Polynomial-exponential 2D data models, Hankel-block-Hankel matrices and zero-dimensional ideals

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Abstract

Subspace-based methods are popular for analysis of two-dimensional data that can be modeled by sums of polynomially modulated exponential (or "polynomial-exponential") functions. In this paper we touch some problems concerning rank properties of Hankel-block-Hankel matrices, which are used in subspace-based methods. We review the correspondence between polynomial-exponential functions and zero-dimensional ideals. Then we demonstrate the usefulness of this correspondence for the problems being considered.

Keywords: 2D data analysis; subspace-based methods; signal subspace; Hankel-block-Hankel matrix; polynomially modulated exponential function; array of finite rank; k-linear recurrent sequence; admissible window sizes; annihilator; zero-dimensional ideal.

1 Introduction

In many problems of 2D data analysis the input data is a two-dimensional function $f : \mathbb{R}^2 \mapsto \mathbb{R}(\mathbb{C})$ measured on a uniform rectangular grid (a digital image is a standard example here). We consider the ubiquitous case when the data is composed into an $N_x \times N_y$ matrix of values $\mathrm{F} = (f_{m,n})_{m,n=0}^{N_x-1,N_y-1}$, which will be called here *2D data array* (or simply *2D array*).

The most common task of data analysis is to decompose the input 2D array into sum

$$\mathrm{F} = \mathrm{F}^{(S)} + \mathrm{F}^{(N)}$$

of a signal component $\mathrm{F}^{(S)}$ and a noise component $\mathrm{F}^{(N)}$ (not necessarily unstructured or random). This decomposition can be motivated by the nature of data (i.e. there is a well-grounded model for the origin of the data) or the input data is being approximated by signals from a certain model class.

An important class of signals is the class of polynomially modulated exponential functions $\mathbb{N}_0^2 \to \mathbb{C}$ (where $\mathbb{N}_0 \stackrel{\mathrm{def}}{=} \mathbb{N} \cup \{0\}$)

$$f^{(S)}_{m,n} = \sum_{k=1}^{r} q_k(m,n)\lambda_k^m \mu_k^n, \tag{1}$$

where $\lambda_k, \mu_k \in \mathbb{C}$, and $q_k(m,n)$ are some complex polynomials. This class of signals is common to various problems such as parameters estimation in radar imaging [3] or analysis of textured images [6].

Along with classical approaches, non-parametric *subspace-based methods* recently received much attention [2, 3, 6]. These methods are based on embedding of the data into a structured (Hankel-block-Hankel) matrix, which has low rank when the noise is absent ($\mathrm{F}^{(N)} = 0$). In presence of noise the signal/noise decomposition can be achieved by approximating a Hankel-block-Hankel matrix with a matrix of low rank (for example, via the SVD).

The Hankel-block-Hankel matrix is generated by a pair of parameters (*window sizes*). Arrays of form (1) have maximal rank for a range of so-called *admissible* window sizes. In this paper we review correspondence between arrays of form (1) and polynomial ideals, we use this correspondence for determining the range of admissible window sizes, thus extending the results of [2].

2 Hankel-block-Hankel matrices and polynomial ideals

In this section we first introduce Hankel-block-Hankel matrices, then proceed to infinite arrays of finite rank and finally describe properties of arrays of finite rank using the language of polynomial ideals.

2.1 Hankel-block-Hankel matrices

Let us consider in detail the construction of a Hankel-block-Hankel matrix from the input array F. Given a pair of parameters (L_x, L_y), $1 \leq L_x \leq N_x$, $1 \leq L_y \leq N_y$ (*window sizes*), we define the following submatrices

$$\mathrm{F}_{k,l}^{(L_x,L_y)} \stackrel{\mathrm{def}}{=} \begin{pmatrix} f_{k,l} & \cdots & f_{k,l+L_y-1} \\ \vdots & \ddots & \vdots \\ f_{k+L_x-1,l} & \cdots & f_{k+L_x-1,l+L_y-1} \end{pmatrix}, \tag{2}$$

where $0 \leq k < K_x$, $0 \leq l < K_y$ and $K_x \stackrel{\mathrm{def}}{=} N_x - L_x + 1$, $K_y \stackrel{\mathrm{def}}{=} N_y - L_y + 1$. Then we compose the matrix

$$\mathbf{W} = [W_1 : \ldots : W_{K_x K_y}],$$

from the vectorizations of (L_x, L_y)-submatrices, i.e.

