Automated Analysis of Weighted Voting Games

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ABSTRACT

Weighted voting games (WVGs) are an important mechanism for modeling scenarios where a group of agents must reach agreement on some issue over which they have different preferences. However, for such games to be effective, they must be well designed. Thus, a key concern for a mechanism designer is to structure games so that they have certain desirable properties. In this context, two such properties are PROPER and STRONG. A game is PROPER if for every coalition that is winning, its complement is not. A game is STRONG if for every coalition that is losing, its complement is not. In most cases, a mechanism designer wants games that are both PROPER and STRONG. To this end, we first show that the problem of determining whether a game is PROPER or STRONG is, in general, NP-hard. Then we determine those conditions (that can be evaluated in polynomial time) under which a given WVG is PROPER and those under which it is STRONG. Finally, for the general NP-hard case, we discuss two different approaches for overcoming the complexity: a deterministic approximation scheme and a randomized approximation method.

1. INTRODUCTION

Weighted voting games (WVGs) provide a way of modeling scenarios where a group of agents must reach agreement on some issue over which they have different preferences [8]. They have long been studied in game theory, and have more recently been used by researchers in multi-agent systems [3]. In a WVG, the individual players are assigned weights (a player's weight is the number of votes he or she has). In such a game, the players resolve an issue by voting for or against it. Those in favor of the issue form one coalition and those against form another. The issue is then resolved in favor of the coalition whose cumulative weight of players exceeds a given quota. Such a coalition is a winning coalition. A coalition that is not winning is a losing coalition.

A WVG game can be viewed from two perspectives: that of an individual voter and that of a game designer. From the former's perspective, a key concern is how much influence it has on turning a losing coalition into a winning one. This influence is measured in terms of power indices such as the Shapley value [9] and the Banzhaf index [2]. But from the game designer's perspective, a key concern is to structure games with specific properties such as ensuring that a coalition and its complement[1] cannot simultaneously be winning, or that a coalition and its complement cannot simultaneously be losing. In this context, two important properties are PROPER and STRONG. A game is PROPER if for every coalition that is winning, its complement is not. A game is STRONG if for every coalition that is losing, its complement is not. Games that are not PROPER are typically of little practical interest because they allow disjoint winning coalitions, giving rise to the peculiar possibility of two winning coalitions forming independently and simultaneously. This can lead to contradictory decisions being made by a voting body. In the same way, games that are not STRONG allow for two disjoint losing coalitions to form simultaneously and independently, thereby giving rise to the possibility of leaving some issues unresolved. A mechanism designer therefore wants games that are PROPER and STRONG.

To date, however, the analysis of games from a mechanism designer's perspective has received little attention. This paper therefore aims to fill this gap by asking the following key questions:

- is a given WVG PROPER?, and
- is it STRONG?

Finding answers to the above questions is crucial because most existing work on WVGs (including the very definitions of main power indices like the Shapley value and the Banzhaf index) is based on the assumption that games are PROPER and STRONG. This is a very strong assumption because, as we will show, it is computationally hard to answer the above questions.

This paper makes a number of important contributions to the state of the art. First, and most importantly, for the first time, we show that the problem of finding whether a given WVG satisfies any of the above mentioned properties is, in general, computationally hard. Second, we determine those problem instances for which it can be decided in polynomial time whether the game is PROPER or STRONG. Finally, for the general NP-hard case, we discuss two different approximation methods for overcoming the complexity.

The rest of the paper is structured as follows. Section 2 provides background. Section 3 shows that the problem of determining whether a WVG is PROPER or STRONG is NP-hard. Section 4 determines

[1]The complement of a coalition is the set of players that are in the game but not in the coalition.

those instances for which this problem is computationally easy. Section 5 discusses approximation methods. Section 6 concludes.

2. BACKGROUND

For a set $N = \{1, \ldots, n\}$ of players, a WVG is represented as a tuple of the form $\langle q; w_1, \ldots, w_n \rangle$. Here, $w_i \in R_+$ denotes player i's weight, and q the quota. The value of a coalition X is given by a function v defined as follows:

$$v(X) = \begin{cases} 1 & \text{If } w(X) = \sum_{x \in X} w_x \geq q \\ 0 & \text{otherwise} \end{cases}$$

A coalition with value zero is a *losing* coalition and with value one a *winning* coalition.

A common initial intuition from the above context may well be that there are only two kinds of subsets of N: the winning coalitions and the losing ones. Moreover, one may tend to think (appropriately) that the former are the *large* subsets of N, and the latter are the *small* subsets of N. However, there is a secondary type of largeness that plays an important role in both the real world and in theory. Specifically, coalition X is said to be a *blocking coalition* [10] if X' (the complement of X) is losing. Thus, X is blocking if it corresponds to a collection of voters that can prevent an issue from being passed.

