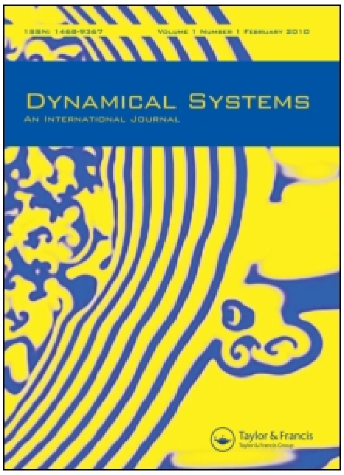

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Discontinuity geometry for an impact oscillator

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Abstract. We use methods of singularity theory to classify the local geometry of the discontinuity set, together with associated local dynamics, for a discrete dynamical system that represents a natural class of oscillator with one degree of freedom impacting against a fixed obstacle. We also include descriptions of the generic transitions that occur in the discontinuity set as the position of the obstacle is smoothly varied. The results can be applied to any choice of restitution law at impact. The analysis provides a general setting for the study of local and global dynamics of discontinuous systems of this type, for example giving a geometric basis for the possible construction of Markov partitions in certain cases.

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1. Introduction

An impact oscillator (sometimes called a 'vibro-impact system') here consists of a system of ordinary differential equations in which the 'free' dynamics are interrupted by an obstacle, together with a given 'restitution rule' for describing the dynamics immediately after impact with the obstacle. A typical system of this kind with one degree of freedom takes the form

$$\ddot{x} + f(x, \dot{x}) = g(t) \tag{1}$$

for $x \in \mathbf{R}$, where we assume f and g are smooth functions (meaning C^∞ or analytic) and g is periodic with period $T > 0$, and where the obstacle is placed at $x = c$ with restitution rule that $\dot{x}$ is replaced by $-r\dot{x}$ for constant $0 < r \leqslant 1$ whenever a trajectory reaches $x = c$. As usual the dot denotes differentiation with respect to t. An important particular case is the linear system

$$\ddot{x} + x = \cos \omega t \tag{2}$$

with $0 < \omega \neq 1$.

The geometric analysis that we give in this paper can be applied to quite general types of interaction of the obstruction with the 'free' dynamics of (2) such as, for example, the presence of a thin resistant layer at $x = c$ which does not cause the velocity v to change sign but merely replaces v by rv with $r \geqslant 0$. The purpose of our analysis is to bring out the primary geometric role of the 'obstacle' at $x = c$ in the overall phase space dynamics: after this, the imposition of any particular rule for interaction with the obstacle is a second step which determines how our geometric

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information is to be used to help describe the dynamics. Indeed, taking $r = 1$ allows us to reconstruct most of the original free dynamics, an observation worth keeping in mind in what follows. Our results apply to fixed values of the 'clearance' c and also describe the transitions that take place as c is varied.

The main ideas of the paper are as follows. After straightening out the flow for (1) we describe in sections 2 and 3 a 2-manifold (almost) in $\mathbf{R}^3$ that represents the impacts of the trajectories of (1) with the obstacle $x = c$: we call this the 'impact surface'. The dynamics of the impact oscillator are then represented by a discrete dynamical system obtained by choosing initial data (velocity and phase) at a point where $x = c$, proceeding parallel to the t-axis until meeting the impact surface, re-setting the time to $t = 0$, applying the restitution rule and then repeating the process. This involves studying (sections 4 and 5) the geometry of two maps from the impact surface to the plane of initial data, namely the projection parallel to the t-axis (for which we need to understand its discontinuous inverse) and the time 're-set' map. We use methods of singularity theory to analyse the geometry of both these maps under explicit generic assumptions on (1). The dynamics for the impact oscillator can then in principle be constructed (sections 6 and 7) from this information together with the restitution rule. A list of notation is given in section 8.

There is an extensive literature on impact oscillators which we do not attempt to survey here. Useful references include Nordmark (1991, 1997), Ivanov (1993, 1994), Bishop (1994), Budd and Dux (1994 a,b), Chin *et al.* (1994), Casas *et al.* (1996), Babitsky (1998), Foale and Bishop (1992) with their own bibliographies to which the interested reader is referred. The first author to approach the geometry of discontinuities in a systematic way was Whiston (1987, 1992) whose pioneering approach was the inspiration for the present work. In this paper we outline the main geometric results; in a further paper (Chillingworth 2003) some examples and specific applications to dynamics are given.

2. The impact surface in phase space

The natural phase space for the study of the system (1) is $\mathbf{R}^3$ with coordinates $(x, y, u) = (x, \dot{x}, t)$ in which the system becomes a first-order system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -f(x, y) + g(u) \\ \dot{u} &= 1 \end{aligned}\right\}. \quad (3)$$

The standard way to investigate the dynamics is to study the time-T map $(x_0, y_0) \mapsto (x(T), y(T))$ where $(x(t), y(t)) = (x(t), \dot{x}(t))$ denotes the solution to (1) with initial data $(x(0), y(0)) = (x_0, y_0)$. However, as observed by Shaw and Holmes (1983), Whiston (1987), Foale and Bishop (1992), Bud and Dux (1994a) and others, in the case of an impact oscillator it is more useful instead to work in the plane

$$\Sigma_c = \{(x, \dot{x}, t) \in \mathbf{R}^3 : x = c\}$$

and then to study the discrete dynamical system on (most of) Σ_c generated by the composition

$$\tilde{I} = \tilde{R} \circ \tilde{G} : \Sigma_c \to \Sigma_c$$

where $\tilde{G} : \Sigma_c \to \Sigma_c$ is given by following the dynamics of (3) with initial data on Σ_c until the trajectory hits Σ_c the next time (if at all), and $\tilde{R}$ is the 'restitution map' taking y to $-ry$ or other appropriate rule.

The map $\tilde{G}$ is discontinuous at points of $\tilde{S} = \tilde{G}^{-1}(\Sigma_c^0)$ where $\Sigma_c^0 = \Sigma_c \cap \{y = 0\}$: if a trajectory 'grazes' Σ_c at z (that is if it meets Σ_c at $z \in \Sigma_c^0$ where $\ddot{x} \neq 0$) then nearby trajectories may miss Σ_c in a neighbourhood of z and not hit Σ_c until a considerable time later (if at all). Therefore, as emphasized by Whiston (1987), the dynamical behaviour of the system is crucially dependent on the nature and position of the 'discontinuity set' $\tilde{S}$ and its inverse iterates $\tilde{I}^{-n}(\tilde{S})$ for $n = 1, 2, 3, \ldots$. In what follows we give a local and general global geometric description of the discontinuity set $\tilde{S}$ (in a slightly different setting) for the system (1) under generic assumptions on f and g. For the linear system (2) we give more specific information. We also describe the changes that the discontinuity structure undergoes as the clearance parameter c is varied. Some of the ways in which this geometric information gives insight into various aspects of dynamical behaviour are also analysed.

Our approach is based on viewing the dynamics as being generated not by families of curved trajectories meeting the plane Σ_c in $\mathbf{R}^3$ but rather by families of straight lines intersecting a corresponding curved surface. This allows us to use methods of singularity theory as applied to the study of apparent outlines (apparent contours). We now give the details of this 'straightened-out' interpretation.

Let $x(c, v, \tau; t)$ denote the solution to (1) with initial data $(x, \dot{x}) = (c, v)$ when $t = \tau$. For fixed $c \in \mathbf{R}$ define the 'impact surface' V_c as

$$V_c = \{(v, \tau; t) \in \mathbf{R}^3 : x(c, v, \tau; \tau + t) = c\}.$$

Observe that the plane

$$\Pi = \{(v, \tau; t) \in \mathbf{R}^3 : t = 0\}$$

is automatically part of V_c. The projection

$$p : \mathbf{R}^3 \to \mathbf{R}^2 : (v, \tau; t) \mapsto (v, \tau)$$

plays a major role in this paper, and we identify Π with its image under p, that is the (v, τ)-plane $\mathbf{R}^2$. We write

$$p_c = p|V_c : V_c \to \Pi$$

and in what follows we use known generic properties of maps from a 2-manifold to a plane in order to shed light on the structure of p_c.

The straight line $p^{-1}(v, \tau)$ intersects V_c at points whose t-coordinates are the times $t \in \mathbf{R}$ at which the trajectory of (1) starting at $x = c$ with initial velocity v at time τ meets the obstacle at $x = c$.

