Refined and l-adic Euler Characteristics of Nearly Perfect Complexes

by

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Abstract. We lift the Euler characteristic of a nearly perfect complex to a relative algebraic K-group by passing to its l-adic Euler characteristics.

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Introduction

Let X be a smooth projective, geometrically connected surface over a finite field k and assume that the Brauer group $H^2_{\mathrm{et}}(X, \mathbb{G}_m)$ is finite (as is widely believed to be true). Let G be a finite group acting on X in such a way that the canonical projection $\pi : X \to X/G$ is étale.

It is well-known (see [Li1], [Li2], or §3 in [CKPS]) that the étale cohomology group $H^i_{\mathrm{et}}(X, \mathbb{G}_m)$ is finitely generated for every $i \in \mathbb{N}$ except for $i = 3$ (in fact, it vanishes for $i \geq 5$ and it is finite for $i = 0$, 2 and 4) and that $H^3_{\mathrm{et}}(X, \mathbb{G}_m)$ is canonically isomorphic to the Pontryagin dual of $H^1_{\mathrm{et}}(X, \mathbb{G}_m)$. Furthermore, the assumption that π is étale ensures that $R\Gamma_{\mathrm{et}}(X, \mathbb{G}_m)$ can be represented by a bounded complex of cohomologically trivial $\mathbb{Z}[G]$-modules C^* (cf. Proposition 3.2 in [CKPS]).

We now let L_3 denote the quotient of $H^1_{\mathrm{et}}(X, \mathbb{G}_m)$ by its torsion subgroup, and set $L_i := 0$ for $i \neq 3$. We let τ_i denote the obvious isomorphism from $\mathrm{Hom}(L_i, \mathbb{Q}/\mathbb{Z})$ to the maximal divisible subgroup $H^i(C^*)_{\mathrm{div}}$ of $H^i(C^*)$ (for any i). Then the triple $(C^*, (L_i)_{i\in\mathbb{Z}}, (\tau_i)_{i\in\mathbb{Z}})$ constitutes a 'nearly perfect complex' in the sense of [CKPS]. That is, C^* is a bounded complex of cohomologically trivial $\mathbb{Z}[G]$-modules and, in each degree i, the quotient $H^i(C^*)_{\mathrm{codiv}} := H^i(C^*)/H^i(C^*)_{\mathrm{div}}$ is finitely generated, the module L_i is finitely generated and $\mathbb{Z}$-free, and the map

$$\tau_i : \mathrm{Hom}_{\mathbb{Z}}(L_i, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} H^i(C^*)_{\mathrm{div}}$$

is a $\mathbb{Z}[G]$-equivariant isomorphism.

We recall that the main algebraic result of the paper [CKPS] is that to any nearly perfect complex $(C^*, (L_i)_i, (\tau_i)_i)$ as above one can associate a canonical *Euler characteristic* element $\chi(C^*)$ in the Grothendieck group $K_0(\mathbb{Z}[G])$ of all finitely generated projective $\mathbb{Z}[G]$-modules which has image in the Grothendieck group $G_0(\mathbb{Z}[G])$ of all finitely generated $\mathbb{Z}[G]$-modules equal to

$$\chi_{\mathrm{coh}}(C^*) := \big([H^+(C^*)_{\mathrm{codiv}}] + [\mathrm{Hom}_{\mathbb{Z}}(L_-, \mathbb{Z})]\big) - \big([H^-(C^*)_{\mathrm{codiv}}] + [\mathrm{Hom}_{\mathbb{Z}}(L_+, \mathbb{Z})]\big)\,.$$

(Here, we write $H^+(C^*)_{\mathrm{codiv}}$ for $\oplus_{i\,\mathrm{even}} H^i(C^*)_{\mathrm{codiv}}$, L_- for $\oplus_{i\,\mathrm{odd}} L_i$, and so on.)

We further recall that the intersection pairing on $\mathrm{CH}^1(X) = \mathrm{Pic}(X) = H^1_{\mathrm{et}}(X, \mathbb{G}_m) = H^1(C^*)$ induces a $\mathbb{Z}[G]$-equivariant homomorphism

$$\mu : H^1(C^*)_{\mathrm{codiv}} = H^1(C^*) \to \mathrm{Hom}(H^1(C^*), \mathbb{Z}) = \mathrm{Hom}(L_3, \mathbb{Z})$$

such that $\mu_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \mu$ is bijective. In particular, the element

$$\chi^{\mathrm{rel}}_{\mathrm{coh}}(C^*, \mu_{\mathbb{Q}}) := [H^0(C^*)] - [\ker(\mu)] + [H^2(C^*)] - [H^3(C^*)_{\mathrm{codiv}}] + [\mathrm{coker}(\mu)] + [H^4(C^*)]$$

belongs to the Grothendieck group $G_0T(\mathbb{Z}[G])$ of all finite $\mathbb{Z}[G]$-modules and is a preimage of $\chi_{\mathrm{coh}}(C^*)$ under the canonical map $G_0T(\mathbb{Z}[G]) \to G_0(\mathbb{Z}[G])$ (cf. Definition 3.5 in [CKPS]).

The main result of this paper, Theorem (3.9), implies that for any nearly perfect complex $(C^*, (L_i)_i, (\tau_i)_i)$ and any $\mathbb{Q}[G]$-equivariant 'trivialization' isomorphism

$$\lambda : \mathbb{Q} \otimes H^-(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_+, \mathbb{Q}) \xrightarrow{\sim} \mathbb{Q} \otimes H^+(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_-, \mathbb{Q})$$

one can define a canonical *Euler characteristic* element $\chi^{\mathrm{rel}}(C^*, \lambda)$ in the Grothendieck group $K_0T(\mathbb{Z}[G])$ of all finite $\mathbb{Z}[G]$-modules of finite projective dimension which satisfies the following key properties: in the obvious commutative diagram

$$\begin{array}{ccc} K_0T(\mathbb{Z}[G]) & \longrightarrow & K_0(\mathbb{Z}[G]) \\ \downarrow & & \downarrow \\ G_0T(\mathbb{Z}[G]) & \longrightarrow & G_0(\mathbb{Z}[G]), \end{array}$$

$\chi^{\mathrm{rel}}(C^*, \lambda)$ is a preimage of $\chi(C^*)$ under the upper horizontal map and, in the case that λ is equal to the map $\mu_{\mathbb{Q}}$ described above, $\chi^{\mathrm{rel}}(C^*, \lambda)$ is also a preimage of $\chi^{\mathrm{rel}}_{\mathrm{coh}}(C^*, \lambda)$ under the left vertical map.

To define the element $\chi^{\mathrm{rel}}(C^*, \lambda)$ we proceed as follows. We first choose a bounded complex P^* of projective $\mathbb{Z}[G]$-modules together with a quasi-isomorphism $P^* \to C^*$. (This is possible by Lemma (1.1).) Then, for every prime l, the l-adic completion $\hat{P}^*$ of P^* is a bounded complex of cohomologically trivial $\mathbb{Z}_l[G]$-modules (by Lemma (2.1)). Furthermore, for all $i \in \mathbb{Z}$, there is a natural short exact sequence

$$0 \to H^i(C^*)_{\mathrm{codiv}} \otimes \mathbb{Z}_l \to H^i(\hat{P}^*) \to \mathrm{Hom}(L_{i+1}, \mathbb{Z}_l) \to 0$$

(by Proposition (2.2)) which we view as a 2-step filtration on the cohomology group $H^i(\hat{P}^*)$. In particular the $\mathbb{Z}_l[G]$-module $H^i(\hat{P}^*)$ is finitely generated (for all $i \in \mathbb{Z}$).

Hence, we can find a perfect complex $\tilde{P}^*$ of $\mathbb{Z}_l[G]$-modules together with a quasi-isomorphism $\tilde{P}^* \to \hat{P}^*$ (again by Lemma (1.1)). Moreover, the given isomorphism λ induces an isomorphism

$$\lambda_l : \mathrm{Gr}(H^-(\tilde{P}^*))_{\mathbb{Q}_l} \tilde{\to} \mathrm{Gr}(H^+(\tilde{P}^*))_{\mathbb{Q}_l}$$

between the graded objects associated with the 2-step filtrations on $H^-(\tilde{P}^*)_{\mathbb{Q}_l}$ and $H^+(\tilde{P}^*)_{\mathbb{Q}_l}$ mentioned above. We then develop a refined version of the classical construction of Reidemeister-Whitehead torsion and use this construction to define an Euler characteristic element $\chi^{\mathrm{rel}}(\tilde{P}^*, \lambda_l)$ in the Grothendieck group $K_0T(\mathbb{Z}_l[G])$ of all finite $\mathbb{Z}_l[G]$-modules of finite projective dimension (see §3). One key property of the element $\chi^{\mathrm{rel}}(\tilde{P}^*, \lambda_l)$ is that the canonical map $K_0T(\mathbb{Z}_l[G]) \to K_0(\mathbb{Z}_l[G])$ sends it to the Euler characteristic $\chi(\tilde{P}^*) \in K_0(\mathbb{Z}_l[G])$ of the perfect complex $\tilde{P}^*$ which in turn is equal to the image of the Euler characteristic element $\chi(C^*) \in K_0(\mathbb{Z}[G])$ (defined in [CKPS]) under the canonical map $K_0(\mathbb{Z}[G]) \to K_0(\mathbb{Z}_l[G])$ (by Theorem (2.4)). Finally, we define $\chi^{\mathrm{rel}}(C^*, \lambda)$ to be equal to the image of the infinite tuple $(\chi^{\mathrm{rel}}(\tilde{P}^*, \lambda_l))_{l\,\mathrm{prime}}$ under the canonical decomposition isomorphism

$$\bigoplus_{l\,\mathrm{prime}} K_0T(\mathbb{Z}_l[G]) \cong K_0T(\mathbb{Z}[G])$$

(see §3).

We remark that in several natural arithmetical contexts the element $\chi^{\mathrm{rel}}(C^*, \lambda)$ can be directly related to the leading terms of associated L-functions. For example, in the context described at the beginning of this introduction, if k is of characteristic p, then in [Bu2] the first named author has shown that the image of $\chi^{\mathrm{rel}}(C^*, \mu_{\mathbb{Q}})$ in $K_0T(\mathbb{Z}[G])/K_0T(\mathbb{Z}_p[G])$ can be explicitly computed in terms of the leading term of the G-equivariant L-function of X at $s = 1$. (The result in [Bu2] is actually for a quasi-projective variety X of arbitrary dimension). With a view to obtaining analogous results for varieties over number fields, we have therefore phrased our definition of $\chi^{\mathrm{rel}}(C^*, \lambda)$ in §3 in terms of isomorphisms λ which are defined over an arbitrary field of characteristic zero.

