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From External to Internal System Decompositions

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Abstract: The recently obtained approach to the construction of state maps, which is directly based on the linear differential operator describing the system, is shown to lead to an immediate and insightful relation between external and internal decompositions and symmetries of a linear system. This is applied to the decomposition of a system into its controllable and uncontrollable part (in the state space representation commonly referred to as the Kalman decomposition), and to the correspondence between external and internal symmetries.

Keywords: State maps, integration by parts, bilinear differential forms, factorization, Kalman decomposition, controllable part, symmetries

1. INTRODUCTION

A common thread in systems and control theory is the relation between properties of the *external* (input-output) behavior of a system and properties of the *internal* behavior of a (minimal) state space representation. Although there are various ways to construct a state space representation of a linear system, they are mostly computationally oriented, and are not always suitable to translate properties of the external behavior into properties of the resulting state space representation, and vice versa. In fact, the most commonly employed approach for deriving properties of a state space representation from the properties of the external behavior is the *state space isomorphism theorem*, which has the drawback that it is mainly an *existence* result ('there exists a unique linear mapping from a minimal state space representation to any other minimal state space representation of the same external behavior').

Recently (van der Schaft, Rapisarda (2010)) we have developed a novel approach to the construction of a state map for finite-dimensional linear systems given by higher-order linear differential equations in the external variables (inputs and outputs), which can be regarded as a canonical construction. Indeed, we have shown that a (minimal) state map can be constructed directly from the differential operator describing the external behavior of the linear system, and in fact corresponds to a factorization of the 'remainders' in an integration by parts procedure. This provides an explicit and algebraically very simple way of deriving state maps for finite-dimensional linear systems. Since the construction of the state map is directly in terms of the original data (the linear differential operator describing the higher-order differential equations) this allows us to immediately transfer properties of the external behavior (encoded in the differential operator) into properties of the state space representation. Furthermore, this approach has the potential to be extendable to other system classes, including linear pde systems.

2. CONSTRUCTION OF STATE MAPS BASED ON INTEGRATION BY PARTS

We start with a brief summary of the construction of state maps as recently developed in van der Schaft, Rapisarda (2010), see also Rapisarda, van der Schaft (2010). Consider a linear time-invariant system given by a set of higher-order differential equations

$$R\left(\frac{d}{dt}\right) w(t) = 0, \quad w(t) \in \mathcal{W} := \mathbb{R}^q\,, \tag{1}$$

where $R(\xi) = R_0 + R_1\xi^1 + \ldots + R_N\xi^N \in \mathbb{R}^{p\times q}[\xi]$, with $\mathbb{R}^{p\times q}[\xi]$ the space of $p \times q$ polynomial matrices in the indeterminate ξ. Of course, this includes the case of input-output linear systems given by

$$D\left(\frac{d}{dt}\right) y(t) = N\left(\frac{d}{dt}\right) u(t) \tag{2}$$

with external variables w split into an input vector u and output vector y, and $D(s)$ a square polynomial matrix whose determinant is a non-zero polynomial in the complex variable $s \in \mathbb{C}$. Denote the space of locally integrable trajectories from $\mathbb{R}$ to $\mathbb{R}^q$ by $\mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q)$. Recall that $w \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q)$ is a *weak solution* of (1) if

$$\int_{-\infty}^{\infty} w^T(t) R^T\left(-\frac{d}{dt}\right)\varphi(t)dt = 0 \tag{3}$$

for all $\mathcal{C}^\infty$ test functions $\varphi : \mathbb{R} \to \mathbb{R}^p$ with compact support. The set of all weak solutions of (1), sometimes called its *behavior* (see e.g. Rapisarda, Willems (1997), Polderman, Willems (1997)), will be denoted by $\mathcal{B}_R$; i.e.,

$$\mathcal{B}_R := \{w \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^q) \mid (1) \text{ is satisfied weakly}\} \tag{4}$$

Sometimes we will also denote this as $\mathcal{B}_R = \ker R\left(\frac{d}{dt}\right)$.