EXAMPLE 1. *Let the voting game be $\langle 4; w_1 = 1, w_2 = 2, w_3 = 2, w_4 = 3 \rangle$. Then $\{1, 4\}$ is a winning coalition that is not blocking because its complement $\{2, 3\}$ is a winning coalition. On the other hand, $\{3, 4\}$ is a winning coalition that is blocking because its complement $\{1, 2\}$ is losing.*

Given this background, a more appropriate intuition, then, is that there are four kinds of coalitions of N, arising from a consideration of both X and its complement. These are as follows [10, p 15]:

1. A coalition X is LARGE if X is both winning and blocking (i.e., if X is winning and X' is losing).
2. A coalition X is W-MEDIUM if X is winning but not blocking (i.e., if X is winning and X' is winning).
3. A coalition X is B-MEDIUM if X is blocking but not winning (i.e., if X is losing and X' is losing).
4. A coalition X is SMALL if X is neither winning nor blocking (i.e., if X is losing and X' is winning).

Based on the above coalition types, a voting game is said to have the property STRONG or PROPER. These properties are defined as follows [10, p 15]:

DEFINITION 1. *A WVG is* PROPER *if it has no* W-MEDIUM *coalitions, and* STRONG *if it has no* B-MEDIUM *coalitions.*

EXAMPLE 2. *Consider three games $G_1 = \langle 4; 1, 1, 1, 1, 1 \rangle$, $G_2 = \langle 2; 1, 1, 1, 1, 1 \rangle$, and $G_3 = \langle 3; 1, 1, 1, 1, 1 \rangle$. G_1 is* PROPER *because any winning coalition must have at least four players and the complement of such a coalition is losing. G_1 is not* STRONG *because any two player coalition is losing and its complement (i.e., a three player coalition) is also losing. G_2 is* STRONG*: the only losing coalitions are those containing a single player, and the complement of any single player coalition is winning. G_2 is not* PROPER *because any two player coalition is winning and its complement is also winning. Finally, G_3 is both* PROPER *and* STRONG.

3. TIME COMPLEXITY

We begin by defining key terms and then introduce problems (called WMC, CLW, BMC, and CLL) that will be used to determine whether a game is PROPER or STRONG.

DEFINITION 2. *Least winning coalition: Let $\mathcal{W}$ denote the set of all winning coalitions. The least winning coalition is a winning coalition with least weight (i.e., $S \in \mathcal{W}$ is a least winning coalition if, for all $s \in \mathcal{W}$, $w(S) \leq w(s)$).*

DEFINITION 3. *Largest losing coalition: Let $\mathcal{L}$ denote the set of all losing coalitions. The largest losing coalition is a losing coalition with maximum weight (i.e., $L \in \mathcal{L}$ is a largest losing coalition if, for all $l \in \mathcal{L}$, $w(L) \geq w(l)$).*

WMC:

Instance: A WVG (defined in terms of N, q and w).
Question: Is there a W-MEDIUM coalition in the game?

CLW:

Instance: A WVG (defined in terms of N, q and w).
Question (decision version): Is the complement of a least winning coalition winning?
Question (optimization version): Find a least winning coalition.

BMC:

Instance: A WVG (defined in terms of N, q and w).
Question: Is there a B-MEDIUM coalition in the game?

CLL:

Instance: A WVG (defined in terms of N, q and w).
Question (decision version): Is the complement of a largest losing coalition losing?
Question (optimization version): Find a largest losing coalition.

We first prove that the WMC problem is equivalent to CLW (Lemmas 1 and 2 together prove this), and BMC is equivalent to CLL (Lemmas 3 and 4 together prove this). Then in Theorem 1, we prove that WMC and BMC are NP-hard. In what follows, S denotes a least winning coalition, L a largest losing coalition, S' and L' their respective complements, and $|X|$ denotes the number of elements in a set X. We let $W = \sum_{i=1}^{n} w_i$.

