Time-relevant $2D$ behaviors

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***Abstract*— In this work we present some results on $2D$ behaviors described by linear constant coefficient partial difference equations where one of the independent variables, "time", is distinguished and plays a special role. We call such systems 'time-relevant'. We first give a test to check time-relevance starting from an arbitrary description of the behavior. Then, we introduce a stability notion for these systems and provide an algebraic test in terms of the location of the zeros of the determinant of a polynomial matrix describing the system.**

I. INTRODUCTION

Systems described by linear constant coefficient PDE's or partial difference equations have received a great deal of attention in the last decades, and many interesting results have been obtained using the behavioral approach, see for instance [29], [20], [31]. Most of the contributions concern the case where all the independent variables are treated on an equal footing. However, it can be argued that in many applications, one of the independent variables is time, and the others are spatial variables.

In this paper we study discrete two dimensional ($2D$) systems with time being one of the independent variables. For this reason we call these systems *time-relevant* $2D$ *systems*.

Considering time-relevant $2D$ systems clearly implies the definition of a new type of stability, now naturally associated with the passage of "time".

After introducing some preliminaries notions in section II, in section III we present a definition of a time-relevant behavior and provide an algebraic test to check this property. We concentrate on autonomous behaviors, i.e., behaviors in which the trajectories are completely determined by their "initial conditions" in a special subset $\mathcal{S}$ of the independent variable domain. Roughly speaking, we shall say that a $2D$ system is time-relevant whenever it is autonomous and the special subset $\mathcal{S}$ is contained in the strict "past". The corresponding notion of time-relevant stability is then introduced in section IV. Our definition is motivated by the principle that a time-relevant system should be classified as unstable only when "finite energy" initial conditions give rise to trajectories whose "instant energy" does not go to zero as time goes to infinity. In this section we also provide an algebraic characterization of square autonomous [25] time-relevant stable systems in terms of the location of the zeroes of the determinant of any square polynomial matrix describing the system.

In the spirit of the approach used in this paper, some inspiring work has been already presented in [23], [24], [30], [27] for continuous independent variables. Note that this approach is quite different from the framework used for infinite-dimensional systems in, e.g., [1].

II. PRELIMINARIES

In this section we briefly cover preliminaries and some definitions in the behavioral approach. The notational aspects are also presented here.

Following the behavioral formalism, we denote with $\mathbb{W}^{\mathbb{T}}$ the set consisting of all maps from a set $\mathbb{T}$ to a set $\mathbb{W}$ and call $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ a $2D$ *linear shift-invariant partial difference behavior* if $\mathfrak{B}$ is the set of solutions of a finite system of constant-coefficient partial difference equations. We denote with $\mathfrak{L}^{\mathtt{w}}(\mathbb{Z}^2, \mathbb{R}^{\mathtt{w}})$ the set of all $2D$ linear shift-invariant partial difference behaviors with $\mathtt{w}$ variables, often denoted simply with $\mathfrak{L}^{\mathtt{w}}$.

The system of constant-coefficient partial difference equations describing $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ can be efficiently represented using polynomial matrices in two variables as follows. Denote with σ_i the *i-th shift operator*, defined for $i = 1$ as

$$\sigma_1 : (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \to (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$$
$$(\sigma_1 w)(k_1, k_2) := w(k_1 + 1, k_2) \; ,$$

and analogously for σ_2; the inverse shift operators σ_1^{-1} and σ_2^{-1} are defined in the obvious way. Then $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ if and only if there exist nonnegative integers M and L and matrices $R_{ij} \in \mathbb{R}^{p \times \mathtt{w}}$, $i, j = -L, \ldots, M$, such that

$$[w \in \mathfrak{B}] \Longleftrightarrow \left[\sum_{i,j=-L}^{M} R_{ij} \sigma_1^i \sigma_2^j w = 0 \right] \; .$$

Define the two-variable Laurent polynomial matrix $R(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1})$ in the indeterminates ξ_1 and ξ_2 as $R(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}) := \sum_{i,j=-L}^{M} R_{ij} \xi_1^i \xi_2^j$; then we can write

$$\mathfrak{B} = \ker \, R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}) \; , \tag{1}$$

expressing $\mathfrak{B}$ as the kernel of a polynomial operator in the shifts. We call (1) a *kernel representation* of $\mathfrak{B}$.