The fundamental system-theoretic notion of *state* of a linear system (1) can be defined as follows. Consider two solutions $w_a, w_b \in \mathcal{B}_R$, and define the *concatenation* of w_a and w_b at time 0 as the time-function

$$(w_a \wedge_0 w_b)(t) := \begin{cases} w_a(t) \ , & t < 0 \\ w_b(t) \ , & t \geq 0 \end{cases} , \quad t \in \mathbb{R} \ . \tag{5}$$

We say that $w_1, w_2 \in \mathcal{B}_R$ are *equivalent at time* 0, denoted as $w_1 \sim_0 w_2$, if for all $w \in \mathcal{B}_R$:

$$w_1 \wedge_0 w \in \mathcal{B}_R \Leftrightarrow w_2 \wedge_0 w \in \mathcal{B}_R \tag{6}$$

Thus $w_1 \sim_0 w_2$ if and only if w_1 and w_2 have the same continuations from $t = 0$ within $\mathcal{B}_R$. (Note that in this linear case w_1 and w_2 have the same continuations within $\mathcal{B}_R$ if they have at least *one* shared continuation in $\mathcal{B}_R$.)

Let now $X \in \mathbb{R}^{n \times q}[\xi]$. Then the associated differential operator

$$X\left(\frac{d}{dt}\right) : \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^q) \rightarrow \mathcal{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^n)$$
$$w \mapsto x := X\left(\frac{d}{dt}\right) w$$

is said to be a *state map* (Rapisarda, Willems (1997)) for the system (1) with behavior $\mathcal{B}_R$ defined in (4) if for all $w_1, w_2 \in \mathcal{B}_R$ and corresponding $x_i := X\left(\frac{d}{dt}\right) w_i$, $i = 1, 2$, the following property (the *state property*) holds:

$$\begin{aligned} &[x_1(0) = x_2(0)] \ \text{ and } \ [x_1, x_2 \text{ continuous at } t = 0] \\ &\qquad \Longrightarrow \ [w_1 \sim_0 w_2] \end{aligned} \tag{7}$$

We call the vector $x(0) = X\left(\frac{d}{dt}\right) w(0)$ a state of the system at time 0 corresponding to the time-function w, and we call $\mathcal{X} = \mathbb{R}^n$ the state space for the system. If n is minimal among all the state vector dimensions, then the state map is called a *minimal* state map. We then call n the *McMillan degree* of the system.

The starting point for the direct construction of state maps taken in (van der Schaft, Rapisarda (2010)), see also (Rapisarda, van der Schaft (2010)), is the basic *integration by parts* formula. Take any N-times differentiable functions $w : \mathbb{R} \rightarrow \mathbb{R}^q$ and $\varphi : \mathbb{R} \rightarrow \mathbb{R}^p$, and denote $w^{(i)} := \frac{d^i}{dt^i} w$, $i \in \mathbb{N}$, and analogously for φ. For each pair of time instants $t_1 \leq t_2$ integration by parts yields

$$\begin{aligned} &\int_{t_1}^{t_2} w^T(t) R^T\left(-\frac{d}{dt}\right) \varphi(t) dt = \\ &\int_{t_1}^{t_2} \varphi^T(t) R\left(\frac{d}{dt}\right) w(t) dt + B_\Pi(\varphi, w)|_{t_1}^{t_2} \ , \end{aligned} \tag{8}$$

where the expression $B_\Pi(\varphi, w)(t)$ is the *remainder*, which has the form

$$\left[\varphi^T(t) \ \varphi^{(1)T}(t) \ \ldots \ \varphi^{(N-1)^T}(t) \ \ldots\right] \tilde{\Pi} \begin{bmatrix} w(t) \\ w^{(1)}(t) \\ \vdots \\ w^{(N-1)}(t) \\ \vdots \end{bmatrix} , \tag{9}$$

for some constant infinite matrix $\tilde{\Pi}$ with a finite number of nonzero entries. In fact, the expression $B_\Pi(\varphi, w)(t)$ only depends on φ, w and their time-derivatives up to order $N - 1$.

