Black polarization sandwiches are square roots of zero

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Abstract

In the $2 \times 2$ matrices representing retarders and ideal polarizers, the eigenvectors are orthogonal. An example of the opposite case, where eigenvectors collapse onto one, is matrices $\mathbf{M}$ representing crystal plates sandwiched between a crossed polarizer and analyser. For these familiar combinations, $\mathbf{M}^2 = 0$, so black sandwiches can be regarded as square roots of zero. Black sandwiches illustrate physics associated with degeneracies of non-Hermitian matrices.

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Because of its unsymmetrical form (polarizer and analyser different), $\mathbf{M}$ is non-Hermitian, and explicit calculation shows that both of its eigenvalues are zero. Therefore the two eigenvectors, that are different for the general case, have here collapsed onto one, namely $|+\rangle$. A polarization $|\psi\rangle$ entering the black sandwich emerges in the state

$$ \mathbf{M}|\psi\rangle = (\langle+|\mathbf{A}|-\rangle\langle-|\psi\rangle)|+\rangle. \quad (3) $$

In contrast to the ideal polarizer $\mathbf{P}_+$, from which the emerging light is also in the state $|+\rangle$ but which extinguishes incident light in the orthogonal state $|\rangle$, the black sandwich extinguishes its own eigenstate. This is obvious when $\mathbf{M}$ is written in the form

$$ \mathbf{M} = (\langle+|\mathbf{A}|-\rangle)|+\rangle\langle-|. \quad (4) $$

It follows immediately that

$$ \mathbf{M}^2 = 0, \quad (5) $$

i.e. $\mathbf{M}$ is nilpotent, reflecting the obvious fact that the combination of two black sandwiches extinguishes all light. Therefore black sandwiches (1), incorporating general matrices $\mathbf{A}$, can be regarded as nontrivial square roots of zero. (The trivial square root $\mathbf{M} = 0$ corresponds to $\mathbf{A} = 1$—simply a crossed polarizer and analyser, i.e. a sandwich with no filling.)

A familiar black sandwich consists of a transparent crystal plate between a linear polarizer and analyser. If the polarizer and analyser are

$$ |\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{(polarizer)}, $$

$$ |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{(analyser)}, $$

are the projection matrices corresponding to the ideal polarizer $(-)$ and analyser $(+)$ that select orthogonal states represented by the column (e.g. Jones) vectors $|\pm\rangle$, and $\mathbf{A}$, representing the specimen, can be any $2 \times 2$ matrix.
and the orthogonal eigenpolarizations and eigenvalues of the crystal are

\[
|\phi_1\rangle = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{eigenvalue } \lambda_0 + \lambda,
\]

\[
|\phi_2\rangle = \begin{pmatrix} v^* \\ -u^* \end{pmatrix}, \quad \text{eigenvalue } \lambda_0 - \lambda,
\]

with \(|u|^2 + |v|^2 = 1\), then \(\mathbf{A}\) is the unitary matrix

\[
\mathbf{A} = \exp(i\lambda_0)[|\phi_1\rangle \langle \phi_1| + |\phi_2\rangle \langle \phi_2|] \exp(-i\lambda_0)]
\]

\[
= \exp(i\lambda_0) \begin{bmatrix} \cos \lambda + i \sin \lambda \left( |u|^2 - |v|^2 \right) & 2uv^* \\ 2u^*v & |v|^2 - |u|^2 \end{bmatrix},
\]

and the black sandwich matrix is

\[
\mathbf{M} = (\langle + | \mathbf{A} | - \rangle \langle + | - ) = 2iuv^* \exp(i\lambda_0) \sin \lambda \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (9)
\]

For an anisotropic material, the polarization components \(u\) and \(v\) depend on direction, and any diffuse light (e.g. the sky, or white paper) viewed through the sandwich will exhibit the familiar conoscopic figures [7, 8]. An easy way to see the conoscopic figures displaying the polarization singularity at the optic axis of a biaxial material is with a ‘crystal’ consisting of a sheet of overhead-projector transparency film [9], so this ‘black plastic sandwich’ is a square root of zero. For conoscopic figures corresponding to more general polarization singularities, with \(\mathbf{A}\) representing crystals that are gyrotropic and dichroic as well as birefringent, see [10].

The de Lang class [6] of nilpotent devices, mentioned at the end of the first paragraph, is more restricted, being of the form \(\mathbf{M} = \mathbf{U} \mathbf{P} \mathbf{U},\) where \(\mathbf{U}\) is a quarter-wave retarder whose eigenpolarizations are directions on the Poincaré sphere perpendicular to that selected by \(\mathbf{P}\). This class forms a four-parameter family of devices (including overall phase), whereas the black sandwiches form a ten-parameter family (or, if \(\mathbf{A}\) is restricted to the class of retarders, a six-parameter family).

The polarization optics of black sandwiches joins a growing class of physics associated with the collapse of two eigenvectors onto one at a degeneracy of eigenvalues of non-Hermitian matrices. Other examples occur in the diffraction of atoms by ‘crystals of light’ [11–13], in nuclear physics [14–17], and in the linewidths of unstable lasers [18]. Such degeneracies also occur in the optics of absorbing crystals [8, 10, 19–21], for light travelling along a ‘singular axis’ (to avoid confusion, we emphasize that in (1) these degeneracies occur in the crystal matrix \(\mathbf{A}\), not the black sandwich matrix \(\mathbf{M}\)).

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References

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