The proof of our main result, Theorem (3.9), relies upon a certain mapping cone construction (explained in §1) and this in fact yields a natural definition of the Euler characteristic $\chi(C^*)$ which avoids the somewhat unsatisfactory inductive procedure used in [CKPS] (see Theorem (1.3)). We remark that the idea of using such a mapping cone construction in this context first occurs in [Bu4]. However, in [Bu4], only those nearly perfect complexes which satisfy a certain natural condition on homology are considered. Under this condition (which is satisfied by all nearly perfect complexes which are known to arise in arithmetic), it is shown in [Bu4] that the mapping cone construction which is described in §1 yields an element of the derived category which depends (to within isomorphism) only upon the given nearly perfect complex, and in addition criteria are described which ensure additivity of the Euler characteristic $\chi(C^*)$ on distinguished triangles of nearly perfect complexes.

In this context we remark that it is also possible to shorten some of the proofs given here by using the language of derived categories. However, we have decided to eschew such formalism in order to make this paper accessible to as wide an audience as possible.

Notations. Throughout this paper, we fix a finite group G, and all rings are assumed to be unital. The group ring of G over a ring R is denoted by $R[G]$. As usual, $G_0(R)$ and $K_0(R)$ denote the Grothendieck group of all finitely generated R-modules and the Grothendieck group of all projective finitely generated R-modules, respectively. For any abelian group A, the maximal divisible subgroup of A is denoted by A_{div} and the codivisible quotient A/A_{div} of A by A_{codiv}. For any $n \in \mathbb{N}$, the subgroup of n-torsion elements in A is denoted by ${}_nA$. For a fixed prime l, we set $\hat{A} := \lim_{\leftarrow n} A/l^nA$ (the *l-adic completion of* A) and $T_l(A) := \lim_{\leftarrow n}\, {}_{l^n}A$ (the *l-adic Tate module of* A). By a complex M^* in an abelian category we mean a cochain complex, and the differential from M^i to M^{i+1} is denoted by $d^i := d^i(M^*)$. In each degree i the modules of coboundaries, cocyles and cohomology of M^* are denoted by $B^i(M^*)$, $Z^i(M^*)$ and $H^i(M^*)$ respectively. For any morphism $f^* : M^* \to N^*$ between complexes, the *mapping cone of* f is the complex C^* with $C^i := M^{i+1} \oplus N^i$ and differential $d^i(C^*)$ given by $(x, y) \mapsto (d^{i+1}(x), d^i(y) + (-1)^{i+1} f^{i+1}(x))$. By a *perfect complex* of R-modules we shall mean a bounded complex of finitely generated projective R-modules. For any $\mathbb{Z}$-graded object M^*, resp. N_*, we shall write M^+ and M^-, resp. N_+ and N_-, for the direct sum of the objects M^i, resp. N_i, as i runs over all even and odd integers respectively.

§1 The Cone Construction

In the paper [CKPS], Chinburg, Kolster, Pappas and Snaith introduced the notion of a nearly perfect complex, and they defined its Euler characteristic in $K_0(\mathbb{Z}[G])$ by using a certain inductive procedure. The object of this section is to describe this Euler characteristic in a more natural way, namely as the Euler characteristic of a certain mapping cone which turns out to be a bounded complex of cohomologically trivial $\mathbb{Z}[G]$-modules whose cohomology groups are finitely generated. In the first two lemmas, we will therefore recall how to associate an Euler characteristic in $K_0(\mathbb{Z}[G])$ to any such complex.

(1.1) Lemma. *Let R be a Dedekind domain and let C^* be a bounded complex of cohomologically trivial $R[G]$-modules. Then there exists a* bounded *complex P^* of projective $R[G]$-modules and a quasi-isomorphism $P^* \to C^*$. If, moreover, all cohomology groups $H^i(C^*)$, $i \in \mathbb{Z}$, are finitely generated, then P^* can be chosen to be a perfect complex of $R[G]$-modules.*

Proof. It is well-known that there is a complex Q^* of projective $R[G]$-modules which is bounded to the right together with a quasi-isomorphism $Q^* \to C^*$. Let $m \in \mathbb{N}$ with $C^i = 0$ for $i < m$. Then the induced homomorphism

$$\tau_{\geq m}(Q^*) := (\cdots \to 0 \to Q^m/d(Q^{m-1}) \to Q^{m+1} \to \cdots) \longrightarrow C^*$$

is a quasi-isomorphism, too. Hence, $Q^m/d(Q^{m-1})$ is cohomologically trivial since the corresponding mapping cone is acyclic and the modules C^i for $i \in \mathbb{Z}$ and Q^j for $j \geq m+1$, are cohomologically trivial. So, by Proposition 4.1(b) on p. 457 in [Ch], there is a short exact sequence $0 \to P^{m-1} \to P^m \to Q^m/d(Q^{m-1}) \to 0$ where P^{m-1} and P^m are projective $R[G]$-modules. Now the complex

$$P^* := (\cdots \to 0 \to P^{m-1} \to P^m \to Q^{m+1} \to \cdots)$$

where $P^m \to Q^{m+1}$ is the composition $P^m \twoheadrightarrow Q^m/d(Q^{m-1}) \longrightarrow Q^{m+1}$ is as required. The second assertion of the lemma is equivalent to asserting that, under the stated condition on cohomology, each projective module P^i can be chosen to be finitely generated, and this follows by a standard argument (see, for example, the proof of Theorem 1.1 on p. 447 in [Ch] or Proposition 2.2.4 on p. 21 in [Sn]). □

(1.2) Lemma. *Let*

$$\begin{array}{ccc} P^* & \xrightarrow{\alpha^*} & C^* \\ & & \downarrow f^* \\ Q^* & \xrightarrow{\beta^*} & D^* \end{array}$$

be a diagram of complexes in an abelian category $\mathcal{M}$ where P^* is a complex of projective objects which is bounded to the right and where β^* is a quasi-isomorphism. Then there exists a homomorphism of complexes $h^* : P^* \to Q^*$ such that $f^*\alpha^*$ is homotopic to β^*h^*. In particular, in the situation of the second statement of Lemma (1.1), the element

$$\chi(C^*) := \sum_{i\in\mathbb{Z}} (-1)^i [P^i] \in K_0(R[G])$$

does not depend on the chosen quasi-isomorphism $P^* \to C^*$, and we call it the Euler characteristic *of C^*.*

Proof. The first statement is a standard result (see, for example, the proof of Proposition 2.2.4(ii) on p. 21 in [Sn]). If, in the situation of the second statement of Lemma (1.1), we choose another perfect complex Q^* together with a quasi-isomorphism $Q^* \to C^*$, then we obtain a quasi-isomorphism $P^* \to Q^*$ by the first statement of Lemma (1.2). The corresponding mapping cone being acyclic, we obtain

$$\sum_{i\in\mathbb{Z}} (-1)^i [P^i] - \sum_{i\in\mathbb{Z}} (-1)^i [Q^i] = \sum_{i\in\mathbb{Z}} (-1)^i [P^i \oplus Q^{i-1}] = 0 \quad \text{in} \quad K_0(R[G]),$$

as was to be shown. □

Let now $(C^*, (L_i)_i, (\tau_i)_i)$ be a fixed nearly perfect complex: thus, C^* is a bounded complex of cohomologically trivial $\mathbb{Z}[G]$-modules and, for each $i \in \mathbb{Z}$, the $\mathbb{Z}[G]$-module $H^i(C^*)_{\text{codiv}}$ is finitely generated , L_i is a $\mathbb{Z}$-free, finitely generated $\mathbb{Z}[G]$-module, and $\tau_i : \text{Hom}_{\mathbb{Z}}(L_i, \mathbb{Q}/\mathbb{Z}) \to H^i(C^*)_{\text{div}}$ is a $\mathbb{Z}[G]$-equivariant isomorphism.

We recall that a uniquely divisible $\mathbb{Z}[G]$-module is cohomologically trivial and therefore (as observed earlier) admits a $\mathbb{Z}[G]$-projective resolution of length at most 1. For each i, we fix such a resolution

$$0 \longrightarrow R^{i-1} \hookrightarrow Q^i \xrightarrow{\varepsilon^i} \text{Hom}(L_i, \mathbb{Q}) \longrightarrow 0$$

of $\mathrm{Hom}(L_i, \mathbb{Q})$. Furthermore, we choose a map of complexes

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & R^{i-1} & \longrightarrow & Q^i & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \beta^{i-1}} & & \downarrow{\scriptstyle \alpha^i} & & \downarrow & & \\
\cdots & \longrightarrow & C^{i-2} & \longrightarrow & C^{i-1} & \longrightarrow & C^i & \longrightarrow & C^{i+1} & \longrightarrow & \cdots
\end{array}$$

such that the following diagrams commute:

$$\begin{array}{ccc}
Q^i & \xrightarrow{\varepsilon^i} & \mathrm{Hom}(L_i, \mathbb{Q}) \\
 & & \downarrow \\
\vdots{\scriptstyle \alpha^i} & & \mathrm{Hom}(L_i, \mathbb{Q}/\mathbb{Z}) \\
 & & \wr\downarrow{\scriptstyle \tau_i} \\
 & & H^i(C^*)_{\mathrm{div}} \\
\downarrow & & \downarrow \\
Z^i(C^*) & \longrightarrow & H^i(C^*)
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccccc}
R^{i-1} & \hookrightarrow & Q^i & & \\
\vdots{\scriptstyle \beta^{i-1}} & \searrow & & \searrow{\scriptstyle \alpha^i} & \\
C^{i-1} & \twoheadrightarrow & B^i(C^*) & \hookrightarrow & Z^i(C^*).
\end{array}$$

These maps induce a homomorphism of complexes

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & Q^{i-1} \oplus R^{i-1} & \longrightarrow & Q^i \oplus R^i & \longrightarrow & Q^{i+1} \oplus R^{i+1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle (\alpha^{i-1}, \beta^{i-1})} & & \downarrow{\scriptstyle (\alpha^i, \beta^i)} & & \downarrow{\scriptstyle (\alpha^{i+1}, \beta^{i+1})} & & \\
\cdots & \longrightarrow & C^{i-1} & \longrightarrow & C^i & \longrightarrow & C^{i+1} & \longrightarrow & \cdots
\end{array}$$

where the upper complex (in the sequel denoted by $(Q \oplus R)^*$) is the direct sum of the complexes $\cdots \to 0 \to R^{i-1} \to Q^i \to 0 \to \cdots$, $i \in \mathbb{Z}$. Let Cone_C^* denote the mapping cone of this homomorphism. Then we have a short exact sequence of complexes

$$0 \to C^* \to \mathrm{Cone}_C^* \to (Q \oplus R)^*[1] \to 0$$

which yields the long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}(L_i, \mathbb{Q}) \xrightarrow{\partial} H^i(C^*) \longrightarrow H^i(\mathrm{Cone}_C^*) \longrightarrow \mathrm{Hom}(L_{i+1}, \mathbb{Q}) \longrightarrow \cdots$$

where the connecting homomorphism ∂ is the composition

$$\mathrm{Hom}(L_i, \mathbb{Q}) \twoheadrightarrow \mathrm{Hom}(L_i, \mathbb{Q}/\mathbb{Z}) \xrightarrow[\sim]{\tau_i} H^i(C^*)_{\mathrm{div}} \hookrightarrow H^i(C^*) .$$

Therefore we obtain natural short exact sequences

$$(1) \qquad 0 \to H^i(C^*)_{\mathrm{codiv}} \to H^i(\mathrm{Cone}_C^*) \to \mathrm{Hom}(L_{i+1}, \mathbb{Z}) \to 0, \quad i \in \mathbb{Z}.$$

In particular, the $\mathbb{Z}[G]$-module $H^i(\mathrm{Cone}_C^*)$ is finitely generated for all i. Since all of the $\mathbb{Z}[G]$-modules Cone_C^i, $i \in \mathbb{Z}$, are cohomologically trivial, we obtain an Euler characteristic $\chi(\mathrm{Cone}_C^*) \in K_0(\mathbb{Z}[G])$ by using Lemma (1.2).

