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LETTER TO THE EDITOR

Phase critical point densities in planar isotropic random waves

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Abstract

The densities of critical points of phase (extrema and saddles), which play an important role in the theory of phase singularities (wave dislocations) in two dimensions, are calculated in isotropic plane wave superpositions. Critical points and dislocations are put on an equal footing as zeros of the two-dimensional current (Poynting vector), and the results, depending only on the second and fourth moments of the wave spectrum (distribution of wavenumbers), are related to the corresponding dislocation density. Explicit results for several spectra are derived, discussed and related to previous results.

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Critical points of phase in two-dimensional wavefields are places (generically points) where the gradient of phase vanishes, and can either be phase extrema (maxima or minima) or saddle points. They are topologically related to the points, called phase singularities, wave dislocations or optical vortices (Nye and Berry 1974, Berry 1998a, Nye 1999), where the phase is singular (phase gradient undefined), and saddles play an important role in the two-dimensional theory of dislocations (e.g. Freund 1995, Berry 1998b, Nye 1998). It is recent work in the statistical properties of dislocations in Gaussian random waves (Berry and Dennis 2000, hereafter referred to as BD, Saichev *et al* 2000, Halperin 1981), that has motivated this study of statistical densities of phase critical points, realized (along with dislocations) as zeros of the current (Poynting vector). I shall present calculations of the densities of both saddles (equation (26)) and extrema (equation (27)) in superpositions of plane waves, isotropically distributed in direction and with random phases, a model often used in statistical optics (being a good model for laser speckle patterns (Ochoa and Goodman 1983, Goodman 1985)), and for which the dislocation density has been calculated (BD, Halperin 1981, Berry 1978).

The calculations here are complementary to those of Weinrib and Halperin (1982), who find the densities of critical points of intensity (where dislocations appear as zeros/minima) in the same statistical model. Although not phase critical points, dislocations, being intensity minima, are intensity critical points, and although phase and intensity critical points occur in general in different places, their configurations are related (Shvartsman and Freund 1995).

Also, although important for two-dimensional waves, phase and intensity critical points in plane sections of three-dimensional waves play no special role in space, and I shall discuss this further at the end. The notation and approach follow closely that of BD, to which the reader is referred for further details and references. The two-dimensional complex wavefield, dependent on $\boldsymbol{R} \equiv (x, y)$, is denoted by $\psi = \psi(\boldsymbol{R}) = \sqrt{I}\exp(\mathrm{i}\chi) = \xi + \mathrm{i}\eta$, where I is the (real, positive) wave intensity, χ is the (real) phase and ξ, η are the real and imaginary parts of the field.

Critical points of phase occur when the gradient $\nabla\chi$ vanishes, and may be phase maxima, minima or saddles. The phase can also be singular ($\nabla\chi$ undefined), at points of vanishing amplitude, $\psi = I = 0$; these are the dislocations, around which the phase increases by an integer multiple of 2π, the sign of the dislocation depending on the direction of increase. Dislocations and critical points may be put on an equal footing by considering the planar current (Poynting vector) $\boldsymbol{J}$, (where subscripts after scalars denote partial derivatives)

$$\boldsymbol{J} \equiv \mathrm{Im}\,\psi^*\nabla\psi = (\xi\eta_x - \eta\xi_x, \xi\eta_y - \eta\xi_y) \tag{1}$$

which is zero at dislocations (where $\xi = \eta = 0$) and at critical points, where the fraction

$$\frac{\xi}{\eta} = \frac{\xi_x}{\eta_x} = \frac{\xi_y}{\eta_y} \tag{2}$$

is finite and nonzero.

Sufficiently close to dislocations, the current circulates in perfect circles (BD, Dennis 2001); the phase maxima are sinks, minima are sources and phase saddles are saddle points. The sources, sinks and circulations have Poincaré index +1, and the saddles have index -1 (Berry 1998a). All singularities considered have unit Poincaré index or dislocation strength, since only these are structurally stable and found generically.

