Simple and Accurate Error Probability Evaluation of Multiple-Input–Multiple-Output Systems Using Optimum Linear Combining

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***Abstract*—This correspondence motivates simple and accurate evaluation of the error probability of multiantenna multiple-input–multiple-output (MIMO) systems using optimum linear combining (OLC). Using both the first- and second-order statistics of the eigenvalues involved, we propose an Order-2 approximation approach (i.e., Gamma-approximation), which approximates the related eigenvalues as independent gamma-distributed random variables. Our analysis shows that the proposed Order-2 approximation results in simple formulas that are convenient to evaluate. The performance examples show that the error probability evaluated using the derived formulas is accurate and usually indistinguishable from that obtained by simulations.**

***Index Terms*—Digital modulation, maximum signal-to-interference-plus-noise ratio (MSINR), minimum mean-square error (MMSE), multiple-input multiple-output (MIMO), optimum linear combining (OLC), Rayleigh fading, space-division multiple access.**

I. Introduction

In wireless communications, linear combining (LC), which is called *linear multiuser detection* (MUD) when detection of multiple users is considered, is highly attractive, owing to its low complexity of implementation relative to nonlinear combining or nonlinear MUD [1]. Depending on the knowledge available to the receiver, LC may be implemented based on various optimization criteria [1]–[3], such as maximal ratio combining (MRC) or matched filtering, zero forcing (ZF) or decorrelating, minimum mean-square error (MMSE), maximum signal-to-interference-plus-noise ratio (MSINR), minimum variance distortionless response (MVDR), and minimum power distortionless response (MPDR). In these optimization criteria, the MRC scheme maximizes the signal-to-noise ratio (SNR) without being aware of the cochannel interference (CCI) [or multiuser interference (MUI)] [1]. By exploiting the knowledge about the cross correlation existing among the cochannels (users), the ZF scheme completely removes the CCI (MUI) but at a cost of noise amplification [1], [4]. The ZF scheme does not exploit the noise variance knowledge. Except for the preceding two criteria, all the other optimization criteria, as previously mentioned, make use of both the knowledge about the cross correlation among the cochannels (users) and that about noise variance [1], [2], [4], [5]. Although the starting points (cost functions) of these optimization criteria are different, all of them are capable of reaching the MSINR target and attaining the same error performance, as shown, e.g., in [2]. Since this family of LC schemes attain the MSINR and outperform both the MRC and ZF schemes, we usually refer to them as the family of optimum linear combining (OLC) schemes [6]–[11].

It has been shown that determining the closed-form expressions for evaluating the exact error performance of OLC is highly challenging. The exact bit error rate (BER) performance of OLC has been investigated in [12] in the context of direct-sequence code-division multiple-access systems using MMSE MUD, demonstrating that the detector's outputs can usually be closely approximated by independent Gaussian random variables. By contrast, in most other references, e.g., in [5]–[11] and [13], the error performance of the OLC has been studied under the assumption that the OLC's outputs can be approximated as independent Gaussian random variables and can independently be decoded.

In this paper, we motivate to analyze the error probability of multiantenna MIMO systems using OLC and to derive simple formulas that are convenient to evaluate but provide results that are as accurate as possible. As shown in [9]–[13], determining the closed-form expressions for evaluating the exact error probability of multiantenna MIMO systems using OLC is still highly involving, even under the aforementioned Gaussian-approximation. For this sake, approximation methodologies [9], [11] have been proposed for obtaining formulas that are simple to evaluate. Specifically, Pham and Balmain [11] suggested to approximate the eigenvalues of the interfering signals' autocorrelation matrix by their average values obtained by simulations. It has been shown [9], [11] that this approach results in simple expressions that are convenient to evaluate and generally yields error performance that is close to that obtained by simulations. However, the error probability evaluated by this approach is not always accurate, particularly, when the SNR is relatively high.

In this paper, we propose Gamma-approximation for the eigenvalues of the interfering signals' autocorrelation matrix, i.e., approximate the related eigenvalues as independent gamma-distributed [14] random variables. Since the gamma distribution makes use of both the first- and second-order statistics of eigenvalues, we hence refer to our approach as the Order-2 approximation. By contrast, the approach proposed in [11] only uses the first-order statistics of eigenvalues; it is hence correspondingly referred to as the "Order-1 approximation." As our forthcoming discourse shows, the Order-2 approximation results in simple formulas that are very easy to evaluate; furthermore, the error probability evaluated is indistinguishable from that evaluated by simulations for the scenarios considered. Additionally, in this paper, we explain in detail the principles behind the Order-1 approximation and also explain why the Order-1 approximation is effective for some cases.

The rest of this paper is organized as follows: Section II forms the problems to solve. Section III considers the corresponding solutions. In Section IV, some performance results are illustrated, and finally, in Section V, we summarize the conclusion.