(1.3) Theorem. *The class* $\chi(\mathrm{Cone}^*_C) \in K_0(\mathbb{Z}[G])$ *is equal to the class* $\chi(C^*) := \chi((C^*, (L_i)_i, (\tau_i)_i))$ *which is defined in ([CKPS], Definition 2.11).*

Proof. First, we assume that at most one module $C := C^n$ in the complex C^* is non zero. Then $\chi(C^*)$ is defined in the following way. We choose a finitely generated projective $\mathbb{Z}[G]$-module F together with a $\mathbb{Z}[G]$-module homomorphism $F \to C$ such that the composition $F \to C \to C_{\mathrm{codiv}}$ is surjective. This homomorphism yields the following commutative diagram with exact rows and columns (where $L := L_n$ and M and N are defined in such a way that the two right columns become exact):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}(L,\mathbb{Z}) & \to & M & \to & N & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}(L,\mathbb{Q}) & \to & \mathrm{Hom}(L,\mathbb{Q}) \oplus F & \to & F & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}(L,\mathbb{Q}/\mathbb{Z}) & \to & C & \to & C_{\mathrm{codiv}} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then M is a finitely generated projective $\mathbb{Z}[G]$-module (see Corollary 2.8 in [CKPS]) and, by definition, $\chi(C^*) = (-1)^n([F] - [M])$ in $K_0(\mathbb{Z}[G])$ or, in other words, $\chi(C^*)$ is the Euler characteristic of the perfect complex $\cdots \to 0 \to M \to F \to 0 \to \cdots$ (with F in degree n, of course) resulting from the diagram above. This diagram also shows that this complex is quasi-isomorphic to the complex $\cdots \to 0 \to \mathrm{Hom}(L,\mathbb{Q}) \to C \to 0 \to \cdots$. Since we furthermore have an obvious quasi-isomorphism from the complex Cone^*_C to the complex $\cdots \to 0 \to \mathrm{Hom}(L,\mathbb{Q}) \to C \to 0 \to \cdots$, we obtain a quasi-isomorphism from the complex $\cdots \to 0 \to M \to F \to 0 \to \cdots$ to the complex Cone^*_C (by Lemma (1.2)) which proves that $\chi(C^*) = \chi(\mathrm{Cone}^*_C)$, as was to be shown.
We now proceed by induction on the length of C^*. Let $n \in \mathbb{Z}$ such that $C^n \neq 0$ and $C^r = 0$ for all $r > n$. We choose a finitely generated projective $\mathbb{Z}[G]$-module F^n together with a homomorphism $F^n \to C^n$ such that the composition

$$F^n \longrightarrow C^n \twoheadrightarrow H^n(C^*) \twoheadrightarrow H^n(C^*)_{\mathrm{codiv}}$$

is surjective. Furthermore, we choose a projective $\mathbb{Z}[G]$-module S^n together with an epimorphism $S^n \twoheadrightarrow B^n(C^*)$ Then we obtain the following commutative diagram

$$\begin{array}{ccc}
P^n := S^n \oplus Q^n \oplus F^n & \xrightarrow{0 \oplus \varepsilon^n \oplus \mathrm{id}} & \mathrm{Hom}(L_n,\mathbb{Q}) \oplus F^n \\
\downarrow & & \downarrow \\
C^n & \twoheadrightarrow & H^n(C^*).
\end{array}$$

Let the $\mathbb{Z}[G]$-modules B^n_P and M^n be defined by the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & M^n & \longrightarrow & \mathrm{Hom}(L_n, \mathbb{Q}) \oplus F^n & \twoheadrightarrow & H^n(C^*) & \longrightarrow & 0 \\
 & & \uparrow & & \twoheaduparrow & & \| & & \\
0 & \longrightarrow & B^n_P & \longrightarrow & P^n & \twoheadrightarrow & H^n(C^*) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & S^n \oplus R^{n-1} & = & S^n \oplus R^{n-1} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0. & & & &
\end{array}$$

Let E^{n-1} be defined by the following pull-back diagram where the epimorphism $B^n_P \twoheadrightarrow B^n(C^*)$ is induced by the epimorphism $P^n = S^n \oplus Q^n \oplus F^n \twoheadrightarrow C^n$ introduced above:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z^{n-1}(C^*) & \longrightarrow & E^{n-1} & \longrightarrow & B^n_P & \longrightarrow & 0 \\
 & & \| & & \twoheaddownarrow & & \downarrow & & \\
0 & \longrightarrow & Z^{n-1}(C^*) & \longrightarrow & C^{n-1} & \longrightarrow & B^n(C^*) & \longrightarrow & 0.
\end{array}$$

Since $S^n \oplus R^{n-1}$ is $\mathbb{Z}[G]$-projective, we can lift the inclusion $S^n \oplus R^{n-1} \hookrightarrow B^n_P$ to an inclusion $S^n \oplus R^{n-1} \hookrightarrow E^{n-1}$ and define $D^{n-1} := E^{n-1}/(S^n \oplus R^{n-1})$. Then we obtain the following exact sequence:

$$0 \to Z^{n-1}(C^*) \to D^{n-1} \to M^n \to 0.$$

Putting all diagrams together, we obtain the following commutative diagram with exact rows and vertical epimorphisms:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & Z^{n-1}(C^*) & \longrightarrow & D^{n-1} & \longrightarrow & \mathrm{Hom}(L_n, \mathbb{Q}) \oplus F^n & \longrightarrow & H^n(C^*) & \longrightarrow & 0 \\
 & & \| & & \twoheaduparrow & & \twoheaduparrow & & \| & & \\
0 & \longrightarrow & Z^{n-1}(C^*) & \longrightarrow & E^{n-1} & \longrightarrow & P^n & \longrightarrow & H^n(C^*) & \longrightarrow & 0 \\
 & & \| & & \twoheaddownarrow & & \twoheaddownarrow & & \| & & \\
0 & \longrightarrow & Z^{n-1}(C^*) & \longrightarrow & C^{n-1} & \longrightarrow & C^n & \longrightarrow & H^n(C^*) & \longrightarrow & 0.
\end{array} \tag{2}$$

Since the kernel of the epimorphism $E^{n-1} \twoheadrightarrow C^{n-1}$ is isomorphic to the kernel of the epimorphism $P^n \twoheadrightarrow C^n$, E^{n-1} is cohomologically trivial; hence D^{n-1} is also cohomologically trivial. The diagram (2) furthermore shows that the 1-extension

$$0 \to Z^{n-1}(C^*) \to D^{n-1} \to M^n \to 0$$

is a preimage (which is unique by the proof of Lemma 2.4 in [CKPS]) of the tautological 2-extension

$$0 \to Z^{n-1}(C^*) \to C^{n-1} \to C^n \to H^n(C^*) \to 0$$

under the connecting homomorphism

$$\mathrm{Ext}^1_{\mathbb{Z}[G]}(M^n, Z^{n-1}(C^*)) \to \mathrm{Ext}^2_{\mathbb{Z}[G]}(H^n(C^*), Z^{n-1}(C^*))$$

which is associated with the short exact sequence

$$0 \to M^n \to \mathrm{Hom}(L_n, \mathbb{Q}) \oplus F^n \to H^n(C^*) \to 0.$$

Let D^* denote the complex

$$\cdots \to C^{n-3} \to C^{n-2} \to D^{n-1} \to 0 \to \cdots$$

where $C^{n-2} \to D^{n-1}$ is the composition $C^{n-2} \to Z^{n-1}(C^*) \to D^{n-1}$. Then, by definition, we have

$$\chi(C^*) = \chi(D^*) + (-1)^n [F^n] \quad \text{in} \quad K_0(\mathbb{Z}[G])$$

where we view D^* as a nearly perfect complex (of a smaller length than C^*) as in Corollary 2.10 in [CKPS]. Let D^*_{aug} respectively E^* denote the complexes

$$\cdots \to C^{n-3} \to C^{n-2} \to D^{n-1} \to \mathrm{Hom}(L_n, \mathbb{Q}) \oplus F^n \to 0 \to \cdots$$

respectively

$$\cdots \to C^{n-3} \to C^{n-2} \to E^{n-1} \to P^n \to 0 \to \cdots$$

resulting from the diagram (2). From the construction we obtain natural homomorphisms of complexes

$$(Q \oplus R)^* \to E^* \quad \text{and} \quad (Q \oplus R)^* \to D^*_{\mathrm{aug}}$$

such that the following triangles commute:

$$\begin{array}{ccc} & (Q \oplus R)^* & \\ \swarrow & & \searrow^{(\alpha^i, \beta^i)_i} \\ E^* & \longrightarrow & C^* \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & (Q \oplus R)^* & \\ \swarrow & & \searrow \\ E^* & \longrightarrow & D^*_{\mathrm{aug}} \end{array}$$

Therefore, we obtain quasi-isomorphisms between the corresponding cones:

$$\mathrm{Cone}^*_C \leftarrow \mathrm{Cone}^*_E \to \mathrm{Cone}^*_{D_{\mathrm{aug}}}$$

hence:

$$\chi(\mathrm{Cone}^*_C) = \chi(\mathrm{Cone}^*_{D_{\mathrm{aug}}}) \quad \text{in} \quad K_0(\mathbb{Z}[G]).$$

Furthermore, the natural epimorphism from $\mathrm{Cone}^*_{D_{\mathrm{aug}}}$ to the mapping cone M^* of the homomorphism of complexes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & Q^{n-2} \oplus R^{n-2} & \longrightarrow & Q^{n-1} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & D^{n-2} & \longrightarrow & D^{n-1} & \longrightarrow & F^n & \longrightarrow & \cdots \end{array}$$

is a quasi-isomorphism since the kernel

$$\cdots \to 0 \to R^{n-1} \to Q^n \to \mathrm{Hom}(L_n, \mathbb{Q}) \to 0 \to \cdots$$

is acyclic. Finally, by using the inductive hypothesis it is easy to see that the Euler characteristic of M^* is equal to $\chi(D^*) + (-1)^n[F^n]$. So, Theorem (1.3) is proved:

$$\chi(\mathrm{Cone}^*_C) = \chi(\mathrm{Cone}^*_{D_{\mathrm{aug}}}) = \chi(M^*) = \chi(D^*) + (-1)^n[F^n] = \chi(C^*) \quad \text{in} \quad K_0(\mathbb{Z}[G]).$$

□

§2 The l-adic Euler Characteristic of a Nearly Perfect Complex

In this section we fix a prime l. We will show that the image of the Euler characteristic $\chi(C^*)$ of a nearly perfect complex C^* in the Grothendieck group $K_0(\mathbb{Z}_l[G])$ of all finitely generated projective $\mathbb{Z}_l[G]$-modules is equal to the Euler characteristic of any complex which is obtained by replacing C^* with a quasi-isomorphic complex of projective $\mathbb{Z}[G]$-modules and then passing to l-adic completions.