LEMMA 1. *There is a* W-MEDIUM *coalition iff the complement of a least winning coalition is winning.*

PROOF 1. *If the complement of a least winning coalition is winning, then clearly there is a* W-MEDIUM *coalition. We now prove that if there is a* W-MEDIUM *coalition, the complement of a least winning coalition is winning. Let X and X' denote a* W-MEDIUM *coalition and its complement respectively. So both X and X' are winning, and so there are two possibilities with respect to $w(X)$: $w(X) = w(S)$ or $w(X) > w(S)$. For the former case ($w(X) = w(S)$), we get $w(X') = w(S')$. Since X' is winning, $w(S')$ must also be winning (i.e., the complement of a least winning coalition is winning). Now consider the latter case $w(X) > w(S)$. This implies $w(X') < w(S')$ because $w(X') = W - w(X)$. Since X' is winning, S' must also be winning. This proves that if there is a* W-MEDIUM *coalition, the complement of a least winning coalition is winning.* □

LEMMA 2. *There is no* W-MEDIUM *coalition iff the complement of a least winning coalition is losing.*

PROOF 2. *We first prove that if the complement of a least winning coalition is losing, there is no* W-MEDIUM *coalition. The weight of the complement of every winning coalition that is not a least wining coalition is lower than $w(S')$. So if S' is losing, then the complement of every winning coalition that is not a least winning coalition is losing. This proves that, if the complement of the least winning coalition is losing, there is no* W-MEDIUM *coalition.*

We now prove that if there is no W-MEDIUM *coalition, the complement of a least winning coalition is losing. If there is no* W-MEDIUM *coalition, then the complement of every winning coalition (including least winning) is losing.* □

LEMMA 3. *There is a* B-MEDIUM *coalition iff the complement of a largest losing coalition is losing.*

PROOF 3. *If the complement of a largest losing coalition is losing, then it is obvious that there is a* B-MEDIUM *coalition. We now prove that if there is a* B-MEDIUM *coalition, the complement of a largest losing coalition is losing. Let X and X' denote a* B-MEDIUM *coalition and its complement respectively such that $w(X) \geq w(X')$. So both X and X' are losing, and there are two possibilities with respect to $w(X)$: $w(X) = w(L)$ or $w(X) < w(L)$. For the former case, we get $w(X') = w(L')$ (where $w(X') = W - w(X)$ and $w(L') = W - w(L)$). Since X' is losing, L' must also be losing. Thus, the complement of a largest losing coalition is losing. For the latter case, we get $w(X') > w(L')$ (because $w(X) < w(L)$ and $w(X') = W - w(X)$ and $w(L') = W - w(L)$). Thus, if X' is losing, then so is L'. This proves that if there is a* B-MEDIUM *coalition, the complement of a largest losing coalition is losing.* □

LEMMA 4. *There is no* B-MEDIUM *coalition iff the complement of a largest losing coalition is winning.*

PROOF 4. *We first prove that if the complement of a largest losing coalition is winning, then there is no* B-MEDIUM *coalition. The weight of the complement of every losing coalition that is not a largest losing coalition is greater than $w(L')$. So the complement of every losing coalition that is not a largest losing coalition is winning. This proves that, if the complement of a largest losing coalition is winning, there is no* B-MEDIUM *coalition.*

We now prove that if there is no B-MEDIUM *coalition, then the complement of a largest losing coalition is winning. If there is no* B-MEDIUM *coalition, then the complement of every losing coalition (including a largest losing coalition) is winning.* □

THEOREM 1. *The decision problems* WMC, CLW, BMC, *and* CLL *are* NP-*hard.*

PROOF 1. *As per Lemmas 1 and 2, the problem* WMC *is equivalent to* CLW. *Also, as per Lemmas 3 and 4,* BMC, *is equivalent to* CLL. *Hence we will now prove that* CLW *and* CLL *are* NP-*hard. The optimization version of* CLW, *can be formulated as the following integer programming problem:*

$$\text{CLW:} \quad \textit{minimize} \quad \sum_{i=1}^{n} w_i x_i$$
$$\textit{subject to} \quad \sum_{i=1}^{n} w_i x_i \geq q \qquad x_i \in \{0, 1\}$$

Thus CLW *is nothing but the minimization version of the standard* SUBSET-SUM *problem which is already known to be* NP-*complete [4, 6, 7].*

Now consider the problem CLL *, which can be formulated as the following integer programming problem:*

$$\text{CLL:} \quad \textit{maximize} \quad \sum_{i=1}^{n} w_i x_i$$
$$\textit{subject to} \quad \sum_{i=1}^{n} w_i x_i < q \qquad x_i \in \{0, 1\}$$

Thus CLL *is clearly the maximization version of the standard* SUBSET-SUM *problem which is already known to be* NP-*complete [4, 6, 7]. Thus,* WMC, CLW, BMC, *and* CLL *are* NP-*hard.* □

As per Theorem 1, the problem of determining whether a WVG is STRONG or PROPER is, in general, NP-hard. So it is almost certainly not possible to devise computationally feasible ‘exact’ solutions to these problems. Hence, we will present ‘approximate’ solutions to them. Before doing so, however, we find those problem instances for which it is computationally easy to determine whether a voting game is PROPER or STRONG.