II. Problem Formulation

Let us consider a MIMO system that employs K transmit antennas representing the inputs and N receive antennas denoting the outputs. We assume that the channel from any of the transmit antennas to any of the receive antennas experiences independent identically distributed (i.i.d.) flat Rayleigh fading. Then, the MIMO equation describing the output–input relation can be expressed as

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{n} \tag{1}$$

where $\boldsymbol{y}$ and $\boldsymbol{n}$ are N-length complex-valued observation and noise vectors, respectively. We assume that the noise vector $\boldsymbol{n}$ obeys the

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multivariate Gaussian distribution with mean zero and a covariance matrix $E[\boldsymbol{n}\boldsymbol{n}^H] = \sigma^2 \boldsymbol{I}_N$, where $\boldsymbol{I}_N$ is an $(N \times N)$ identity matrix and $\sigma^2 = 1/(N\gamma_s)$, with γ_s denoting the average SNR per receive antenna. In (1), $\boldsymbol{x} = [x_1, x_2, \ldots, x_K]^T$ contains the K symbols transmitted, and it is assumed that $E[x_k] = 0$ and $E[|x_k|^2] = 1$. Finally, in (1), $\boldsymbol{H}$ is the $(N \times K)$ channel matrix given by

$$\boldsymbol{H} = [\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_K]. \tag{2}$$

We assume that each element of $\boldsymbol{H}$ obeys i.i.d. complex Gaussian distribution with zero mean and a variance of $1/2N$ per dimension, implying that the signature $\boldsymbol{h}_k$ for x_k has been normalized to satisfy $E[\boldsymbol{h}_k^H \boldsymbol{h}_k] = 1$. Note that this assumption, in turn, explains why there is a factor of N associated with σ^2.

When LC is considered, the decision variable for x_k can be expressed as

$$z_k = \boldsymbol{w}_k^H \boldsymbol{y}, \qquad k = 1, 2, \ldots, K \tag{3}$$

where, when OLC is employed, the weight vector $\boldsymbol{w}_k$ can be expressed as [6]

$$\boldsymbol{w}_k = \alpha \boldsymbol{R}_I^{-1} \boldsymbol{h}_k, \qquad k = 1, 2, \ldots, K \tag{4}$$

where α is a positive constant, and $\boldsymbol{R}_I$ is the $(N \times N)$ autocorrelation matrix of the interfering signals plus background noise, which can be expressed as

$$\boldsymbol{R}_I = \sum_{j \neq k}^{K} \boldsymbol{h}_j \boldsymbol{h}_j^H + \sigma^2 \boldsymbol{I}_N. \tag{5}$$

Note that the solution of (4) may be obtained based on various optimization criteria [2], [3]. As shown in [2], the solutions derived based on the MMSE, MVDR, MPDR, and MSINR criteria can all be expressed in the form of (4) associated with some specific values for α.

It has been shown in [5], [6], and [12] that the decision variable z_k obtained using OLC can closely be approximated as a Gaussian random variable distributed with mean x_k and a variance of $1/\boldsymbol{h}_k^H \boldsymbol{R}_I^{-1} \boldsymbol{h}_k$. Hence, the instantaneous signal-to-interference-plus-noise ratio (SINR) for detection of x_k is given by [6], [9]

$$\gamma = \boldsymbol{h}_k^H \boldsymbol{R}_I^{-1} \boldsymbol{h}_k, \qquad k = 1, 2, \ldots, K \tag{6}$$

owing to the assumption of $E[|x_k|^2] = 1$. Since the average error probability is the same for any of the K data streams, hence no subscript is attached with γ in (6).

Given the instantaneous SINR of (6), the symbol error rate (SER) of the MIMO systems using various coherent baseband modulation schemes may be evaluated through the formula [15]

$$T[a, b, g; \gamma] = a \int_0^{b\pi} \exp\left(-\frac{g\gamma}{\sin^2\theta}\right) d\theta, \qquad a, b, g > 0 \tag{7}$$

where the parameters a, b, and g are determined by the specific baseband modulation scheme considered. Specifically, for the binary phase-shift keying (BPSK), binary frequency-shift keying, multiple phase-shift keying (MPSK), and M-ary quadrature amplitude modulation (MQAM), which are coherent modulation schemes, the corresponding values for the parameters a, b, and g can be found in [15].

To evaluate the average SER of the MIMO systems using OLC, the instantaneous SINR seen in (7) must be averaged out. This can be done by integrating (7) with respect to the distribution $f(\gamma)$ of the instantaneous SINR of γ, which can be formulated as

$$\begin{aligned} T[a, b, g] &= \int_0^{\infty} T[a, b, g; \gamma]; f(\gamma)\, d\gamma \\ &= \int_0^{\infty} \left[a \int_0^{b\pi} \exp\left(-\frac{g\gamma}{\sin^2\theta}\right) d\theta \right] f(\gamma)\, d\gamma \\ &= a \int_0^{b\pi} \left[\int_0^{\infty} \exp\left(-\frac{g\gamma}{\sin^2\theta}\right) f(\gamma)\, d\gamma \right] d\theta \\ &= a \int_0^{b\pi} I(g; \theta)\, d\theta \end{aligned} \tag{8}$$

where, by definition

$$I(g; \theta) = \int_0^{\infty} \exp\left(-\frac{g\gamma}{\sin^2\theta}\right) f(\gamma)\, d\gamma. \tag{9}$$

As shown, e.g., in [9], [11], [13], and [16], in the error performance analysis for the MIMO systems using OLC, the most challenging problem to solve is to find efficient ways to evaluate (9), which is also the main objective to achieve in our forthcoming discourse.