Associating behaviors with Laurent polynomial matrices allows the development of a calculus of representations in which properties of a behavior are reflected in algebraic properties of the polynomial matrices representing it. A thorough introduction to this calculus is given in the literature; we now

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briefly review only those notions necessary for the results presented in this paper.

First, we introduce some notation. We denote with $\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1,\xi_2]$ (respectively, with $\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1}]$) the set of all $\mathtt{r}\times\mathtt{w}$ matrices with entries in the ring $\mathbb{R}[\xi_1,\xi_2]$ of polynomials in 2 indeterminates, with real coefficients (respectively in the ring $\mathbb{R}[\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1}]$ of Laurent polynomials in 2 indeterminates with real coefficients). For simplicity in the following we often omit an explicit indication of the indeterminates when referring to (Laurent) polynomial matrices. When one of the dimensions of a matrix is not specified (but finite), we denote it with a bullet; for example, $\mathbb{R}^{\bullet\times\mathtt{w}}$ is the set of matrices with real entries and $\mathtt{w}$ columns.

Inclusion and equality of behaviors are reflected in properties of the Laurent polynomial matrices associated with their kernel representations as follows. If two behaviors are represented as $\mathfrak{B}_i := \ker\ R_i(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})$, with $R_i \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1}]$, $i=1,2$, then $\mathfrak{B}_1\subseteq\mathfrak{B}_2$ if and only if there exists $L\in\mathbb{R}^{\bullet\times\bullet}[\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1}]$ such that $R_2 = LR_1$. Also, $\mathfrak{B}_1=\mathfrak{B}_2$ if and only if there exist $L_1,L_2\in\mathbb{R}^{\bullet\times\bullet}[\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1}]$ such that $R_2=L_1R_1$ and $R_1 = L_2R_2$. If the polynomial matrices R_1 and R_2 have full row rank, then $\mathfrak{B}_1=\mathfrak{B}_2$ if and only if there exists a unimodular matrix $L\in\mathbb{R}^{\bullet\times\bullet}[\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1}]$, i.e. a matrix whose determinant is a unit in $\mathbb{R}[\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1}]$, such that $R_1 = LR_2$. Note that since the determinant of a unimodular matrix is a unit, a unimodular matrix is invertible in the ring it belongs to.

A set $\mathcal{K}\subset\mathbb{R}\times\mathbb{R}$ is a *cone* if $\alpha\mathcal{K}\subset\mathcal{K}$ for all $\alpha\geq 0$; a cone is *convex* if it contains, with any two points, also the line segment between them; a convex cone is *solid* if it contains an open ball of $\mathbb{R}\times\mathbb{R}$.

We denote with $\ell_2(\mathbb{Z},\mathbb{Z}^{\mathtt{w}})$ (often abbreviated with ℓ_2 when the trajectory dimension is evident from the context) the set of square summable trajectories:

$$\ell_2(\mathbb{Z},\mathbb{Z}^{\mathtt{w}}) :=$$
$$\{w\in(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}} \mid \sum_{k=-\infty}^{+\infty} w(k)^\top w(k) = \sum_{k=-\infty}^{+\infty}\|w(k)\|_2^2 < \infty\}\ .$$

III. Time-relevant $2D$ systems

In this section we introduce the notion of time-relevant system. The idea is that a time-relevant system is an autonomous system having time as one of the independent variables and whose "past" determines its "future". This brings up the notion of characteristic sets, already used in this context in [4], [22], [26], [30], [24]. We conclude this section with a result that provides an algebraic characterization of time-relevant systems.

Definition 1: Let $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}}$. A subset $\mathcal{S}\subseteq\mathbb{Z}^2$ is *characteristic* for $\mathfrak{B}$ if

$$\left\{[w_1,w_2\in\mathfrak{B}] \text{ and } \left[w_{1|\mathcal{S}} = w_{2|\mathcal{S}}\right]\right\}\Longrightarrow[w_1=w_2]\ .$$

The following result is a straightforward consequence of this definition and of the linearity of $\mathfrak{B}$.