The *differential version* of the integration by parts formula (8) is

$$w^T(t) R^T(-\frac{d}{dt})\varphi(t) - \varphi^T(t) R(\frac{d}{dt}) w(t) = \frac{d}{dt} B_\Pi(\varphi, w)(t) \tag{10}$$

Both sides of this equality define a *bilinear differential form*, cf. Willems, Trentelman (1998). In general, a bilinear differential form is defined as an expression of the form

$$B_\Phi(\varphi, w)(t) = \sum_{k,l=0}^{M-1} \left[\frac{d^k}{dt^k}\varphi(t)\right]^T \Phi_{k,l} \frac{d^l}{dt^l} w(t) \tag{11}$$

for certain constant $p \times q$ matrices $\Phi_{k,l}$, $k, l = 0, \cdots, M-1$. The infinite matrix $\tilde{\Phi}$ whose (k, l)-th block is the matrix $\Phi_{k,l}$ for $k, l = 0, \ldots, M-1$, and is zero everywhere else, is called the *coefficient matrix* of the bilinear differential form B_Φ. The coefficient matrix of the bilinear differential form B_Π appearing in (10) is precisely the matrix $\tilde{\Pi}$ as defined before in (9).

There is a useful one-to-one correspondence between the bilinear differential form B_Φ and the *two-variable polynomial matrix* $\Phi(\zeta, \eta)$ defined as

$$\Phi(\zeta, \eta) := \sum_{k,l=0}^{M-1} \Phi_{k,l} \zeta^k \eta^l \ . \tag{12}$$

An important fact in the calculus of bilinear differential forms is the following (cf. Willems, Trentelman (1998)). The *time-derivative* of a bilinear differential form B_Φ defines the bilinear differential form

$$B_\Psi(\varphi, w)(t) := \frac{d}{dt}\left(B_\Phi(\varphi, w)\right)(t) \ , \tag{13}$$

which corresponds to the two-variable polynomial matrix

$$\Psi(\zeta, \eta) = (\zeta + \eta)\Phi(\zeta, \eta) \ . \tag{14}$$

Hence the differential version of the integration by parts formula (10) corresponds to the *two-variable polynomial matrix equality*

$$R(-\zeta) - R(\eta) = (\zeta + \eta)\Pi(\zeta, \eta) \tag{15}$$

From here we see how $\Pi(\zeta, \eta)$ and its coefficient matrix $\tilde{\Pi}$ can be easily computed from $R(\xi)$. Indeed, since $R(-\zeta) - R(\eta)$ is zero for $\eta = -\zeta$, it follows that $R(-\zeta) - R(\eta)$ contains a factor $\zeta + \eta$, and thus we can *define* $\Pi(\zeta, \eta)$ as

$$\Pi(\zeta, \eta) := \frac{R(-\zeta) - R(\eta)}{\zeta + \eta} \ . \tag{16}$$

It turns out that state maps are obtained by *factorizing* the two-variable polynomial matrix $\Pi(\zeta, \eta)$ as $\Pi(\zeta, \eta) = Y^T(\zeta) X(\eta)$. Such a factorization corresponds in a one-to-one manner to a factorization $\tilde{\Pi} = \tilde{Y}^T \tilde{X}$ of the coefficient matrix:

Proposition 2.1. Let $\Phi \in \mathbb{R}^{p \times q}(\zeta, \eta)$, with $\tilde{\Phi}$ its coefficient matrix. Any factorization $\Phi(\zeta, \eta) = F(\zeta)^\top G(\eta)$ corresponds to a factorization $\tilde{\Phi} = \tilde{F}^T \tilde{G}$, where $\tilde{F}, \tilde{G}$ are the coefficient matrices of $F(\zeta)$, respectively $G(\eta)$.

Factorizations which correspond to the minimal value $n = \text{rank}(\tilde{\Phi})$, are called *minimal.* They are unique up to premultiplication by a constant nonsingular matrix (Willems, Trentelman (1998)):

Proposition 2.2. Given a minimal factorization $\tilde{\Phi} = \tilde{F}^T \tilde{G}$, every other minimal factorization $\tilde{\Phi} = \tilde{F}'^T \tilde{G}'$ can be obtained from it by premultiplication of $\tilde{F}$ and $\tilde{G}$ by a nonsingular $n \times n$ matrix S, respectively S^{-T}. In view of Proposition 2.1 this implies that $\Phi(\zeta,\eta) = F(\zeta)^T G(\eta) = F'(\zeta)^T G'(\eta)$ with $F'(\xi) := SF(\xi)$, $G'(\xi) := S^{-T}G(\xi)$.

The fundamental theorem concerning state maps obtained in (van der Schaft, Rapisarda (2010)), see also (Rapisarda, van der Schaft (2010)), is the following.