As external parameters are varied, (such as the z coordinate if the wave is a plane section of a three-dimensional field), dislocations may be created or destroyed in pairs of opposite sign, and by conservation of Poincaré index, such a process must also involve two saddles (Nye *et al* 1988), or a maximum and a minimum, with saddle spectator (as in Freund and Kessler 2001, figure 4). Thus for stationary random fields, the mean density of zeros of $\boldsymbol{J}$, weighted by Poincaré index, is zero, implying that for dislocation density D, extrema density E and saddle density S,

$$D + E - S = 0. \tag{3}$$

This will be verified later for the Gaussian random waves considered here. With $\boldsymbol{M} = \partial_\alpha J_\beta$ ($\alpha, \beta = x, y$) the number of zero points of $\boldsymbol{J}$ in an area $\mathcal{A}$, $\#_{\mathcal{A}}$, is

$$\#_{\mathcal{A}} = \int_{\mathcal{A}} \mathrm{d}^2\boldsymbol{R}\,\delta^2(\boldsymbol{J})|\det\boldsymbol{M}| \tag{4}$$

$$= \int_{\mathcal{A}} \mathrm{d}^2\boldsymbol{R}\,\delta^2(\boldsymbol{J})|\det\boldsymbol{M}_{\mathrm{sym}}| + \int_{\mathcal{A}} \mathrm{d}^2\boldsymbol{R}\,\delta^2(\boldsymbol{J})|\det\boldsymbol{M}_{\mathrm{asym}}| \tag{5}$$

where $\boldsymbol{M}_{\mathrm{sym}}, \boldsymbol{M}_{\mathrm{asym}}$ are the symmetric and antisymmetric parts of $\boldsymbol{M}$ respectively, and the separation in (5) requires justification. If modulus signs are removed from (4), the zeros are weighted by their Poincaré index, and the integral counts the total index in $\mathcal{A}$. $\boldsymbol{M}_{\mathrm{sym}}, \boldsymbol{M}_{\mathrm{asym}}$ are

$$\det\boldsymbol{M}_{\mathrm{sym}} = \tfrac{1}{2}(\nabla\cdot\boldsymbol{J})^2 - (\xi\eta_{xy} - \eta\xi_{xy})^2 - \tfrac{1}{2}(\xi\eta_{xx} - \eta\xi_{xx})^2 - \tfrac{1}{2}(\xi\eta_{yy} - \eta\xi_{yy})^2 \tag{6}$$

$$\det\boldsymbol{M}_{\mathrm{asym}} = (\xi_x\eta_y - \xi_y\eta_x)^2 \tag{7}$$

$\boldsymbol{M}_{\mathrm{asym}}$ contributes only to the dislocations, since $\boldsymbol{M}_{\mathrm{asym}}$ and circulations/dislocations exchange sign under an exchange of x, y, and $\det\boldsymbol{M}_{\mathrm{sym}}$ is zero at a dislocation, since $\xi = \eta = 0$ there.

$\boldsymbol{M}_{\rm sym}$ contributes only to extrema and saddles, being invariant under exchange of x, y, and $\det \boldsymbol{M}_{\rm asym} = 0$ there by (2). This justifies (5), since the zeros of $\boldsymbol{J}$ are separated into two disjoint sets, the critical points and dislocations, at which $\det \boldsymbol{M}$ reduces to its symmetric and antisymmetric parts respectively. Thus the second term in (5) may be interpreted as the number of dislocations in the area $\mathcal{A}$, and is easily seen to be equal to the form used in earlier calculations (cf BD equation (2.5), Halperin 1981 equation (6.19)). The first term provides the number of critical points in $\mathcal{A}$, and $\det \boldsymbol{M}_{\rm sym}$ is positive for extrema, negative for saddles (removing the modulus signs in (4), (5) gives the total Poincaré index in $\mathcal{A}$, analogous to (3) on average). The form for $\det \boldsymbol{M}_{\rm sym}$ in (6) confirms that extrema are places of nonzero divergence of phase; for solutions of the planar Helmholtz equation $\psi_{\rm H}$, that is

