techniques than linear programming[8]. Additionally, since SDP is a subclass of convex optimization, it does not suffer from getting trapped in local minima[9].

B. DVA-SDPR Solving Method

The SDP problem of (24) is solved using the efficient primal-dual interior-point algorithm of [14], which guarantees a polynomial-

[7]This virtual MIMO system is not exactly equivalent to a real $(2N_T \times N_R)$-element MIMO system, because the left half and the right half of the virtual channel matrix $[2\mathbf{H}\ \ \mathbf{H}]$ are fully correlated, both relying on $\mathbf{H}$.

[8]Several standard optimization problems, such as linear and quadratic programming can be unified under the framework of SDP [4].

[9]This does not mean that the SDPR detector is always capable of achieving the optimal ML performance, because the problem of (24) is a relaxed version of the original ML optimization problem of (19).

time[10] worst-case complexity. The Lagrange dual problem associated with (24) is formulated as

$$\begin{aligned} \max \quad & \mathbf{e}_{4N_T+1}^T \mathbf{v} \\ \text{s.t.} \quad & \mathbf{Z} = \mathbf{Q} - \mathrm{Diag}(\mathbf{v}) \succeq 0, \end{aligned} \tag{25}$$

where Diag($\mathbf{v}$) represents a diagonal matrix with its diagonal elements being $\mathbf{v}$.

When the objective function values of the primal problem (24) and of its dual problem (25) satisfy

$$\mathrm{Trace}\left(\mathbf{XQ}\right) - \mathbf{e}_{4N_T+1}^T \mathbf{v} \leq \max\left[1.0, \mathrm{abs}\left(\mathbf{e}_{4N_T+1}^T \mathbf{v}\right)\right] \times \epsilon, \tag{26}$$

the primal-dual interior-point algorithm is deemed to have converged, where the so-called convergence tolerance $\epsilon = 10^{-k}$ associated with an integer $k \geq 1$, controls the accuracy of convergence.

After obtaining the solution matrix $\mathbf{X}^\star$ of the problem (24), the solution vector $\mathbf{p}^\star$ of the problem (19) may be derived with the aid of several post-processing techniques [8], among which the simplest one is

$$\mathbf{p}^\star = \mathrm{sgn}(\mathbf{X}_{1:4N_T,4N_T+1}), \tag{27}$$

with $\mathbf{X}_{1:4N_T,4N_T+1}$ denoting the first $4N_T$ elements of the last column of $\mathbf{X}$. As shown by (12), the vector $\hat{\mathbf{u}} = (\mathbf{p}^\star + \mathbf{e}_{4N_T})/2$ contains half of the original information bit vector $\mathbf{u}^\star$. The remaining half of $\mathbf{u}^\star$ may be obtained from $\hat{\mathbf{u}}$ with the aid of the element-wise XOR operations of (15).

C. Performance Refinement Using Bit-Flipping

The proposed DVA-SDPR detector exhibits an unequal error protection for the bits in different positions of a single 16-QAM symbol[11]. This may be explained with the aid of (12), where the bits u_1 and u_3 are mapped to a 4-QAM constellation having a doubled amplitude. Inspired by this observation, corresponding to (17), each time we may flip the sign of the ith bit $p_i^\star$ of $\mathbf{p}^\star$, $i = 2N_T + 1, \cdots, 4N_T$, to obtain a modified solution vector $\tilde{\mathbf{p}}_i^\star$. There will be a total of $2N_T$ modified solution vectors. The final solution vector is chosen as the one, which minimizes $\|\mathbf{y} - \mathbf{Gp}\|_2^2$, when considering $\mathbf{p}^\star$ and $\tilde{\mathbf{p}}_i^\star$.

D. Complexity Analysis

The SDP problem of (24) involves a matrix variable $\mathbf{X}$ of size $(4N_T + 1) \times (4N_T + 1)$, which entails a computational complexity of $O\left[(4N_T + 1)^{3.5}\right]$, when employing the primal-dual interior-point algorithm of [14]. The complexity of the sgn($\cdot$) operations of (27), the XOR operations of (15) and the operations of the bit-flipping as well as the $2N_T$ Euclidean distance computations do not affect the complexity order. Hence the overall complexity of recovering the original information bit vector is on the order of $O\left[(4N_T + 1)^{3.5}\right]$.