There is a natural map $\Phi : \mathbf{R}^4 \to \mathbf{R}^4$ corresponding to 're-setting the initial data', namely

$$\Phi(c, v, \tau; t) = (x(c, v, \tau; \tau + t), \dot{x}(c, v, \tau; \tau + t), \tau + t; 0).$$

Then $\Phi(\{c\} \times V_c) \subset \{c\} \times \mathbf{R}^2 \times \{0\}$ and therefore writing $\Phi(c, z) = (c, \varphi_c(z); 0)$ for $z \in V_c$ gives a map

$$\varphi_c : V_c \to \mathbf{R}^2$$

which we call the 're-set map' as it represents re-setting $(\dot{x}, \tau + t)$ to the new initial data for velocity and phase at points of V_c (where x is automatically re-set to c). Thus $\varphi_c = \Phi_c|V_c$ where

$$\Phi_c : \mathbf{R}^3 \to \mathbf{R}^2 : (v, \tau; t) \mapsto (\dot{x}(c, v, \tau; t), \tau + t).$$

For $(v, \tau) \in \Pi$ let $F_c(v, \tau)$ be the point of $p^{-1}(v, \tau) \cap V_c$ with the smallest positive t-coordinate, provided this exists. We assume that $p^{-1}(v, \tau) \cap V_c$ is a discrete set: we disregard systems (1) where f and g are bizarre enough to give an accumulation point of times t at which a solution $x(v, \tau; t)$ takes the value c. Thus $F_c(v, \tau)$ is defined for $(v, \tau) \in \Pi_c$ where

$$\Pi_c = p(V_c \setminus \Pi) \subset \Pi$$

and F_c is a right inverse for the projection $p_c = p|V_c$. We call F_c the 'first-hit map'. In many cases (and certainly in the case of (2) as we see below) Π_c will be almost all of Π, as trajectories leaving $x = c$ can be expected to return. Observe that the map $G_c = \varphi_c \circ F_c : \Pi_c \to \Pi$ corresponds to the map $\tilde{G} : \Sigma_c \to \Sigma_c$ previously described. Finally, write

$$R : \Pi \to \Pi : (v, \tau) \mapsto (\rho(v), \tau)$$

where $\rho(v) = -rv$ or some other appropriate formula. Using V_c we can now give the following geometrical characterization of the dynamics of the impact oscillator (1).

- Start at $x = c$ at time τ with initial velocity v.
- Proceed along the t-axis in the positive direction until the first point of intersection with V_c; that is apply the map F_c.
- Re-set initial data by applying the map φ_c.
- Replace the new v by $\rho(v)$; that is apply the map R.
- Repeat the process.

In other words our model for the dynamics of the impact oscillator is the (discontinuous) discrete system

$$I_c = R \circ G_c : \Pi_c \to \Pi. \tag{4}$$

The points of discontinuity in the resulting discrete dynamical system are among the points (v, τ) for which the straight line $p^{-1}(v, \tau)$ is tangent to (that is, fails to be transverse to) the impact surface V_c. In visual terms, these are the points such that the line of sight along the t-direction is somewhere tangent to V_c, that is, they form the 'apparent outline' or 'apparent contour' P_c of V_c viewed in the t-direction. Technically, they are the 'singular values' of the projection $p_c = p|V_c : V_c \to \Pi$. The tangency points themselves are the 'singular points' of p_c (points where the derivative fails to have rank 2). We call the set H_c of singular points the 'horizon' of V_c viewed in the t-direction; thus $P_c = p(H_c)$. See figure 1 for a schematic illustration.

The points of H_c correspond to points in $(x, \dot{x}, t)$-space where the trajectories of (1) 'graze' (that is, are tangent to) the plane Σ_c, and these points are central to understanding the nonlinear behaviour of an impact oscillator. Many authors such as Foale and Bishop (1992), Chin *et al.* (1994) and Nordmark (1997) have studied aspects of the dynamical behaviour close to graze points. Our interpretation of these points in terms of singularities of smooth maps sheds new light on some existing results and extends these to situations with more complicated local phase space geometry. We study the structure of the horizon $H_c \subset V_c$ and the outline $P_c \subset \Pi$, and describe their role in the dynamics both for a given fixed clearance c and as c is allowed to vary. We give some key results for the general system (1), while for the basic linear system (2) we provide a fairly complete analysis.

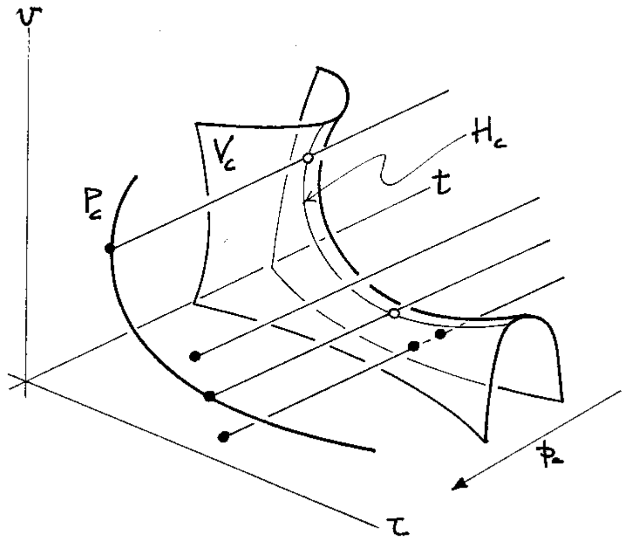

Figure 1. Viewing the impact surface V_c.

3. Geometry of the impact surface

We have boldly called V_c a 'surface' but have yet to see whether it deserves the name: is V_c indeed a 2-manifold? By definition $V_c = x_c^{-1}(c)$ where

$$x_c(v, \tau; t) = x(c, v, \tau; \tau + t) \tag{5}$$

and so V_c is a particular level set of the function $x_c : \mathbf{R}^3 \to \mathbf{R}$. The Implicit Function Theorem implies that V_c will indeed be a smooth manifold in a neighbourhood of every point where grad x_c does not vanish, i.e. the regular points of the function x_c. We call these points 'regular points' of V_c. As we now see, all points of V_c where $v \neq 0$ are regular points.

Let V_c^* and V_c^0 denote the subsets of V_c where $v \neq 0$ and $v = 0$, respectively.

Lemma 1. *The set V_c^* is a smooth 2-manifold in* $\mathbf{R}^3$.

Proof. Consider the flow $\{\Psi_t\}$ on $\mathbf{R}^3$ generated by the autonomous system of equations (3). Write $q = (x, y, u) \in \mathbf{R}^3$. Since Ψ_t takes trajectories to trajectories its derivative $D\Psi_t(q_0)$ takes the vector $\dot{q}_0$ to $\dot{q}_t$ (where $q_t = \Psi_t(q_0)$); thus if $q_0 = (c, v, \tau)$ and $\dot{q}_0 = (v, a, 1)$ we have

$$D\Psi_t(q_0).\begin{pmatrix} v & 0 & 0 \\ a & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \dot{x} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial \tau} + \dot{x} \\ \ddot{x} & \dfrac{\partial \dot{x}}{\partial v} & \dfrac{\partial \dot{x}}{\partial \tau} + \ddot{x} \\ 1 & 0 & 1 \end{pmatrix} \tag{6}$$

where x stands for $x(c, v, \tau; \tau + t)$ and $\partial/\partial\tau$ denotes partial differentiation with respect to the third variable. Since $D\Psi_t(q_0)$ is invertible it follows that the second matrix is non-singular when $v \neq 0$ and so in particular

$$\operatorname{grad} x_c(v,\tau;t) = \left(\frac{\partial x_c}{\partial v}, \frac{\partial x_c}{\partial \tau}, \frac{\partial x_c}{\partial t}\right) = \left(\frac{\partial x}{\partial v}, \frac{\partial x}{\partial \tau} + \dot{x}, \dot{x}\right)$$

is not zero. □

There is another consequence of (6) that we shall use later.