We begin with the following easy observation.

(2.1) Lemma. *Let P be a $\mathbb{Z}$-torsion-free cohomologically trivial $\mathbb{Z}[G]$-module. Then its l-adic completion $\hat{P}$ is also cohomologically trivial.*

Proof. Since the transition maps in the inverse system $(P/l^nP)_{n\geq 0}$ are surjective, we have a short exact sequence

$$0 \to \varprojlim_n P/l^nP \to \prod_n P/l^nP \to \prod_n P/l^nP \to 0$$

where the right homomorphism maps a tuple $(x_n)_{n\in\mathbb{N}}$ to the tuple $(x_n - \overline{x_{n+1}})_{n\in\mathbb{N}}$. Now, let $\hat{H}$ denote Tate cohomology with respect to some subgroup of G. Then we obtain a long exact sequence

$$\cdots \to \hat{H}^s(\hat{P}) \to \prod_n \hat{H}^s(P/l^nP) \to \prod_n \hat{H}^s(P/l^nP) \to \hat{H}^{s+1}(\hat{P}) \to \cdots$$

Since P is $\mathbb{Z}$-torsion-free and cohomologically trivial, $\hat{H}^s(P/l^nP)$ vanishes for all $s \in \mathbb{Z}$ and $n \in \mathbb{N}$. Thus, $\hat{H}^s(\hat{P}) = 0$ for all $s \in \mathbb{Z}$, as was to be shown. □

(2.2) Proposition. *Let P^* be a complex of $\mathbb{Z}$-torsion-free $\mathbb{Z}[G]$-modules such that the l^n-torsion subgroup ${}_{l^n}H^i(P^*)$ and the l^n-quotient group $H^i(P^*) \otimes \mathbb{Z}/l^n\mathbb{Z}$ are finite for all $i \in \mathbb{Z}$ and all $n \in \mathbb{N}$. Then there is a natural short exact sequence*

$$0 \to H^i\widehat{(P^*)_{\mathrm{codiv}}} \to H^i(\hat{P}^*) \to T_l(H^{i+1}(P^*)_{\mathrm{div}}) \to 0$$

of $\mathbb{Z}_l[G]$-modules for all $i \in \mathbb{Z}$.

Proof. The exact sequences

$$0 \to P^* \to P^* \to P^* \otimes \mathbb{Z}/l^n\mathbb{Z} \to 0, \quad n \geq 0,$$

yield the familiar short exact sequences

$$0 \to H^i(P^*) \otimes \mathbb{Z}/l^n\mathbb{Z} \to H^i(P^* \otimes \mathbb{Z}/l^n\mathbb{Z}) \to {}_{l^n}H^{i+1}(P^*) \to 0, \quad n \in \mathbb{N}, \quad i \in \mathbb{Z}.$$

Since these are short exact sequences of finite groups, they remain exact upon completion by the Mittag-Leffler criterion. Thus we have short exact sequences

(3) $$0 \to \widehat{H^i(P^*)} \to \varprojlim_n H^i(P^* \otimes \mathbb{Z}/l^n\mathbb{Z}) \to T_l(H^{i+1}(P^*)) \to 0, \quad i \in \mathbb{Z}.$$

The exact sequence

$$0 \to H^i(P^*)_{\mathrm{div}} \to H^i(P^*) \to H^i(P^*)_{\mathrm{codiv}} \to 0$$

yields an isomorphism $H^i(P^*) \otimes \mathbb{Z}/l^n\mathbb{Z} \cong H^i(P^*)_{\mathrm{codiv}} \otimes \mathbb{Z}/l^n\mathbb{Z}$ for all n, hence

(4) $$\widehat{H^i(P^*)} \cong H^i\widehat{(P^*)_{\mathrm{codiv}}} \text{ for all } i \in \mathbb{Z};$$

furthermore, we obtain left exact sequences

$$0 \to {}_{l^n}(H^i(P^*)_{\mathrm{div}}) \to {}_{l^n}H^i(P^*) \to {}_{l^n}(H^i(P^*)_{\mathrm{codiv}}), \quad i \in \mathbb{Z}, \quad n \in \mathbb{N};$$

hence

(5) $$T_l(H^i(P^*)_{\mathrm{div}}) \cong T_l(H^i(P^*)) \text{ for all } i \in \mathbb{Z}$$

since, for any abelian group A, $T_l(A_{\mathrm{codiv}})$ vanishes. (Proof: if $T_l(A_{\mathrm{codiv}})$ contained a non-zero element $(x_n)_{n\in\mathbb{N}}$, then the subgroup U of A_{codiv} generated by the l^∞-torsion elements x_n, $n \in \mathbb{N}$, would be non-zero and divisible (by *all* primes), hence the preimage of U under the canonical epimorphism $A \to A_{\mathrm{codiv}}$ would be a divisible subgroup of A bigger than A_{div}.)

Furthermore, we have a short exact sequence of complexes

$$0 \to \hat{P}^* \to \prod_n P^* \otimes \mathbb{Z}/l^n\mathbb{Z} \to \prod_n P^* \otimes \mathbb{Z}/l^n\mathbb{Z} \to 0$$

where the right map is defined as in the proof of Lemma (2.1). Thus we obtain the long exact sequence

$$\cdots \to H^i(\hat{P}^*) \to \prod_n H^i(P^* \otimes \mathbb{Z}/l^n\mathbb{Z}) \to \prod_n H^i(P^* \otimes \mathbb{Z}/l^n\mathbb{Z}) \to H^{i+1}(\hat{P}^*) \to \cdots$$

which yields the exact Milnor sequence

$$0 \to \varprojlim_n{}^1 H^{i-1}(P^* \otimes \mathbb{Z}/l^n\mathbb{Z}) \to H^i(\hat{P}^*) \to \varprojlim_n H^i(P^* \otimes \mathbb{Z}/l^n\mathbb{Z}) \to 0.$$

Since $H^{i-1}(P^* \otimes \mathbb{Z}/l^n\mathbb{Z})$ is finite for all n, the inverse system $(H^{i-1}(P^* \otimes \mathbb{Z}/l^n\mathbb{Z}))_{n\geq 0}$ satisfies the Mittag-Leffler condition, so $\varprojlim_n{}^1 H^{i-1}(P^* \otimes \mathbb{Z}/l^n\mathbb{Z}) = 0$, and hence

$$(6) \qquad H^i(\hat{P}^*) \cong \varprojlim_n H^i(P^* \otimes \mathbb{Z}/l^n\mathbb{Z}) \text{ for all } i.$$

Upon combining (3), (4), (5) and (6) we obtain Proposition (2.2). ☐

(2.3) Corollary. *Let $P^* \to Q^*$ be a quasi-isomorphism of complexes as in Proposition (2.2). Then the induced morphism $\hat{P}^* \to \hat{Q}^*$ is also a quasi-isomorphism.*

Proof. Obvious. ☐

Now, let $C^* = (C^*, (L_i)_i, (\tau_i)_i)$ be a nearly perfect complex of $\mathbb{Z}[G]$-modules. We choose a bounded complex P^* of projective $\mathbb{Z}[G]$-modules together with a quasi-isomorphism $P^* \to C^*$ as in Lemma (1.1). Then the l-adic completion $\hat{P}^*$ is a complex of cohomologically trivial $\mathbb{Z}_l[G]$-modules by Lemma (2.1). By definition, we have short exact sequences

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(L_i, \mathbb{Q}/\mathbb{Z}) \to H^i(P^*) \to H^i(C^*)_{\mathrm{codiv}} \to 0, \quad i \in \mathbb{Z},$$

where L_i is a $\mathbb{Z}$-free finitely generated $\mathbb{Z}[G]$-module and the $\mathbb{Z}[G]$-module $H^i(C^*)_{\mathrm{codiv}}$ is finitely generated. So, the conditions of Proposition (2.2) are satisfied and we obtain the short exact sequences

$$0 \to H^i(C^*)_{\mathrm{codiv}} \otimes \mathbb{Z}_l \to H^i(\hat{P}^*) \to \mathrm{Hom}(L_{i+1}, \mathbb{Z}_l) \to 0, \quad i \in \mathbb{Z},$$

since $T_l(\mathbb{Q}/\mathbb{Z}) = \mathbb{Z}_l$ and since l-adic completion of a finitely generated $\mathbb{Z}[G]$-module is the same as tensoring with $\mathbb{Z}_l$. In particular, the $\mathbb{Z}_l[G]$-modules $H^i(\hat{P}^*)$, $i \in \mathbb{Z}$, are finitely generated. Thus, by Lemma (1.1), there is a perfect complex $\tilde{P}^*$ of $\mathbb{Z}_l[G]$-modules together with a quasi-isomorphism $\tilde{P}^* \to \hat{P}^*$. If we choose another quasi-isomorphism $Q^* \to C^*$ as in Lemma (1.1) and a corresponding quasi-isomorphism $\tilde{Q}^* \to \hat{Q}^*$ as above, then we obtain a quasi-isomorphism $\tilde{P}^* \to \tilde{Q}^*$ by applying Lemma (1.2) twice and by using Corollary (2.3). In particular therefore, the class

$$\chi_l(C^*) := \sum_{i\geq 0} (-1)^i [\tilde{P}^i] \in K_0(\mathbb{Z}_l[G])$$

does not depend on the above choices. We call $\chi_l(C^*) \in K_0(\mathbb{Z}_l[G])$ the *l-adic Euler characteristic of the nearly perfect complex $(C^*, (L_i)_i, (\tau_i)_i)$.*

(2.4) Theorem. *Under the canonical homomorphism $K_0(\mathbb{Z}[G]) \to K_0(\mathbb{Z}_l[G])$, the element $\chi(C^*) = \chi((C^*, (L_i)_i, (\tau_i)_i))$ of $K_0(\mathbb{Z}[G])$ maps to $\chi_l(C^*)$ in $K_0(\mathbb{Z}_l[G])$.*

