Technical Report 2: Mathematical Analysis for Shape Registration based on Signed Distance Functions

Sasan Mahmoodi,

School of Electronic and Computer Science, Building 1, Southampton University, Southampton, SO17 1BJ, UK,

In this technical report, we prove some theorems leading to new mathematical expressions. These expressions then can be exploited to form a fast and reliable similarity registration technique based on signed distance functions. Let us initially define signed distance function of shape p as:

$$\phi_p(x,y)=\left\{\begin{array}{ll} D_E((x,y),B), & (x,y)\in I_p, \\ -D_E((x,y),B), & (x,y)\in \Omega - I_p, \end{array}\right\} \tag{1}$$

where $\phi_B(x,y):\Omega\rightarrow\Re$ is a Lipschitz function and Ω is the bounded domain, D_E stands for minimum Euclidean distance between perimeter B of the shape and the domain Ω, and I_p is the subset of Ω representing the interior of the shape.

We basically start from the following dissimilarity measure introduced by Paragois *et al.* [1]:

$$E=\iint_{\Omega}\left|\phi_p(x,y)-\frac{1}{s}\phi_q\left(sR_\theta\left(x+T_x, y+T_y\right)\right)\right|^2 dxdy \tag{2}$$

where $\phi_p(x,y):\Omega\rightarrow\Re$ and $\phi_q(x,y):\Omega\rightarrow\Re$ are signed distance functions of shapes p and q:

Registration between two shapes aims to retrieve transform parameters s, θ, T_x and T_y (scaling, rotation, and translations along x and y axes respectively) minimizing dissimilarity measure (2) between ϕ_p and ϕ_q, such that

$$\left(\hat{\theta},\hat{s},\hat{T}_x,\hat{T}_y\right)=\underset{\theta, s\, T_x, T_y}{\arg\min}\, E \tag{3}$$

where Ω, $\hat{\theta},\hat{s},\hat{T}_x,\hat{T}_y$ and R_θ are image domain, the estimated angle, scale, translations parameters, and a conventional rotation (transform) matrix $R_\theta=\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}$. respectively.

a) *Rotation*

Let us now assume that the two shapes are centered at the origin of the coordinate system and we would like to find the optimal rotation angle minimizing the following dissimilarity measure.

$$E_\theta=\iint_{\Omega}\left|\phi_p(x,y)-\phi_q\left(R_\theta(x,y)\right)\right|^2 dxdy \tag{4}$$

Centralized shapes are mapped to polar coordinates i.e., $\hat{\phi}_p(\rho,\omega)$ and $\hat{\phi}_q(\rho,\omega)$ such that $x = \rho\cos\omega$ and $y = \rho\sin\omega$. In theorem 1, we prove that the rotation angle minimizing term (5) minimizes dissimilarity measure (4):

$$E_\theta = \int_0^{+\infty}\int_0^{2\pi}\left|\hat{\phi}_p(\rho,\omega) - \hat{\phi}_q((\rho,\omega+\theta))\right|^2 \mathrm{d}\rho\,\mathrm{d}\omega, \tag{5}$$

Theorem 1: *The minimizer of dissimilarity term (4) is the minimizer of term (5) where* $\Lambda = R^2$.

Proof:

Dissimilarity measure (4) can be written in polar coordinate system, i.e.:

$$E_\theta = \iint_{R^2}\left|\phi_p(x,y) - \phi_q(R_\theta(x,y))\right|^2 dxdy = \int_0^{+\infty}\int_0^{2\pi}\left|\phi_p(\rho,\omega) - \phi_q(R_\theta(\rho,\omega))\right|^2 \rho\, d\rho\, d\omega \tag{6}$$

where ρ and ω are polar coordinates so that $x = \rho\cos\omega$, and $y = \rho\sin\omega$.

In a polar coordinate system, term (6) can be written as:

$$E_\theta = \lim_{L\to+\infty}\int_0^{L}\int_0^{2\pi}\left|\phi_p(\rho,\omega) - \phi_q(\rho,\theta+\omega)\right|^2 \rho d\rho d\omega < \lim_{L\to+\infty} L\int_0^{L}\int_0^{2\pi}\left|\phi_p(\rho,\omega) - \phi_q(\rho,\theta+\omega)\right|^2 d\rho d\omega \tag{7}$$

It is easy to see from (6) and (7) that a parameter $\hat{\theta}$ minimizing $\int_0^{+\infty}\int_0^{2\pi}\left|\phi_p(\rho,\omega) - \phi_q(\rho,\theta+\omega)\right|^2 d\rho d\omega$ is a minimzer of the term $\iint_{R^2}\left|\phi_p(x,y) - \phi_q(R_\theta(x,y))\right|^2 dxdy$.

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b) ***Scaling***

The dissimilarity measure (2) is reduced to the following term when two shapes are different in scaling:

$$E_s = \iint\left|\phi_p(x,y) - \frac{1}{s}\phi_q(s(x,y))\right|^2 dx\,dy, \tag{8}$$

where the relation between two shapes' SDFs which have different scales is well known to be:

$$s\hat{\phi}_p(x,y) = \hat{\phi}_q(sx,sy), \tag{9}$$

In Theorem 2, we prove that the scaling parameter s minimizing the following term is a minimizer of term (8).

$$E'_s = \left|\sum_{m=0}^{M}\sum_{n=0}^{N}\left(M_{mn}^{p} - \frac{M_{mn}^{q}}{s^{m+n+3}}\right)\right|^2 \tag{10}$$

where M_{mn}^p and M_{mn}^q are respectively the $(m+n)^{\text{th}}$ order geometrical moments of $\hat{\phi}_p(x,y)$ and $\hat{\phi}_q(x,y)$ defined as:

$$M_{mn}^{p} = \iint_{\Omega} x^m y^n \hat{\phi}_p(x, y)\,dx\,dy$$
$$M_{mn}^{q} = \iint_{\Omega} x^m y^n \hat{\phi}_q(x, y)\,dx\,dy$$

Theorem 2: *The scaling parameter minimizing term (8) is also the minimizer of term (10).*

Before proving theorem 2, we need to visit theorem 3:

Theorem 3: *Let a geometrical moment with orders m and n of signed distance function (SDF)* $\phi_q(x, y): \Omega \to R$ *be* M_{mn}^{q}. *The geometrical moment with orders m and n of the scaled SDF with scaling parameter s is* $\dfrac{M_{mn}^{q}}{s^{m+n+3}}$.