Proposition 2: Let $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}}$. A subset $\mathcal{S}\subseteq\mathbb{Z}^2$ is characteristic for $\mathfrak{B}$ if and only if

$$\left\{[w\in\mathfrak{B}] \text{ and } \left[w_{|\mathcal{S}}=0\right]\right\}\Longrightarrow[w=0]\ .$$

Of course the trivial set $\mathcal{S}=\mathbb{Z}^2$ is characteristic for every behavior $\mathfrak{B}$; however, in the following we are only interested in those behaviors with nontrivial characteristic sets. We call these systems autonomous (see Def. 1 p. 1503 of [4] and also Def. 2.2 p. 292 of [26]).

Definition 3: A behavior $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}}$ is called *autonomous* if it admits a characteristic set $\mathcal{S}\subset\mathbb{Z}\times\mathbb{Z}$ whose complementary set $(\mathbb{Z}\times\mathbb{Z})\setminus\mathcal{S}$ includes the intersection $\mathcal{K}\cap(\mathbb{Z}\times\mathbb{Z})$ of a closed solid convex cone $\mathcal{K}$ of $\mathbb{R}\times\mathbb{R}$ with $\mathbb{Z}\times\mathbb{Z}$.

It is well known (see for example [4]) that every $2D$ autonomous behavior admits a kernel representation (1) with $R\in\mathbb{R}^{\bullet\times\mathtt{w}}[\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1}]$ of full column rank. Moreover, autonomous behaviors $\mathfrak{B}$ can be decomposed (non-uniquely) as the sum of a finite-dimensional part and of an infinite-dimensional, "square" part, the latter being unique, i.e.

$$\mathfrak{B}=\mathfrak{B}^{\text{fd}}+\ker\ S(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})\ , \tag{2}$$

where $\mathfrak{B}^{\text{fd}}$ a behavior that has finite dimension (as a vector space over $\mathbb{R}$), $S\in\mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1}]$ is a non-singular square polynomial matrix and $\mathfrak{B}^{\text{sq}} := \ker\ S(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})$ is uniquely determined by $\mathfrak{B}$, as follows, see [25, Proposition 2.3]. Let $R\in\mathbb{R}^{\bullet\times\mathtt{w}}[\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1}]$ be a kernel representation of $\mathfrak{B}$. Since $\mathfrak{B}$ is autonomous, $\mathbb{R}$ has full column rank, and can be factorized as $R(\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1}) = P(\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1})S(\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1})$ with $P(\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1})$ factor right prime and $S(\xi_1,\xi_1^{-1},\xi_2,\xi_2^{-1})$ square and non-singular. Then, $\mathfrak{B}^{\text{sq}} := \ker\ S(\sigma_1,\sigma_1^{-1},\sigma_2,\sigma_2^{-1})$. These behaviors $\mathfrak{B}^{\text{sq}}$ are known as *square autonomous behaviors*.

We now introduce time-relevant behaviors; of special importance in this case are the sets

$$\mathcal{S}_{t_1} := \{(k_1,k_2)\in\mathbb{Z}^2 \mid k_1\leq t_1\}\ , \tag{3}$$

and their subsets

$$\mathcal{S}_{t_0,t_1} := \{(k_1,k_2)\in\mathbb{Z}^2 \mid t_0\leq k_1\leq t_1\}\ . \tag{4}$$

These are illustrated in Fig.s 1 and 2, respectively. Often in the following we call a set $\mathcal{L}_t=\mathcal{S}_{t,t}$ a *vertical line*.

The definition of time-relevant behavior is the following.

Definition 4: $\mathfrak{B}\in\mathfrak{L}^{\mathtt{w}}$ is *time-relevant* if for all $t\in\mathbb{Z}$ the sets S_t of the form (3) are characteristic.