Proof. As in the mapping cone construction of §1, for each i, we choose a $\mathbb{Z}[G]$-projective resolution

$$0 \longrightarrow R^{i-1} \hookrightarrow Q^i \xrightarrow{\varepsilon^i} \mathrm{Hom}(L_i, \mathbb{Q}) \longrightarrow 0$$

and a map of complexes $(Q \oplus R)^* \to P^*$ such that the composition with the quasi-isomorphism $P^* \to C^*$ chosen above is of the form considered in §1. Let Cone_P^* denote the mapping cone of the map $(Q \oplus R)^* \to P^*$. The corresponding short exact sequence of complexes

$$0 \to P^* \to \mathrm{Cone}_P^* \to (Q \oplus R)^*[1] \to 0$$

yields a quasi-isomorphism between the complexes $P^*/l^n P^*$ and $\mathrm{Cone}_P^*/l^n\mathrm{Cone}_P^*$ for all $n \geq 0$ since the cohomology of $(Q \oplus R)^*$ is uniquely divisible. Hence we obtain a quasi-isomorphism $\hat{P}^* \to \widehat{\mathrm{Cone}}_P^*$ between the l-adically completed complexes (by means of the isomorphism (6) in the proof of Proposition (2.2)). Furthermore, we choose a quasi-isomorphism $P_{\mathrm{Cone}}^* \to \mathrm{Cone}_P^*$ from a perfect complex P_{Cone}^* to Cone_P^* (which exists by Lemma (1.1)). Then the induced homomorphism $\widehat{P_{\mathrm{Cone}}}^* \to \widehat{\mathrm{Cone}}_P^*$ between the completed complexes is a quasi-isomorphism by Corollary (2.3). Hence, by Lemma (1.2), we obtain a quasi-isomorphism from the perfect complex $\widehat{P_{\mathrm{Cone}}}^*$ to the complex $\hat{P}^*$. Therefore, $\chi_l(C^*) \stackrel{\mathrm{def}}{=} \chi(\hat{P}^*) \stackrel{\mathrm{def}}{=} \chi(\widehat{P_{\mathrm{Cone}}}^*)$ is the image of $\chi(P_{\mathrm{Cone}}^*)$ under the canonical map $K_0(\mathbb{Z}[G]) \to K_0(\mathbb{Z}_l[G])$. Finally, by using the composition of the quasi-isomorphism $P_{\mathrm{Cone}}^* \to \mathrm{Cone}_P^*$ with the obvious quasi-isomorphism $\mathrm{Cone}_P^* \to \mathrm{Cone}_C^*$ we deduce from Theorem (1.3) that $\chi(P_{\mathrm{Cone}}^*)$ is equal to the Euler characteristic of the nearly perfect complex C^*. So, Theorem (2.4) is proved. □

§3 Refined Euler Characteristics

Let $C^* = (C^*, (L_i)_i, (\tau_i)_i)$ be a nearly perfect complex. In this section we also assume given a field E of characteristic 0 and an $E[G]$-equivariant *trivialization* isomorphism

$$(7) \qquad \lambda : E \otimes H^-(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_+, E) \xrightarrow{\sim} E \otimes H^+(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_-, E).$$

We shall refer to any such pair (C^*, λ) as an *E-trivialized nearly perfect complex.*

The main aim of this section (achieved in Theorem (3.9)) is to associate to each such pair (C^*, λ) a canonical Euler characteristic element $\chi^{\mathrm{rel}}(C^*, \lambda)$. This element belongs to the relative algebraic K-group $K_0(\mathbb{Z}[G], E)$ (whose definition is recalled below) and constitutes a natural refinement of the Euler characteristic $\chi(C^*)$ of C^* which has been discussed in §1.

Before proceeding we reassure the reader that E-trivialized nearly perfect complexes (C^*, λ) arise naturally in many arithmetical contexts (with the isomorphism λ arising via regulator maps or height pairings which are defined over $E = \mathbb{R}$ or $E = \mathbb{Q}_l$). Further, in many such cases it can be shown that the element $\chi^{\mathrm{rel}}(C^*, \lambda)$ provides an important means of relating natural Euler characteristics to special values of associated L-functions (for more details in this direction see [Bu4]).

The initial constructions and results of this section are modeled on those of §1 in [Bu1] (as reviewed in §1.2 of [Bu3]). However, it will be convenient for us to consider the following more general context. Let $\varphi : R \to S$ be a homomorphism between unital rings. We assume that the tensor functor $S \otimes_R -$ from the category of R-modules to the category of S-modules is exact. (We will simply write M_S for $S \otimes_R M$ for any R-module M.) We further assume that the ring S is semisimple, i.e., that all S-modules are projective (and hence also injective).

We recall (see §5, Chapter VII in [Ba]) that the *relative Grothendieck group* $G_0(R, \varphi)$ *of coherent* R*-modules* is defined to be the abelian group with generators $[A, g, B]$ where A, B are finitely generated R-modules and g is an S-module isomorphism from A_S to B_S and the following relations:
(Ri) $[A, g, B] = [A', g', B'] + [A'', g'', B'']$ whenever there is a short exact sequence of triples (with the obvious meaning)

$$0 \to (A', g', B') \to (A, g, B) \to (A'', g'', B'') \to 0.$$

(Rii) $[A, hg, C] = [A, g, B] + [B, h, C]$.
In particular, we have $[A, \mathrm{id}, A] = 0$ and $[A, g, B] = -[B, g^{-1}, A]$ in $G_0(R, \varphi)$.

We now describe the main example which we have in mind.

(3.1) Example. Let E be any field of characteristic 0, and let $\varphi : \mathbb{Z}[G] \to E[G]$ be the canonical inclusion of group rings. In this case, we also write $G_0(\mathbb{Z}[G], E)$ for $G_0(\mathbb{Z}[G], \varphi)$. It is well-known (see Theorem (5.4) on p. 423 in §5, Chapter VIII of [Ba]) and easy to prove that the association $[M] \mapsto [0, 0, M]$ induces a well-defined isomorphism

$$G_0T(\mathbb{Z}[G]) \xrightarrow{\sim} G_0(\mathbb{Z}[G], \mathbb{Q})$$

where $G_0T(\mathbb{Z}[G])$ denotes the Grothendieck group of all $\mathbb{Z}$-torsion $\mathbb{Z}[G]$-modules; its inverse is given by sending a generator $[A, g, B]$ as above to the element

$$[\mathrm{coker}(h)] - [\ker(h)] - [B/nB] + [{}_nB]$$

where $h \in \mathrm{Hom}_{\mathbb{Z}[G]}(A, B)$ and $n \in \mathbb{Z}$ are chosen in such a way that $h/n = g$ in $\mathrm{Hom}_{\mathbb{Q}[G]}(A_\mathbb{Q}, B_\mathbb{Q}) = \mathrm{Hom}_{\mathbb{Z}[G]}(A, B)_\mathbb{Q}$. (Here, we write $A_\mathbb{Q}$ for $\mathbb{Q} \otimes_\mathbb{Z} A \cong \mathbb{Q}[G] \otimes_{\mathbb{Z}[G]} A$.) (This statement can of course be generalized to the situation in which $\mathbb{Z}$ is replaced by any Dedekind domain and $\mathbb{Q}$ by the corresponding field of fractions.)

Let now M^* be a bounded complex of finitely generated R-modules. We assume that, for each i, we are given a (finite, exhaustive and separated) decreasing filtration $(F^nH^i)_{i\in\mathbb{Z}}$ on the cohomology modules $H^i := H^i(M^*)$ and that we are given an isomorphism

$$\lambda : \mathrm{Gr}(H^-)_S \xrightarrow{\sim} \mathrm{Gr}(H^+)_S$$

from the module $\mathrm{Gr}(H^-)_S := \oplus_{i \text{ odd}} \oplus_{n\in\mathbb{Z}} \mathrm{Gr}^n(H^i)_S := \oplus_{i \text{ odd}} \oplus_{n\in\mathbb{Z}} (F^nH^i/F^{n+1}H^i)_S$ to the module $\mathrm{Gr}(H^+)_S := \oplus_{i \text{ even}} \oplus_{n\in\mathbb{Z}} (F^nH^i/F^{n+1}H^i)_S$. By the assumption on φ and S, we have split short exact sequences

$$0 \to Z^i_S \to M^i_S \to B^{i+1}_S \to 0, \quad 0 \to B^i_S \to Z^i_S \to H^i_S \to 0$$

and

$$0 \to (F^{n+1}H^i)_S \to (F^nH^i)_S \to \mathrm{Gr}^n(H^i)_S \to 0, \quad n \in \mathbb{Z},$$

where we write $B^i := B^i(M^*)$, $Z^i := Z^i(M^*)$ and $H^i := H^i(M^*)$ for brevity. So we obtain an S-module isomorphism

$$\begin{aligned}\lambda_{M^*} : M_S^- := \underset{i \text{ odd}}{\oplus} M_S^i &\cong \underset{i \text{ odd}}{\oplus} (Z_S^i \oplus B_S^{i+1}) \\ &\cong \underset{i \text{ odd}}{\oplus} (H_S^i \oplus B_S^i \oplus B_S^{i+1}) \cong \oplus_{i\,\mathrm{odd}}(\oplus_{n\in\mathbb{Z}}(\mathrm{Gr}^n(H^i)_S) \oplus B_S^i \oplus B_S^{i+1}) \\ &\overset{\iota}{\cong} \oplus_{i\,\mathrm{even}}(\oplus_{n\in\mathbb{Z}}(\mathrm{Gr}^n(H^i)_S) \oplus B_S^i \oplus B_S^{i+1}) \cong \oplus_{i \text{ even}}(H_S^i \oplus B_S^i \oplus B_S^{i+1}) \\ &\cong \underset{i \text{ even}}{\oplus} (Z_S^i \oplus B_S^{i+1}) \cong \underset{i \text{ even}}{\oplus} M_S^i =: M_S^+\end{aligned}$$

where all isomorphisms except ι are induced by splittings of the short exact sequences above and the isomorphism ι is the direct sum of the isomorphism λ with the identity on $\oplus_{i \text{ odd}}(B_S^i \oplus B_S^{i+1}) = \oplus_{i \text{ even}}(B_S^i \oplus B_S^{i+1})$.

(3.2) Lemma. *We have:*

$$[M^-, \lambda_{M^*}, M^+] = [\mathrm{Gr}(H^-), \lambda, \mathrm{Gr}(H^+)] \quad \text{in} \quad G_0(R, \varphi).$$

In particular, the class $[M^-, \lambda_{M^*}, M^+] \in G_0(R, \varphi)$ does not depend on the chosen splittings.

Proof. By relation (Rii), the class $[M^-, \lambda_{M^*}, M^+]$ can be written as the sum of the 7 classes T_i, $i = 1, \ldots, 7$, of the triples corresponding to the 7 isomorphisms in the definition of λ_{M^*}. We have $T_4 = [\mathrm{Gr}(H^-), \lambda, \mathrm{Gr}(H^+)]$ by the relation (Ri) and the relation $[A, \mathrm{id}, A] = 0$. Thus, it suffices to show that $T_1 = T_2 = T_3 = 0 = T_5 = T_6 = T_7$. This immediately follows from the relation (Ri) and the fact that, for any short exact sequence $0 \to A' \to A \to A'' \to 0$ of finitely generated R-modules, we have $[A, \alpha, A' \oplus A''] = 0$ in $G_0(R, \varphi)$ where the isomorphism $\alpha : A_S \to A'_S \oplus A''_S$ is induced by any splitting of the short exact sequence $0 \to A'_S \to A_S \to A''_S \to 0$. To prove this fact, we merely apply relation (Ri) to the obvious short exact sequence of triples

$$0 \to (A', \mathrm{id}, A') \to (A, \alpha, A' \oplus A'') \to (A'', \mathrm{id}, A'') \to 0$$

and use the relations $(A', \mathrm{id}, A') = 0 = (A'', \mathrm{id}, A'')$. □

(3.3) Corollary. *Let $f : \mathbb{Z}[G] \hookrightarrow \mathbb{Q}[G]$ be the canonical inclusion as in Example (3.1). We assume that $M^*_{\mathbb{Q}}$ is acyclic. Then, under the canonical isomorphism $G_0(\mathbb{Z}[G], \mathbb{Q}) \cong G_0T(\mathbb{Z}[G])$, the class $[M^-, 0_{M^*}, M^+]$ is mapped to $[H^+(M^*)] - [H^-(M^*)]$.*

Proof. This immediately follows from Lemma (3.2). □

We now recall (see Chapter 15 in [Sw]) that the *relative Grothendieck group $K_0(R, \varphi)$ of coherent projective R-modules* is defined to be the abelian group with generators $[P, \psi, Q]$ where P and Q are finitely generated projective R-modules and $\psi : P_S \to Q_S$ is an S-module isomorphism and relations which are analogous to the relations (Ri) and (Rii) described above. This group $K_0(R, \varphi)$ is also often referred to as a relative algebraic K-group.