4. COMPUTATIONALLY EASY INSTANCES

Here we focus on those games where, $q > 0$, and for all i, $w_i > 0$. Also, we focus on those games that are neither *dictatorial* – this occurs if, for an i, $w_i \geq q$ – nor those where only the grand coalition wins (since such games are of little practical interest). For such games, depending on the relation between q and W, Theorem 2 classifies games into three types: those that are PROPER, those that are not PROPER, and those that may or may not be PROPER. Theorem 3 does the same for the property STRONG. Based on Theorems 2 and 3, Theorem 4 finds the conditions under which a game may be both PROPER and STRONG. Corollary 1 shows the time complexity of evaluating the conditions in each of these theorems.

THEOREM 2. *If, for a voting game $\langle q; w_1, \ldots, w_n \rangle$, $q > \frac{W}{2}$ the game is* PROPER. *If $q \leq \frac{W}{3}$, the game is not* PROPER. *If $\frac{W}{3} < q \leq \frac{W}{2}$, the game may or may not be* PROPER.

PROOF 2. *We first prove that games with $q > \frac{W}{2}$ are* PROPER. *For such games, $w(S) > \frac{W}{2}$. Since $w(S') = W - w(S)$, we have $w(S') < \frac{W}{2}$. So S' is a losing coalition. Thus, as per Lemma 2, such a game is* PROPER.

Next, we prove that games with $q \leq \frac{W}{3}$ are not PROPER. *Let $q = kW/3$ where $0 < k \leq 1$. For such a game, we have $w(S) \geq \frac{kW}{3}$ and so $w(S') \leq (W - \frac{kW}{3})$. We assume that such a game is* PROPER *and prove by contradiction that it is not. If the game is* PROPER, *then (as per Lemma 2) S' must be a losing coalition, i.e., $w(S') < q$ or $w(S') < \frac{kW}{3}$. This means that $w(S) > (W - \frac{kW}{3})$ which is a lower bound for $w(S)$. We will now obtain an upper bound for it. Consider S which must have at least two players, i.e., $|S| \geq 2$ since any winning coalition must have at least two players. Now, for $j \in S$, we have $w(S - \{j\})) < q$ (since S is a least winning coalition and so any proper subset of it must be a losing coalition). We sum this inequality for all subsets (of size $|S| - 1$) of S. There are $|S|$ such subsets. So the summed inequality is $\sum_{j \in S} w(S - \{j\}) < q|S|$. Simplifying the terms in $\sum_{j \in S} w(S - \{j\})$, we get $\sum_{j \in S} w(S - \{j\}) = (|S| - 1)w(S)$. This gives us $(|S| - 1)w(S) < q|S|$ or $w(S) < \frac{q|S|}{|S|-1}$. Since $|S| \geq 2$, we get $w(S) < 2q$ or $w(S) < 2kW/3$. So the lower and upper bounds for $w(S)$ are $w(S) > (W - \frac{kW}{3})$ and $w(S) < 2kW/3$. But the lower bound is always greater than the upper bound, i.e., $(W - \frac{kW}{3}) > 2kW/3$ since $0 < k \leq 1$. Thus, there is a contradiction with the assumption that games with $q \leq \frac{W}{3}$ are* PROPER. *So such games are not* PROPER.

Finally, we prove that games with $\frac{W}{3} < q \leq \frac{W}{2}$ may or may not be PROPER. *We do this by giving examples of those games that are* PROPER *and those that are not. We first give examples for the former. Consider games with $n = 3$ players and weights $w_1 = kq$ and $w_2 = w_3 = \frac{W-w_1}{2}$ where $\frac{W}{q} - 2 < k < 1$. We first prove that such a game*[2] *exists and then prove that it is* PROPER. *Given that $\frac{W}{3} < q \leq \frac{W}{2}$, we have $2 \leq \frac{W}{q} < 3$ or $0 \leq \frac{W}{q} - 2 < 1$. This proves that a k satisfying $\frac{W}{q} - 2 < k < 1$ exists. Next, we prove that none of the three players can win individually, i.e., for $1 \leq i \leq 3$, $w_i < q$. Since $k < 1$ and $w_1 = kq$, we get $w_1 < q$. Now, we are given that $\frac{W}{q} - 2 < k$, which can be rearranged as $\frac{W-kq}{2} < q$. Since $w_2 = w_3 = \frac{W-w_1}{2} = \frac{W-kq}{2}$, we get $w_2 < q$ and $w_3 < q$. Thus a game with $w_1 = kq$ and $w_2 = w_3 = \frac{W-w_1}{2}$ where $\frac{W}{q} - 2 < k < 1$ exists.*

We will now prove that S' is losing. For players 2 and 3, we have $w_2 + w_3 = W - w_1 = W - kq$. Given that $k < 1$ and $q \leq W/2$, we get $kq < W/2$. This implies that $W - kq > W/2$ or $W - kq > q$. Hence, $w_2 + w_3 > q$. This, together with the fact that none of the three players can win individually and all weights are greater than zero, gives us $|S| = 2$. But irrespective of which two player coalition is wining, we know that the complement of any two player coalition contains a single player and is therefore losing. Hence, the complement of least winning coalition is losing. So As per Lemma 2, the above defined game is PROPER.