$$W_{1+k+lK_x} = \mathrm{vec}(\mathrm{F}_{k,l}^{(L_x,L_y)}) \quad \text{for} \quad 0 \leq k < K_x,\ 0 \leq l < K_y, \tag{3}$$

where

$$\mathrm{vec}(a_{mn})_{m,n=1}^{M,N} \stackrel{\mathrm{def}}{=} (a_{11}, \ldots, a_{M1}; a_{12}, \ldots, a_{M2}; \ldots; a_{1N}, \ldots, a_{MN})^{\mathrm{T}} \in \mathbb{C}^{MN}.$$

The matrix $\mathbf{W}$ is called Hankel-block-Hankel since it is a *block Hankel matrix* [2], i.e. it can be represented in the form:

$$\mathbf{W} = \mathbf{W}^{(L_x,L_y)}(\mathrm{F}) = \begin{pmatrix} \mathbf{H}_0 & \mathbf{H}_1 & \mathbf{H}_2 & \cdots & \mathbf{H}_{K_y-1} \\ \mathbf{H}_1 & \mathbf{H}_2 & \mathbf{H}_3 & \cdots & \mathbf{H}_{K_y} \\ \mathbf{H}_2 & \mathbf{H}_3 & \iddots & \iddots & \vdots \\ \vdots & \vdots & \iddots & \iddots & \vdots \\ \mathbf{H}_{L_y-1} & \mathbf{H}_{L_y} & \cdots & \cdots & \mathbf{H}_{N_y-1} \end{pmatrix}, \tag{4}$$

and, in addition, each block is a *Hankel matrix*

$$\mathbf{H}_n \stackrel{\mathrm{def}}{=} \begin{pmatrix} f_{0,n} & f_{1,n} & \cdots & f_{K_x-1,n} \\ f_{1,n} & f_{2,n} & \cdots & f_{K_x,n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{L_x-1,n} & f_{L_x,n} & \cdots & f_{N_x-1,n} \end{pmatrix}. \tag{5}$$

The rank of the Hankel-block-Hankel matrix $\mathbf{W}^{(L_x,L_y)}$ is equal to the dimension of the space $\mathcal{L}^{(L_x,L_y)}(\mathrm{F}) \stackrel{\mathrm{def}}{=} \mathrm{span}(\{\mathrm{F}_{k,l}^{(L_x,L_y)}\}_{k,l=0}^{K_x-1,K_y-1})$.

2.2 Arrays of finite rank

Consider an infinite 2D-array with complex entries $\mathcal{F} = (f_{m,n})_{m,n=0}^{+\infty} \in \mathbb{C}^{\mathbb{N}_0^2}$, where $\mathbb{C}^{\mathbb{N}_0^2}$ denotes the space of infinite arrays. The (k,l)-*shift* of the array $\mathcal{F}$ is defined as the infinite subarray starting from element (k,l): $\mathcal{F}_{k,l} \stackrel{\mathrm{def}}{=} (f_{m+k,n+l})_{m,n=0}^{+\infty}$. The *space of shifts* is, by definition, $\mathcal{L}(\mathcal{F}) \stackrel{\mathrm{def}}{=} \mathrm{span}(\{\mathcal{F}_{k,l}\}_{k,l=0}^{+\infty}) \subseteq \mathbb{C}^{\mathbb{N}_0^2}$. The dimension $\mathrm{r}(\mathcal{F}) \stackrel{\mathrm{def}}{=} \dim \mathcal{L}(\mathcal{F})$ is also called *linear complexity* elsewhere [1].

Consider the space spanned by finite windows as well. Let $\mathrm{F}_{k,l}^{(L_x,L_y)} \stackrel{\text{def}}{=} (f_{m+k,n+l})_{m=0,n=0}^{L_x-1,L_y-1}$ denote an $L_x \times L_y$ submatrix of the infinite array $\mathcal{F}$ and define the $(L_x, L_y)-$*trajectory space*

$$\mathcal{L}^{(L_x,L_y)}(\mathcal{F}) = \mathrm{span}(\{\mathrm{F}_{k,l}^{(L_x,L_y)}\}_{k,l=0}^{+\infty}).$$

The following trivial lemma relates the (L_x, L_y)-trajectory space of windows with the space of shifts.