Observe that linear shift-invariant finite-dimensional $2D$ behaviors $\mathfrak{B}^{\text{fd}}$ are time-relevant: indeed, it can be shown that any sufficiently large (finite) rectangle in $\mathbb{Z}^2$ is characteristic for $\mathfrak{B}^{\text{fd}}$, and consequently all the sets S_t are characteristic for $\mathfrak{B}^{\text{fd}}$. From the decomposition (2) it follows then that a behavior $\mathfrak{B}$ is time-relevant if and only if its square part

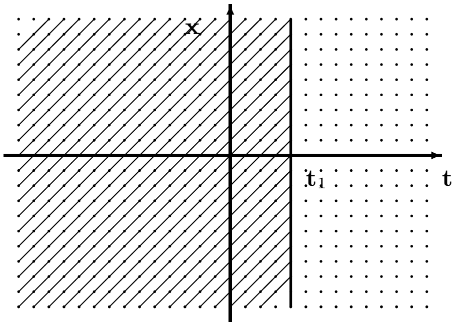

Fig. 1. A set $\mathcal{S}_{t_1}$, see formula (3).

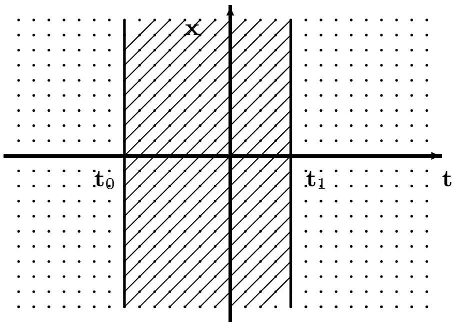

Fig. 2. A set $\mathcal{S}_{t_0,t_1}$, see formula (4).

$\mathfrak{B}^{\text{sq}}$ is time-relevant. Therefore, in the rest of this section we concentrate on square autonomous behaviors.

We next give a result, whose proof we omit, that provides a characterization of time-relevance for square autonomous systems; namely, we show that a time-relevant (square, autonomous) behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ has a special kernel representation; this will be useful in proving several important results later on in the paper.

Proposition 5: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be a square autonomous behavior. Then $\mathfrak{B}$ is time-relevant if and only if there exists $R^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1^{-1}, \xi_2, \xi_2^{-1}]$ such that $\mathfrak{B} = \ker R^*(\sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$ and

$$R^*(\xi_1^{-1}, \xi_2, \xi_2^{-1}) = I_{\mathtt{w}} + R_1^*(\xi_2, \xi_2^{-1})\xi_1^{-1} + \dots + R_L^*(\xi_2, \xi_2^{-1})\xi_1^{-L}, \qquad (5)$$

where $L \in \mathbb{N}$, and $R_i^* \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_2, \xi_2^{-1}]$, $i = 1, \dots, L$.

This result shows that $\mathfrak{B}$ is time-relevant if and only if the restriction of $w \in \mathfrak{B}$ to a vertical line $\mathcal{L}_{t_1}$ where $t_1 \in \mathbb{Z}$, is a linear combination of the restrictions of w and its shifts $\sigma_2^k w$ to a finite number of similar lines $\mathcal{L}_{t_0}$ with $t_0 < t_1$. The minimal number of such lines will be called the *time-lag* of $\mathfrak{B}$.

The following provides an easy way to check whether a given square autonomous behavior is time-relevant.

Proposition 6: Let $\mathfrak{B} = \ker R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}) \in \mathfrak{L}^{\mathtt{w}}$ be a square autonomous behavior with $R(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$. Write $\det R(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}) =: p(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}) = \sum_{i=M}^{N} p_i(\xi_2, \xi_2^{-1})\xi_1^i$ with $M \le N$ and $p_N(\xi_2, \xi_2^{-1}) \neq 0$. Then, $\mathfrak{B}$ is time-relevant if and only if $p_N(\xi_2, \xi_2^{-1})$ is invertible in $\mathbb{R}[\xi_2, \xi_2^{-1}]$.