Corollary 1.

$$(v,a) = (0,0) \quad \Leftrightarrow \quad \left(\frac{\partial x}{\partial \tau}, \frac{\partial \dot{x}}{\partial \tau}\right) = (0,0).$$

Proof. Compare first and third columns of the matrices in (6). □

In the above $a = \ddot{x}_c(0,\tau;0)$ which we denote more fully by $a_c(\tau)$. From (1) we have

$$a_c(\tau) = -f(c,0) + g(\tau). \tag{7}$$

Next we study the nature of V_c at points of V_c^0. Denote the τ-axis by L_0: then $L_0 \subset V_c^0$ since $L_0 \subset \Pi \subset V_c$. Moreover, on Π we have $\dot{x}_c(v,\tau;0) = v$ which vanishes on L_0.

Proposition 1. *Assume $r \geqslant 2$. In a neighbourhood of L_0 the impact surface V_c consists of the plane Π together with a 2-manifold V_c' that is the graph of a smooth function $v = v_c(\tau, t)$ satisfying*

$$v_c(\tau,0) = 0, \qquad \dot{v}_c(\tau,0) = -\frac{1}{2}a_c(\tau).$$

Proof. Since $x_c = c$ on Π we can write

$$x_c(v,\tau;t) = c + t y_c(v,\tau;t)$$

in a neighbourhood of Π, where $y_c = \dot{x}_c = v$ and $\dot{y}_c = \frac{1}{2}\ddot{x}_c$ on Π (i.e. $t = 0$). Solutions to $x_c = c$ not on Π are given by $y_c = 0$. Since $\partial y_c/\partial v = 1 \neq 0$ we can (by the Implicit Function Theorem) express $y_c = 0$ close to L_0 as $v = v_c(\tau, t)$ for a smooth function v_c with $v_c(\tau,0) = -\dot{y}_c(\tau,0)$, which gives the result. □

Corollary 2. *The contribution to the outline P_c from V_c' intersects L_0 at the points τ where $a_c(\tau) = 0$.*

We shall study the behaviour of P_c close to L_0 in more detail in section 4 below.

The structure of V_c^0 away from L_0 seems harder to pin down. Instead, we make a conjecture on generic behaviour. Let M be a smooth manifold and $h_c : M \to \mathbf{R}$ a smooth function varying smoothly with a real parameter c. By a 'Morse point' we mean a non-degenerate critical point $x_0 \in M$ of h_c for some particular $c = c_0$, and we say that the level set $h_c^{-1}(0)$ undergoes a 'Morse transition' at x_0 as c passes through c_0 if $(\partial/\partial_c)h_c(x_0) \neq 0$. We apply this to $h_c = x_c - c : \Pi \times \mathbf{R}^2 \to \mathbf{R}$ where $V_c = h_c^{-1}(0)$.

Conjecture 1. *It is a generic property of the pair of functions f, g in (1) that $c \in \mathbf{R}$ is a regular value of the function $x_c : \mathbf{R}^3 \setminus \Pi \to \mathbf{R}$ for all $c \in \mathbf{R}$ except possibly for a discrete set $\{c_i\}$. Each c_i is the value of x_{c_i} at one or more nondegenerate critical points of x_{c_i}, with V_c undergoing one or more Morse transitions as c passes through c_i.*

The conjecture is supported by the correspondence between Morse points of V_c on V_c^0 and points of quadratic tangency of separate branches of P_c which we discuss further below, and the fact that the latter occur as generic codimension-1 phenomena in 1-parameter families of apparent outlines. As we see later, the conjecture holds for the linear system (2).

3.1. *The linear system (2)*

In this subsection we look at the precise structure of V_c for the specific system (2). We can do this since we have an explicit general solution.

Proposition 2. *The solution of* (2) *gives the following formula for* x_c*:*

$$x_c(v, \tau; t) = A \cos t + B \sin t + \gamma \cos \omega(\tau + t) \tag{8}$$

where

$$A = c - \gamma \cos \omega\tau, \qquad B = v + \omega\gamma \sin \omega\tau \tag{9}$$

and $\gamma = (1 - \omega^2)^{-1}$.

Proof. Immediate verification.

Corollary 3. *The domain of definition* Π_c *for the first hit map* F_c *in the case of the linear system* (2) *is given by*

$$\Pi_c = \begin{cases} \Pi, & \text{if } \omega \in \mathbf{Q} \text{ or } |c| < \gamma \\ \Pi \setminus \{(0, 2n\pi/\omega) : n \in \mathbf{Z}\}, & \text{if } \omega \notin \mathbf{Q} \text{ and } c > \gamma \\ \Pi \setminus \{(0, (2n+1)\pi/\omega) : n \in \mathbf{Z}\}, & \text{if } \omega \notin \mathbf{Q} \text{ and } c < -\gamma \end{cases}$$

Proof. The only circumstances under which a trajectory $(x_c, \dot{x}_c)$ can leave $x = c$ at $t = 0$ never to return (although it will return arbitrarily closely) are when $\omega/2\pi$ is irrational and the initial point is an extreme point on the x-axis, that is $\sin \omega\tau = 0$ and A has the same sign as $\cos \omega\tau$. These conditions characterize Π_c as stated. □

For (2) there are important symmetries in the system that manifest themselves as symmetries of V_c.

Corollary 4. *For* (2) *the impact surface* V_c *has the following symmetries:*

(*i*) *The surface* V_c *is invariant under translation* $\tau \mapsto \tau + 2\pi/\omega$. *If* $\omega = p/q \in \mathbf{Q}$ *where* $p, q \in \mathbf{N}$ *then* V_c *is also invariant under translation* $t \mapsto t + 2q\pi$.
(*ii*) *The surface* V_{-c} *is obtained from* V_c *by translating* $\tau \mapsto \tau + \pi/\omega$ *and changing the sign of* v.

Proof. Clear from the expression (8), because (i) $2q\pi$ is an integer multiple of both 2π and $2\pi/\omega$, and (ii) $x_{-c}(-v, \tau + \pi/\omega; t) = -x_c(v, \tau; t)$. □

Corollary 5. *If* $\omega = p/q$ *where* $p, q \in \mathbf{N}$ *then the line segment in* $\mathbf{R}^3$ *parallel to the* t-*axis from* $(v, \tau; 0)$ *to* $(v, \tau; 2\pi q)$ *intersects* V_c *an odd number of times between these points.*

Proof. Any periodic trajectory meets $x = c$ an even number of times. □

For (2) we find from Proposition 2 that

$$\left.\begin{aligned}\frac{\partial x_c}{\partial v} &= \sin t \\ \frac{\partial x_c}{\partial t} - \frac{\partial x_c}{\partial \tau} &= a_c(\tau)\sin t + v\cos t\end{aligned}\right\} \tag{10}$$

where

$$a_c(\tau) = \cos\omega\tau - c = \ddot{x}_c(v, \tau; 0). \tag{11}$$

Therefore, grad $x_c = 0$ precisely where

$$\sin t = a_c(\tau)\sin t + v\cos t = \dot{x}_c(v, \tau; t) = 0,$$

that is, $v = 0$ (of course), $t = k\pi$, $k \in \mathbf{Z}$ and $\dot{x}_c(0, \tau; k\pi) = 0$. Investigating the geometry of V_c in the neighbourhood of these singular points will occupy the remainder of this section.

For $k \in \mathbf{Z}$ let L_k denote the line $t = k\pi$ in the $(\tau; t)$-plane, that is,

$$L_k = \{(v, \tau; t) \in \mathbf{R}^3 : v = 0,\ t = k\pi\}.$$

At points of L_k we have

$$x_c(0, \tau; k\pi) = (c - \gamma\cos\omega\tau)(-1)^k + \gamma\cos\omega(\tau + k\pi) \tag{12a}$$

$$\dot{x}_c(0, \tau; k\pi) = \alpha\sin\omega\tau(-1)^k - \alpha\sin\omega(\tau + k\pi) \tag{12b}$$

where $\alpha = \omega\gamma$, so the points $(0, \tau; k\pi) \in V_c \cap L_k$ are points where the right-hand side of (12a) is equal to c, and those where V_c may be singular are those where also the right-hand side of (12b) vanishes.

Recall from Proposition 1 that when $k = 0$ (that is $t = 0$) we have $L_0 \subset V_c$ and V_c fails to be a 2-manifold at all points of the line L_0: there it is the transverse intersection of two 2-manifolds, namely the plane Π and the manifold V_c' that is a graph $v = v_c(\tau, t)$. Therefore, we turn to cases where $k \neq 0$.

Case 1. k even.