(3.4) Example. In the situation of Example (3.1) we again write $K_0(\mathbb{Z}[G], E)$ for $K_0(\mathbb{Z}[G], \varphi)$. It is well-known (see the proof of Theorem (5.8) in §5, Chapter VIII

of [Ba]) and easy to prove that the relative Grothendieck group $K_0(\mathbb{Z}[G], \mathbb{Q})$ is isomorphic to the Grothendieck group $K_0T(\mathbb{Z}[G])$ of all finite $\mathbb{Z}[G]$-modules of projective dimension at most 1. Here, the class of a finite $\mathbb{Z}[G]$-module M of projective dimension 1 is mapped to the class of the triple $(P, \alpha_{\mathbb{Q}}, Q)$ where

$$0 \longrightarrow P \xrightarrow{\alpha} Q \longrightarrow M \longrightarrow 0$$

is any resolution of M by finitely generated projective modules. The inverse map sends the class of a triple (P, ψ, Q) to the element $[\mathrm{coker}(\alpha)] - [P/nP]$ where $\alpha \in \mathrm{Hom}_{\mathbb{Z}[G]}(P, Q)$ and $n \in \mathbb{Z}$ are chosen in such a way that $\psi = \alpha/n$ in $\mathrm{Hom}_{\mathbb{Q}[G]}(P_{\mathbb{Q}}, Q_{\mathbb{Q}}) = \mathrm{Hom}_{\mathbb{Z}[G]}(P, Q)_{\mathbb{Q}}$. (Again, this statement can be generalized in the obvious way to the case in which $\mathbb{Z}$ is replaced by any Dedekind domain and $\mathbb{Q}$ by the corresponding field of fractions.)

By Theorem 15.5 on p. 216 in [Sw], we have a natural exact sequence

$$K_1(R) \to K_1(S) \xrightarrow{\partial} K_0(R, \varphi) \to K_0(R) \to K_0(S) \tag{8}$$

where the connecting homomorphism ∂ is (uniquely) given by the following rule (see Lemma 15.7 on p. 217 in [Sw]): Let P be a finitely generated projective R-module and ψ an S-module automorphism of P_S; then ∂ maps the class of the pair (P_S, ψ) in $K_1(S)$ to the class of the triple $[P, \psi, P]$ in $K_0(R, \varphi)$.

Let now P^* be a perfect complex of R-modules. We assume that, for each i, we are given a (finite, exhaustive and separated) decreasing filtration $(F^n H^i)_{n\in\mathbb{Z}}$ on the cohomology module $H^i(P^*)$ and that we are also given an S-module isomorphism

$$\lambda : \mathrm{Gr}(H^-)_S \xrightarrow{\sim} \mathrm{Gr}(H^+)_S$$

from $\mathrm{Gr}(H^-)_S := \oplus_{i\,\mathrm{odd}} \oplus_{n\in\mathbb{Z}} (F^n H^i / F^{n+1} H^i)_S$ to $\mathrm{Gr}(H^+)_S$. As above, we form the isomorphism

$$\lambda_{P^*} : P_S^- \xrightarrow{\sim} P_S^+$$

which depends on splittings of the corresponding natural short exact sequences. The following result states that this isomorphism gives rise to a well-defined class in the relative Grothendieck Group $K_0(R, \varphi)$ and records the basic properties of this class. (We remark that this result is an obvious generalization of a special case of results proved in §1 of [Bu1]. However, as that paper is not published, we shall repeat the relevant arguments here.)

(3.5) Proposition.
(a) The class $\chi^{\mathrm{rel}}(P^*, \lambda) := [P^-, \lambda_{P^*}, P^+] \in K_0(R, \varphi)$ *does not depend upon the chosen splittings.*
(b) If the complex P^* *is acyclic, then* $\chi^{\mathrm{rel}}(P^*, 0) = 0$ *in* $K_0(R, \varphi)$.
(c) Let Q^* *be another perfect complex with a filtration on the cohomology groups as above and let* $\alpha^* : Q^* \to P^*$ *be a quasi-isomorphism such that* $H^i(\alpha^*)$ *is compatible with the given filtrations for all* i*. Then one has*

$$[P^-, \lambda_{P^*}, P^+] = [Q^-, (\mathrm{Gr}(H^+(\alpha^*))^{-1} \circ \lambda \circ \mathrm{Gr}(H^-(\alpha^*)))_{Q^*}, Q^+]$$

in $K_0(R,\varphi)$.
(d) Let $\lambda' : \mathrm{Gr}(H^-)_S \xrightarrow{\sim} \mathrm{Gr}(H^+)_S$ *be any other* S*-module isomorphism. Then one has*

$$\chi^{\mathrm{rel}}(P^*,\lambda') - \chi^{\mathrm{rel}}(P^*,\lambda) = \partial([\mathrm{Gr}(H^-)_S, \lambda^{-1}\circ\lambda']) \quad \textit{in} \quad K_0(R,\varphi).$$

Proof.
(a) Let $0 \to W' \xrightarrow{i} W \xrightarrow{\varepsilon} W'' \to 0$ be a short exact sequence of finitely generated S-modules. Then it is easy to see that the composition $W \cong W' \oplus W'' \cong W$ of two isomorphisms induced by any splittings of the given sequence is given by $w \mapsto w + ih\varepsilon(w)$ for some S-module homomorphism $h : W'' \to W'$. Therefore, if we choose, for example for one odd i, a different splitting in one of the short exact sequences

$$0 \to Z^i_S \to P^i_S \to B^{i+1}_S \to 0, \qquad 0 \to B^i_S \to Z^i_S \to H^i_S \to 0$$

or

$$0 \to (F^{n+1}H^i)_S \to (F^nH^i)_S \to \mathrm{Gr}^n(H^i)_S \to 0, \quad n \in \mathbb{Z},$$

and denote the corresponding isomorphism by $\lambda^{\dagger}_{P^*}$, then we can find a short exact sequence

$$0 \to U \xrightarrow{j} P^-_S \xrightarrow{\eta} V \to 0$$

of S-modules such that the composition $(\lambda^{\dagger}_{P^*})^{-1}\lambda_{P^*}$ is given by $p \mapsto p + jZ\eta(p)$ for some S-module homomorphism $Z : V \to U$. In particular, we obtain a short exact sequence of pairs

$$0 \to (U, \mathrm{id}_U) \to (P^-_S, (\lambda^{\dagger}_{P^*})^{-1}\lambda_{P^*}) \to (V, \mathrm{id}_V) \to 0.$$

Hence in $K_0(R,\varphi)$ we have

$$\begin{aligned}[P^-, \lambda_{P^*}, P^+] - [P^-, \lambda^{\dagger}_{P^*}, P^+] &= [P^-, (\lambda^{\dagger}_{P^*})^{-1}\circ\lambda_{P^*}, P^-] \\ &= \partial([P^-_S, (\lambda^{\dagger}_{P^*})^{-1}\circ\lambda_{P^*}]) = \partial([U,\mathrm{id}_U] + [V,\mathrm{id}_V]) = 0,\end{aligned}$$

as was to be shown.
(b) If P^* is acyclic, then for each i, the R-module $B^i = Z^i$ is projective and so the short exact sequence of R-modules $0 \to Z^i \to P^i \to B^{i+1} \to 0$ splits. Using such splittings we obtain an isomorphism

$$\iota : P^- \cong \oplus_{i\,\mathrm{odd}}(Z^i \oplus B^{i+1}) = \oplus_{i\,\mathrm{even}}(Z^i \oplus B^{i+1}) \cong P^+$$

and, by claim (a), we may take ι_S for 0_{P^*}. Now, as in the proof of Lemma (3.2), we see that $\chi^{\mathrm{rel}}(P^*, 0) = [P^-, \iota_S, P^+] = 0$ in $K_0(R,\varphi)$.
(c) By Lemma (3.6) below, we may assume that, in each degree i, both the map $\alpha^i : Q^i \to P^i$ and the induced map $Z^i(\alpha^*) : Z^i(Q^*) \to Z^i(P^*)$ are surjective. Then the kernel K^* of the epimorphism $\alpha^* : Q^* \to P^*$ is an acyclic perfect complex and, in

each degree i, we have the following commutative diagram of short exact sequences of S-modules

$$\begin{array}{ccccc}
Z^i(K^*)_S & \hookrightarrow & K^i_S & \twoheadrightarrow & B^{i+1}(K^*)_S \\
\downarrow & & \downarrow & & \downarrow \\
Z^i(Q^*)_S & \hookrightarrow & Q^i_S & \twoheadrightarrow & B^{i+1}(Q^*)_S \\
\downarrow & & \downarrow & & \downarrow \\
Z^i(P^*)_S & \hookrightarrow & P^i_S & \twoheadrightarrow & B^{i+1}(P^*)_S
\end{array}$$

and

$$\begin{array}{ccccc}
B^i(K^*)_S & \hookrightarrow & Z^i(K^*)_S & \twoheadrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
B^i(Q^*)_S & \hookrightarrow & Z^i(Q^*)_S & \twoheadrightarrow & H^i(Q^*)_S \\
\downarrow & & \downarrow & & \downarrow \\
B^i(P^*)_S & \hookrightarrow & Z^i(P^*)_S & \twoheadrightarrow & H^i(P^*)_S
\end{array} .$$

By Lemma (3.7) below, we can choose compatible splittings in these diagrams. In particular, we obtain a short exact sequence of triples

$$[K^-, 0_{K^*}, K^+] \hookrightarrow [Q^-, (\mathrm{Gr}(H^+(\alpha))^{-1} \circ \lambda \circ \mathrm{Gr}(H^-(\alpha)))_{Q^*}, Q^+] \twoheadrightarrow [P^-, \lambda_{P^*}, P^+].$$

Now, relation (Ri) and claim (b) together imply claim (c).
(d) In $K_0(R, \varphi)$ we have

$$\chi^{\mathrm{rel}}(P^*, \lambda') - \chi^{\mathrm{rel}}(P^*, \lambda) = [P^-, (\lambda_{P^*})^{-1} \circ \lambda'_{P^*}, P^-] = \partial([P^-_S, (\lambda_{P^*})^{-1} \circ \lambda'_{P^*}]).$$