Proof: Assume that $\mathfrak{B}$ is time-relevant. By Proposition 5 it follows that R is unimodularly equivalent to a representation R^* of the form (5). Therefore, $\det R(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}) = u(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}) \det R^*(\sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$ where the polynomial $u(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1})$ is a unit in $\mathbb{R}(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1})$, i.e., $u(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}) = c\ \xi_1^{N_1}\xi_2^{N_2}$ for some $N_1, N_2 \in \mathbb{Z}$ and $c \in \mathbb{R}$. But, taking (5) into account, we have that $\det R^*(\xi_1^{-1}, \xi_2, \xi_2^{-1}) = d_0(\xi_2, \xi_2^{-1}) + d_1(\xi_2, \xi_2^{-1})\xi_1^{-1} + \dots + d_K(\xi_2, \xi_2^{-1})\xi_1^{-K}$, with $d_0(\xi_2, \xi_2^{-1}) = 1$ and for suitable polynomial $d_j(\xi_2, \xi_2^{-1})$, $j = 1, \dots, K$, and $K \in \mathbb{N}$. Therefore, $\det R(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}) = c\ \xi_2^{N_2}\xi_1^{N_1} + c\ \xi_2^{N_2}\xi_1^{N_1-1}d_1(\xi_2, \xi_2^{-1}) + \dots + c\ \xi_2^{N_2}\xi_1^{N_1-K}d_K(\xi_2, \xi_2^{-1})$ which is obvious of the desired form $p_N(\xi_2, \xi_2^{-1})\xi_1^N + \dots + p_M(\xi_2, \xi_2^{-1})\xi_1^M$ where $N = N_1$, $M = N_1 - K$, $p_N(\xi_2, \xi_2^{-1}) = c\ \xi_2^{N_2}$ is a unit and the other $p_j(\xi_2, \xi_2^{-1})$, $j = M, \dots, N-1$, are defined accordingly.

For the converse implication, note first that every square autonomous $2D$ behaviors $\mathfrak{B}$ can be represented as $\mathfrak{B} = \ker R(\xi_1^{-1}, \xi_2, \xi_2^{-1}) = R_0(\xi_2, \xi_2^{-1}) + \dots + R_L(\xi_2, \xi_2^{-1})\xi_1^{-L}$, with $R_0(\xi_2, \xi_2^{-1})$ non-singular. Assume now that $\mathfrak{B}$ is not time-relevant. Then, by Proposition 5 $\det R_0(\xi_2, \xi_2^{-1})$ is not unimodular, and hence $p_0(\xi_2, \xi_2^{-1}) := \det R_0(\xi_2, \xi_2^{-1})$ is not invertible in $\mathbb{R}[\xi_2, \xi_2^{-1}]$. Since $\det R(\sigma_1^{-1}, \sigma_2, \sigma_2^{-1}) = p_0(\sigma_2, \sigma_2^{-1}) + p_1(\sigma_2, \sigma_2^{-1})\xi_1^{-1} + \dots + p_M(\sigma_2, \sigma_2^{-1})\xi_1^{-M}$ for some $M \in \mathbb{N}$ and suitable L-polynomials $d_j(\sigma_2, \sigma_2^{-1}) \in \mathbb{R}[\sigma_2, \sigma_2^{-1}]$, $j = 1, \dots, M$, which is written as in the statement of the proposition, with $N = 0$, we conclude that the condition of the proposition is not satisfied. This concludes the proof. ∎

We shall say that a polynomial $p(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}) = p_N(\xi_2, \xi_2^{-1})\xi_1^N + \dots + p_M(\xi_2, \xi_2^{-1})\xi_1^M$, $N \ge M$, is a *time-relevant polynomial* if it satisfies the condition of the previous proposition.

Since, as mentioned earlier, a behavior is time-relevant if and only if its square part is time-relevant, Proposition 6 together with [25, Proposition 2.3] yields the following characterization for time-relevance.

Corollary 7: Let $\mathfrak{B} = \ker R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1}) \in \mathfrak{L}^{\mathtt{w}}$ be an autonomous $2D$ behavior with $R(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}) \in \mathbb{R}^{g\times\mathtt{w}}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$. Let further $\mathfrak{B}^{\text{fd}} = \ker S(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$ be its square part. Then the following statements are equivalent.

1) $\mathfrak{B}$ is time-relevant,
2) $\det S(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1})$ is a time-relevant polynomial,
3) the greatest common divisor of the $\mathtt{w} \times \mathtt{w}$ minors of $\mathbb{R}(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1})$ is a time-relevant polynomial.