With $(x_c, \dot{x}_c) = (c, 0)$ equations (12) become

$$\cos\omega\tau = \cos(\omega\tau + k\omega\pi) \tag{13a}$$

$$\sin\omega\tau = \sin(\omega\tau + k\omega\pi) \tag{13b}$$

respectively, which imply $k\omega \in 2\mathbf{Z}$. If $\omega \notin \mathbf{Q}$ this can never be satisfied, while if $\omega = p/q \in \mathbf{Q}$ where $p, q \in \mathbf{N}$ then (13a), (13b) are satisfied simultaneously for all τ when k is an integer multiple of $2q$. In view of Corollary 4 we summarize this as follows.

Proposition 3. *There are no points on L_k for even $k \neq 0$ at which V_c fails to be regular, apart from (when $\omega \in \mathbf{Q}$) the images of L_0 under the translations in the t-direction which leave V_c invariant.* □

Case 2. k odd.

The equations to solve now are

$$\cos\omega\tau + \cos(\omega\tau + k\omega\pi) = 2c\omega/\alpha \tag{14a}$$

$$\sin\omega\tau + \sin(\omega\tau + k\omega\pi) = 0. \tag{14b}$$

From (14b) we have one or other of the equalities

$$\omega\tau + k\omega\pi = -\omega\tau + 2l\pi \tag{15a}$$

$$\omega\tau + k\omega\pi = \omega\tau + (2l+1)\pi \tag{15b}$$

for $l \in \mathbf{Z}$. Now (15a) gives

$$\omega\tau = l\pi - k\omega\frac{\pi}{2} \tag{16}$$

and then from (14a)

$$\cos\omega\tau = c\omega/\alpha = c/\gamma. \tag{17}$$

On the other hand, (15b) gives

$$k\omega\pi = (2l+1)\pi$$

and then from (14a)

$$c = 0.$$

Thus (14a) and (15b) can have simultaneous solutions τ only when ω is the ratio of two odd integers, and then when $c = 0$. However, (14a) and (15a) will have solutions τ for every ω for certain choices of c.

Therefore, we have two possible settings for solutions to (14) as follows:

$$\text{I}: \quad k \in 2\mathbf{Z}+1, \quad k\omega \notin 2\mathbf{Z}+1;$$

$$\text{II}: \quad k \in 2\mathbf{Z}+1, \quad k\omega \in 2\mathbf{Z}+1, \quad c = 0.$$

We consider these each in turn.

Type I: $k \in 2\mathbf{Z}+1, \quad k\omega \notin 2\mathbf{Z}+1$

From (15a), (17) we have $\omega\tau_* = l\pi - k\omega(\pi/2)$ and $c = c_{k,l} = \gamma\xi$ where $\xi = \cos\omega\tau_*$. We then find

$$x_c(v, \tau_* + \sigma; k\pi + u) = -vu - \frac{1}{2}c\omega^2(2\sigma^2 + 4\sigma u + u^2) + O(3).$$

The quadratic terms are non-degenerate in $(v, \sigma; u)$ so the Morse Lemma implies that (up to a local diffeomorphism which is the identity map to first order) the structure of V_c is locally that of a cone tangent to the v-axis. Notice moreover that for all $\tau \in \mathbf{R}$ we have

$$x_c(v, \tau; k\pi) = \gamma(\cos\omega\tau + \cos\omega(\tau + k\pi)) - c$$

which does not involve v. Since $|\cos\omega\tau + \cos\omega(\tau + k\pi)|$ has a local maximum value of $2|\xi|$ at $\tau = \tau_*$ it follows that the impact surface V_c contains a pair of lines intersecting L_k and parallel to the v-axis that are created as c passes through the value $c_{k,l}$ from above if $c_{k,l} > 0$ or from below if $c_{k,l} < 0$. When $c = c_{k,l}$ the line $\tau = \tau_*$, $t = k\pi$ is a line of tangency of V_c with the plane $t = k\pi$, so the local cone structure is not only tangent to the v-axis but contains it entirely. As c passes through $c_{k,l}$ the structure of V_c undergoes a Morse transition at $(0, \tau_*; k\pi)$ between (locally) a hyperboloid of one sheet and a hyperboloid of two sheets. See figure 2.

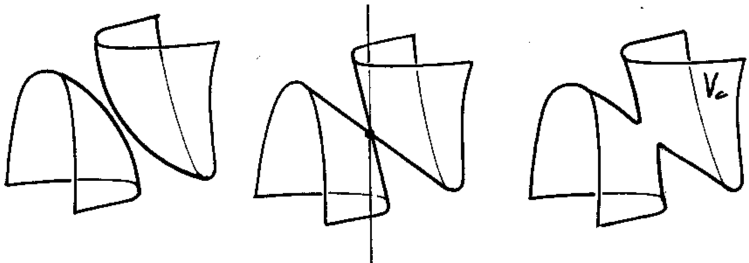

Figure 2. Morse transition for (2) at $c = c_{k,l}$; note that the vertical line lies in the surface V_c.

Type II: $k \in 2\mathbf{Z}+1, \quad k\omega \in 2\mathbf{Z}+1, \quad c = 0$

Here we find

$$x_c(v,\tau;k\pi) = -c$$

$$\dot{x}_c(v,\tau;k\pi) = -v)$$

$$\ddot{x}_c(v,\tau;k\pi) = c - \cos\omega\tau$$

$$\dddot{x}_c(v,\tau;k\pi) = v + \omega\sin\omega\tau$$

and so for all $(v,\tau) \in \mathbf{R}^2$ and $c \in \mathbf{R}$ we have

$$x_c(v,\tau;k\pi+u) = -c\cos u - v\sin u + u\left(-\frac{u}{2}\cos\omega\tau + \frac{u^2}{6}(\omega\sin\omega\tau) + O(u^3)\right).$$

Therefore in a neighbourhood of the plane $t = k\pi$ we may write $V_c = h^{-1}(c)$ where $h : \mathbf{R}^3 \to \mathbf{R}$ is given by

$$h(v,\tau;u) = (1+\cos u)^{-1}\left(-v\sin u + u\left(-\frac{u}{2}\cos\omega\tau + \frac{u^2}{6}\omega\sin\omega\tau + O(u^3)\right)\right).$$

Thus V_0 consists of the plane $t = k\pi$ (that is $u = 0$) itself together with a 2-manifold V_0'' intersecting this plane along the line L_k (that is $u = v = 0$) and locally the graph of an analytic function $v = v_0(\tau, u)$ given by

$$\psi(u)v_0(\tau,u) = -\frac{u}{2}\cos\omega\tau + \frac{u^2}{6}(\omega\sin\omega\tau) + O(u^3)$$

where again $\psi(u) = u^{-1}\sin u$ $(u \neq 0)$, $\psi(0) = 1$. Clearly for $\omega\tau \neq \pm(\pi/2) \pmod{2\pi}$ we see $(\partial v_0/\partial u)(\tau,0) \neq 0$ and the projection $p|V_0''$ is regular at $(\tau,0)$. When $\omega\tau = \pm(\pi/2) \pmod{2\pi}$ we write $\tau = \pm(\pi/2\omega) + \sigma \pmod{2\pi/\omega}$ and find

$$\psi(u)v_0\left(\pm\frac{2\pi}{\omega} + \sigma, u\right) = \pm\omega\left(\frac{1}{2}\sigma u + \frac{u^2}{6} + O(u^3)\right)$$

so v_0 has a saddle point at $(\sigma,u) = (0,0)$, that is $(\tau,t) = (\pm\pi/2\omega, k\pi) \in L_k$. The singular points of $p|V_0''$ are given by $\partial v_0/\partial u = 0$, that is, (to first order) $3\sigma + 2u = 0$, giving the apparent outline P_0 locally as $v_0 = \mp 3/8\omega\sigma^2 + O(\sigma^3)$ and showing that P_0 locally is a curve with quadratic tangency to $v = 0$ from the $v < 0$ side at $\tau = \pi/2\omega$ and from the $v > 0$ side at $\tau = -\pi/2\omega$.