Theorem 2.3. For any factorization $\Pi(\zeta,\eta) = Y(\zeta)^T X(\eta)$ the map

$$w \mapsto x := X\left(\frac{d}{dt}\right) w$$

is a state map. The equivalence

$$w_1 \sim_0 w_2 \Leftrightarrow X\left(\frac{d}{dt}\right) w_1(0) = X\left(\frac{d}{dt}\right) w_2(0)$$

holds if and only if the factorization is minimal. Hence a necessary condition for the state map $x = X\left(\frac{d}{dt}\right) w$ to be minimal is that the factorization is minimal. If $R(\xi)$ is row-reduced [1] then the state map is minimal *if and only if* the factorization is minimal. Furthermore, the map

$$x_a = Y\left(\frac{d}{dt}\right) \varphi$$

defines a state map for the *adjoint system* $w_a = R^T\left(-\frac{d}{dt}\right)\varphi$ (given in image representation), which is minimal if and only if the factorization is minimal.

3. EXTERNAL AND INTERNAL DECOMPOSITIONS

In this section we discuss how a decomposition of the behavior $\mathcal{B}_R$ of a linear system (1) corresponding to a decomposition of $R(\xi)$ directly leads to a decomposition of the state map. Consider a behavior (set of weak solutions) $\mathcal{B}_R$, where $R(\xi)$ is factorized as

$$R(\xi) = R_2(\xi)R_1(\xi) \tag{17}$$

with $R_1(\xi)$ an $k \times q$ and $R_2(\xi)$ an $p \times k$ polynomial matrix. Corresponding to R_1 and R_2 we define two other behaviors $\mathcal{B}_{R_1}$ and $\mathcal{B}_{R_2}$. The behavior $\mathcal{B}_{R_1}$ is simply defined as

$$\mathcal{B}_{R_1} := \{w \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^q) \mid R_1\left(\frac{d}{dt}\right) w(t) = 0 \text{ weakly }\} \tag{18}$$

It immediately follows that

$$\mathcal{B}_{R_1} \subset \mathcal{B}_R \tag{19}$$

Conversely (Polderman, Willems (1997)), (19) implies the existence of a polynomial matrix $R_2(\xi)$ such that (17) holds. Thus to any subbehavior there corresponds a factorization (17).

The second behavior $\mathcal{B}_{R_2}$ is defined as

$$\mathcal{B}_{R_2} := \{v \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^k) \mid R_2(\frac{d}{dt})v(t) = 0 \text{ weakly}\} \tag{20}$$

(and thus is defined on a different space of variables than $\mathcal{B}_R$ and $\mathcal{B}_{R_1}$).

The relation between the three behaviors is given by

$$\mathcal{B}_R / \mathcal{B}_{R_1} = \mathcal{B}_{R_2} \tag{21}$$

in the sense that to every equivalence class in the quotient space on the left-hand side there corresponds in a one-to-one way an element in $\mathcal{B}_{R_2}$.

[1] Note that any polynomial matrix $R(\xi)$ may be transformed, - by premultiplication with a unimodular matrix and up to zero rows-, into a row-reduced matrix; see e.g. Polderman, Willems (1997).

We will show how the decomposition $R(\xi) = R_2(\xi)R_1(\xi)$ immediately leads to a corresponding decomposition of a state map $x = X(\frac{d}{dt}w)$ for the full behavior $\mathcal{B}_R$. Consider as above for each $R_1(\xi), R_2(\xi)$ the factorizations

$$R_i(-\zeta) - R_i(\eta) = (\zeta+\eta)Y_i(\zeta)^T X_i(\eta)\,, i = 1,2\,, \tag{22}$$

with

$$\begin{aligned} x_1 &= X_1(\frac{d}{dt})w \\ x_2 &= X_2(\frac{d}{dt})v \end{aligned} \tag{23}$$

defining state maps for $\mathcal{B}_{R_1}$, respectively $\mathcal{B}_{R_2}$. This leads to the following factorization of $R(-\zeta) - R(\eta)$:

$$\begin{aligned} &R(-\zeta) - R(\eta) = R_2(-\zeta)R_1(-\zeta) - R_2(\eta)R_1(\eta) = \\ &R_2(-\zeta)[R_1(-\zeta) - R_1(\eta)] + [R_2(-\zeta) - R_2(\eta)]R_1(\eta) = \\ &(\zeta+\eta)[R_2(-\zeta)\Pi_1(\zeta,\eta) + \Pi_2(\zeta,\eta)R_1(\eta)] = \\ &(\zeta+\eta)[R_2(-\zeta)Y_1^T(\zeta)X_1(\eta) + Y_2^T(\zeta)X_2(\eta)R_1(\eta)] = \\ &(\zeta+\eta)\left[R_2(-\zeta)Y_1^T(\zeta)\ Y_2^T(\zeta)\right]\begin{bmatrix} X_1(\eta) \\ X_2(\eta)R_1(\eta)\end{bmatrix} \end{aligned} \tag{24}$$