Our analysis starts with the spectral decomposition of $\boldsymbol{R}_I^{-1}$ of the inverse autocorrelation matrix, yielding [3]

$$\boldsymbol{R}_I^{-1} = \sum_{n=1}^{K-1} \frac{\boldsymbol{\phi}_n \boldsymbol{\phi}_n^H}{\lambda_n + \sigma^2} + \sum_{n=K}^{N} \frac{\boldsymbol{\phi}_n \boldsymbol{\phi}_n^H}{\sigma^2} \tag{10}$$

where $\{\boldsymbol{\phi}_n\}$ are the orthogonal eigenvectors corresponding to the eigenvalues

$$\eta_n = \begin{cases} \lambda_n + \sigma^2, & \text{when } 1 \leq n \leq K-1 \\ \sigma^2, & K \leq n \leq N. \end{cases} \tag{11}$$

Furthermore, in (10) and (11), λ_n for $n = 1, 2, \ldots, K-1$ are the $(K-1)$ nonzero eigenvalues of $\sum_{j \neq k}^{K} \boldsymbol{h}_j \boldsymbol{h}_j^H$, as shown in (5). Upon substituting (10) into (6), we can express the instantaneous SINR as

$$\gamma = \sum_{n=1}^{K-1} \frac{|s_n|^2}{\lambda_n + \sigma^2} + \sum_{n=K}^{N} \frac{|s_n|^2}{\sigma^2} \tag{12}$$

where we have defined $s_n = \boldsymbol{h}_k^H \boldsymbol{\phi}_n$. Note that, in (12), s_n obeys the same distribution as h_{nk}, i.e., it obeys the i.i.d. complex Gaussian distribution with mean zero and a variance of $1/2N$ per dimension. This is because the vector $\boldsymbol{s} = [s_1, s_2, \ldots, s_N]^T$ is obtained by a unitary transform $\boldsymbol{\Phi} = [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_N]$ on $\boldsymbol{h}_k$, i.e., $\boldsymbol{s} = \boldsymbol{\Phi}^H \boldsymbol{h}_k$ [9].

Furthermore, when an overload MIMO system with $K \geq N+1$ is considered, the N eigenvalues of $\sum_{j \neq k}^{K} \boldsymbol{h}_j \boldsymbol{h}_j^H$, as shown in (5), are all nonzero eigenvalues. In this case, (10) is reduced to $\boldsymbol{R}_I^{-1} = \sum_{n=1}^{N} \boldsymbol{\phi}_n \boldsymbol{\phi}_n^H / (\lambda_n + \sigma^2)$, and (12) is simply represented by $\gamma = \sum_{n=1}^{N} |s_n|^2 / (\lambda_n + \sigma^2)$. Note that our following derivations are carried out based on (12) under the assumption of $N \geq K$. The expressions for the overload MIMO systems with $K \geq N+1$ can be obtained by simply setting $K - 1 = N$ in the corresponding expressions derived.

Let

$$\gamma_n = \frac{|s_n|^2}{\eta_n}, \qquad n = 1, 2, \ldots, N. \tag{13}$$

Then, we have

$$\gamma = \sum_{n=1}^{N} \gamma_n. \tag{14}$$

With the aid of (14), we can modify (9) to

$$I(g;\theta) = \underbrace{\int_0^\infty \cdots \int_0^\infty}_{N \text{ terms}} \prod_{n=1}^{N} \exp\left(-\frac{g\gamma_n}{\sin^2\theta}\right) \times f(\gamma_1, \gamma_2, \ldots, \gamma_N) d\gamma_1 d\gamma_2 \ldots \gamma_N. \tag{15}$$

Furthermore, owing to s_n for $n = K, \ldots, N$ being i.i.d. complex Gaussian random variables, γ_n for $n = K, \ldots, N$, as shown in (13), are i.i.d. exponentially distributed random variables obeying the common probability density function (pdf) $f(\gamma_n) = \bar{\gamma}_0^{-1} e^{-\gamma_n/\bar{\gamma}_0}$, where $\bar{\gamma}_0 = 1/(N\sigma^2)$. Applying this pdf in terms of $n = K, \ldots, N$ into (15) and completing correspondingly the integrations, we can simplify (15) to

$$I(g;\theta) = \left(\frac{\sin^2\theta}{g\bar{\gamma}_0 + \sin^2\theta}\right)^{N-K+1} \times \underbrace{\int_0^\infty \cdots \int_0^\infty}_{(K-1) \text{ terms}} \prod_{n=1}^{K-1} \exp\left(-\frac{g\gamma_n}{\sin^2\theta}\right) \times f(\gamma_1, \gamma_2, \ldots, \gamma_{K-1}) d\gamma_1 d\gamma_2 \ldots d\gamma_{K-1}. \tag{16}$$

As shown in (16), there are still $(K-1)$ terms corresponding to the $(K-1)$ nonzero eigenvalues to simplify. It has been illustrated, e.g., in [9], [10], and [13], that, without approximation, the closed forms in terms of these nonzero eigenvalues are very hard to derive. For this sake, approximation methodologies have been proposed for the simplification of (16) [9], [11]. Here, we try to simplify (16) by considering two approximation approaches, i.e., the Order-1 and Order-2 approximations, for the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{K-1}$, as shown in (10) and (11), respectively. Note that the Order-1 approximation is proposed in [11], which will be used in Section IV as a benchmark for the Order-2 approximation proposed in this contribution. Furthermore, along with the Order-2 approximation, we explain in detail the properties of the Order-1 approximation.

