GEOMETRIC FORMULATION OF EDGE END NODAL FINITE ELEMENT EQUATIONS IN ELECTROMAGNETICS

Andrzej Demenko¹ and Jan. K. Sykulski²,³

¹Poznań University of Technology, ul Piotrowo 3a, 60-965 Poznań, Poland, andrzej.demenko@put.poznan.pl
²University of Southampton, Southampton SO17 1BJ, UK, jks@soton.ac.uk
³Technical University of Łódź, ul. Stefanowskiego 18/22, 90-924 Łódź, Poland

Abstract – Finite element equations for electromagnetic fields are examined, in particular nodal elements using scalar potential formulation and edge elements for vector potential formulation. It is shown how the equations usually obtained via variational approach may be more conveniently derived using integral methods employing a geometrical description of the interpolating functions of edge and facet elements. Moreover, the resultant equations describe the equivalent multi-branch circuit models.

Introduction

Finite element equations are commonly derived using a variational approach, including weak forms (the Galerkin weighted residual method) and/or a strong formulation via a functional (the Rayleigh-Ritz method). These equations have various geometrical interpretations [1] and may be explained using the language of circuit theory by considering the nodal or loop descriptions of equivalent magnetic or electric circuits [2, 3]. The classical equivalent circuits, however, arise from integral formulations. In this work we will demonstrate that by applying appropriate geometrical forms to the interpolating functions of an edge or a facet element the finite element equations may also be derived via integral methods. We will show that by applying approximate integration in the finite element formulation for a mesh with rectangular parallelepiped elements classical expressions for magnetic and electric networks emerge. Both magnetic and electric fields are considered, while for the electric fields conducting and displacement currents may be present.

Geometrical representation of interpolating functions of finite elements

Consider the interpolating functions of an 8-node, 6-facet element (Fig 1a), which allow for the field to be determined at an arbitrary point $P$ within the element. These functions are related to geometrical forms of eight hexahedra $v_i$ defined by drawing straight lines through the point $P$ as shown in Fig. 1. It is well known that the ratio of the volume of the $i$th hexahedron to the element volume $V_e$ is related to the $i$th interpolating function of the nodal element, but it is rarely appreciated that the facets and edges of the volume $v_i$ inside the element represent interpolating functions of the associated edge or facet elements. For example, the interpolating function $w_{e4,8}$ of the edge element for the edge $P_4P_8$ expresses the ratio of the facet vector $s_{4,8}$ to the volume $V_e$ (Fig. 1a), while the ratio $r/V_e$ describes the interpolating function $w_f$ of the facet element for the facet $S$. In a similar manner the interpolating functions for triangular prisms and pentahedron elements may be expressed. As an example, for the pentahedron of Fig. 1b, the expressions for the interpolating functions of the edge element for the edges $P_4P_5$ and $P_2P_5$ take the form $w_{e4,5}=s_{4,5}/(2V_e)$ and $w_{e2,5}=s_{2,5}/V_e$, respectively, while the interpolating functions of the facet element for the facets $S_3$ and $S_4$ are given by $w_{f3}=r_3/(2V_e)$ and $w_{f4}=r_4/V_e$, respectively.
By analysing the relevant integrals, where the geometrical forms provided are integrands, it may be easily inferred that the volume integral in $V$ of the product of $w_{ij}$ and the current density vector, or flux density vector, represents current, or flux, associated with the region next to $P_i P_j$. At the same time the volume integral of the product of $w_i$ and the magnetic field strength represents the average value of voltage.

In 2D systems, where the $z$ component of the field is absent, e.g. $H_z = 0$, and the other components are functions of $x$ and $y$ only, the element depicted in Fig. 1b is reduced to a triangle with vertices $P_1 P_2 P_3$. The four-sided facets of the pentahedron then ‘collapse’ to the sides of a triangle, e.g. the facet $S_1$ becomes the edge $P_2 P_3$. The edges $P_i P_{i+3}$, ($i=1, 2, 3$) of a prism, parallel to the $z$ axis, are represented in 2D by nodes $P_i$. The functions of the edge element for these edges $P_i P_{i+3}$ become in 2D the functions of the nodal element for the nodes $P_i$, while the functions of the facet element for the edges of the prism become similar to the functions of the edge element for the edge of the triangular facets. This similarity arises due to the specific properties of the function describing the pentahedron when the point $P$ lies on the triangular surface of the facet, and thus its $z$ coordinate of point $P$ equals zero. The consequence of this similarity is the fact that the scalar and vector potential formulations in 2D are similar.