Proof:

$$M_{mn}^{q} = \iint_{\Omega} x^m y^n \phi_q(x, y)\,dxdy \tag{11}$$

The m^{th} and n^{th} order moment of the scaled SDF $\frac{1}{s}\phi_q(sx, sy)$ is therefore calculated as:

$$M'^{q}_{mn} = \iint_{\Omega} x^m y^n \left(\frac{1}{s}\phi_q(sx, sy)\right) dxdy \tag{12}$$

By changing the variables $X = sx$ and $Y = sy$, equation (12) is rewritten as:

$$M'^{q}_{mn} = \iint_{\Omega} \left(\frac{X}{s}\right)^m \left(\frac{Y}{s}\right)^n \left(\frac{1}{s}\phi_q(X, Y)\right) \frac{dXdY}{s^2} = \frac{1}{s^{m+n+3}} \iint_{\Omega} X^m Y^n \phi_q(X, Y)\,dXdY = \frac{M_{mn}^{q}}{s^{m+n+3}} \tag{13}$$

$\blacksquare$

Proof of theorem 2: The signed distance functions $\phi_p(x, y)$ and $\phi_q(x, y)$ can be approximated in terms of their geometrical moments, i.e.:

$$\phi_i(x, y) \approx \sum_{m=0}^{M} \sum_{n=0}^{N} M_{mn}^{i} x^m y^n \tag{14}$$

where $M_{mn}^{i} = \iint_{\Omega} x^m y^n \phi_i(x, y)\,dxdy$ and i can be p or q. M and N are the total number of geometrical moments around the axes x and y. If equations (14) are substituted in equation (8), by using the result of theorem 3, one can obtain:

$$E_s \approx \iint_{\Omega} \left| \sum_{m=0}^{M} \sum_{n=0}^{N} M_{mn}^{p} x^m y^n - \sum_{m=0}^{M} \sum_{n=0}^{N} \frac{M_{mn}^{q}}{s^{m+n+3}} x^m y^n \right|^2 dxdy = \iint_{\Omega} \left| \sum_{m=0}^{M} \sum_{n=0}^{N} \left(M_{mn}^{p} - \frac{M_{mn}^{q}}{s^{m+n+3}} \right) x^m y^n \right|^2 dx\,dy \tag{15}$$

By letting a_{mn} denote $\left(M_{mn}^{p} - \dfrac{M_{mn}^{q}}{s^{m+n+3}} \right)$, equation (15) can be written as:

$$E_s = \iint_{\Omega} \left| \sum_{m=0}^{M}\sum_{n=0}^{N} a_{mn} x^m y^n \right|^2 dxdy = \iint_{\Omega} \left(\sum_{m=0}^{M}\sum_{n=0}^{N} a_{mn}^2 x^{2m} y^{2n} + 2\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{\substack{m'=0\\ m'\neq m}}^{M}\sum_{\substack{n'=0\\ n'\neq n}}^{N} a_{mn} a_{m'n'} x^{m+m'} y^{n+n'} \right) dxdy \tag{16}$$

Without loss of generality, for $\Omega = [0, L] \times [0, H]$, E_s in (16) can be calculated as:

$$E_s = \left(\sum_{m=0}^{M}\sum_{n=0}^{N} \left(a_{mn}^2 \frac{L^{2m+1}}{2m+1} \frac{H^{2n+1}}{2n+1} \right) + 2\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{\substack{m'=0\\ m'\neq m}}^{M}\sum_{\substack{n'=0\\ n'\neq n}}^{N} \left(a_{mn} a_{m'n'} \frac{L^{m+m'+1}}{m+m'+1} \frac{H^{n+n'+1}}{n+n'+1} \right) \right) \tag{17}$$

It is easy to conclude from (17) that

$$E_s < L^{2M+1} H^{2N+1} \left(\sum_{m=0}^{M}\sum_{n=0}^{N} \left(a_{mn}^2 \right) + 2\sum_{m=0}^{M}\sum_{n=0}^{N}\sum_{\substack{m'=0\\ m'\neq m}}^{M}\sum_{\substack{n'=0\\ n'\neq n}}^{N} \left(a_{mn} a_{m'n'} \right) \right) = L^{2M+1} H^{2N+1} \left| \sum_{m=0}^{M}\sum_{n=0}^{N} a_{mn} \right|^2 \tag{18}$$

By recalling that the term $a_{mn} = \left(M_{mn}^{p} - \frac{M_{mn}^{q}}{s^{m+n+3}} \right)$ is the only term which is a function of the scaling parameter s and from inequality (18), it is straightforward to see that the scaling parameter s minimizing $\left| \sum_{m=0}^{M}\sum_{n=0}^{N} \left(M_{mn}^{p} - \frac{M_{mn}^{q}}{s^{m+n+3}} \right) \right|^2$ is a minimizer of E_s given in equation (8).

$\blacksquare$

References:

[1] N., Paragios, M. Rousson, and V. Ramesh, "Non-rigid registration using distance functions" *Computer Vision and Image Understanding*, **89**(2-3) pp. 142-165, 2003.