III. Approximation on Error Probability Analysis

In this section, we consider two approximation approaches, i.e., the Order-1 and Order-2 approximations, for simplifying (16). Both approaches result in simple formulas for (16), which are convenient to evaluate and yield the error performance having good agreement with that obtained by simulations.

A. Order-1 Approximation

The Order-1 approximation approximates the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{K-1}$ by their first-order statistics, i.e., by their average values expressed as $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_{K-1}$ [11]. In this case, γ_n in (13) can be expressed as $\gamma_n = |s_n|^2/(\bar{\lambda}_n + \sigma^2)$, and its pdf is given by $\gamma_n = \bar{\gamma}_n^{-1} e^{-\gamma_n/\bar{\gamma}_n}$, where $\bar{\gamma}_n = E[\gamma_n] = 1/N(\bar{\lambda}_n + \sigma^2)$. Upon applying this pdf into (16), it can readily be shown that [9]

$$I(g;\theta) = \left(\frac{\sin^2\theta}{g\bar{\gamma}_0 + \sin^2\theta}\right)^{N-K+1} \prod_{n=1}^{K-1} \left(\frac{\sin^2\theta}{g\bar{\gamma}_n + \sin^2\theta}\right). \tag{17}$$

Finally, substituting (17) into (8) yields

$$T[a,b,g] = a \int_0^{b\pi} \left(\frac{\sin^2\theta}{g\bar{\gamma}_0 + \sin^2\theta}\right)^{N-K+1} \times \prod_{n=1}^{K-1} \left(\frac{\sin^2\theta}{g\bar{\gamma}_n + \sin^2\theta}\right) d\theta \tag{18}$$

which is simple to evaluate.

Note that the Order-1 approximation is capable of obtaining a good approximation for the actual SER. However, the Order-1 approximation is far away from sufficient to embrace the statistics of the eigenvalues. The reason for these arguments will be stated in our forthcoming discourse.

B. Order-2 Approximation: Gamma-Approximation

As shown in [9] and [11], as well as in Section IV, the Order-1 approximation is capable of achieving good agreement with the actual SER performance, particularly when the SNR of γ_s is relatively low. However, the SER evaluated with the aid of (18) is not very accurate when γ_s is relatively high. The reason for this can be stated as follows: Observe from (12) that the term $\lambda_n + \sigma^2$, where $\sigma^2 = 1/N\gamma_s$, is dominated by σ^2 when the SNR γ_s is low. In this case, the SER evaluated using (18) should be very accurate. By contrast, when the SNR γ_s is high, resulting in that the term $\lambda_n + \sigma^2$ in (12) is dominated by λ_n, the statistics of λ_n then generate a relatively high impact on the SER performance of the MIMO system. Consequently, (18), which is obtained using only the first-order statistics of λ_n, may not provide accurate evaluation of the SER. In this case, higher order statistics of λ_n are required to achieve more accurate evaluation for the SER of the MIMO systems using OLC. Therefore, we propose to make use of both the first- and second-order statistics of λ_n, which are the mean $\bar{\lambda}_n$ and variance $\sigma^2_{\lambda_n}$.

Straightforwardly, when both the first- and second-order statistics are available, we may use the Gaussian-approximation on λ_n for $n = 1, 2, \ldots, K-1$. Unfortunately, the Gaussian-approximation does not lead to simple and closed-form solutions for $I(g, \theta)$ of (16). For this sake, we employ the Gamma-approximation and approximate $\eta_n = \lambda_n + \sigma^2$ in (12) as an independent random variable obeying the gamma distribution [14]

$$f(\eta_n) = \frac{1}{\Gamma(m_n)} \left(\frac{m_n}{\Omega_n}\right)^{m_n} \eta_n^{m_n - 1} \exp\left(-\frac{m_n \eta_n}{\Omega_n}\right), \quad \eta_n \geq 0, \quad n = 1, 2, \ldots, K-1 \tag{19}$$

where $\Omega_n = E[\eta_n] = \bar{\lambda}_n + \sigma^2$, and $m_n = \Omega_n^2/E[(\eta_n - \Omega_n)^2] = (\bar{\lambda}_n + \sigma^2)^2/\sigma^2_{\lambda_n}$. Note that $\sqrt{\eta_n}$ obeys the Nakagami-m distribution [17] associated with the parameters (m_n, Ω_n).