**Integral formulations of the finite element equations**

We consider both vector potential ($A$ for the magnetic field and $T$ for the electric field) and scalar potential formulations ($\Omega$ for the magnetic field and $V$ for the electric field). It was noted in [2] that the finite element equations formulated for nodal elements and scalar potentials are related to nodal equations of an edge network (EN) with branches assigned to element edges. The equations describing the edge values of vector potentials represent loop (mesh) equations of a facet network (FN) with nodes positioned in element centres and branches passing through the facets. Figure 2 shows the edge and facet models of single hexahedron and pentahedron elements. By appropriate connections between elements we create the network model of the discretised volume. In the case of the edge network we make parallel connections between branches associated with common element edges. The facet network, on the other hand, involves connecting in series the branches of the facet models of elements with a common facet. We first consider the equations for the edge model, that is the scalar potential formulation using nodal elements.

**Integral formulations for scalar potentials and nodal elements**

A single nodal equation of the edge network results from the summation of currents or fluxes in the branches having common nodes. The currents $i_{ij}$ and fluxes $\phi_{ij}$ in the branch $P_i P_j$ associated with the element edge are described by
\[ i_{i,j} = \int_{e} w_{ei,j} J \, dv, \quad \phi_{i,j} = \int_{e} w_{ei,j} B \, dv. \] (1a,b)

Fig. 2. Edge and facet models of hexahedron (a) and 9-edge pentahedron (b).

Next the constitutive equations are imposed \( \mathbf{J} = \sigma \mathbf{E} + d(\mathbf{eE})/dt \) and \( \mathbf{B} = \mathbf{\mu H} \) and the \( \mathbf{E} \) and \( \mathbf{H} \) vectors are expressed in terms of the functions of the edge element, hence
\[
i_{i,j} = \sum_{p,q} u_{ep,q} \int_{e} w_{ei,j} \gamma w_{ep,q} \, dv, \quad \phi_{i,j} = \sum_{p,q} u_{Hp,q} \int_{e} w_{ei,j} \mu w_{ep,q} \, dv. \] (2a,b)

In the above equations the summation includes all edges \( P_i P_q \) of the element, where \( \gamma = \sigma + \mathbf{pc} \) (p=d/dt), and \( u_{ep,q} \) and \( u_{Hp,q} \) are the edge values of vectors \( \mathbf{E} \) and \( \mathbf{H} \), respectively. These values represent the voltages on the elements of the equivalent element model (see Fig. 2), \( u_{Hp,q} \) is the voltage across the permeance and \( u_{Ep,q} \) the voltage across the parallel connection of the capacitance and conductance. The relationship (2) suggests that in the equivalent models of the element there exist mutual couplings between branches, as in the expressions for current and flux in the branch \( P_i P_j \) we have not only voltages in this branch but also in other branches of the element. Consequently the mass matrices for nodal element method are non-diagonal.

By using the substitutions
\[
\text{grad} \, V = \mathbf{E} + dA/dt, \quad \text{grad} \, \Omega = \mathbf{H} - \mathbf{T}, \] (3a,b)

the edge values of vectors \( \mathbf{H} \) and \( \mathbf{E} \) may be expressed as
\[
u_{ep,q} = V_q - V_p + e_{p,q}, \quad u_{Hp,q} = \Omega_q - \Omega_p + \theta_{p,q}, \] (4a,b)

where \( \Omega_q, \Omega_p, V_p, V_q \) are the nodal values of scalar potential for nodes \( P_q, P_p \) and \( \theta_{p,q} \) and \( e_{p,q} \) are the edge values of \( \mathbf{T} \) and \(-dA/dt\), respectively, and represent the sources of magnetic and electric field. Substituting (4) to (2) yields
\[
i_{i,j} = i_{ii,j} + \sum_{p,q} (V_q - V_p) \int_{e} w_{ei,j} \gamma w_{ep,q} \, dv, \quad \phi_{i,j} = \phi_{ii,j} + \sum_{p,q} (\Omega_q - \Omega_p) \int_{e} w_{ei,j} \mu w_{ep,q} \, dv, \] (5a,b)