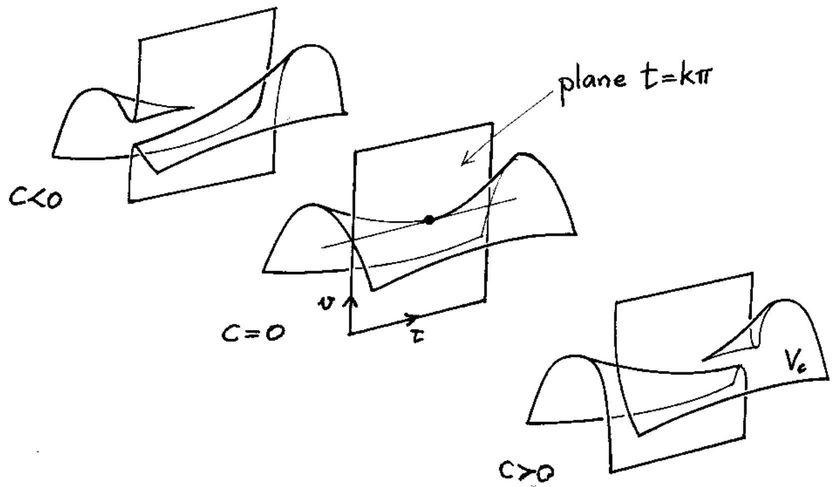

Figure 3. Type II transition as c passes through 0.

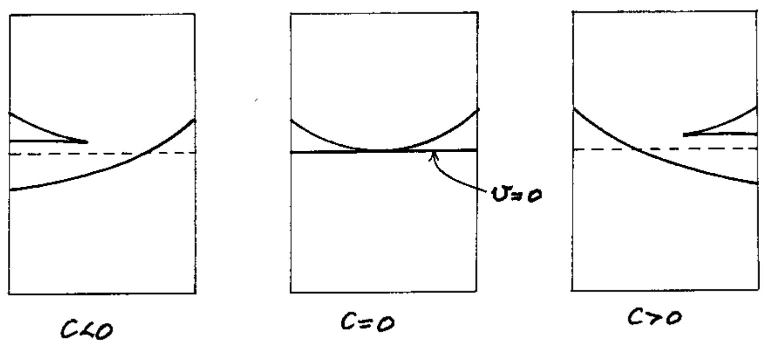

Figure 4. Change in outline P_c at a Type II transition.

As c varies away from $c = 0$ observe that V_c no longer contains the plane $t = k\pi$; contrast this with $k = 0$ where the plane Π remains automatically part of V_c. See figure 3.

Closer analysis shows that P_c unfolds into the disjoint union (locally) of a regular curve and a cusped curve as c moves away from zero in either sense: see figure 4.

An analogous description applies in the neighbourhood of $\tau = -\pi/2\omega$, related to the above by the symmetry described in Corollary 4.

The full set of local transitions of V_c that occur as c varies can therefore be summarized as follows:

Theorem 1. *There are no transitions for* $|c| > |\gamma|$.

(*i*) *As c descends through positive values* $c_{k,l} = \gamma\cos(k\omega(\pi/2) - l\pi)$, *where* $k \in 2\mathbf{Z} + 1$ *and* $k\omega \notin 2\mathbf{Z} + 1$ *and* $l \in \mathbf{Z}$, *there are simultaneous Morse (Type I) transitions from two sheets to one sheet at corresponding points* $(v, \tau; t) = (0, -k(\pi/2) + l(\pi/\omega); k\pi)$.

(*ii*) *If* $\omega = p/q$ *where* $p, q \in 2\mathbf{N} + 1$ *then as c descends through the value* 0 *there is a Type II transition at* $(v, \tau; t) = (0, \pi/2\omega; q\pi)$ *modulo* $t = 2q\pi$.

(*iii*) *As c descends through negative values* $c_{k,l} = \gamma\cos(k\omega(\pi/2) - l\pi)$ *as in (ii) there are simultaneous Morse (Type I) transitions from one sheet to two sheets at corresponding points* $(v, \tau; t) = (0, -k(\pi/2) + l(\pi/\omega); k\pi)$.

4. Geometry of the projection $p_c = p|V_c$

In section 2 we analyzed the local geometry of the impact surface V_c for the general system (1), in particular at points where it fails to be a smooth 2-manifold, and in Theorem 1 we classified the local changes in topology for the system (2) as c passes through certain special values.

We now turn to look closely at the set of singular points and the set of singular values of the projection $p_c = p|V_c : V_c \to \Pi$, at points where v may or may not be zero. We give some results for the general system (1), and a more specific analysis for (2).

First, some facts about apparent outlines in general. See for example Bruce (1984a, b).

Theorem 2. *Typically the only singularities exhibited by the projection into a plane of a smooth* 2*-manifold in* $\mathbf{R}^3$ *are curves of 'folds' with isolated 'cusps'. See figure 5.*

Typically the only transitions that occur in the projection of a 1*-parameter family of surfaces are 'lips', 'beaks' and 'swallowtails'. See figure 6.*

The word 'typically' can be expressed precisely in terms of openness and density of relevant properties in certain function spaces: we omit a more formal statement. The interpretation is that these phenomena are structurally stable (robust) under sufficiently small perturbations, and that any other singular phenomena that may arise can be perturbed into collections of these alone.

If the manifold is compact and the above conditions prevail then the singular set will be compact and will in particular contain only finitely many cusps. In our case

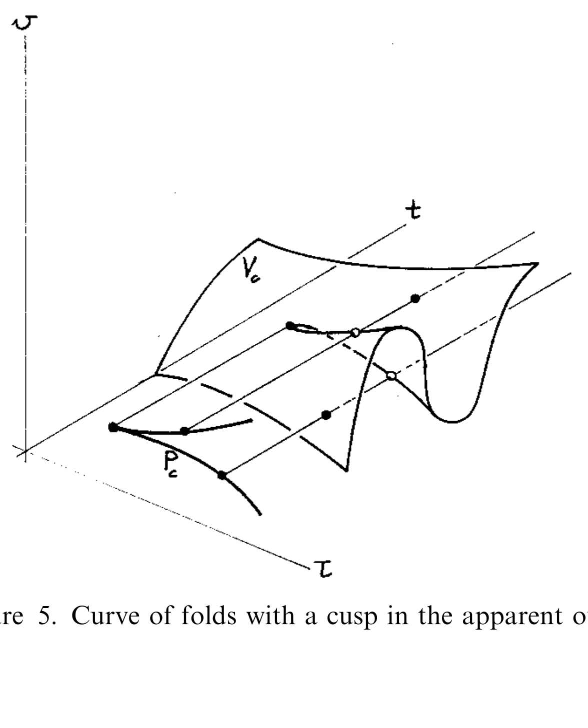

Figure 5. Curve of folds with a cusp in the apparent outline P_c.

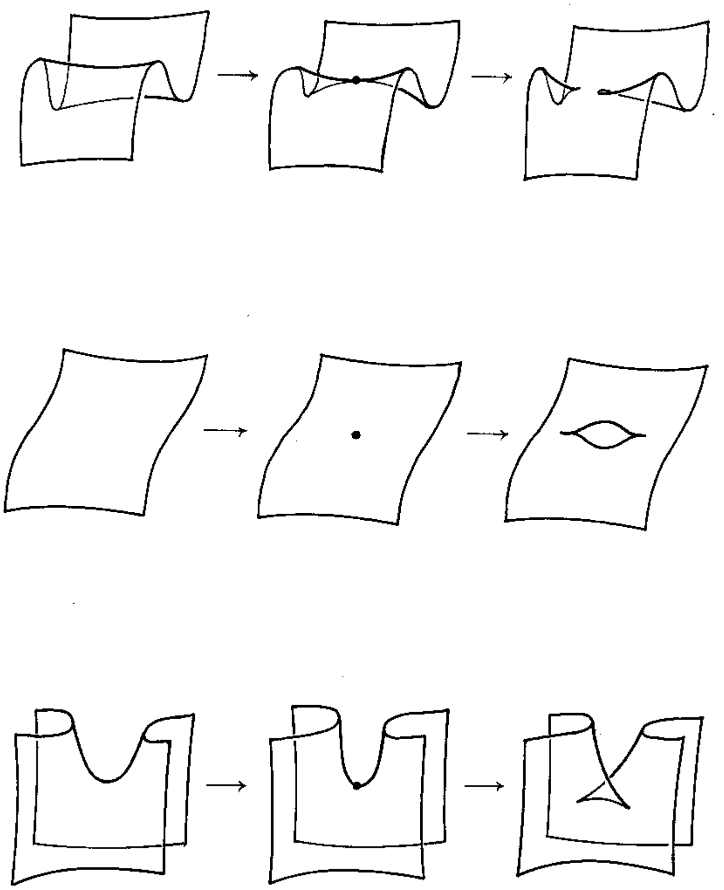

Figure 6. Beaks, lips and swallowtail transitions.

the manifold V_c is not compact. However, for the system (2) with $\omega \in \mathbf{Q}$ it is periodic in t and hence compact modulo this period. The apparent outline P_c is thus compact. For (1) in general and for (2) with $\omega \notin \mathbf{Q}$ the apparent outline will not be compact and may contain infinitely many cusps.