Since we approximate η_n as an independent random variable and $|s_n|^2$ is also an independent random variable, γ_n of (13) is hence an independent random variable. Using this fact, (16) can be simplified to

$$I(g;\theta) = \left(\frac{\sin^2\theta}{g\bar{\gamma}_0 + \sin^2\theta}\right)^{N-K+1} \prod_{n=1}^{K-1} G_n(g, \theta, \bar{\gamma}_n) \tag{20}$$

associated with defining

$$G_n(g, \theta, \bar{\gamma}_n) = \int_0^\infty \exp\left(-\frac{g\gamma_n}{\sin^2\theta}\right) f(\gamma_n)\, d\gamma_n \tag{21}$$

where $\bar{\gamma}_n = 1/N\Omega_n$, as shown later. To derive the closed-form expression for $G_n(g, \theta, \bar{\gamma}_n)$, we first need to find the pdf $f(\gamma_n)$ of $\gamma_n = |s_n|^2/\eta_n$, where $|s_n|^2$ has the pdf of $f_{|s_n|^2}(y) = Ne^{-Ny}$, $y \geq 0$, and η_n follows the pdf of (19). According to [18], the pdf of γ_n can be derived using the formula

$$f(\gamma_n) = \int_0^\infty \eta_n f_{|s_n|^2}(\eta_n \gamma_n) f(\eta_n)\, d\eta_n. \tag{22}$$

Upon applying the related pdfs into (22) and simplifying it, we obtain

$$f(\gamma_n) = \frac{m_n(\bar{\gamma}_n m_n)^{m_n}}{(\gamma_n + \bar{\gamma}_n m_n)^{m_n+1}}, \qquad \gamma_n \geq 0 \tag{23}$$

where, by definition, $\bar{\gamma}_n = 1/N\Omega_n$, as previously mentioned.

It can be shown that, when $m_n \to \infty$, implying that η_n is a constant, as expected, the pdf of (23) is reduced to

$$\lim_{m_n \to \infty} f(\gamma_n) = \frac{1}{\bar{\gamma}_n} e^{-\gamma_n/\bar{\gamma}_n}, \qquad \gamma_n \geq 0 \tag{24}$$

which is a scaled pdf of $|s_n|^2$.

When substituting (23) into (21) and simplifying it, we obtain

$$G_n(g, \theta, \bar{\gamma}_n) = m_n(\bar{\gamma}_n m_n)^{m_n} \int_0^\infty (\gamma_n + \bar{\gamma}_n m_n)^{-(m_n+1)} \times \exp\left(-\frac{g\gamma_n}{\sin^2\theta}\right) d\gamma_n \tag{25}$$

which, using [19], can further be simplified to

$$G_n(g, \theta, \bar{\gamma}_n) = m_n \left(\frac{g\bar{\gamma}_n m_n}{\sin^2\theta}\right)^{m_n} \Gamma\left(-m_n, \frac{g\bar{\gamma}_n m_n}{\sin^2\theta}\right) \times \exp\left(\frac{g\bar{\gamma}_n m_n}{\sin^2\theta}\right), \qquad n = 1, 2, \ldots, K-1 \tag{26}$$

where $\Gamma(t, x)$ is the (upper) incomplete Gamma-function [19]. Finally, using the relation (8.358) in [19] for the incomplete Gamma-function, we can attain a simple form

$$G_n(g, \theta, \bar{\gamma}_n) = m_n \varphi\left(-m_n, \frac{g\bar{\gamma}_n m_n}{\sin^2\theta}\right), \qquad n = 1, 2, \ldots, K-1 \tag{27}$$

where $\varphi(t, x)$ is defined by

$$\varphi(t, x) = \cfrac{1}{x + \cfrac{1-t}{1 + \cfrac{1}{x + \cfrac{2-t}{1 + \cfrac{2}{x + \cdots}}}}} \tag{28}$$

which can usually be evaluated efficiently.

From (27) and (28), it can readily be shown that, when $m_n \to \infty$, we have

$$\lim_{m_n \to \infty} G_n(g, \theta, \bar{\gamma}_n) = \left(\frac{\sin^2\theta}{g\bar{\gamma}_n + \sin^2\theta}\right), \qquad n = 1, 2, \ldots, K-1 \tag{29}$$

which is the solution when treating λ_n as a constant.

Finally, a desired formula for evaluation of the MIMO systems' SER can be obtained by substituting (20) associated with (27) into

TABLE I
VALUES OF MEAN $\bar{\lambda}_n$, VARIANCE $\sigma^2_{\lambda_n}$, AND m_n OBTAINED BY SIMULATIONS OF 100 000 REALIZATIONS AT THE SNR OF $\gamma_s = 10$ dB

$N = 4,\ K = 4$			$N = 8,\ K = 8$		
$\bar{\lambda}_n$	$\sigma^2_{\lambda_n}$	m_n	$\bar{\lambda}_n$	$\sigma^2_{\lambda_n}$	m_n
0.197429	0.0187015	2.64549	0.0454903	0.000972805	3.45689
0.789125	0.103751	6.38836	0.172967	0.00470041	7.31806
2.01608	0.409897	10.1635	0.390438	0.0124527	13.0381
			0.711059	0.026164	20.0098
			1.15841	0.0501299	27.3495
			1.78386	0.0960667	33.5902
			2.74147	0.221071	34.3072