V. Simulation Results

In this section, we characterize the achievable performance versus the computational complexity imposed by the proposed DVA-SDPR MIMO detector for the classic Gray-mapping aided 16-QAM modulation using Monte Carlo (MC) simulations. The average SNR per receive antenna is defined as

$$\mathrm{SNR} \triangleq 10\log_{10}\left(E\left\{\|\mathbf{Hs}\|^2/N_R\right\}/2\sigma^2\right) = 10\log_{10}\left(N_T/2\sigma^2\right). \tag{28}$$

The computational complexity is quantified in terms of the number of equivalent additions, denoted as N_{add}, required for decoding a single transmitted MIMO symbol vector. More explicitly, we have $N_{\text{add}} \triangleq \mathrm{E}\{T_{\text{tot}}\}/\mathrm{E}\{T_{\text{add}}\}$, where T_{tot} is the average time required for decoding a MIMO symbol vector, while T_{add} is the average computation-time per addition operation. Compared to the "execution-time" metric used in [9], this complexity metric has the advantage of being independent of different simulation platforms[12]. An (8×8)-element flat Rayleigh fading MIMO channel is considered, where the MIMO channel-matrix entries are chosen as independent and identically distributed (i.i.d.), zero mean, unit-variance complex-valued Gaussian random variables. Hence the system's total throughput is $8 \times 4 = 32$ bits/MIMO symbol vector. A new realization of the channel matrix is drawn for each transmitted symbol vector. Each element of the noise vector $\mathbf{n}$ is i.i.d. and $\mathcal{CN}(0, 2\sigma^2)$. Since it has been shown that the SDPR detectors of [8], [9] and [11] are equivalent in performance, below we will consider the index-bit-based VA-SDPR (IVA-SDPR) of [11] as one of the benchmarkers.

In Fig. 3, we contrasted the BER performance of the proposed DVA-SDPR (with or without bit-flipping) to that of these benchmarkers, namely to that of the IVA-SDPR of [11], of the minimum-mean-square-error–ordered-successive-interference-cancelation (MMSE-OSIC), and of the SD relying on an adaptive sphere radius for the sake of achieving the exact ML performance[13]. Observe in Fig. 3 that the proposed DVA-SDPR detector operating without bit-flipping achieves a BER performance identical to that of the IVA-SDPR benchmarker. By contrast, the bit-flipping aided DVA-SDPR outperforms the IVA-SDPR by about 2dB at BER $= 10^{-3}$ and BER $= 10^{-4}$. As expected, all the SDPR detectors considered exhibit a superior BER performance compared to the MMSE-OSIC detector. However, unlike in the BPSK scenario, where the SDPR detector achieves the maximum attainable diversity [6], in the 16-QAM scenario considered, the DVA-SDPR and IVA-SDPR detectors suffer from a considerable performance degradation in the high SNR region compared to the SD. This indicates that the SDPR detectors considered might not be able to achieve full diversity for the Gray-coded 16-QAM aided (8×8)-element MIMO fading channel.

In Fig. 4, we compared the complexity of the detectors considered in Fig. 3. It is readily seen that the SD imposed a significantly higher computational complexity in the low-SNR region than in the high-SNR region, which is consistent with the theoretical results of [3]. By comparison, the computational complexities of both the proposed DVA-SDPR detectors operating with and without bit-flipping as well as the IVA-SDPR detector are near-constant. More specifically, the DVA-SDPR dispensing with bit-flipping has a slightly lower complexity than the IVA-SDPR benchmarker, since the IVA-SDPR detector requires the computation of Eq. (4) plus the computation of 16 Euclidean distances for deciding upon each transmitted 16-QAM symbol, before proceeding to the information-bit decisions. On the other hand, the DVA-SDPR using bit-flipping imposes a computational complexity near-identical to that of the IVA-SDPR. Furthermore, the complexity of both the IVA-SDPR and the DVA-SDPR detectors is considerably lower than that of the SD detector, but still higher than that of the MMSE-OSIC detector.

To the best of our knowledge, in the uncoded Gray-mapping aided 16-QAM (8×8)-element MIMO scenario considered, the DVA-SDPR using bit-flipping achieves the best BER performance result among the known SDPR-aided MIMO detectors, while still maintaining a polynomially increasing worst-case complexity order of $O\left[(4N_T + 1)^{3.5}\right]$. Additionally, since the proposed DVA-SDPR detector directly generates the information-bit decisions without

[10] The complexity increases as a polynomial function of the problem size, which is determined by the number of rows (or columns) of the symmetric cost matrix $\mathbf{Q}$ of (24) in the considered context. Here $\mathbf{Q}$ is the input argument of the primal-dual interior-point algorithm of [14].

[11] Observe in Fig. 2 of Section V, the first and the third bits (resp. the second and the fourth bits), namely u_1 and u_3 (resp. u_2 and u_4) in a single 16-QAM symbol exhibit an identical BER performance, which is better (resp. worse) than the overall BER performance. Due to space limitation, Fig. 2 will not be explained again in Section V.

[12] For a given algorithm, both T_{tot} and T_{add} should be measured in the same simulation platform, where T_{add} serves as a normalizing unit.