To obtain more than the most basic results for (1) we need to make some generic assumptions about the functions f and g.

Definition 1. The pair of functions f, g in (1) is 'generic' if the following holds: $f(c, 0)$ is a regular value of the function g except possibly for a finitely many values c_i of c, these being regular points of $f(\cdot, 0)$. Moreover, all critical points of g with value c_i are non-degenerate. Thus

$$g(t) = f(c, 0), \quad c \neq c_i \qquad \Rightarrow \qquad g'(t) \neq 0 \tag{18}$$

and

$$g(t) = f(c_i, 0), \quad g'(t) = 0 \quad \Rightarrow \quad g''(t) \neq 0, \quad \frac{\partial f}{\partial c}(c_i, 0) \neq 0. \tag{19}$$

If the pair f, g is generic then as c varies the set of solutions t to $g(t) = f(c, 0)$ undergoes no worse than quadratic (Morse) transitions.

Recall that for (1) the set V_c^* is a smooth 2-manifold. We begin by studying singularities of $p_c|V_c^*$.

4.1. *Behaviour of p_c at points of V_c^**

Let $H_c^* = H_c \cap V_c^*$ be the set of singular points of the map $p_c|V_c^*$, thus

$$H_c^* = \{(v, \tau; t) \in \Pi \times \mathbf{R} : v \neq 0, x_c(v, \tau; t) - c = \dot{x}_c(v, \tau; t) = 0\}.$$

Theorem 3.

(*1*) *For the system* (1) *the horizon H_c^* is a smooth* 1-*manifold.*

(*2*) *For the system* (1) *with the pair f, g generic, for $c \neq c_i$ there is a fold singularity at every point of H_c^* except possibly for isolated cusp singularities. There are swallowtail transitions as c passes through c_i.*

Proof. Let $z = (v, \tau; t) \in \mathbf{R}^3$ and write

$$X_c(z) = (x_c(z), \dot{x}_c(z)).$$

Then $H_c = X_c^{-1}(c, 0)$, and

$$DX_c(z) = \begin{pmatrix} \dfrac{\partial x_c}{\partial v} & \dfrac{\partial x_c}{\partial \tau} & \dfrac{\partial x_c}{\partial t} \\ \dfrac{\partial \dot{x}_c}{\partial v} & \dfrac{\partial \dot{x}_c}{\partial \tau} & \dfrac{\partial \dot{x}_c}{\partial t} \end{pmatrix} \tag{20}$$

$$= \begin{pmatrix} \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial \tau} + \dot{x} & \dot{x} \\ \dfrac{\partial \dot{x}}{\partial v} & \dfrac{\partial \dot{x}}{\partial \tau} + \ddot{x} & \ddot{x} \end{pmatrix} \tag{21}$$

where here $\dot{x} = 0$. From the proof of Lemma 1 we see that this matrix has rank 2 (regardless of $\dot{x}$) provided $v \neq 0$. Hence H_c^* is a smooth 1-manifold by the Implicit Function Theorem.

The projection p_c^* has a fold singularity at $z \in H_c^*$ precisely when $\ddot{x}_c(z) \neq 0$ and a cusp point at z when $\ddot{x}_c(z) = 0$ but $\dddot{x}_c(z) \neq 0$. From (1) we have

$$\ddot{x}_c(z) = 0 \quad \Leftrightarrow \quad f(c, 0) = g(t) \tag{22}$$

and differentiating (1) with respect to t at z we find that if $\ddot{x}_c(z) = 0$ then

$$\dddot{x}_c = \dot{g}(t). \tag{23}$$

Thus the genericity of f, g guarantees finitely many cusp points (modulo $t = T$) for $c \neq c_i$, and also guarantees $x_c^{(4)} \neq 0$ when $\dot{x}_c = \ddot{x}_c = \dddot{x}_c = 0$ which can occur when $c = c_i$. The fact that swallowtail transitions occur comes from the Morse transitions in solutions of $g(t) = f(c, 0)$ as c passes through c_i: we omit the details which can be found in Chillingworth (2003). □

A fold point $(v, \tau; t) \in V_c$ for p_c corresponds to a non-degenerate 'graze' (quadratic contact) in the trajectory for (1) with initial data (v, τ); a cusp point corresponds to a 'degenerate graze' with cubic contact. A swallowtail transition is the coalescence of two (cubic) degenerate grazes into one with quartic contact. Our expression of these phenomena in terms of singularity theory is a way of describing how they respond to (are 'unfolded by') the initial data v and τ.

Next we look at the behaviour of p_c at points where V_c intersects the plane $v = 0$. From Theorem 3 we know that it is only here that H_c can fail to be a smooth 1-manifold. We first consider points of L_0.

4.2. *Behaviour of p_c at points of L_0*

Recall from (7) that along L_0 the impact surface V_c consists of Π intersected transversally along L_0 by a sheet V_c' of V_c that is the graph of a smooth function $v = v_c(\tau, t)$. Thus for small $|t|$ the horizon H_c consists of L_0 together with points of V_c' where $\dot{v}_c = 0$. By Corollary (2) these occur on L_0 where $a_c(\tau) = 0$.

Proposition 4. *Let $g(\tau_0) = f(c_0, 0)$ so $a_{c_0}(\tau_0) = 0$, and suppose $a_{c_0}'(\tau_0) \neq 0$, that is, $\dot{g}(\tau_0) = k \neq 0$. Then v_{c_0} has a non-degenerate saddle point at $(\tau_0, 0)$. The horizon H_c is parametrized locally as*

$$(v, \tau; t) = (\tfrac{3}{8}k\sigma^2 + O(\sigma^3), \tau_0 + \sigma; -\tfrac{3}{2}\sigma + O(\sigma^2)) \tag{24}$$

so the outline P_c has quadratic contact with L_0 at τ_0 from the side $v > 0$ or $v < 0$ accordingly as $\dot{g}(\tau_0)$ is negative or positive, respectively.

Proof. Since $v_c(\tau, 0) = 0$ we have $(\partial^2/\partial\tau^2)v_c(\tau, 0) = 0$. Implicit differentiation of

$$y_c(v_c(\tau, t), \tau; t) = 0 \tag{25}$$

and use of (1) gives

$$\dot{v}_c(\tau, 0) = -\dot{y}_c(0, \tau; 0) = -\tfrac{1}{2}a_c(\tau),$$

$$\ddot{v}_c(\tau, 0) = -\ddot{y}_c(0, \tau; 0) = -\tfrac{1}{3}x_c^{(3)}(0, \tau; 0) - \tfrac{\partial \ddot{x}_c}{\partial v}(0, \tau; 0)\dot{v}_c(\tau, 0)$$

$$= -\left[\frac{2}{3}\frac{\partial f}{\partial \dot{x}}(c, 0) + \frac{\partial \ddot{x}_c}{\partial v}(0, \tau; 0)\right]\dot{v}_c(\tau, 0) - \tfrac{1}{3}\dot{g}(\tau)$$

from (1), so if $g(\tau_0) = f(c_0, 0)$ then $a_{c_0}(\tau_0) = 0$ and $\ddot{v}_{c_0}(\tau_0, 0) = -\frac{1}{3}k$. Also

$$\frac{\partial \dot{v}_{c_0}}{\partial \tau}(\tau_0, 0) = -\tfrac{1}{2}a_{c_0}'(\tau_0) = -\tfrac{1}{2}k$$

and so the Hessian matrix for v_{c_0} at $(\tau_0, 0)$ is

$$-\frac{k}{6}\begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}.$$

Thus if $\tau = \tau_0 + \sigma$ we have

$$v_c(\tau_0 + \sigma, t) = -\frac{k}{6}(3\sigma t + t^2) + O(3) \tag{26}$$

so V_c' cuts the plane $v = 0$ in a curve through $(\tau_0, 0)$ with tangent direction $(1, -3)$. The horizon H_{c_0} given by $\dot{v}_c = 0$ satisfies $t = -\frac{3}{2}\sigma + O(\sigma^2)$ which yields (24). □

If $g(\tau_0) = f(c_0, 0)$ and $\dot{g}(\tau, 0) = 0$ then the 2-jet of v_c at $(\tau_0, 0)$ vanishes. If the critical point τ_0 of g is non-degenerate and $(\partial f/\partial c)(c_0, 0) \neq 0$ we expect the coalescence or creation of two saddle points as in Proposition 4. This cannot happen without the involvement of other critical points of v_c. As we now show, the geometry is organised by a lips transition.