TABLE II
VALUES OF MEAN $\bar{\lambda}_n$, VARIANCE $\sigma^2_{\lambda_n}$, AND m_n OBTAINED BY SIMULATIONS OF 100 000 REALIZATIONS AT THE SNR OF $\gamma_s = 10$ dB

$N = 16,\ K = 16$			$N = 16,\ K = 16$		
$\bar{\lambda}_n$	$\sigma^2_{\lambda_n}$	m_n	$\bar{\lambda}_n$	$\sigma^2_{\lambda_n}$	m_n
0.0109596	5.75062e-05	5.15026	0.874936	0.00984884	78.8407
0.0412856	0.000266172	8.48938	1.10426	0.0134714	91.5448
0.0917694	0.000674429	14.2458	1.37304	0.0185243	102.7
0.163441	0.00131158	21.9544	1.69041	0.0252647	113.94
0.256353	0.0022044	31.283	2.07091	0.0357103	120.822
0.372291	0.00340697	42.0588	2.54781	0.0547904	119.058
0.512607	0.00500186	53.8225	3.21249	0.102824	100.758
0.679172	0.00711556	66.0249			

(8), yielding

$$T(a, b, g) = a \int_0^{b\pi} \left(\frac{\sin^2\theta}{g\bar{\gamma}_0 + \sin^2\theta}\right)^{N-K+1} \times \prod_{n=1}^{K-1} m_n \varphi\left(-m_n, \frac{g\bar{\gamma}_n m_n}{\sin^2\theta}\right) d\theta. \tag{30}$$

As shown in Section IV, the SER evaluated based on (30) is very accurate and usually indistinguishable from that obtained by simulations.

C. Discussion

First, the average values of the eigenvalues λ_n for $n = 1, \ldots, K-1$ are required to be known to evaluate (18). By contrast, in addition to the average values, the variances of the eigenvalues λ_n for $n = 1, 2, \ldots, K-1$ are also needed to be known before evaluation of (30). In MIMO systems, the average values and variances of the eigenvalues are hard to obtain through analysis. However, for a given MIMO system, they can be obtained *a priori* through simulations. Specifically, for the MIMO systems using the parameters $(N = 4, K = 4)$, $(N = 8, K = 8)$, and $(N = 16, K = 16)$, the average values $\{\bar{\lambda}_n\}$, variance values $\{\sigma^2_{\lambda_n}\}$, and the corresponding values of $\{m_n\}$ at an SNR $\gamma_s = 10$ dB are given in Tables I and II. Note that $\{\bar{\lambda}_n\}$ and $\{\sigma^2_{\lambda_n}\}$ are independent of the SNR of γ_s, whereas $\{m_n\}$ can readily be calculated from the formula $\{m_n = (\bar{\lambda}_n + \sigma^2)^2/\sigma^2_{\lambda_n}\}$, where $\sigma^2 = 1/N\gamma_s$.

Second, (27) can readily be evaluated; considering three to five layers of (28) should be sufficient to give a SER that is indistinguishable from the actual SER.

Third, as shown in Tables I and II, generally, smaller eigenvalues associate with smaller $\sigma^2_{\lambda_n}$ and m_n values, whereas relatively bigger eigenvalues associate with relatively bigger $\sigma^2_{\lambda_n}$ and m_n values.[1]

[1]As shown in Table II, the value of m_n first increases when the eigenvalue increases until $m_n = 120.822$. Then, the value of m_n slightly decreases when the eigenvalue further increases. The reason for this tendency is not explicit. As shown by the formula $m_n = (\bar{\lambda}_n + \sigma^2)^2/\sigma^2_{\lambda_n}$, m_n is jointly determined by $\bar{\lambda}_n$, σ^2, and $\sigma^2_{\lambda_n}$.

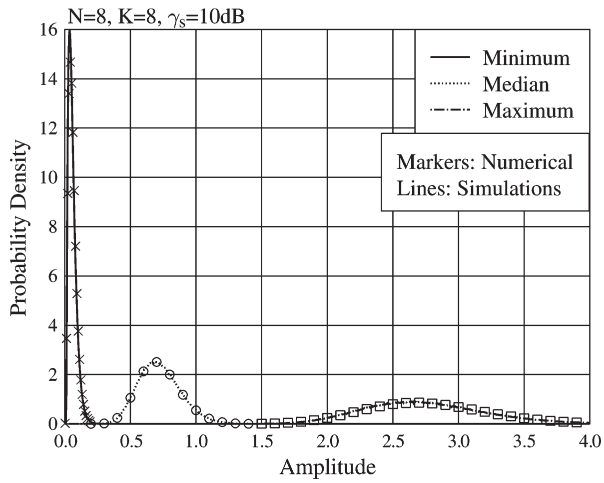

Fig. 1. PDF of η_0, η_3, and η_7 defined in (11), which correspond to the minimum, median, and maximum nonzero eigenvalues for a MIMO system with the parameters $N = 8$ and $K = 8$.