If we use the same splittings for the definition of λ'_{P^*} and λ_{P^*}, then these splittings induce an isomorphism between the pairs $(P^-_S, (\lambda_{P^*})^{-1} \circ \lambda'_{P^*})$ and

$$(\oplus_{i\,\mathrm{odd}}(\oplus_{n\in\mathbb{Z}}(\mathrm{Gr}^n(H^i)_S) \oplus B^i_S \oplus B^{i+1}_S), \lambda^{-1} \circ \lambda' \oplus \mathrm{id}_{\oplus_{i\,\mathrm{odd}}(B^i_S \oplus B^{i+1}_S)}).$$

Hence, we have $[P^-_S, (\lambda_{P^*})^{-1} \circ \lambda'_{P^*}] = [\mathrm{Gr}(H^-)_S, \lambda^{-1} \circ \lambda']$ in $K_1(S)$. This completes the proof of claim (d) of Proposition (3.5). □

(3.6) Lemma. *Let R be a ring and $\alpha^* : Q^* \to P^*$ a quasi-isomorphism between perfect complexes of R-modules. Then there exists a perfect complex T^* of R-modules together with quasi-isomorphisms $\beta^* : T^* \to Q^*$ and $\gamma^* : T^* \to P^*$ such that, in each degree i, we have $H^i(\alpha^* \circ \beta^*) = H^i(\gamma^*)$ and both the maps $\beta^i : T^i \to Q^i$, $Z^i(\beta^*) : Z^i(T^*) \to Z^i(Q^*)$ and the maps $\gamma^i : T^i \to P^i$, $Z^i(\gamma^*) : Z^i(T^*) \to Z^i(P^*)$ are surjective.*

Proof. Let K^* be the acyclic perfect complex with $K^i := P^{i-1} \oplus P^i$ and with differential $K^i \to K^{i+1}$, $(x_{i-1}, x_i) \mapsto (x_i, 0)$. We set $T^* := K^* \oplus Q^*$ and define $\beta^* : T^* \to Q^*$ to be the canonical projection. Then β^* is clearly a quasi-isomorphism and, for each i, the maps β^i and $Z^i(\beta^*)$ are surjective. We define $\gamma^i : T^i = P^{i-1} \oplus P^i \oplus Q^i \to P^i$ by $\gamma^i|_{P^{i-1}} = d^{i-1}(P^*)$, $\gamma^i|_{P^i} = \mathrm{id}_{P^i}$, and $\gamma^i|_{Q^i} = \alpha^i$. Then, γ^* is clearly a homomorphism of complexes such that, in each degree i, we have $H^i(\alpha^* \circ \beta^*) = H^i(\gamma^*)$ and the map γ^i is surjective. One easily checks that the map $Z^i(\gamma^*) : P^{i-1} \oplus Z^i(Q^*) \to Z^i(P^*)$ is also surjective. Thus, the proof of Lemma (3.6) is complete. □

(3.7) Lemma. *Let S be a semisimple ring. Suppose we are given a commutative diagram of short exact sequences of S-modules*

$$\begin{array}{ccccc} K' & \hookrightarrow & K & \overset{\delta}{\twoheadrightarrow} & K'' \\ \downarrow & & \downarrow & & \downarrow \\ V' & \hookrightarrow & V & \overset{\varepsilon}{\twoheadrightarrow} & V'' \\ \downarrow & & \downarrow & & \downarrow \\ W' & \hookrightarrow & W & \overset{\eta}{\twoheadrightarrow} & W''. \end{array}$$

Then one can choose a section $\sigma : V'' \to V$ to ε such that $\sigma|_{K''}$ is a section to δ (and hence σ induces a section to η).

Proof. First, we choose any section $\tilde{\sigma}$ to ε. Then the composition

$$K'' \hookrightarrow V'' \xrightarrow{\tilde{\sigma}} V \twoheadrightarrow W$$

may be considered as a homomorphism θ from K'' to W'. Since K'' is projective, we may lift θ to a homomorphism from K'' to V' and, since V' is injective, this can in turn be extended to a homomorphism from V'' to V' which we view as a homomorphism $\tilde{\theta}$ from V'' to V. Now, one easily checks that the homomorphism $\sigma := \tilde{\sigma} - \tilde{\theta}$ is a section to ε such that $\sigma|_{K''}$ is a section to δ. □

We now return to the context described at the beginning of this section. We thus assume given an E-trivialized nearly perfect complex (C^*, λ) with $C^* = (C^*, (L_i)_i, (\tau_i)_i)$. We choose a quasi-isomorphism $P^* \to C^*$ as in Lemma (1.1). From Proposition (2.2) we then obtain natural exact sequences

$$(9) \qquad 0 \to H^i(C^*)_{\mathrm{codiv}} \otimes \mathbb{Z}_l \to H^i(\hat{P}^*) \to \mathrm{Hom}(L_{i+1}, \mathbb{Z}_l) \to 0, \quad i \in \mathbb{Z},$$

(see §2). Furthermore, we choose resolutions

$$0 \longrightarrow R^{i-1} \longrightarrow Q^i \xrightarrow{\varepsilon^i} \mathrm{Hom}(L_i, \mathbb{Q}) \longrightarrow 0\,, \quad i \in \mathbb{Z},$$

and a homomorphism of complexes $(\alpha^i, \beta^i)_i : (Q \oplus R)^* \to P^*$ such that the composition with the chosen quasi-isomorphism $P^* \to C^*$ is of the form considered in §1. By tensoring the exact sequences (1) in §1 with $\mathbb{Z}_l$, we thus obtain natural exact sequences

$$(10) \quad 0 \to H^i(C^*)_{\mathrm{codiv}} \otimes \mathbb{Z}_l \to H^i(\mathrm{Cone}^*_C) \otimes \mathbb{Z}_l \to \mathrm{Hom}(L_{i+1}, \mathbb{Z}_l) \to 0, \quad i \in \mathbb{Z}.$$

(3.8) Proposition. *For each i, the extension (9) is the negative of the extension (10) in*

$$\mathrm{Ext}^1_{\mathbb{Z}_l[G]}(\mathrm{Hom}(L_{i+1}, \mathbb{Z}_l), H^i(C^*)_{\mathrm{codiv}} \otimes \mathbb{Z}_l).$$

Proof. Let Cone_P^* denote the mapping cone of the map $(\alpha^i, \beta^i)_i : (Q \oplus R)^* \to P^*$. It obviously suffices to show that the extension

$$0 \to H^i(P^*)_{\mathrm{codiv}} \otimes \mathbb{Z}_l \to H^i(\hat{P}^*) \to \mathrm{Hom}(L_{i+1}, \mathbb{Z}_l) \to 0 \tag{11}$$

(resulting from Proposition (2.2)) is the negative of the extension

$$0 \to H^i(P^*)_{\mathrm{codiv}} \otimes \mathbb{Z}_l \to H^i(\mathrm{Cone}_P^*) \otimes \mathbb{Z}_l \to \mathrm{Hom}(L_{i+1}, \mathbb{Z}_l) \to 0 \tag{12}$$

(constructed as in §1). We have the following commutative diagram

$$\begin{array}{ccccc}
H^i\widehat{(P^*)}_{\mathrm{codiv}} & \hookrightarrow & H^i(\hat{P}^*) & \twoheadrightarrow & \mathrm{Hom}(L_{i+1}, \mathbb{Z}_l) \\
\big\downarrow & & \big\downarrow \wr & & \\
H^i\widehat{(\mathrm{Cone}_P^*)} & \xrightarrow{\sim} & H^i(\widehat{\mathrm{Cone}}_P^*) & & \\
\big\downarrow & & & & \\
\mathrm{Hom}(L_{i+1}, \mathbb{Z}_l) & & & &
\end{array}$$

where the top row is the extension (11), the left column is the extension (12), the isomorphism in the second row is a consequence of Proposition (2.2) and the isomorphism in the second column is induced by the canonical inclusion $P^* \hookrightarrow \mathrm{Cone}_P^*$ (see the proof of Theorem (2.4)). The snake lemma applied to this diagram yields a $\mathbb{Z}_l[G]$-module automorphism of $\mathrm{Hom}(L_{i+1}, \mathbb{Z}_l)$. It suffices to show that this automorphism is the multiplication by -1. For this it suffices to check that the connecting homomorphism from the upper right corner in the diagram

$$\begin{array}{ccccc}
 & & & & \mathrm{Hom}(L_{i+1}, \mathbb{Z}/l^n\mathbb{Z}) \\
 & & & & \big\downarrow \tau_{i+1} \\
H^i(P^*)/l^n H^i(P^*) & \hookrightarrow & H^i(P^*/l^n P^*) & \xrightarrow{\partial} & {}_{l^n}H^{i+1}(P^*) \\
\big\downarrow & & \big\downarrow \wr & & \big\downarrow \\
H^i(\mathrm{Cone}_P^*)/l^n H^i(\mathrm{Cone}_P^*) & \hookrightarrow & H^i(\mathrm{Cone}_P^*/l^n \mathrm{Cone}_P^*) & \xrightarrow{\partial} & {}_{l^n}H^{i+1}(\mathrm{Cone}_P^*) \\
\big\downarrow & & & & \\
\mathrm{Hom}(L_{i+1}, \mathbb{Z}/l^n\mathbb{Z}) & & & &
\end{array}$$

(with the obvious maps) to the lower left corner is the multiplication with -1.
Any element of $\mathrm{Hom}(L_{i+1}, \mathbb{Z}/l^n\mathbb{Z})$ can be written as the residue class $\bar{e}$ of some $e \in \mathrm{Hom}(L_{i+1}, \mathbb{Z}) \subseteq \mathrm{Hom}(L_{i+1}, \mathbb{Q})$. We choose an element $q \in Q^{i+1}$ with $\varepsilon^{i+1}(q) = e/l^n$ in $\mathrm{Hom}(L_{i+1}, \mathbb{Q})$. Under the inclusion $\tau_{i+1} : \mathrm{Hom}(L_{i+1}, \mathbb{Z}/l^n\mathbb{Z}) \hookrightarrow {}_{l^n}H^{i+1}(P^*)$, the element $\bar{e}$ is then mapped to $\tau_{i+1}(\overline{\varepsilon^{i+1}(q)})$ where $\overline{\varepsilon^{i+1}(q)}$ denotes the image of $\varepsilon^{i+1}(q)$ in $\mathrm{Hom}(L_{i+1}, \mathbb{Q}/\mathbb{Z})$. Since $l^n \cdot \tau_{i+1}(\overline{\varepsilon^{i+1}(q)}) = 0$ in $H^{i+1}(P^*)$, we can find an element $p \in P^i$ such that $l^n \cdot \alpha^{i+1}(q) = d(p)$ in P^{i+1}. Then the cohomology class