Proposition 5. *Let* (τ_0, c_0) *satisfy* $g(\tau_0) = f(c_0, 0)$. *Suppose*

$$\dot{g}(\tau_0) = 0, \quad \ddot{g}(\tau_0) = m \neq 0, \quad \frac{\partial f}{\partial x}(c_0, 0) \neq 0 \tag{27}$$

so that $f(c, 0)$ *passes with non-zero speed through the non-degenerate critical value* $g(\tau_0)$ *of* g *as* c *passes through* c_0. *Then* $p_c|V_c' : V_c' \to \Pi$ *undergoes a lips creation or annihilation as* c *passes through* c_0.

Proof. Further implicit differentiation of (25) yields

$$v_c^{(3)}(\tau, 0) = -y_c^{(3)}(0, \tau; 0) = -\frac{1}{4}x_c^{(4)}(0, \tau; 0)$$
$$= -\frac{1}{4}\ddot{g}(\tau) = -\frac{1}{4}m \neq 0$$

at $(c, \tau) = (c_0, \tau_0)$, so the 3-jet of $v_c(\tau_0, \cdot)$ does not vanish. We now invoke techniques from singularity theory. After reparametrizing the t-axis (depending on (c, τ)) by $t = \psi(c, \tau; u)$ we may assume $v_{c_0}(\tau_0, u) = u^3$ and moreover (since a versal unfolding of u^3 has the form $\alpha + \beta u + u^3$ and we have $v_c(\tau, 0) = 0$) that

$$v_c(\tau, u) = \beta(c, \tau)u + u^3$$

for (c, τ) near (c_0, τ_0) and small u, where β is a C^∞ function with $\beta(c_0, \tau_0) = 0$. We find $\beta(c, \tau) = (\partial v_c/\partial u)(\tau, 0) = -\frac{1}{2}a_c(\tau)$. Now consider

$$h(v, c, \tau; u) = -v + \beta(c, \tau)u + u^3.$$

The condition for a lips transition in the outline of the graph $v = v_c(\tau, t)$ in the (v, τ)-plane as c passes through c_0 is (see Arnold 1986, Corollary 4.5) that the function c has a non-degenerate critical point when restricted to the 'spine' of the discriminant of h, that is the set $\Psi^{-1}(0, 0)$ where $\Psi : (v, c, \tau) \mapsto (v, \beta(c, \tau))$. This condition is thus that the curve $\beta(c, \tau) = 0$ has quadratic tangency with the τ-axis; this is precisely what the conditions (27) guarantee. □

The geometry of the lips here has a particular form more complicated than the 'standard' picture from Arnold (1976), Bruce (1984b) and Bruce and Giblin (1985), for example. Inspection of the graph of $v_c(\tau, \cdot)$ for (c, τ) near (c_0, τ_0) shows that the cusp points of the lips occur with opposite signs of v, and the lips themselves have two points of quadratic tangency with $v = 0$ from opposite sides as in Proposition 4. See figure 7. Compare Bruce (1989) where the geometry of outlines rather than just

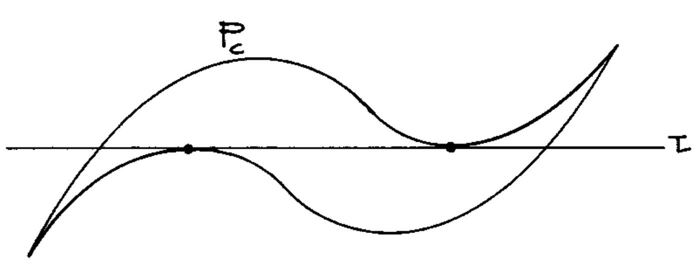

Figure 7. The swan configuration.

their differential topology is explored in some detail. To reflect the slender elegance of this form of the lips we call this a 'swan' configuration.

4.3. *Behaviour of p_c at other points of V_c^0*

Assuming Conjecture (1) the only points away from L_0 that are not regular points of V_c are the Morse points for particular values of c. This still leaves room for points of V_c^0 that are regular points of V_c yet such that p_c has a singularity more degenerate than a fold. We conjecture that for most systems (1) this does not occur. However, it turns out that for (2) this does happen as a result of the special symmetries in (2).

In the terminology of Theorem 3, at a singularity z of p_c on V_c^0 we have

$$DX_c(z) = \begin{pmatrix} \sin t & -a \sin t & 0 \\ \cos t & -a \cos t & \ddot{x}_c \end{pmatrix}$$

and so for the system (2) the point z can fail to be a fold for p_c only if

$$\sin t = 0, \qquad \text{that is}, \qquad z \in L_k \tag{28}$$

or

$$\ddot{x}_c(z) = 0 \qquad \text{with} \qquad z \notin L_k. \tag{29}$$

The first case (28) has already been studied in our earlier analysis of V_c itself at points of L_k, therefore it remains to study H_c at the points z which satisfy (29), corresponding to degenerate graze points. It turns out that the geometry and symmetry of the solutions to (2) imply that if the trajectory of (3) for (2) has a degenerate graze then any other graze in the same trajectory must also be degenerate: this can be seen in Whiston (1992) as a special case of Proposition A1. The consequence is that such $z = (0, \tau; t)$ must satisfy $a_c(\tau) = 0$ and the local geometry of p_c at z and of P_c at $p_c(z)$ become particularly degenerate. This corresponds to the fact that a cusp of P_c^* cannot cross a fold curve of P_c^* but can meet P_c^* only at another cusp, in contrast to the typical behaviour of apparent outlines. See the Correspondence Principle below.

5. Geometry of the re-set map $\varphi_c = \Phi_c|V_c$

The singularity structure of φ_c naturally has close analogies with that of p_c.

Proposition 6. *The map $\varphi_c : V_c \to \Pi$ is a local diffeomorphism at all points of V_c^* and is singular at all (regular) points of V_c^0. If $z = (0, \tau; t) \in V_c^0$ and $a_c(\tau) \neq 0$ then φ_c has a fold singularity at z.*

Proof. At $z = (v, \tau; t) \in \Pi \times \mathbf{R}$ we have

$$D\Phi_c(z) = \begin{pmatrix} \dfrac{\partial \dot{x}_c}{\partial v} & \dfrac{\partial \dot{x}_c}{\partial \tau} & \dfrac{\partial \dot{x}_c}{\partial t} \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \dot{x}}{\partial v} & \dfrac{\partial \dot{x}}{\partial \tau} + \ddot{x} & \ddot{x} \\ 0 & 1 & 1 \end{pmatrix} \tag{30}$$

where $x = x(c, v, \tau; \tau + t)$, and the matrix has kernel dimension 1 or 2 according to whether $D\Phi_c(z)$ has rank 2 or 1, respectively. We consider the two cases separately.

(a) rank $D\Phi(z) = 2$.