Hence by application of Theorem 2.3 we obtain:

Proposition 3.1. Let $R(\xi) = R_2(\xi)R_1(\xi)$ with state maps X_1, X_2 in (23). Then

$$x = X\left(\frac{d}{dt}\right) w := \begin{bmatrix} X_1(\frac{d}{dt})w \\ X_2(\frac{d}{dt})R_1(\frac{d}{dt})w \end{bmatrix} \tag{25}$$

is a state map for $\mathcal{B}_R = \ker R\left(\frac{d}{dt}\right)$. Furthermore,

$$x_a = \begin{bmatrix} Y_1(\frac{d}{dt})R_2^T(-\frac{d}{dt})\varphi \\ Y_2(\frac{d}{dt})\varphi \end{bmatrix} \tag{26}$$

defines a state map for the adjoint system $w_a = R^T\left(-\frac{d}{dt}\right)\varphi$.

In general the state map $x = X(\frac{d}{dt})w$ defined in (25) does not need to be minimal, even if the state maps $x_1 = X_1(\frac{d}{dt})w$ and $x_2 = X_2(\frac{d}{dt})w$ are. An example where X is indeed minimal is discussed in the next section.

4. KALMAN DECOMPOSITION INTO CONTROLLABLE AND UNCONTROLLABLE PART

Consider a behavior $\mathcal{B}_R = \ker R(\frac{d}{dt})$. Assume without loss of generality [2] that $R(\xi)$ has full row-rank. Every full row-rank matrix $R(\xi)$ admits the following factorization, see e.g. Polderman, Willems (1997)

$$R(\xi) = R_{uc}(\xi)R_c(\xi)\,, \tag{27}$$

where $R_c(\xi)$ is such that the complex matrix $R_c(s)$ has full row-rank for all $s \in \mathbb{C}$, while the determinant of the square matrix $R_{uc}(s)$ is a non-zero polynomial. It is well-known that $\ker R_c\left(\frac{d}{dt}\right)$ defines the *controllable part* $\mathcal{B}_R^c$ of the system

$$\mathcal{B}_R^c = \{w \in \mathcal{L}_1^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^q) \mid R_c\left(\frac{d}{dt}\right) w = 0 \text{ weakly }\}\,,$$

[2] Since for every $R(\xi)$ there exists a unimodular matrix $U(\xi)$ such that $U(\xi)R(\xi) = \begin{bmatrix} R'(\xi) \\ 0 \end{bmatrix}$ with $R'(\xi)$ full row-rank.

while the zeros of the polynomial $\det R_{uc}(s)$ correspond to its *autonomous* subbehavior. Application of Proposition 3.1 yields the following result.

Proposition 4.1. Consider the decomposition (27) of a full row-rank matrix $R(\xi)$, and let $x_c = X_c\left(\frac{d}{dt}\right)w$ define a state map of the controllable behavior $\mathcal{B}_R^c = \ker R_c\left(\frac{d}{dt}\right)$ and $x_{uc} = X_{uc}\left(\frac{d}{dt}\right)v$ be a state map of the autonomous behavior $\ker R_{uc}\left(\frac{d}{dt}\right)$, both obtained from factorization. Then

$$x = X(\frac{d}{dt})w := \begin{bmatrix} X_c(\frac{d}{dt})w \\ X_{uc}(\frac{d}{dt})R_c(\frac{d}{dt})w \end{bmatrix} \tag{28}$$

is a state map for $\mathcal{B}_R = \ker R\left(\frac{d}{dt}\right)$, which is minimal in case X_c and X_{uc} are minimal.