Lemma 1

$$\dim \mathcal{L}^{(L_x,L_y)}(\mathcal{F}) = \dim\{\mathcal{F}_{k,l}\}_{k,l=0}^{L_x-1,L_y-1}$$

Immediately, one can derive correspondence between dimensions of $\mathcal{L}^{(L_x,L_y)}(\mathcal{F})$ and $\mathcal{L}(\mathcal{F})$.

Proposition 1 ([5, Proposition 10] or [6, Proposition 4.1.1]) *Let $\mathcal{F}$ be an infinite array.*

- $\dim \mathcal{L}(\mathcal{F}) < +\infty$ *if and only if* $\dim \mathcal{L}^{(L_x,L_y)}(\mathcal{F}) < C$ *for any* $(L_x, L_y) \in \mathbb{N}^2$*, where* $C < +\infty$ *is some constant.*
- *If* $\dim \mathcal{L}(\mathcal{F}) = d < +\infty$*, then there exist* L_{x0}, L_{y0}*, such that* $\dim \mathcal{L}^{(L_x,L_y)}(\mathcal{F}) = d$ *if* $L_x \geq L_{x0}$ *and* $L_y \geq L_{y0}$.

Let us call arrays, which satisfy conditions of Proposition (1) *arrays of finite rank* [5]. They are called *k-linear recurrent sequences* elsewhere [1].

2.3 Arrays of finite rank and zero-dimensional ideals

Next, we describe arrays of finite rank using the language of polynomial ideals (for more details see [5]). Consider the space of complex polynomials $\mathbb{C}[x,y]$ and the dual space (space of linear functionals) $\mathbb{C}^*[x,y] \stackrel{\text{def}}{=} \mathrm{Hom}(\mathbb{C}[x,y],\mathbb{C})$, which is isomorphic to the space $\mathbb{C}^{\mathbb{N}_0^2}$ of infinite arrays. Indeed, each array $\mathcal{F}$ corresponds to the functional $\ell^{(\mathcal{F})}$ defined by $\ell^{(\mathcal{F})}(x^m y^n) \stackrel{\text{def}}{=} f_{m,n}$, and vice versa.

The space $\mathbb{C}^*[x,y]$ is a (left) $\mathbb{C}[x,y]$-module, where the multiplication $p(x,y)\cdot\ell$, $\ell \in \mathbb{C}^*[x,y]$, $p \in \mathbb{C}[x,y]$ is canonically defined as $(p\cdot\ell)(q) \stackrel{\text{def}}{=} \ell(p\cdot q)$. This operation can be better represented through the shifts of the corresponding array. For a polynomial $p(x,y) = \sum\limits_{(\alpha,\beta)\in\mathbb{N}_0^2} a_{(\alpha,\beta)} x^\alpha y^\beta \in \mathbb{C}[x,y]$ and an infinite array $\mathcal{F} \in \mathbb{C}^{\mathbb{N}_0^2}$ we introduce an operation of multiplication:

$$p\cdot\mathcal{F} = \sum_{\alpha,\beta=0}^{+\infty} a_{(\alpha,\beta)}\mathcal{F}_{\alpha,\beta}. \tag{6}$$

Evidently, $\ell^{(p\cdot\mathcal{F})} = p\cdot\ell^{(\mathcal{F})}$. The space of shifts $\mathcal{L}(\mathcal{F})$ of an array $\mathcal{F}$ then is a submodule of $\mathbb{C}^{\mathbb{N}_0^2}$: $\langle\mathcal{F}\rangle_{\mathbb{C}[x,y]} \cong \langle\ell^{(\mathcal{F})}\rangle_{\mathbb{C}[x,y]} \subset \mathbb{C}^*[x,y]$.