$$\nabla_\perp^2 \psi_{\rm H} + K^2 \psi_{\rm H} = 0 \tag{8}$$

(where $\nabla_\perp^2 = \partial_x^2 + \partial_y^2$, and K is a unique wavenumber), the current is divergenceless since

$$\nabla \cdot \boldsymbol{J}_{\rm H} = \operatorname{Im}\{\nabla\psi_{\rm H}^* \cdot \nabla\psi_{\rm H} + \psi_{\rm H}^*\nabla^2\psi_{\rm H}\} = \operatorname{Im}\{-K^2|\psi_{\rm H}|^2\} = 0 \tag{9}$$

so the only critical points are (divergenceless) saddles, whose density must equal the dislocation density by (3). In waves not satisfying (8), there may be extrema, and the saddles are not necessarily divergenceless. This concludes the general remarks about critical points, and a description of the statistical model of Gaussian random waves follows.

The wavefield ψ is taken to be a linear superposition of infinitely many sinusoidal plane waves, with wavevectors $\boldsymbol{K}$,

$$\psi = \sum_{\boldsymbol{K}} a(K) \exp(\mathrm{i}(\boldsymbol{K}\cdot\boldsymbol{R} + \phi_{\boldsymbol{K}})) \tag{10}$$

where the phases $\phi_{\boldsymbol{K}}$ are uniformly distributed and random, and the real amplitude weighting $a(K)$ depends only on the length $K = |\boldsymbol{K}|$ of the wavevector (ensuring that ψ is statistically isotropic). The averaging shall be over different values of $\phi_{\boldsymbol{K}}$ for all $\boldsymbol{K}$, and this ensemble is stationary and ergodic (so spatial averages, such as $\lim_{\mathcal{A}\to\infty} \#_{\mathcal{A}}/\mathcal{A}$ in (4), can be replaced by ensemble averages). Ensemble averages are denoted by angle brackets $\langle\cdot\rangle$. The required critical point density $C \equiv S + E$, from (5), is

$$C = \langle \delta^2(\boldsymbol{J}) |\det \boldsymbol{M}_{\rm sym}|\rangle. \tag{11}$$

The spectral distribution of wavenumbers $\Pi(K)$ is defined by

$$\frac{1}{2}\sum_{\boldsymbol{K}} a(K) \approx \frac{1}{2\pi}\int \mathrm{d}^2\boldsymbol{K}\,\Pi(K)/K \tag{12}$$

in the limit of infinitely many closely spaced $\boldsymbol{K}$. The nth moment of K with respect to the distribution $\Pi(K)$ is denoted K_n, and normalized so that $K_0 = 1$. The critical point density will only involve K_2, K_4. ψ may be a plane section of an isotropic three-dimensional superposition (BD); in this case, the planar spectrum is a projection of a three-dimensional distribution with moments k_n, which are related to the projected moments K_n by $K_n/k_n = (\frac{1}{2})!(\frac{n}{2})!/(\frac{n+1}{2})!$, derived from equation (3.10) of BD.

Explicit results will be given for the following spectra:

(i) *Monochromatic waves in the plane (ring spectrum).* These satisfy the Helmholtz equation (8), and all waves in the superposition (10) have the same wavenumber K_m, wavelength $\Lambda_0 = 2\pi/K_m$. The spectrum $\Pi(K)$ is $\delta(K - K_m)$, implying that

$$K_{n\,\rm ring} = K_m^n. \tag{13}$$

(ii) *Disc spectrum.* $\Pi(K)/K$ is constant for $K \leqslant K_d = 2\pi/\Lambda_d$, and zero otherwise; this is the spectrum for uniform diffuse monochromatic light illuminating a plane in the far field, having passed through a circular aperture, with moments

$$K_{2\,\mathrm{disc}} = \tfrac{1}{2}K_d^2 \qquad K_{4\,\mathrm{disc}} = \tfrac{1}{3}K_d^4. \tag{14}$$