where \( i_{ii,j} \) is the current source and \( \phi_{ii,j} \) is the flux source related to branch \( P_i P_j \),
\[
i_{ii,j} = \sum_{p,q} e_{p,q} \int_{e} w_{ei,j} \gamma w_{ep,q} \, dv, \quad \phi_{ii,j} = \sum_{p,q} \theta_{p,q} \int_{e} w_{ei,j} \mu w_{ep,q} \, dv. \] (6a,b)
It has already been noted that a single equation of the nodal element formulation may be related to a nodal equation of the edge network and is found by equalling to zero the sum of currents or fluxes in branches with a common node. For the node $P_i$ we may therefore write

$$\sum_j i_{i,j} = 0, \quad \sum_j \phi_{i,j} = 0.$$ \hspace{1cm} (7a,b)

where $j$ is the node index $P_j$ ($j=1,2,..n$) of all $n$ branches $P_i P_j$ containing the node $P_i$. Substituting (5) into (7) and some further manipulation results in finite element formulation in terms of the scalar potential, which may be written as

$$\sum_j \sum_{p,q} (V_q - V_p) \int_{e_{ij}} w_{e_{ij},p} \, \nabla \phi_{p,q} \, dv = -\sum_j i_{i,j}, \quad \sum_j \sum_{p,q} (\Omega_q - \Omega_p) \int_{e_{ij}} w_{e_{ij},p} \, \mu \phi_{p,q} \, dv = -\sum_j \Theta_{i,j}.$$ \hspace{1cm} (8a,b)

where $V_{e_{ij}}$ is the volume of the element containing the edge $P_i P_j$.

**Integral formulations for vector potentials and edge elements**

The derivations of the last section referred to the final element equations formulated in terms of the scalar potentials. The substitutions (3) are also used in the vector potential formulations. A volume integral then needs to be considered of the products of the edge element functions and relevant terms in (3). Both sides of (3) are multiplied by a function $w_{ij}$ of the facet element for the $i$th facet and the resultant expressions are integrated over the element volume, which ultimately leads to

$$\int_{e_{ij}} w_{ij} \nabla \Omega \, dv = \int_{e_{ij}} w_{ij} H \, dv - \int_{e_{ij}} w_{ij} T \, dv, \quad \int_{e_{ij}} w_{ij} \nabla V \, dv = \int_{e_{ij}} w_{ij} E \, dv + \int_{e_{ij}} w_{ij} (dA/dt) \, dv.$$ \hspace{1cm} (9a,b)

It can be shown that

$$\int_{e_{ij}} w_{ij} \nabla \Omega \, dv = \Omega_{Si} - \Omega_{Q_i}, \quad \int_{e_{ij}} w_{ij} \nabla V \, dv = V_{Si} - V_{Q_i},$$ \hspace{1cm} (10a,b)

where $\Omega_{Si}$ and $V_{Si}$ are average values of potentials $\Omega$ and $V$ for the $i$th facet, respectively, assigned in Fig. 2 to the node $S_i$, while $\Omega_{Q_i}$ and $V_{Q_i}$ denote average values of potentials $\Omega$ and $V$ in the element, associated with the node $Q_i$. Through using (10) the relationship (9) may be written as

$$\Omega_{Si} - \Omega_{Q_i} = u_{qi} = u_{H_i} - \Theta_i, \quad V_{Si} - V_{Q_i} = u_{qi} = u_{E_i} - \epsilon_i,$$ \hspace{1cm} (11a,b)

where

$$u_{H_i} = \int_{e_{ij}} w_{ij} H \, dv, \quad u_{E_i} = \int_{e_{ij}} w_{ij} E \, dv,$$ \hspace{1cm} (12a,b)

$$\Theta_i = \int_{e_{ij}} w_{ij} T \, dv, \quad \epsilon_i = -d(\int_{e_{ij}} w_{ij} A \, dv)/dt.$$ \hspace{1cm} (13a,b)

The equations (11) describe inter-nodal voltages $u_{qi}$, $u_{qi}$ for the branch $Q_i S_i$ of the facet network (see Fig 2). The terms $u_{H}$ and $u_{E}$ represent voltages across the reluctance and across the impedance of the given branch, respectively, whereas $\Theta_i$ and $\epsilon_i$ are the branch magnetomotive force (mmf) and the electromotive force (emf).