For a set of infinite arrays $\mathfrak{G}$ we introduce the notion of *annihilator* of $\mathfrak{G}$:

$$\mathcal{I}(\mathfrak{G}) \stackrel{\text{def}}{=} \{p \in \mathbb{C}[x,y] :\ p\mathcal{G} = 0 \quad \forall \mathcal{G}\in\mathfrak{G}\}.$$

Clearly, annihilator is a polynomial ideal. The following theorem characterizes arrays of finite rank through their annihilator, defined as $\mathcal{I}(\mathcal{F}) \stackrel{\text{def}}{=} \mathcal{I}(\{\mathcal{F}\}) = \mathcal{I}(\mathcal{L}(\mathcal{F}))$

Theorem 1 ([5, Corollary 2]) $\dim \mathcal{L}(\mathcal{F}) < +\infty$ *if and only if* $\mathcal{I}(\mathcal{F})$ *is zero-dimensional.*

Moreover, the set of zeros $\mathcal{Z}(\mathcal{I}) \stackrel{\text{def}}{=} \{(x,y)\in\mathbb{C}^2 : p(x,y) = 0 \,\text{for all}\, p \in \mathcal{I}\}$ of annihilator ideal gives an explicit form of the array of finite rank.

Theorem 2 ([5, Proposition 7] or [6, Corollary 2.2.2]) *An array $\mathcal{F}$ is of finite rank if and only if it has representation* (1) *where* $\mathcal{Z}(\mathcal{I}(\mathcal{F})) = \{(\lambda_1,\mu_1),\ldots,(\lambda_r,\mu_r)\}$.

3 Admissible window sizes

In this section we demonstrate how the behavior of the rank of Hankel-block-Hankel matrix for finite subarray of array of finite rank can be expressed through the set of admissible window sizes of the infinite array. Finally, we prove the bounds for the infinite array.

3.1 Rank of Hankel-block-Hankel matrix and admissible window sizes

Definition 1 Let $\mathcal{F}$ be an array of finite rank, $\dim \mathcal{L}(\mathcal{F}) = d < +\infty$.
The set of *admissible window sizes* is defined as:

$$\mathfrak{M}(\mathcal{F}) \stackrel{\text{def}}{=} \{(L_x, L_y) \in \mathbb{N}^2 : \dim \mathcal{L}^{(L_x,L_y)}(\mathcal{F}) = d\} \subset \mathbb{N}^2,$$

each pair $(L_x, L_y) \in \mathfrak{M}(\mathcal{F})$ is called admissible window sizes.

By Lemma 1 $\dim \mathcal{L}^{(\alpha,\beta)}(\mathcal{F}) \le \dim \mathcal{L}^{(\delta,\gamma)}(\mathcal{F})$ if $\alpha \le \delta$ and $\beta \le \gamma$. By Proposition 1, $\mathfrak{M}(\mathcal{F})$ is closed with respect to taking greater by partial order elements. Let us write this observation in a compact form.

Remark 3.1 The set $\{x^\alpha y^\beta\}_{(\alpha,\beta)\in\mathfrak{M}(\mathcal{F})}$ is a monomial ideal.

Now we are ready to go back to the finite array and rank of Hankel-block-Hankel matrices. Let $\mathrm{F} = (f_{m,n})_{m,n=0}^{N_x-1,N_y-1}$ be the $N_x \times N_y$ subarray of $\mathcal{F}$, $\dim \mathcal{L}(\mathcal{F}) = d$.

Proposition 2 ([6, Corollary 4.2.4]) *The set of admissible window sizes for* F *can be found as*

$$\mathfrak{M}(\mathrm{F}) \stackrel{\text{def}}{=} \{(L_x, L_y) \in \mathbb{N}^2 : \operatorname{rank} \mathbf{W}^{(L_x,L_y)}(\mathrm{F}) = d\} = \mathfrak{M}(\mathcal{F}) \cap ((N_x+1, N_y+1) - \mathfrak{M}(\mathcal{F})).$$

On Fig. 1 a sample set of admissible window sizes for a finite array is shown.

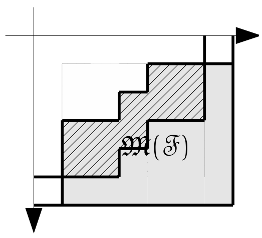

Figure 1: Set of admissible window sizes

3.2 Main results

Proposition 1 states only the existence of a subset of $\mathfrak{M}(\mathcal{F})$ (a monomial subideal). For instance, if $\dim \mathcal{L}(\mathcal{F}) = d$, it is easy to show that $(d,d) + \mathbb{N}_0^2 \subset \mathfrak{M}(\mathcal{F})$, where for a set $\mathfrak{B} \subset \mathbb{N}_0^2$ addition is defined as $\mathfrak{B} + (k,l) \stackrel{\text{def}}{=} \{(\alpha,\beta) \in \mathbb{N}_0^2 : (\alpha - k, \beta - l) \in \mathfrak{B}\}$.
For a finite set of indices $\mathfrak{A} \subset \mathbb{N}_0^2$ denote