(iii) *Gaussian spectrum* (taken as a model of a transverse section of a laser speckle pattern, especially from a Gaussian scatterer), with standard deviation $K_\sigma = 2\pi/\Lambda_\sigma$. The relevant moments are

$$K_{2\,\mathrm{Gauss}} = 2K_\sigma^2 \qquad K_{4\,\mathrm{Gauss}} = 8K_\sigma^4. \tag{15}$$

(iv) *Plane sections of monochromatic waves in space* (shell spectrum). The three-dimensional analogue of the ring spectrum, where all isotropically distributed waves have the same wavenumber k_m:

$$K_{2\,\mathrm{shell}} = \tfrac{2}{3}k_m^2 \qquad K_{4\,\mathrm{shell}} = \tfrac{8}{15}k_m^4. \tag{16}$$

(v) *Plane sections of waves with the Planck spectrum* (scalar caricature of blackbody radiation). Spatial moments are given in BD equation (6.1) in terms of thermal wavenumber $k_T = 2\pi/\lambda_T = k_\mathrm{B}T/\hbar c$, at temperature T, projecting to

$$K_{2\,\mathrm{Planck}} = \tfrac{80}{63}k_T^2 \qquad K_{4\,\mathrm{Planck}} = \tfrac{64}{15}\pi^4 k_T^4. \tag{17}$$

ψ, defined by (10), satisfies circular Gaussian statistics, so, by normalization, the real and imaginary parts ξ, η are independent univariant Gaussian random functions, with probability density function

$$P(\xi,\eta) = \frac{1}{2\pi}\exp(-(\xi^2+\eta^2)/2). \tag{18}$$

Required nonvanishing averages of ξ (identically η) and derivatives (which are also Gaussian random functions), are, where $\alpha \neq \beta$ denote x, y,

$$\begin{aligned}&\langle\xi_\alpha^2\rangle = -\langle\xi\xi_{\alpha\alpha}\rangle = K_2/2 \qquad \langle\xi_{\alpha\alpha}^2\rangle = 3K_4/8\\ &\langle\xi_{\alpha\beta}^2\rangle = \langle\xi_{\alpha\alpha}\xi_{\beta\beta}\rangle = K_4/8\end{aligned} \tag{19}$$

implying that the joint probability density function for $\boldsymbol{X} \equiv (\xi, \xi_x, \xi_y, \xi_{xx}, \xi_{yy}, \xi_{xy})$ is

$$P(\boldsymbol{X}) = \frac{4\sqrt{2}}{(2\pi)^3 K_2\sqrt{K_4}\sqrt{\det\boldsymbol{\Sigma}}}\boldsymbol{X}\exp\left(-(\nabla\xi)^2/K_2 - 4\xi_{xy}^2/K_4 - \boldsymbol{v}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{v}/2\right) \tag{20}$$

where $\boldsymbol{v} = (\xi, \xi_{xx}, \xi_{yy})$, and the components of the correlation matrix $\boldsymbol{\Sigma}$ are $\Sigma_{ij} = \langle v_i v_j\rangle$. The probability density function for $\boldsymbol{Y} \equiv (\eta, \eta_x, \eta_y, \eta_{xx}, \eta_{yy}, \eta_{xy})$ is identical to and independent of (20).