Analysing the geometrical form describing the function of the facet element reveals that the integrals in (13), describing the mmf and emf, may be viewed as the edge values of the potentials $T$ and $A$ for the edge described by a vector $r$, which starts at the centre $S_i$ of the facet and ends in the middle $Q_i$ of the element. The mentioned edge values may be treated as loop currents and fluxes in
a mesh of the edge network, associated with the facet $S_i$. For example, in the model of Fig. 2a such a loop is made up of the branches with nodes $P_j P_j P_j P_j$.

When formulating the expressions for voltages $u_m$ and $u_e$, across the reluctance and impedance, the constitutive equations are used $H=\mu B$ and $E=\gamma J$, whereas to describe the flux density $B$ and current density $J$ the functions of the facet element are used. Substituting yields

$$u_{Hi} = \sum_q \phi_q \int_e w_i \mu^{-1} w_j d\mathbf{v}, \quad u_{Ei} = \sum_q i_q \int_e w_i \gamma^{-1} w_j d\mathbf{v} \quad (14a,b)$$

The summation above refers to all the facets of the element, the integrals (under the summation) describe equivalent reluctances and impedances, respectively, while $\phi_q$ and $i_q$ are the facet values of the flux density $B$ and current density $J$ for the $q$th facet, respectively. Expression (14) suggests that in the facet model of the element couplings between branches may exist, as indeed was the case for the edge model. Consequently the mass matrices for edge element equations are also not diagonal, as is the case for the nodal elements.

When setting up the equations for the edge element method the facet quantities are expressed in terms of the edge values. The following relationships hold for the $q$th facet

$$\phi_q = \sum_{r,j} \phi_{r,j}, \quad i_q = \sum_{r,j} i_{or,j} \quad (15a,b)$$

where $\phi_{r,j}$ and $i_{or,j}$ are the edge values of $A$ and $T$, respectively, for the edge $P_j P_j$ of the facet $S_i$. For example, after applying (15a) to the facet $S_i$ of Fig. 1b we find $\phi_i = \phi_{2,3} + \phi_{3,6} + \phi_{6,5} + \phi_{5,2}$. The edge values of the potentials $A$ and $T$ represent the fluxes and currents in loops around the edge, that is in the loops of the facet network. Therefore (15) expresses the flux/current in the $q$th branch of the facet network in terms of fluxes/currents in loops around element edges.

The equation of the edge element method for the edge $P_j P_j$ is found by summing up the voltages $u_m$ and $u_e$ for all branches $Q_i S_i$ around that edge, that is for those branches for which the node $S_i$ is related to the facet $S_i$ with the edge $P_j P_j$. The resultant sums are then equated to zero

$$\sum_i u_m = 0, \quad \sum_i u_e = 0 \quad (16a,b)$$

Incorporating (11) to (15) into the above results in

$$\sum_i \sum_q \phi_q \int_{e_i} w_i \mu^{-1} w_j d\mathbf{v} = \sum_i \phi_i, \quad \sum_i \sum_q i_{or,j} \int_{e_i} w_i \gamma^{-1} w_j d\mathbf{v} = \sum_i e_i \quad (17a,b)$$

The right hand sides of these equations represent the resultant loop mmf $\theta_{or,j}$ and loop emf $e_{or,j}$. These resultant mmf and emf may be established (a) from edge values of $T$ and $dA/dt$, or (b) facet values of $J$ and $dB/dt$. In the former case the potentials $T$ and $A$ in (13) are expressed in terms of their edge values. After substitution we find

$$\theta_i = \sum_{r,j} i_{or,j} \int_{e_i} w_j w_{er,j} d\mathbf{v}, \quad e_i = -d(\sum_{r,j} \phi_{r,j} \int_{e_i} w_j w_{er,j} d\mathbf{v})/dt \quad (18a,b).$$