$$\mathrm{B}_x(\mathfrak{A}) \stackrel{\text{def}}{=} 1 + \min\{\alpha : (\mathfrak{A} - (\alpha, 0)) \cap \mathbb{N}_0^2 = \varnothing\},$$
$$\mathrm{B}_y(\mathfrak{A}) \stackrel{\text{def}}{=} 1 + \min\{\beta : (\mathfrak{A} - (0, \beta)) \cap \mathbb{N}_0^2 = \varnothing\}.$$

Let also $\mathrm{LT}_\prec(\mathcal{I}) \subset \mathbb{N}_0^2$ denote the set of degrees of leading terms of the ideal $\mathcal{I}$, with respect to the ordering $\prec$.

Theorem 3 ([6, Theorem 4.2.1]) *Let* $\mathfrak{G}_x = \mathbb{N}_0^2 \setminus \mathrm{LT}_{y\succ x}(\mathcal{I})$ *and* $\mathfrak{G}_y = \mathbb{N}_0^2 \setminus \mathrm{LT}_{x\succ y}(\mathcal{I})$.
Then

$$(\mathrm{B}_x(\mathfrak{G}_x), \mathrm{B}_y(\mathfrak{G}_x)), (\mathrm{B}_x(\mathfrak{G}_y), \mathrm{B}_y(\mathfrak{G}_y)) \in \mathfrak{M}(\mathcal{F})$$

and for all $(L_x, L_y) \in \mathfrak{M}(\mathcal{F})$ *the following inequalities hold*

$$L_y \geq \mathrm{B}_y(\mathfrak{G}_x), \quad L_x \geq \mathrm{B}_x(\mathfrak{G}_y).$$

Note that the bounds for admissible window sizes in the case of sum of (not modulated) complex exponents were first proved in [2, Theorem 1]. Let us formulate the bounds from [2] as a simple corollary of Theorem 3. We provide here the proof to show its simplicity and because the proof in [6] was incorrect.

Corollary 1 ([6, Corollary 4.2.3]) *Let*

$$f_{m,n} = c_1 \lambda_1^m \mu_1^n + \ldots + c_r \lambda_r^m \mu_r^n, \tag{7}$$

where $c_l \neq 0$ *and pairs* (λ_l, μ_l) *are different. Denote by* d_x *and* d_y *the number of different values among* $\lambda_1, \ldots, \lambda_r$ *and* $\mu_1, \ldots, \mu_r$ *correspondingly. Denote* m_x *and* m_y *the maximal multiplicity of the same value among* $\lambda_1, \ldots, \lambda_r$ *and* $\mu_1, \ldots, \mu_r$*. Then* $(\mathrm{B}_x(\mathfrak{G}_x), \mathrm{B}_y(\mathfrak{G}_x)) = (d_x, m_x)$ *and* $(\mathrm{B}_x(\mathfrak{G}_y), \mathrm{B}_y(\mathfrak{G}_y)) = (m_y, d_y)$*.*

Proof.
Let us prove, for instance, the first equality. Let $\lambda_1, \ldots, \lambda_k$ be different numbers. Let also $r_1, \ldots, r_k \in \mathbb{N}$, $r_1 \leq \ldots \leq r_k$, $r_1 + \ldots + r_k = r$, and the pairs exponents in (7) be

$$\begin{aligned} \{\ & (\lambda_k, \mu_{k,1}), \ldots, (\lambda_k, \mu_{k,r_k}), \\ & \vdots \\ & (\lambda_1, \mu_{1,1}), \ldots, (\lambda_1, \mu_{1,r_1})\} \subset \mathbb{C}^2. \end{aligned}$$

By [4, Theorem 1], we have

$$\begin{aligned} \mathfrak{G}_x = \{\ & (0,0), \ldots, (0, r_k - 1), \\ & (1,0), \ldots, (1, r_{k-1} - 1), \\ & \vdots \\ & (k-1, 0), \ldots, (k-1, r_1 - 1)\}. \end{aligned}$$

It is left to note that $k = d_x$ and $r_k = m_x$. $\square$

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