In order to evaluate the average (11) using the probability density functions (20) for ξ, η, use must be made of the following Fourier identities (where here and hereafter integrals with ranges not stated are from $-\infty$ to ∞):

$$\delta(\mu) = \frac{1}{2\pi}\int \mathrm{d}t\,\exp(\mathrm{i}t\mu) \qquad |\mu| = -\frac{1}{\pi}⨍\frac{\mathrm{d}s}{s}\partial_s\exp(\mathrm{i}s\mu) \tag{21}$$

where $⨍$ denotes a Cauchy principal value integral with pole at 0. Thus the entire integrand in the average (11) is in the exponent, and the integral can be written

$$C = -\frac{1}{4\pi^3}⨍\frac{\mathrm{d}s}{s}\partial_s\int\mathrm{d}^2\boldsymbol{t}\int\mathrm{d}^6\boldsymbol{X}\int\mathrm{d}^6\boldsymbol{Y}\exp\left(\mathrm{i}\boldsymbol{t}\cdot\boldsymbol{J} + \mathrm{i}s\det\boldsymbol{M}_\mathrm{sym}\right)P(\boldsymbol{X})P(\boldsymbol{Y}). \tag{22}$$

The first few integrals are straightforward Gaussians, and in turn are taken with respect to ξ_x, ξ_y, η_x, η_y, $\boldsymbol{t}$, ξ_{xy}, η_{xy}, $(\xi_{xx}, \xi_{yy}, \eta_{xx}, \eta_{yy})$, where the bracketed terms are integrated as a vector. Once these have been integrated, and having replaced $\xi + \mathrm{i}\eta$ with $\sqrt{I}\exp(\mathrm{i}\chi)$ and rescaled s to $u = Is$, (22) becomes

$$C = -\frac{1}{\pi^3 K_2}\int_0^{2\pi}\mathrm{d}\chi\int_0^{\infty}\mathrm{d}I\exp(-I/2)\,⨍\frac{\mathrm{d}u}{u}\partial_u\frac{1}{(4+\mathrm{i}K_4u)\sqrt{2-\mathrm{i}(K_4-K_2^2)u}}. \tag{23}$$

The χ and I integrals are trivial, giving 4π, and all that remains is the Cauchy principal value integral in u. This may be evaluated as the average of two contour integrals, with the contour displaced in the complex plane both above and below the origin, each of which can be safely integrated by parts. The integrand now has a double pole at the origin, a simple pole in the upper half-plane at $u = 4\mathrm{i}/K_4$, and a branch point in the lower half-plane at $-2\mathrm{i}/(K_4 - K_2^2)$; the branch cut is taken along the imaginary axis to $-\mathrm{i}\infty$. The upper contour can be deformed about the simple pole, giving $-\pi K_4^{3/2}/16\sqrt{3K_4 - 2K_2^2}$; the lower, deformed around the branch cut, can be integrated by elementary means, yielding $-\pi\left(K_4^{3/2}/\sqrt{3K_4-2K_2^2} - K_2^2\right)/16$. The result is

$$C = \frac{K_4^{3/2}}{2\pi K_2\sqrt{3K_4 - 2K_2^2}} - \frac{K_2}{4\pi}. \tag{24}$$

BD, Halperin (1981) showed that the dislocation density D is $K_2/4\pi$, implying that the total density of zeros Z of $\boldsymbol{J}$, that is, the sum of densities of dislocations D, extrema E and saddles S is

$$Z = \frac{K_4^{3/2}}{2\pi K_2\sqrt{3K_4 - 2K_2^2}} \tag{25}$$

and so, by (3),

$$S = \frac{Z}{2} = \frac{K_4^{3/2}}{4\pi K_2\sqrt{3K_4 - 2K_2^2}} \tag{26}$$

$$E = C - S = \frac{K_4^{3/2}}{4\pi K_2\sqrt{3K_4 - 2K_2^2}} - \frac{K_2}{4\pi}. \tag{27}$$

These formulae are the main result of this paper. I make the following observations:

- If (11) is evaluated without modulus signs, the integral is more straightforward, with contributions of $(K_4 - K_2^2)/2\pi K_2$, $-K_4/8\pi K_2$, $-(3K_4 - 2K_2^2)/16\pi K_2$ (twice) from the summands in (6), cancelling the dislocation density $D = K_2/4\pi$ and therefore verifying (3).
- If ψ has ring spectrum statistics (13), (satisfying (8)), then

  $$C_{\mathrm{ring}} = K_m^2/4\pi \qquad E_{\mathrm{ring}} = 0 \tag{28}$$

  confirming that all critical points are saddles in this case, with density equal to the dislocation density.
- For the other spectra, the saddle density S (in appropriate wavelength units) and fraction of dislocation density over positive index current zero density, $f = D/(D + E)$, are:

  $$S_{\mathrm{disc}} = \left(\frac{2}{3}\right)^{3/2}\frac{\pi}{\Lambda_d^2} \qquad f_{\mathrm{disc}} \approx 0.919 \tag{29}$$

$$S_{\mathrm{Gauss}} = \frac{2\sqrt{2}\pi}{\Lambda_\sigma^2} \qquad f_{\mathrm{Gauss}} \approx 0.707 \tag{30}$$

$$S_{\mathrm{shell}} = \frac{2\sqrt{3}\pi}{5\lambda_m^2} \qquad f_{\mathrm{shell}} \approx 0.962 \tag{31}$$

$$S_{\mathrm{Planck}} = \frac{1764\sqrt{3}\pi^3}{25\sqrt{2929}\lambda_T^2} \qquad f_{\mathrm{Planck}} \approx 0.566. \tag{32}$$

(Of course, $f_{\mathrm{ring}} = 1$.) As would be expected, the fraction f decreases with the variance of K_2 (that is, $K_4 - K_2^2$), limiting to zero as the variance approaches ∞. Freund (1998) measured densities of critical points in simulations of waves with the disc spectrum, and found $D/S = 0.935$, $E/S = 0.073$ (Freund 1998, table 3), in good agreement with the values calculated here of $D/S \approx 0.919$, $E/S \approx 0.081$.

- By (23), critical points are uniformly distributed in phase, and negative exponentially distributed in intensity. This implies that phase critical points are most likely to be found in the limit $I \to 0$, that is, near the dislocations; this would certainly be expected near dislocation creations and annihilations (by conservation of Poincaré index), and agrees with observations (Shvartsman and Freund 1995, Freund 1995), as well as the behaviour of intensity critical points (Weinrib and Halperin 1982), although, unlike intensity critical points, the distribution in intensity is not dependent on the spectrum. The probable proximity of saddles, extrema and dislocations implies that there is some type of Poincaré index screening, analogous to dislocation screening (BD, Dennis 2001, Halperin 1981).
- In principle, the correlation functions of critical points/zeros of $\boldsymbol{J}$ could be calculated (weighted or not by Poincaré index), generalizing the calculations here, along the lines of the corresponding calculation for dislocations (BD, Halperin 1981). It is likely that these functions would be very complicated even if analytic (at least as complicated as the dislocation number correlation function, given explicitly in Dennis (2001)). If this calculation (involving the properties of a 24×24 correlation matrix) could be done, it would be possible to confirm whether there is a screening relation for Poincaré index, and examine further the different long-range singularity correlation behaviours for different spectra (BD, Dennis 2001).
- When two-dimensional fields are sections of waves in three dimensions, critical points are dependent on the particular choice of plane section (that is, a saddle point in an xy section need not have any special properties in an xz section) (Berry 1998a, Freund and Kessler 2001). The phase can be stationary in three dimensions also, at places where all three components of the three-dimensional current (defined by appropriate generalization of (1)) vanish. These may be saddles, or possibly extrema in waves not satisfying the three-dimensional Helmholtz equation (the full Laplacian in (8)). The topological interaction of these three-dimensional critical points with dislocations is an interesting problem, particularly as they now have different dimension to dislocations, which are lines in space, and have different classification topology (see, e.g., Mermin 1979).

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