The integral in (18) is dimensionless and may be treated as the weight parameter $\phi^{(i)}_{r,j}$

$$\phi^{(i)}_{r,j} = \int_{e_i} w_j w_{er,j} d\mathbf{v}, \quad (19)$$
This parameter defines the weight of the loop current/flux around the edge \( P_iP_j \) (in the loop of the facet network) in the expression describing the current/flux in the loop of the edge network associated with the facet \( S_q \). The parameter \( g_{i,j}^{(q)} \) is also used in the above mentioned approach (b) where the loop \( \Phi \) and \( \Psi \) are established on the basis of the facet values, leading to

\[
\Theta_{or,j} = \sum_q g_{i,j}^{(q)} \phi_q \quad , \quad e_{or,j} = -d\left( \sum_q g_{i,j}^{(q)} \phi_q / dt \right) \quad (20a,b)
\]

The summation here refers to all facets \( S_q \) of elements sharing the common edge \( P_iP_j \), while the parameter \( g_{i,j}^{(q)} \) is a weight with which the current/flux passing through the loop of the edge network – hence through the facet \( S_q \) – is taken when the current/flux passing through the loop of the facet network (the loop around the edge \( P_iP_j \)) is calculated. As may be seen in Fig. 1 the significant proportion of the vectors representing the functions of the facet element is perpendicular to the vectors representing the functions of the edge elements. Thus the scalar product \( w_{\beta} w_{er,j} \) of such vectors equals zero and the resulting weight parameter is also zero. For the element of Fig. 1a the only non-zero (and in fact equal to 1/8) weights are those which related to transformations between quantities related to facets and edges with parallel vectors.

All the component terms discussed above appearing in the equations for the nodal element method using scalar potential formulations and the edge element method using vector potentials formulations are tabulated and presented in Table 1.

<table>
<thead>
<tr>
<th>Potential, type of network</th>
<th>FE equations</th>
<th>Description of the integrands</th>
<th>FE coefficients</th>
<th>Substitutions, entries to RHS vector</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Omega ), Edge-magnetic</td>
<td>( \sum \phi_{i,j} = 0 )</td>
<td>( \phi_{i,j} = \int_\epsilon w_{\epsilon,i,j} B dv )</td>
<td>( B = aH =</td>
<td>u</td>
</tr>
<tr>
<td>( V ), Edge-electric</td>
<td>( \sum i_{i,j} = 0 )</td>
<td>( i_{i,j} = \int_\epsilon w_{\epsilon,i,j} J dv )</td>
<td>( J = \lambda E = \gamma \sum w_{\epsilon,p,q} \delta E_{p,q} )</td>
<td>( u_{E,p,q} = V_{p,q} + \epsilon_{p,q} \sum \delta_{p,q} )</td>
</tr>
<tr>
<td>( A ), Facet-magnetic</td>
<td>( \sum u_{\beta,m} = 0 )</td>
<td>( u_{\beta,m} = \int_\beta w_{\beta} \text{grad} \Omega dv )</td>
<td>( \text{grad} \Omega = -B - T )</td>
<td>( \sum w_{\beta,j} \phi_{j} - \int_\epsilon w_{\beta,j} \text{grad} \phi_{j} dv )</td>
</tr>
<tr>
<td>( T ), Facet-electric</td>
<td>( \sum i_{\beta,m} = 0 )</td>
<td>( i_{\beta,m} = \int_\beta w_{\beta} \text{grad} \Omega dv )</td>
<td>( \text{grad} \Omega = -E + A )</td>
<td>( \sum w_{\beta,j} \phi_{j} )</td>
</tr>
</tbody>
</table>

Comments: \( \phi_{\beta}, i_{i,j} \) are fluxes and currents associated with edges \( P_iP_j \) of common node; \( u_{H,p,q}, u_{E,p,q} \) are edge values \( H \) and \( E \) for edges \( P_iP_j, u_{h,m}, u_{p,q} \) are voltages associated with facets \( S_q \) of common edge; \( \phi_q, i_q \) are facet values of \( B \) and \( J \) expressed by edge values of \( A \) and \( T \); \( \phi_{\beta}, i_{\beta,j} \) are edge values of \( A \) and \( T \) for \( P_iP_j \); \( \phi = \gamma_{\Omega} + \phi_e \); \( \epsilon = d/dt \) \( \Omega = 1/\gamma \)