We now give examples of games that are not PROPER. *Consider games with $n = 4$ players and $w_1 = w_2 = q/2$ and $w_3 = w_4 = w_1 + \frac{W-2q}{2}$. We prove that no single player can win on its own. Clearly players 1 and 2 cannot win individually. For player 3, we have $w_3 = w_1 + \frac{W-2q}{2} = \frac{W-q}{2}$. Given that $q > W/3$, we get $w_3 < W/3$ or $w_3 < q$. Thus, players 3 and 4 cannot win individually. We now prove that S' winning. Since $w_3 = \frac{W-q}{2}$ and $q \leq W/2$ (or $W \geq 2q$), we get $w_3 \geq q/2$ or $w_3 \geq w_1$. Since no single player can win individually, $w_1 = w_2 = q/2$, $w_3 \geq q/2$, and $w_4 \geq q/2$, we get $S = \{1, 2\}$. This implies $S' = \{3, 4\}$ and $w(S') \geq q$, i.e., S' is winning. So as per Lemma 1, such games are not* PROPER. □

[2] That is, a game that is neither dictatorial nor one in which only the grand coalition wins.

THEOREM 3. *If, for a voting game $\langle q; w_1, \ldots, w_n \rangle$, $q \leq \frac{W}{2}$ the game is* STRONG. *If $q > \frac{2W}{3}$, the game is not* STRONG. *If $\frac{W}{2} < q < \frac{2W}{3}$, the game may or may not be* STRONG.

PROOF *3. Consider games with $q \leq \frac{W}{2}$. To prove that such games are* STRONG, *we will show that L' is winning. Since $q \leq \frac{W}{2}$, we have $w(L) < \frac{W}{2}$, and therefore $w(L') > \frac{W}{2}$ or $w(L') > q$. In order to ensure that L' is a winning coalition, we must show that L' contains at least two players. Since no individual player can have a weight greater than or equal to q, we get $w_i < \frac{W}{2}$ for all i. This means that there must be at least two players in L'. Hence $|L'| \geq 2$ and $w(L') > q$, so, as per Lemma 4, such a game is* STRONG.

Now consider games with $q > \frac{2W}{3}$. We will prove by contradiction that such games are not STRONG. *Assume that the game is* STRONG. *This implies that L' is a winning coalition, i.e., $w(L') > \frac{2W}{3}$ (see Lemma 4). Therefore $w(L) < \frac{W}{3}$. Since no player can win individually, we get $w_i \leq \frac{2W}{3}$ for all i. Moreover, for $j \in L'$, it must be that $w_j > \frac{W}{3}$. Otherwise, (i.e., $w_j \leq \frac{W}{3}$), there will be a contradiction with the fact that L is a largest losing coalition (because $w(L) < \frac{W}{3}$ and $w_j \leq \frac{W}{3}$ means $w(L \cup \{j\}) < \frac{2W}{3}$, i.e., $L \cup \{j\}$ is losing so L cannot be a largest losing coalition). But $w_j > \frac{W}{3}$ means that $|L'| = 2$ because a winning coalition must have at least two players, and $w(L')$ cannot exceed W. Let the weights of the two players in L' be $w_1 = \frac{W}{3} + \delta_1$ and $w_2 = \frac{W}{3} + \delta_2$ where $0 < \delta_1 \leq \frac{W}{3}$ and $0 < \delta_2 \leq \frac{W}{3}$. Then we have $W = w_1 + w_2 + w(L)$ or $w(L) + w_1 = \frac{2W}{3} - \delta_2$. This means $L \cup \{1\}$ is losing which contradicts with the fact that L is a largest losing coalition. Thus $|L'| = 2$ is false. Or $w_j > \frac{W}{3}$ cannot be true for any value of $|L'|$. Hence a contradiction. So games with $q > \frac{2}{3}W$ are not* STRONG.