[13] This SD is based on the classic SD of [1], and the minimum sphere radius was set to 2.

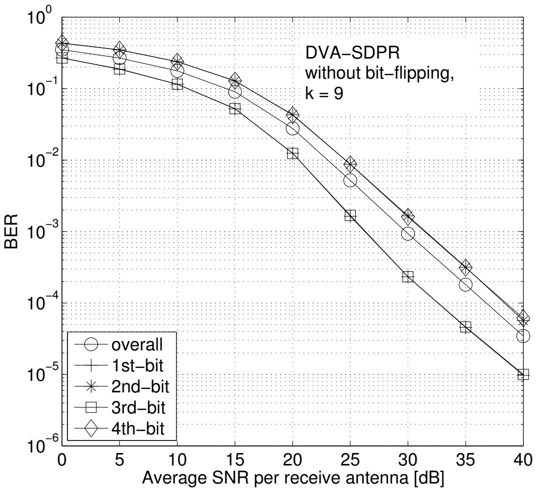

Fig. 2. Unequal error protection of the DVA-SDPR for Gray-coded 16-QAM aided (8×8)-element MIMO over uncorrelated flat Rayleigh fading channels, with the convergence tolerance $\epsilon = 10^{-9}$.

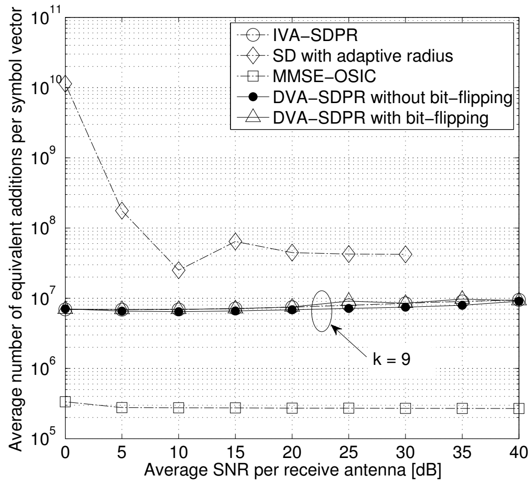

Fig. 4. Complexity comparison of the DVA-SDPR, IVA-SDPR, SD and MMSE-OSIC detectors for 16-QAM aided (8×8)-element MIMO over uncorrelated flat Rayleigh fading channels.

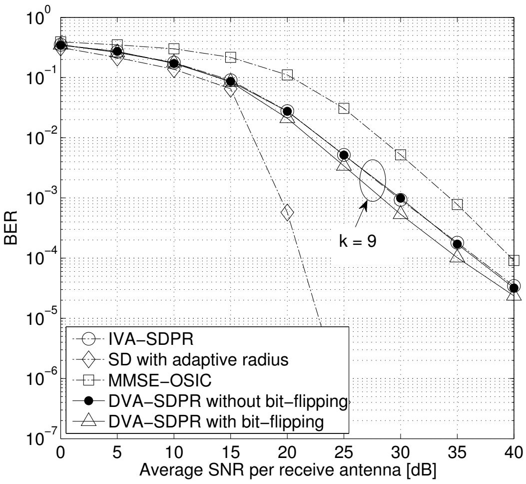

Fig. 3. Performance comparison of the DVA-SDPR, IVA-SDPR, SD and MMSE-OSIC detectors for 16-QAM aided (8×8)-element MIMO over uncorrelated flat Rayleigh fading channels.

first making symbol decisions, it may reduce the hardware cost in practical applications. In general, the DVA-SDPR, the IVA-SDPR and the MMSE-OSIC detectors may serve as efficient alternatives for the SD in the low-SNR region, say below about 15dB in the context considered. The SDPR detectors achieve full-diversity in a BPSK scenario, hence an interesting problem for future research is to conceive efficient SDPR detectors that can approach the ML performance for high-order QAM.

VI. Conclusions

In contrast to the existing IVA-SDPR detector, the proposed DVA-SDPR detector bypasses symbol-decisions and directly generates the information bits of classic Gray-mapping aided 16-QAM by employing a simple linear matrix representation (LMR) of 4-QAM. In principle the proposed method may be extended to general high-order rectangular QAM constellations. Based on this contribution, the MIMO detector and constellation demapper modules of high-order rectangular QAM using either linear natural mapping or nonlinear Gray mapping may be replaced by a single DVA-SDPR detector, which performs detection and demapping jointly. Furthermore, when combined with low-complexity bit-flipping based "hill climbing", the proposed DVA-SDPR detector achieves the best BER performance among the known SDPR-based detectors in the context considered, while still maintaining a polynomial-time worst-case complexity order of $O\left[(4N_T + 1)^{3.5}\right]$.

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