**Approximate description of integrals in FE equations**

The integrals describing coefficients of the finite element method may be easily established using accurate analytical methods, but only for regular multi-sided elements and linear materials. Irregular elements and non-linear material properties inevitably call for approximate numerical methods. The authors of the article recommend the following approximation

\[
\int f dv = \frac{V}{V_{\epsilon}} \sum_{i=1}^{n_{\epsilon}} f(P_i),
\]

where \( n_{\epsilon} \) is the number of element nodes \( P_i \) and \( f(P_i) \) is the value of the function \( f \) in the node \( P_i \) [5]. Application of this approximation results in significant simplification of the description of the coefficients of a mesh made up of parallelepiped elements. As an example, consider the
coefficients of the equations describing the magnetic field in a magnetically non-linear region. When a scalar potential formulation $\Omega$ is employed the equation coefficients represent the permeance

$$\Lambda_{i,j}^{(p,q)} = \int_{e} w_{i,j}^p w_{e,p,q}^q \, dv,$$

(22)

whereas if a vector potential $A$ is used they represent the reluctance

$$R_{i}^{(q)} = \int_{e} w_{j/q} A \, dv,$$

(23)

where $\mu = \mu(H)$ and $\nu = \mu^{-1} = \nu(B)$.

Applying (21) yields

$$\Lambda_{i,j}^{(p,q)} = \mu_{i,j} V_e / l_{ij}^2, \quad \Lambda_{i,j}^{(p,q)} = 0 \text{ for } i \neq j, q,$$

(24a,b)

$$R_{i}^{(q)} = \nu V_e / S_i^2, \quad R_{i}^{(q)} = 0 \text{ for } i \neq q,$$

(25a,b)

where $l_{ij}$ is the length of the edge $P_iP_j$, $S_i$ is the surface area of the facet $S_i$, $\mu_{i,j}$ describes the magnetic permeability in the proximity of the edge $P_iP_j$, and $v_i$ is the reluctivity of the medium in the area close to the facet $S_i$. For a system with magnetic non-linear characteristics

$$\mu_{i,j} = 0.5(\mu(H_i) + \mu(H_j)), \quad v_i = 0.25 \sum_{p=1}^{n} \nu(B_p),$$

(26a,b)

where $H_i$ is the field intensity in the proximity of the node $P_i$ of the edge $P_iP_j$, and $B_p$ the flux density in the proximity of $P_q$ may be calculated as

$$H_i = 1_x \phi_1 (h_x) + 1_y \phi_2 (h_y) + 1_z \phi_3 (h_z)$$

(27)

$$B_p = 1_x \phi_1 (h_x) + 1_y \phi_2 (h_y) + 1_z \phi_3 (h_z)$$

(28)

When deriving the above equations it has been taken into account that the facet vectors are directed 'into' the element, whereas the direction of the edges depends on the sequence of the indices in the description of the edge value (see Figs 1a and 2a).

A closer inspection of (24) and (25) reveals that in the models following from application of the recommended approximating formula (21) there are no couplings between branches; hence the resulting mass matrix is diagonal and the coefficients express only self permeances/reluctances. Expressions (24a) and (25a) describing these permeances/reluctances are identical to those obtained form a classical formulation using magnetic networks, whose parameters may be established via the tubes and slices method [7]. For a hexahedron the geometrical forms are as shown in Fig. 3; it is taken into account that the permeance of the branch $P_iP_j$ of the edge network is related to the 'magnetic conductance' of a block of length $l_{ij}$ and cross section $0.25h_xh_y$ (Fig. 3a). The reluctance in the branch $Q,S_i$ of the facet network, on the other hand, represents the 'magnetic resistance' of a block of length $0.5l_{ij}$ and cross section $h_xh_y$. 
Conclusion

Finite element equations may be derived from equivalent circuit models without the need for variational formulation. By exploiting the geometrical properties of interpolating functions the relevant parameters may be established using integral methods. The presented approach is valid not only for hexahedra and pentahedra but also for tetrahedra [1] and mixed finite elements [6]. The proposed approach promises to be very beneficial in teaching, especially to students well familiar with circuit methods, to whom the analogy of the finite element formulation to loop or nodal magnetic or electric networks may be appealing and easier to understand. Thus the teaching of computational electromagnetics may be seen as supplementing the circuit theory by the relevant information about the integral methods of calculating network model parameters, as argued in this paper. The presented methods are also helpful when formulating classical network models, such as described in [4].

References