Finally, consider games with $\frac{W}{2} < q < \frac{2W}{3}$. We prove that such games may or may not be STRONG *by giving examples of those games that are* STRONG *and also those that are not* STRONG. *Consider games with $n = 4$ players and weights $w_1 = w_2 = q/2$ and $w_3 = w_4 = w_1 + \frac{W-2q}{2}$. We first show that none of the players can win individually. Clearly, this is true for player 1 nor 2. Given that $w_3 = w_1 + \frac{W-2q}{2} = \frac{q}{2} + \frac{W-2q}{2} = \frac{W-q}{2}$ and the fact that $q > W/2$ (i.e., $W < 2q$), we get $w_3 < q/2$. Hence neither player 3 nor 4 can win individually. We will now prove that L' is losing. To do so, we obtain a lower bound for w_3. Since $w_3 = \frac{W-q}{2}$ and we are given that $q < \frac{2W}{3}$, we get $w_3 > q/4$. This together with the fact that $w_1 = w_2 = q/2$ and $w_3 = w_4$ proves that every three player coalition is winning. So a possible largest losing coalition is $L = \{1, 3\}$. This gives $L' = \{2, 4\}$ which is a losing coalition. Thus the complement of L is losing, so as per Lemma 3, such a game is not* STRONG.

Next, we give an example for games that are STRONG. *Consider games with $n = 3$ players and weights $w_1 = w_2 = q/2$ and $w_3 = W - 2w_1$. We first prove that no individual player can win. Clearly, this is true for players 1 and 2. For player 3, we have $w_3 = W - 2w_1 = W - q$. This together with the fact that $q > W/2$ (i.e, $W < 2q$), gives $w_3 < q$. Thus no single player can win individually. We now prove that L' is winning. Given that $w_3 = W - q$ and $q < \frac{2W}{3}$ (i.e., $W > \frac{3q}{2}$), we get $w_3 > q/2$. This, together with the fact that $w_1 = w_2 = q/2$ shows that any two* *player coalition is winning. Hence $|L| = 1$, and since player 3 has the highest weight among all three players, we get $L = \{3\}$. Thus, $L' = \{1, 2\}$ which is a winning coalition. So, as per Lemma 4, such a game is* STRONG. $\square$

THEOREM 4. *If for a WVG, $q > \frac{2W}{3}$, the game is* PROPER *but not* STRONG. *If $q \leq \frac{W}{3}$, it is* STRONG *but not* PROPER. *If $\frac{W}{2} < q < \frac{2W}{3}$, it may be both* PROPER *and* STRONG.

PROOF *4. For $q > \frac{2W}{3}$ and $q \leq \frac{W}{3}$, the proof follows directly from Theorems 2 and 3. For $\frac{W}{2} < q < \frac{2W}{3}$, we know from Theorem 3 that a game may or may not be* STRONG. *So we will now show that for $\frac{W}{3} < q < \frac{2W}{3}$, a game may be both* PROPER *and* STRONG. *Theorem 3 showed that games with $n = 3$, $w_1 = w_2 = q/2$, and $w_3 = W - 2w_1$ are* STRONG. *We now show that these games are also* PROPER. *From Theorem 3, we know that $w_3 > q/2$. So $S = \{1, 2\}$, and $S' = \{3\}$. Since S' is losing, as per Lemma 2, such a game is* PROPER. $\square$

COROLLARY 1. *The conditions in Theorems 2, 3, and 4 can be evaluated in $\mathcal{O}(n)$ time.*

PROOF *1. It takes $\mathcal{O}(n)$ time to compute $W = \sum_{i=1}^{n} w_i$; so it is possible to evaluate the conditions in linear time.* $\square$

5. APPROXIMATION METHODS

We explore two possible approaches for determining whether a game is PROPER or STRONG. The first approach is to solve CLW and CLL. The second approach is to count the number of W-MEDIUM and B-MEDIUM coalitions in a game. For the latter approach, let CP and CS be two boolean functions that map any subset (i.e., a coalition) to 0 or 1. There are 2^n possible coalitions. If $\mathcal{N}$ denotes the set of all possible coalitions of N, we have CP : $\mathcal{N} \rightarrow \{0, 1\}$ and CS : $\mathcal{N} \rightarrow \{0, 1\}$. For $X \subseteq N$, CP$(X) = 1$ if both X and X' are winning, otherwise CP$(X) = 0$ (i.e., CP$(X) = v(X) \wedge v(X'))$. Also, CS$(X) = 1$ if both X and X' are losing, otherwise CS$(X) = 0$ (i.e., CS$(X) = \neg v(X) \wedge \neg v(X'))$. Then, we define two real valued functions P and S on the interval [0,1] as follows: P$(X) = (1/2^n) \sum_{s \subset N}$ CP(X) and S$(X) = (1/2^n) \sum_{s \subset N}$ CS(X). So P gives the average number of W-MEDIUM coalitions in a game and S gives the average numberof B-MEDIUM coalitions. Thus, if P$(X) = 0$, the game is PROPER and if P$(X) = 0$, the game is STRONG.