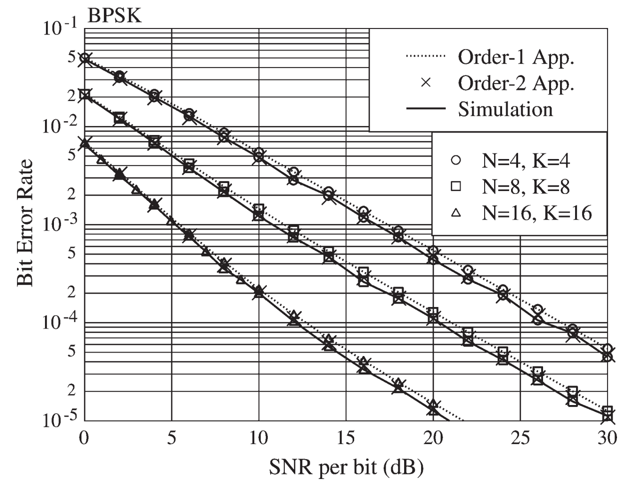

Fig. 2. BER performance of various MIMO systems using BPSK baseband modulation evaluated with the aid of the Order-1 approximation of (18), Order-2 approximation of (30), and simulations.

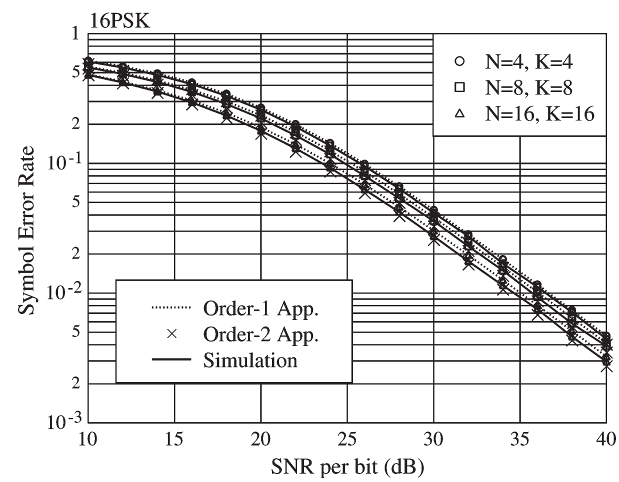

Fig. 3. SER performance of various MIMO systems using 16PSK baseband modulation evaluated with the aid of the Order-1 approximation of (18), Order-2 approximation of (30), and simulations.

Using this property and (12), we can explain why the Order-1 approximation is capable of attaining good agreement with the exact SER of the MIMO systems [9], [11]. Observe from (12) that only the relatively small-valued eigenvalues may generate noticeable effect on the SER of the MIMO systems. This is because the SNR $\gamma_n = |s_n|^2/(\lambda_n + \sigma^2)$ is small and may be ignorable when λ_n is big, in comparison with the other terms of $\gamma_i = |s_i|^2/\sigma^2$ or $\gamma_j = |s_j|^2/(\lambda_j + \sigma^2)$ associated with small λ_j values, as shown in (12). By contrast, for a small-valued eigenvalue λ_n having relatively small $\sigma^2_{\lambda_n}$ and m_n values, the distribution of λ_n, in principle, should impose an explicit impact on the SER performance. However, when σ^2 is small or the SNR γ_s is big, due to the fact that $\eta_n = \lambda_n + \sigma^2$ and that the variance $\sigma^2_{\lambda_n}$ of λ_n is small relative to σ^2, the variation of λ_n is overwhelmed by $(\bar{\lambda}_n + \sigma^2)$, making the effect imposed by the variation of λ_n insignificant. In this case, λ_n can also be approximated by its average $\bar{\lambda}_n$. Consequently, as shown in [9], [11], and Section IV, the Order-1 approximation presents us the SER that is also very close to the exact SER.

However, it can be implied from (12) and the preceding analysis that the small-valued eigenvalues will generate more impact on the SER performance, as the value of σ^2 decreases or the SNR γ_s increases. Hence, we may predict that the SER evaluated using the Order-1 approximation will become less accurate as the SNR γ_s goes higher.

Fourth, the Gamma-approximation yields near-exact approximations to the marginal pdfs of the eigenvalues λ_n or $\eta_n = \lambda_n + \sigma^2$ for $n = 1, \ldots, K-1$, as shown in Fig. 1. This is, in fact, not a coincidence. According to the matrix theory [20], $X = \sum_{n=1}^{K-1} \lambda_n = \sum_{k=1}^{K-1} \boldsymbol{h}_k^H \boldsymbol{h}_k = \sum_{k=1}^{K-1} \sum_{n=1}^{N} |h_{nk}|^2$. Hence, the sum X of the eigenvalues obeys the central χ^2-distribution—a special gamma distribution with an integer m_n value—with $m_n = 2(K-1)N$ degrees of freedom [21] when $\{h_{nk}\}$ are independent complex Gaussian random variables. As each of the eigenvalues constitutes a portion of X, it is likely that each individual eigenvalue also obeys the gamma distribution, owing to the property that the sum of the gamma-distributed random variables obeys gamma distribution [20]. Since the eigenvalues' distributions can be near-exactly approximated by the gamma distributions, as shown in Fig. 1, we can expect that the Order-2 approximation will result in more accurate performance evaluation than the Order-1 approximation.