Proof. The only thing left to be proved is the claim regarding minimality. However, it is well-known [3] (see e.g. Polderman, Willems (1997)) that the McMillan degree of $R(\xi)$ is equal to the sum of the McMillan degrees of $R_c(\xi)$ and $R_{uc}(\xi)$. In case $x_c = X_c(\frac{d}{dt})(w)$ and $x_{uc} = X_{uc}(\frac{d}{dt})v$ are minimal, the dimensions of x_c and x_{uc} are equal to these McMillan degrees, and thus the dimension of $x = \begin{bmatrix} x_c \\ x_{uc} \end{bmatrix}$ is equal to the McMillan degree of $R(\xi)$. ■

The subvector $x_c = X_c(\frac{d}{dt})w$ corresponds to the (uniquely defined) controllable subspace in the Kalman decomposition of the state space representation $\dot{x} = Ax + Bu$ corresponding to the state map $x = X(\frac{d}{dt})w$, while the subvector $x_{uc} = X_{uc}(\frac{d}{dt})R_c(\frac{d}{dt})w$ corresponds to the 'uncontrollable subspace'. Notice that this last subvector is *not* uniquely defined, since the factorization $R(\xi) = R_{uc}(\xi)R_c(\xi)$ is not unique. Of course, this is in accordance with the fact that in the Kalman decomposition for a linear system $\dot{x} = Ax + Bu$ the controllable subspace is intrinsically defined (and is given as $\operatorname{im}\left(B\vdots AB\vdots\cdots\vdots A^{n-1}B\right)$), while the uncontrollable subspace is *not* (but instead corresponds to *some* complement of the controllable subspace, or better, to the quotient of the total state space by the controllable subspace).

Remark 4.2. Using the state space realization theory developed in (van der Schaft, Rapisarda (2010)) it can be directly seen that the controllable behavior is *invariant* within the total behavior.

5. FROM EXTERNAL SYMMETRIES TO INTERNAL SYMMETRIES

As another application of Proposition 3.1 we study the correspondence between external symmetries for the behavior $\mathcal{B}_R$ and internal symmetries for its state space realization. For further background on this topic we refer to e.g. van der Schaft (1984); Fagnani, Willems (1993).

Consider a behavior $\mathcal{B}_R$ which is invariant under a group $\mathcal{G}$ of (static) linear symmetries. This means that for each $g \in \mathcal{G}$ there exists a linear invertible map on the space of external variables $w \in \mathcal{W} = \mathbb{R}^q$

$$T_g : \mathcal{W} \to \mathcal{W}$$

(with the obvious extension to functions $w : T \to \mathcal{W}$) such that

$$T_g\mathcal{B} = \mathcal{B}\,, \tag{29}$$

with the group property that for all $g_1, g_2 \in \mathcal{G}$

$$T_{g_2g_1} = T_{g_2}T_{g_1}$$

In other words, we have a group representation [4]

$$\mathcal{G} \to Gl(\mathcal{W}) \tag{30}$$

given by $g \mapsto T_g$.

Now let $\mathcal{B}_R = \ker R\left(\frac{d}{dt}\right)$, where, without loss of generality, $R(\xi)$ has full row-rank. Property (29) then implies $\ker T_g R\left(\frac{d}{dt}\right) = \ker R\left(\frac{d}{dt}\right)$ for all $g \in \mathcal{G}$, and hence, cf. Polderman, Willems (1997), for every $g \in \mathcal{G}$ there exists a unimodular matrix $U_g(\xi)$ such that

$$R(\xi)T_g = U_g(\xi)R(\xi) \tag{31}$$

Consider now a minimal state map X for $\ker R\left(\frac{d}{dt}\right)$ obtained from factorization, and a state map $X_{U_g}\left(\frac{d}{dt}\right)$ for $\ker U_g\left(\frac{d}{dt}\right)$. Then by Proposition 17

$$X_g\left(\frac{d}{dt}\right) := \begin{bmatrix} X(\frac{d}{dt}) \\ X_{U_g}(\frac{d}{dt})R(\frac{d}{dt}) \end{bmatrix} \tag{32}$$

is a state map for $\ker U_g\left(\frac{d}{dt}\right)R\left(\frac{d}{dt}\right)$. On the other hand, since $R(\xi)T_g = U_g(\xi)R(\xi)$ it follows that also $X\left(\frac{d}{dt}\right)T_g$ is a state map for $\ker U_g\left(\frac{d}{dt}\right)R\left(\frac{d}{dt}\right)$, which is minimal since $X\left(\frac{d}{dt}\right)$ is minimal.