The problem of finding the number of W-MEDIUM/B-MEDIUM coalitions in a game is #P-hard (i.e., computing P or S is a #P-hard problem) because it is the counting[3] version of the NP-hard problem CLW/CLL. Hence we can only aim to find an approximation for P and S.

For the former approach (i.e., solving the problems CLW and CLL), we discuss a *deterministic* approximation scheme. For the latter (i.e., computing P or S), we discuss a sampling based *randomized* approximation method.

[3]Note that the function P/S gives the 'average' number of W-MEDIUM/B-MEDIUM coalitions in a game, the average taken over all possible coalitions. We consider average instead of the 'total' because a game is PROPER/STRONG if the total or average number of W-MEDIUM/B-MEDIUM coalitions is zero, but taking the average offers of ease of discussion of approximation method.

A Deterministic Approximation Scheme

As shown in Section 3, WMC is equivalent to CLW, and BMC is equivalent to CLL. Also, CLW is the minimization version of the standard SUBSET-SUM problem while CLL is its maximization version. Thus we will focus on finding an approximate solution to CLL, i.e., finding an approximate weight for a largest losing coalition.

A polynomial time approximation scheme was proposed in [7] for the standard SUBSET-SUM problem. For performance ratio $r = a/(a+1)$ where $a \geq 6$, this scheme has time complexity $\mathcal{O}(n^{a-3})$. This means that an approximate solution to the optimization version of CLW (or CLL) can also be found in the same time and with the same performance ratio. Given an approximate weight of a smallest winning coalition, one can determine whether its complement is winning (i.e., the game is PROPER) or not (i.e., the game is not PROPER). Likewise, one can determine whether a game is STRONG or not.

A Randomized Approximation Method

Here, we seek to find approximate solutions to the functions P and S defined above. The proposed approximation algorithm builds upon the sampling method proposed in [1] for finding a player's approximate Banzhaf index (BI) for a WVG. A player's BI depends on the number of coalitions in which it is critical, out of all possible coalitions that contain the player. Specifically, player i's BI is $\beta_i = \frac{1}{2^{n-1}} \sum_{X \subset N | i \in X} [v(X) - v(X \backslash \{i\})]$. We first describe how $\hat{\beta}_i$ (an approximate for β_i) is computed in [1], and then show how this method can be extended to compute $\hat{\text{P}}$ (an approximate for P) and $\hat{\text{S}}$ (an approximate for S).

In [1], coalitions X containing i are randomly sampled and $\hat{\beta}_i$ is computed based on the proportion of sampled coalitions where $v(X) - v(X \backslash \{i\})$ is one. When sampling coalitions, each sample has a probability β_i of being a coalition where $v(X) - v(X \backslash \{i\})$ is one. So β_i can be approximated by taking several such samples. The number of samples determines the accuracy of the procedure: for a given $\epsilon > 0$, the probability δ of missing the correct β_i by more than ϵ depends on the number of samples taken. The method determines the number of samples (k) according to the required confidence level δ and the accuracy ϵ (i.e., the correct β_i lies in the interval $[\hat{\beta}_i - \epsilon, \hat{\beta}_i + \epsilon]$). Specifically, for confidence interval[4] $[\hat{\beta}_i - \sqrt{\frac{1}{2k} ln \frac{2}{\delta}}, \hat{\beta}_i + \sqrt{\frac{1}{2k} ln \frac{2}{\delta}}]$, the required number of samples is $k \geq \frac{1}{2\epsilon^2} ln \frac{2}{\delta}$. Although this method is simple, their empirical studies have shown that it performs well in terms of running time, accuracy, and confidence. Specifically, they showed that no deterministic algorithm can achieve comparable accuracy with a polynomial number of queries, and no randomized algorithm can achieve superpolynomial accuracy. Given this, and the similarity between the definitions of β_i and P (or S), we now show how this method can be extended to find $\hat{\text{P}}$ and $\hat{\text{S}}$.

[4]This a a conservative confidence interval because it is based on Hoeffding's [5] bound which is an exact rather than an approximate bound.