$[\bar{p}] \in H^i(P^*/l^nP^*)$ of the cocycle $\bar{p} \in P^i/l^nP^i$ is mapped to $\tau_{i+1}(\overline{\varepsilon^{i+1}(q)})$ under the connecting homomorphism ∂ in the upper row. Furthermore, the cohomology class of the cocycle $(p, -l^nq, 0) \in P^i \oplus Q^{i+1} \oplus R^{i+1} = \mathrm{Cone}^i_P$ is obviously mapped to the cohomology class of $(\bar{p}, 0, 0)$ under the canonical inclusion

$$H^i(\mathrm{Cone}^*_P)/l^nH^i(\mathrm{Cone}^*_P) \hookrightarrow H^i(\mathrm{Cone}^*_P/l^n\mathrm{Cone}^*_P).$$

Finally, the cohomology class of $(p, -l^nq, 0)$ is mapped to $-l^n\varepsilon^{i+1}(q) = -e$ under the epimorphism $H^i(\mathrm{Cone}^*_P) \twoheadrightarrow \mathrm{Hom}(L_{i+1}, \mathbb{Z})$. Therefore, $\bar{e} \in \mathrm{Hom}(L_{i+1}, \mathbb{Z}/l^n\mathbb{Z})$ is mapped to $-\bar{e}$ under the connecting homomorphism in the diagram above, as was to be shown. □

We next observe that, by the classical Noether-Deuring theorem (see Theorem (29.7) on p. 200 in [CR]), the existence of an E-trivialization λ as in (7) implies that there also exists a (non-canonical) $\mathbb{Q}[G]$-equivariant isomorphism

(13) $$\tilde{\lambda} : \mathbb{Q} \otimes H^-(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_+, \mathbb{Q}) \tilde{\to} \mathbb{Q} \otimes H^+(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_-, \mathbb{Q}).$$

For each prime l we may therefore tensor $\tilde{\lambda}$ with $\mathbb{Q}_l$ in order to obtain a $\mathbb{Q}_l[G]$-equivariant isomorphism

$$\tilde{\lambda}_l : \mathbb{Q}_l \otimes H^-(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_+, \mathbb{Q}_l) \tilde{\to} \mathbb{Q}_l \otimes H^+(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_-, \mathbb{Q}_l).$$

As in §2, we choose a quasi-isomorphism $\tilde{P}^* \to \hat{P}^*$ from a perfect complex $\tilde{P}^*$ of $\mathbb{Z}_l[G]$-modules to the l-adically completed complex $\hat{P}^*$. For each i, we consider the extension (9) as a 2-step filtration on $H^i(\tilde{P}^*)$. Then, from the construction above, we obtain a well-defined element

$$\chi^{\mathrm{rel}}(C^*, \tilde{\lambda}_l) := [\tilde{P}^-, (\tilde{\lambda}_l)_{\tilde{P}^*}, \tilde{P}^+] \in K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)$$

which, by Lemma (1.2), Corollary (2.3) and Proposition (3.5)(c), does not depend on the chosen quasi-isomorphisms $P^* \to C^*$ and $\tilde{P}^* \to \hat{P}^*$. Finally we recall that there are canonical isomorphisms

$$\bigoplus_{l\,\mathrm{prime}} K_0(\mathbb{Z}_l[G], \mathbb{Q}_l) \cong \bigoplus_{l\,\mathrm{prime}} K_0T(\mathbb{Z}_l[G]) \cong K_0T(\mathbb{Z}[G]) \cong K_0(\mathbb{Z}[G], \mathbb{Q}),$$

and a canonical injective homomorphism $i_E : K_0(\mathbb{Z}[G], \mathbb{Q}) \to K_0(\mathbb{Z}[G], E)$.

As promised at the beginning of this section, we now define a canonical refined Euler characteristic $\chi^{\mathrm{rel}}(C^*, \lambda)$ for the E-trivialized nearly perfect complex (C^*, λ).

(3.9) Theorem.
(a) *The infinite tuple*

$$\chi^{\mathrm{rel}}(C^*, \tilde{\lambda}) := (\chi^{\mathrm{rel}}(C^*, \tilde{\lambda}_l))_{l\,\mathrm{prime}} \in \prod_{l\,\mathrm{prime}} K_0(\mathbb{Z}_l[G], \mathbb{Q}_l)$$

belongs to the direct sum $\oplus_{l\,\mathrm{prime}} K_0(\mathbb{Z}_l[G], \mathbb{Q}_l) \cong K_0(\mathbb{Z}[G], \mathbb{Q})$.
(b) *The element*

$$\chi^{\mathrm{rel}}(C^*, \lambda) := i_E(\chi^{\mathrm{rel}}(C^*, \tilde{\lambda})) + \partial([E \otimes H^-(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_+, E), \tilde{\lambda}_E^{-1} \circ \lambda])$$

of $K_0(\mathbb{Z}[G], E)$ depends only upon C^* and λ.
(c) *The image of $\chi^{\mathrm{rel}}(C^*, \lambda)$ under the canonical map $K_0(\mathbb{Z}[G], E) \to K_0(\mathbb{Z}[G])$, $[P, \psi, Q] \mapsto [Q] - [P]$, is equal to the Euler characteristic $\chi(C^*)$.*
(d) *The image of $\chi^{\mathrm{rel}}(C^*, \lambda)$ under the forgetful map $K_0(\mathbb{Z}[G], E) \to G_0(\mathbb{Z}[G], E)$ is equal to the element*

$$[H^-(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_+, \mathbb{Z}), \lambda, H^+(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_-, \mathbb{Z})].$$

Proof. As in the proof of Theorem (2.4), we choose a quasi-isomorphism $P^*_{\mathrm{Cone}} \to \mathrm{Cone}^*_P$ from a perfect complex P^*_{Cone} of $\mathbb{Z}[G]$-modules to the mapping cone Cone^*_P of the homomorphism $(\alpha^i, \beta^i)_i : (Q \oplus R)^* \to P^*$ chosen above. We recall from §1 that we have short exact sequences

$$0 \to H^i(C^*)_{\mathrm{codiv}} \to H^i(P^*_{\mathrm{Cone}}) \to \mathrm{Hom}(L_{i+1}, \mathbb{Z}) \to 0, \quad i \in \mathbb{Z}.$$

From the construction above, we therefore obtain an element

$$[P^-_{\mathrm{Cone}}, \tilde{\lambda}_{P^*_{\mathrm{Cone}}}, P^+_{\mathrm{Cone}}] \in K_0(\mathbb{Z}[G], \mathbb{Q}). \tag{14}$$

To prove claim (a) it therefore suffices to show that this element is equal to $\chi^{\mathrm{rel}}(C^*, \tilde{\lambda})$. In other words, it suffices to show that for all primes l the element

$$[P^-_{\mathrm{Cone}} \otimes \mathbb{Z}_l, (\tilde{\lambda}_{P^*_{\mathrm{Cone}}})_{\mathbb{Q}_l}, P^+_{\mathrm{Cone}} \otimes \mathbb{Z}_l] = [\widehat{P_{\mathrm{Cone}}}^-, (\tilde{\lambda}_l)_{\widehat{P_{\mathrm{Cone}}}^*}, \widehat{P_{\mathrm{Cone}}}^+]$$

coincides with $\chi^{\mathrm{rel}}(C^*, \tilde{\lambda}_l)$. But the natural inclusion $P^* \hookrightarrow \mathrm{Cone}^*_P$ induces a quasi-isomorphism $\hat{P}^* \to \widehat{\mathrm{Cone}_P}^*$ (see the proof of Theorem (2.4)). So, by Lemma (1.2), we obtain a quasi-isomorphism between the perfect complexes $\tilde{P}^*$ and $\widehat{P_{\mathrm{Cone}}}^*$ which, by (the proof of) Proposition (3.8), is compatible with the filtrations on the cohomology given by the short exact sequences (9) and (10). Thus the desired equality follows from Proposition (3.5)(c).
We now turn to claim (b). Here it suffices to show that the element $\chi^{\mathrm{rel}}(C^*, \lambda)$ is independent of the choice of $\mathbb{Q}[G]$-equivariant isomorphism $\tilde{\lambda}$ as in (13). But if $\lambda^\dagger$ is any other isomorphism as in (13), then Proposition (3.5)(d) implies that

$$i_E(\chi^{\mathrm{rel}}(C^*, \lambda^\dagger)) = i_E(\chi^{\mathrm{rel}}(C^*, \tilde{\lambda})) + \partial([E \otimes H^-(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_+, E), \tilde{\lambda}_E^{-1} \circ \lambda_E^\dagger])$$

in $K_0(\mathbb{Z}[G], E)$. This leads directly to the required equality since in $K_1(E[G])$ one has

$$[E \otimes H^-(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_+, E), (\lambda_E^\dagger)^{-1} \circ \lambda] + [E \otimes H^-(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_+, E), \tilde{\lambda}_E^{-1} \circ \lambda_E^\dagger]$$

$$= [E \otimes H^-(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_+, E), \tilde{\lambda}_E^{-1} \circ \lambda].$$

Next we observe that the exact sequence of relative K-theory (8) implies that the elements $\chi^{\mathrm{rel}}(C^*, \lambda)$ and $\chi^{\mathrm{rel}}(C^*, \tilde{\lambda})$ have the same image under the homomorphism $K_0(\mathbb{Z}[G], E) \to K_0(\mathbb{Z}[G])$. Claim (c) therefore follows from the fact that $\chi^{\mathrm{rel}}(C^*, \tilde{\lambda})$ is equal to the element which occurs in (14). Indeed, the latter element clearly maps

to $\chi(P^*_{\mathrm{Cone}})$ in $K_0(\mathbb{Z}[G])$ which is equal to $\chi(C^*)$ by Theorem (1.3).
To prove claim (d), we first observe that the element (14) maps to the element $[P^-_{\mathrm{Cone}}, \tilde{\lambda}_{P_{\mathrm{Cone}}}, P^+_{\mathrm{Cone}}]$ in $G_0(\mathbb{Z}[G], \mathbb{Q})$. Further, Lemma (3.2) implies that the latter element is equal to

$$\begin{aligned}&[\mathrm{Gr}(H^-(\mathrm{Cone}^*_C)), \tilde{\lambda}, \mathrm{Gr}(H^+(\mathrm{Cone}^*_C))]\\ &\quad = \ [H^-(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_+, \mathbb{Z}), \tilde{\lambda}, H^+(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_-, \mathbb{Z})].\end{aligned}$$

It therefore follows that the image of $\chi^{\mathrm{rel}}(C^*, \lambda)$ in $G_0(\mathbb{Z}[G], E)$ is equal to

$$\begin{aligned}&[H^-(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_+, \mathbb{Z}), \tilde{\lambda}_E, H^+(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_-, \mathbb{Z})]\\ &\qquad + [H^-(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_+, \mathbb{Z}), \tilde{\lambda}_E^{-1} \circ \lambda, H^-(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_+, \mathbb{Z})]\\ &\quad = \ [H^-(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_+, \mathbb{Z}), \lambda, H^+(C^*)_{\mathrm{codiv}} \oplus \mathrm{Hom}(L_-, \mathbb{Z})],\end{aligned}$$

as required.
This completes the proof of Theorem (3.9). □

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