Finally, we note that the Order-2 approximation assumes that the $(K-1)$ nonzero eigenvalues are independent random variables. However, the ordered eigenvalues are correlated [10], [16]. Nevertheless, as shown by the SER results in Section IV, there is no noticeable deviation from the SER obtained by Monte Carlo simulations, when without taking into account of the correlation among the ordered eigenvalues.

IV. Performance Results

In this section, a few examples are provided to illustrate the SER performance of the MIMO schemes using OLC when communicating over Rayleigh fading channels. The parameter values for N and K concerning the system size are shown in the corresponding figures.

In Figs. 2–4, the error performance for different-sized MIMO systems is evaluated when the baseband modulations of BPSK (Fig. 2), 16-state phase-shift keying (16PSK, Fig. 3), and 16-state QAM (16QAM, Fig. 4) are employed. As the results of these figures show, the Order-1 approximation is capable of providing good approximation of the actual error performance for all the MIMO schemes considered. By contrast, for all the MIMO and modulation schemes considered, the error performance evaluated based on the Order-2 approximation is usually indistinguishable from the actual achievable error performance.

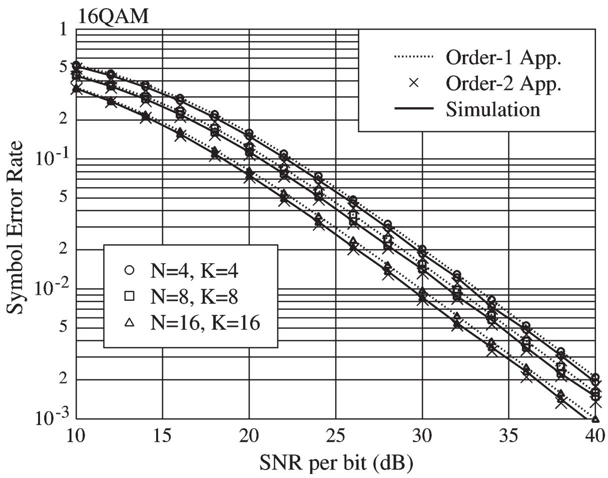

Fig. 4. SER performance of various MIMO systems using 16QAM baseband modulation evaluated with the aid of the Order-1 approximation of (18), Order-2 approximation of (30), simulations.

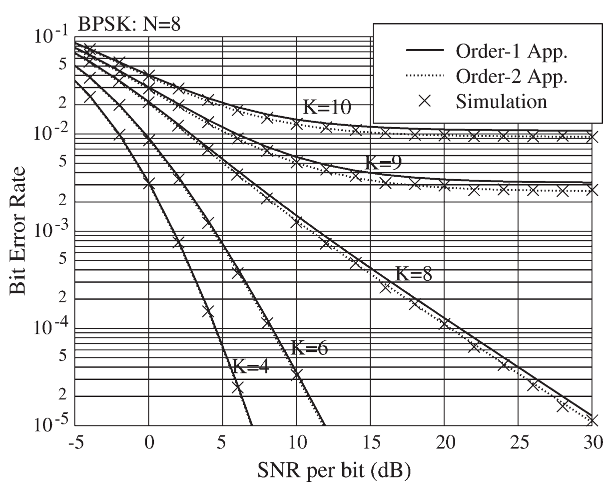

Fig. 5. BER performance of the MIMO systems using BPSK baseband modulation evaluated with the aid of the Order-1 approximation of (18), Order-2 approximation of (30), or simulations.

In the preceding figures, the full-loaded $(N = K)$ MIMO systems are considered. By contrast, in Fig. 5, the error performance of the MIMO systems invoking different loads of $K = 10, 9, 8, 6, 4$ is evaluated. The results show that the BER evaluated using the Order-1 and Order-2 approximations converges as the value of K decreases. However, as the value of K increases from the underload $(K < N)$ cases to the full-load and overload $(K > N)$ cases, the Order-1 approximation becomes less accurate. However, as shown in Fig. 5, the BER evaluated for these cases by the Order-2 approximation is still indistinguishable from that obtained by simulations.

V. Conclusion

An Order-2 approximation approach has been proposed for attaining simple expressions, which are convenient for evaluating the error performance of multiantenna MIMO systems using OLC and various digital modulation schemes. It has been illustrated that the error probability evaluated by the derived formulas is very accurate, compared with that obtained by simulations. Furthermore, in this paper, the principles behind the Order-1 approximation have been analyzed in detail. It can be implied from the analysis that the Order-1 approximation can only be applied to the scenarios where the eigenvalues impose insignificant impact since the Order-1 approximation does not reflect the eigenvalues' distributions. By contrast, owing to the fact that the Order-2 approximation is capable of closely approximating the distributions of the eigenvalues, it may hence be applied to various communications and signal-processing scenarios. Our future related work will endeavor to extend the Order-2 approximation to the MIMO systems where correlation exists among the transmit/receive antennas.

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