In order to relate $X_g\left(\frac{d}{dt}\right)$ to $X\left(\frac{d}{dt}\right)T_g$ we recall the following lemma from Rapisarda, Willems (1997).

Lemma 5.1. Let $x = X\left(\frac{d}{dt}\right)w$ be a *minimal* state map. Then for any other state map $x = X'\left(\frac{d}{dt}\right)w$ there exist a constant matrix $S \in \mathbb{R}^{\bullet\times\bullet}$ and a polynomial matrix $F \in \mathbb{R}^{\bullet\times\bullet}[s]$ such that

$$X(\xi) = SX'(\xi) + F(\xi)R(\xi) \tag{33}$$

Application of this lemma to the minimal state map $X\left(\frac{d}{dt}\right)T_g$ for $\ker R\left(\frac{d}{dt}\right)T_g$ and the alternative state map $X'\left(\frac{d}{dt}\right) = X_g\left(\frac{d}{dt}\right)$ yields the existence of S_g, P_g, F_g such that (leaving out arguments ξ for simplicity of notation)

$$\begin{aligned} XT_g = [S_g \;\; P_g]\begin{bmatrix} X \\ X_{U_g}R \end{bmatrix} + F_gRT_g = \\ S_gX + [P_gX_{U_g}R + F_gRT_g] \end{aligned} \tag{34}$$

This means that restricted to $\mathcal{B}_R = \ker R\left(\frac{d}{dt}\right) = \ker R\left(\frac{d}{dt}\right)T_g$ we have the equality

$$X\left(\frac{d}{dt}\right)T_g = S_gX\left(\frac{d}{dt}\right) \tag{35}$$

Thus we have obtained the following result.

Proposition 5.2. Consider $\mathcal{B}_R = \ker R\left(\frac{d}{dt}\right)$ with $R(\xi)$ full row-rank. Let $\mathcal{G} \to Gl(\mathcal{W})$ given by $g \mapsto T_g$ be a representation of the symmetry group $\mathcal{G}$ satisfying (29), or equivalently (31). Let $x = X\left(\frac{d}{dt}\right)w$ be a minimal state

[3] The McMillan degree of a polynomial matrix can be computed as the maximal degree of its minors. Clearly, for a polynomial matrix $R_{uc}(\xi)R_c$ this is equal to the degree of $\det R_{uc}(\xi)$ times the maximal degree of the minors of $R_c(\xi)$.

[4] $Gl(\mathcal{W})$ denotes the matrix group of invertible linear maps from $\mathcal{W}$ to $\mathcal{W}$.

map obtained from factorization of Π with corresponding state space $\mathcal{X} = \mathbb{R}^n$. Then there exists a unique representation $\mathcal{G} \to Gl(\mathcal{X})$ given by $g \mapsto S_g$ such that on $\mathcal{B}_R$ the equality (35) holds for all $g \in \mathcal{G}$.

The above proposition shows how the existence of an external group of symmetries immediately translates into an equivalent group of symmetries on the state space obtained from factorization of the two-variable polynomial matrix $\Pi(\zeta, \eta)$, at least when the state map is minimal.

6. CONCLUSIONS

We have elaborated on further applications of the state map construction for linear systems as recently obtained in (van der Schaft, Rapisarda (2010)). Since this state map construction remains very close to the original data, i.e., the polynomial matrix $R(\xi)$, we have been able to relate in a simple manner decomposition properties of $R(\xi)$ (and therefore of the external behavior) to decomposition properties of the state map and the corresponding internal behavior. This has been applied to the decomposition of the external and internal behavior into its controllable and uncontrollable (autonomous) part, and to the correspondence of an external representation of a group of symmetries to an internal one.

Although all results in this paper have been stated for finite-dimensional linear systems given by linear differential operators $R(\frac{d}{dt})$ corresponding to higher-order linear differential equations, the framework is in principle extendable to infinite-dimensional linear systems given by partial differential equations

$$R(\frac{\partial}{\partial t}, \frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_s}) w(t, z_1, \cdots, z_s) = 0,$$

involving, next to the time-variable t, the spatial variables $z_1, \cdots, z_s$. This is a topic of current investigations. Another research avenue is the extension of the results to (classes of) nonlinear systems; see also (van der Schaft (1998)) for some preliminary results obtained in this direction.

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