Algorithm 1 $\hat{\text{P}}(n, q, w, \epsilon, \delta)$

1: $count \Leftarrow 0$; $k \Leftarrow 0$;
2: **while** $k \leq \frac{ln\frac{2}{\delta}}{2\epsilon^2}$ **do**
3: Choose a random coalition X;
4: $k \Leftarrow k + 1$;
5: **if** $(v(X) \wedge v(X')) = 1$ **then**
6: $count \Leftarrow count + 1$
7: **end if**
8: **end while**
9: $\hat{\text{P}} \Leftarrow count/k$;
10: ConfidenceInterval $\Leftarrow [\hat{\text{P}} - \sqrt{\frac{1}{2k}ln\frac{2}{\delta}}, \hat{\text{P}} + \sqrt{\frac{1}{2k}ln\frac{2}{\delta}}]$
11: **return** $\hat{\text{P}}$ and ConfidenceInterval

Comparing β_i with P, we see that the former requires computing $v(X) - v(X \backslash \{i\})$ for all possible coalitions containing i, while the latter requires computing $v(X) \wedge v(X')$ or $\neg v(X) \wedge \neg v(X')$ for all possible coalitions. Also, for a given coalition X containing i, it takes $\mathcal{O}(n)$ time to compute $v(X) - v(X \backslash \{i\})$. This is also the time it takes to compute $v(X) \wedge v(X')$ or $\neg v(X) \wedge \neg v(X')$ for any X. Thus, by using the method of [1] to compute $\hat{\text{P}}$, we get the same confidence interval and the same number of required samples as that for $\hat{\beta}_i$. Specifically, the confidence interval for $\hat{\text{P}}$ is $[\hat{\text{P}} - \sqrt{\frac{1}{2k}ln\frac{2}{\delta}}, \hat{\text{P}} + \sqrt{\frac{1}{2k}ln\frac{2}{\delta}}]$, and that for $\hat{\text{S}}$ is analogous. Likewise, the number of required samples for $\hat{\text{P}}$ or $\hat{\text{S}}$ is $k \geq \frac{1}{2\epsilon^2}ln\frac{2}{\delta}$.

The method for computing $\hat{\text{P}}$ is described in Algorithm 1. The while loop in line 2 is repeated for k samples. Each time a sample coalition X is drawn and $count$ is incremented if X is W-MEDIUM. Finally, line 9 gives the average number of W-MEDIUM coalitions as an approximate for $\hat{\text{P}}$. The method for computing $\hat{\text{S}}$ is analogous to Algorithm 1.

Comparing the approximation scheme with approximation by sampling, we note the following key differences. First, the former method is deterministic in that it is guaranteed to have performance ratio $r = a/(a+1)$ for $a \geq 6$, while the latter method is randomized and may, with probability δ, generate an approximate that lies outside the confidence interval. Second, in terms of accuracy, the two methods can be made to achieve comparable performance. By varying a for the deterministic method, it is possible to achieve an accuracy that is comparable to that of the randomized method (although the running time for the former is $\mathcal{O}(n^{a-3})$ while that for the latter is $\mathcal{O}(\frac{ln\frac{2}{\delta}}{2\epsilon^2})$).

6. CONCLUSIONS AND FUTURE WORK

This paper focussed on two desirable properties of WVGs: PROPER and STRONG and showed that the problem of determining whether a game has either of these properties is, in general, an NP-hard. Then, we determined those conditions (that can be evaluated in polynomial time) under which a game is PROPER and those under which it is STRONG. We also determined conditions under which a game may be both PROPER and STRONG. Finally, for the general NP-hard case, we discussed two different approximation methods for overcoming the complexity.

Possible avenues for future work include extending this work to more general k-vector WVGs, which are intersections of k different WVGs.

References

[1] Y. Bachrach, E. Markakis, E. Resnick, A. Procaccia, J. Rosenschein, and A. Saberi. Approximating power indices: Theoretical and empirical analysis. *Journal of Autonomous Agents and Multi-Agent Systems: To appear*, 20(2):105–122, 2010.

[2] J. F. Banzhaf. Weighted voting doesn't work: A mathematical analysis. *Rutgers Law Review*, 19:317–343, 1965.

[3] S. S. Fatima, M. Wooldridge, and N. R. Jennings. A linear approximation method for the Shapley value. *Artificial Intelligence Journal*, 172(14):1673 – 1699, 2008.

[4] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman and Company, 1979.

[5] W. Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of American Statistical Association*, 58(301):13–30, 1963.

[6] H. Kellerer, U. Pferschy, and D. Pisinger. *Knapsack Problems*. Springer, 2004.

[7] S. Martello and P. Toth. Worst-case analysis of greedy algorithms for the subset sum problem. *Mathematical Programming*, 28:198–205, 1984.

[8] M. J. Osborne and A. Rubinstein. *A Course in Game Theory*. The MIT Press, 1994.

[9] L. S. Shapley. A value for n person games. In A. E. Roth, editor, *The Shapley value*, pages 31–40. University of Cambridge Press, Cambridge, 1988.

[10] A. D. Taylor and W. Z. Zwicker. *Simple games: Desirability relations, trading, pseudoweightings*. Princeton University Press, 1999.