Proof: The equivalence between (1) and (2) is stated in Proposition 6. The equivalence between (2) and (3) follows from that fact that $\det S(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1})$ equals the greatest common divisor of the $\mathtt{w} \times \mathtt{w}$ minors of $R(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1})$ (up to a unimodular factor), since it can be shown that $R(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}) = F(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1})S(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1})$ with $F(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}) \in \mathbb{R}^{g\times\mathtt{w}}[(\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1})]$ having coprime maximal order minors. ∎

IV. Time-relevant stability, and its algebraic characterization

In the $1D$ case, the notion of stability is obvious: a $1D$ behavior is stable if all its trajectories go to zero as time goes to infinity. For multidimensional systems these exist several definitions of stability. In this section, we propose a notion of stability of time-relevant systems based on the idea that a time-relevant behavior $\mathfrak{B}$ is "unstable" if it contains trajectories with finite-energy initial conditions, whose instant energy does not go to zero as time goes to infinity. We also provide an algebraic test in terms of the location of the zeros of the determinant of a polynomial matrix describing $\mathfrak{B}$. We begin with the following result, stated without proof.

Proposition 8: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be a time-relevant square autonomous behavior with time-lag L. Assume that there exist $t_0 \in \mathbb{Z}$ and $w \in \mathfrak{B}$ such that, for all $k \in \mathbb{Z} \cap [t_0, t_0+L)$, $w_{|\mathcal{L}_k} := w(k,\cdot) \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ is square-summable. Then $w_{|\mathcal{L}_k} \in \ell_2(\mathbb{Z}, \mathbb{R}^{\mathtt{w}})$ for all $k \geq t_0 + L$.

This result supports the definition of time-relevant stability that follows: a system is time-relevant stable if whenever $w \in \mathfrak{B}$ has 'initial conditions' of finite energy in a set $\mathcal{S}_{t_0,t_0+N-1} = \cup_{k=t_0,\ldots,t_0+N-1}\mathcal{L}_k$, then the instant energy, i.e., energy of the restrictions of w along vertical lines goes to zero with time.

Definition 9: A time-relevant behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is *time-relevant stable* if there exists $N \in \mathbb{N}$ such that

$$\begin{aligned}
&\{[w \in \mathfrak{B}] \text{ and} \\
&[w(k,\cdot) \in \ell_2(\mathbb{Z}, \mathbb{R}^{\mathtt{w}}) \text{ for all } 0 \leq k \leq N-1]\} \\
&\quad \Longrightarrow \left[\lim_{k\to\infty} \|w(k,\cdot)\|_{\ell_2} = 0\right] .
\end{aligned}$$

Note that, if it exists, the integer N of the previous definition must be greater than or equal to the time lag of the behavior.

It is easy to see that if $\mathfrak{B}$ is finite-dimensional, the only trajectory which is square summable along a vertical line is the zero trajectory. Thus, it follows from Definition 9 that a finite-dimensional behavior is always time-relevant stable. From the decomposition of autonomous $2D$ behaviors explained after Definition 3, it then also follows that an autonomous behavior $\mathfrak{B}$ is time-relevant stable if and only if its square part is time-relevant stable. Consequently, in the rest of this paper we will be focusing on square autonomous behaviors $\mathfrak{B}$.

The following result, whose proof we omit, gives an algebraic test for the stability of a time-relevant system.

Theorem 10: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be a time-relevant square autonomous behavior, and let $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$ induce a kernel representation of $\mathfrak{B}$. The behavior $\mathfrak{B}$ is time-relevant stable if and only if for all $\omega \in \mathbb{R}$ the Laurent polynomial $\det\ R(\xi_1, \xi_1^{-1}, e^{i\omega}, e^{-i\omega})$ has all its roots in the open unit disk.

V. Concluding remarks

In this work we have presented the definitions of time-relevant $2D$ system and time-relevant stability. We have derived necessary and sufficient algebraic conditions for these properties to hold.

Our results suggest to verify the time-relevant stability of a behavior $\ker\ R(\sigma_1, \sigma_1^{-1}, \sigma_2, \sigma_2^{-1})$ by checking the location of the roots of the ω-dependent Laurent polynomial $\det\ R(\xi_1, \xi_1^{-1}, e^{i\omega}, e^{-i\omega})$ as ω varies in $\mathbb{R}$. Although this condition seems rather difficult to check, since it involves determining the location of the roots of a parameter-dependent polynomial, it turns out that it can be translated into an easily checkable LMI condition. Results on this issue will be reported elsewhere.

An interesting and important research direction is the extension of the results presented in this paper to continuous systems and to ND systems for $N > 2$.

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