Here $(\partial \dot{x}/\partial v, \partial \dot{x}/\partial \tau) \neq (0, 0)$ and ker $D\Phi_c(z)$ is spanned by the non-zero vector

$$u = \left(\frac{\partial \dot{x}}{\partial \tau}, -\frac{\partial \dot{x}}{\partial v}, \frac{\partial \dot{x}}{\partial v}\right)$$

so $\varphi_c = \Phi_c | V_c$ is singular precisely when u is orthogonal to $\operatorname{grad} x_c(z)$, that is

$$\frac{\partial \dot{x}}{\partial \tau}\frac{\partial x_c}{\partial v} - \frac{\partial \dot{x}}{\partial v}\frac{\partial x_c}{\partial \tau} + \frac{\partial \dot{x}}{\partial v}\frac{\partial x_c}{\partial t} = 0$$

which reduces to $\det W = 0$ where

$$W = W(z) = \begin{pmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial \tau} \\ \frac{\partial \dot{x}}{\partial v} & \frac{\partial \dot{x}}{\partial \tau} \end{pmatrix}, \tag{31}$$

evaluated at z. From (6) we see this determinant vanishes if and only if the matrix on the left-hand side of (6) is also singular, that is $v = 0$. We have $\dim \ker D\varphi_c(z) = \dim \ker \Phi_c(z) = 1$, and the conditions for a fold are that V_c be transverse to the plane $\{v = 0\}$ and that $\ker D\varphi_c(z)$ not lie in that plane. Now transversality fails only if $\partial x_c/\partial \tau = \dot{x}_c = 0$, i.e. $\partial x/\partial \tau = \dot{x} = 0$. At a regular point of V_c we then must have $\partial x/\partial v \neq 0$, so $\det W$ vanishes only if $\partial \dot{x}/\partial \tau = 0$; by Corollary (1) this happens only if $a_c(\tau) = 0$. Likewise $\ker D\varphi_c(z)$ lies in $\{v = 0\}$ only if $\partial \dot{x}/\partial \tau = a_c(\tau) = 0$.

(b) $\operatorname{rank} D\Phi(z) = 1$.
Here $(\partial \dot{x}/\partial v, \partial \dot{x}/\partial \tau) = (0, 0)$ and $\ker D\Phi_c(z)$ is spanned by $\{(1,0,0),(0,1,-1)\}$. Clearly $\det W = 0$ so $v = 0$. Now $\dim \ker D\varphi_c(z) > 1$ if and only if $\ker D\Phi_c(z)$ is the tangent space to V_c at z, that is if $(1,0,0)$ and $(0,-1,1)$ are both orthogonal to $\operatorname{grad} x_c$, that is

$$\frac{\partial x_c}{\partial v} = 0 = \frac{\partial x_c}{\partial \tau} - \frac{\partial x_c}{\partial t}$$

so $(\partial x/\partial v, \partial x/\partial \tau) = (0, 0)$. This cannot occur (in particular, $\partial x/\partial v \neq 0$) as the second columns of both matrices in (6) are non-zero. Thus $\ker D\varphi_c(z) = 1$ has dimension 1 and is spanned by $\tilde{u} = (\partial x/\partial \tau, -\partial x/\partial v, \partial x/\partial v)$. It fails to be a fold only if $\partial x/\partial \tau = 0$ which (since $\partial \dot{x}/\partial \tau = 0$) occurs only if $a_c(\tau) = 0$. □

Remark 1. At singular points z of φ_c the vectors $\{u, \tilde{u}\}$ are linearly dependent since $\det W(z) = 0$; either vector (if non-zero) can be taken as a basis for $\ker D\varphi_c(z)$.

Next we identify the cusp singularities of φ_c.

Proposition 7. *At regular points $z = (v, \tau; t)$ of $V_c \setminus H$ where $v = 0$, $a_c(\tau) = 0$ and $a_c'(\tau) \neq 0$ the map $\varphi_c : V_c \to \Pi$ has cusp singularities.*

Proof. From the proof of Proposition 6 we know that φ_c has a singularity at z that is not a fold since $(v, a_c(\tau)) = (0, 0)$ implies $(\partial x/\partial \tau, \partial \dot{x}/\partial \tau) = (0, 0)$.

The condition for a cusp is that the tangency of $\ker D\varphi_c(z)$ with $\{v = 0\}$ be as non-degenerate as possible, meaning here that the v-component of $\ker D\varphi_c(z)$ have non-zero derivative along V_c^0 at z. As $\operatorname{grad} x_c(z) = (\partial x/\partial v, \dot{x}, \dot{x})$ with $\dot{x} \neq 0$ the tangent space to V_c^0 at z is spanned by $(0, 1, -1)$, so the cusp condition is $\partial m/\partial \tau - \dot{m} \neq 0$ where m denotes $(\partial \dot{x}/\partial \tau)(c, 0, \tau; \tau + t)$ in case (a) or $(\partial x/\partial \tau)(c, 0, \tau; \tau + t)$ in case (b). These reduce to $\partial^2 \dot{x}/\partial \tau^2 \neq 0$ (case (a)) or $\partial^2 x/\partial \tau^2 \neq 0$ (case (b)). Now differentiating (6) with respect to τ at z yields

$$D\psi_t(q)\begin{pmatrix} 0 \\ a_c'(\tau) \\ 0 \end{pmatrix} = -\begin{pmatrix} \dfrac{\partial^2 x}{\partial\tau^2} + \dfrac{\partial \dot{x}}{\partial\tau} \\ \dfrac{\partial^2 \dot{x}}{\partial\tau^2} + \dfrac{\partial \ddot{x}}{\partial\tau} \end{pmatrix}.$$

We have $\partial\dot{x}/\partial\tau = 0$ as $a_c(\tau) = 0$, and also $\partial\ddot{x}/\partial\tau = 0$ as follows from differentiating (1) since $(\partial x/\partial\tau, \partial\dot{x}/\partial\tau) = (0, 0)$; thus $a_c'(\tau) \neq 0$ implies

$$\left(\frac{\partial^2 x}{\partial\tau^2}, \frac{\partial^2 \dot{x}}{\partial\tau^2}\right) \neq (0, 0). \tag{32}$$

Also, differentiating $\det W = 0$ along V_c^0 gives

$$\det\begin{pmatrix} \dfrac{\partial^2 \dot{x}}{\partial\tau^2} & \dfrac{\partial \dot{x}}{\partial v} \\ \dfrac{\partial^2 x}{\partial\tau^2} & \dfrac{\partial x}{\partial v} \end{pmatrix} = 0.$$

In case (a) we have $\partial\dot{x}/\partial v \neq 0$, so if the cusp condition fails so that $\partial^2\dot{x}/\partial\tau^2 = 0$ then $\partial^2 x/\partial\tau^2 = 0$, contradicting (32). In case (b) we have $\partial\dot{x}/\partial v = 0$ and $\partial x/\partial v \neq 0$ so $\partial^2\dot{x}/\partial\tau^2 = 0$ which by (32) implies $\partial^2 x/\partial\tau^2 \neq 0$. □

Finally, we can identify swallowtail transitions, using the generic assumptions (18), (19) on f and g. We omit the proof; see Chillingworth (2003).

Proposition 8. *Assume the pair f, g is generic. Then $a_c(\tau) = a_c'(\tau) = 0$ precisely when $c = c_i$, and as c passes through c_i there are creations or annihilations of pairs of cusps of φ_c at swallowtail transitions.*

Using φ_c we can show that swallowtails are the only transitions that occur where $v \neq 0$, in other words there are no lips or beaks transitions in this context.

Theorem 4. *As c varies there are no lips or beaks transitions occurring for p_c on V_c^*.*

Proof. On V_c^* the map φ_c is everywhere a local diffeomorphism, and takes H_c to the line L_c. At a lips transition H_c would be a single point, while at a beaks transition it would be a pair of mutually tangent arcs: in neither case is it a 1-manifold. □

6. The composition $G_c = \varphi_c \circ F_c$

Recall that $P_c = p(H_c)$ is the apparent outline of V_c viewed in the t-direction, including the τ-axis L_0, and corresponds to initial data (c, v, τ) that give rise to grazing trajectories. Clearly $P_c \subset \Pi_c$ except possibly for some points of L_0. Write $\Pi_c \setminus P_c = \Pi_c \setminus (\Pi_c \cap P_c)$. The first result is a straightforward consequence of the definitions.

Proposition 9. *For the general system* (1) *the map $G_c = \varphi_c \circ F_c : \Pi_c \to \Pi$ has the following properties:*

(*i*) *G_c is injective;*
(*ii*) $G_c(\Pi_c \cap P_c) \subset P_c$;
(*iii*) $G_c(\Pi_c \setminus P_c) \subset \Pi \setminus P_c$.

Proof. Statement (i) follows from uniqueness of solutions to (1): two solutions with $x = c$ and different (v, τ) cannot next pass through $x = c$ at the same time and with the same velocity. Statements (ii), (iii) record the fact that a given trajectory of (1) either is or is not a grazing trajectory regardless of the choice of initial point. □

Since p_c is a local diffeomorphism away from H_c (which includes L_0), and in the previous section we have seen that φ_c is a local diffeomorphism away from $\{v = 0\}$, we immediately have the following regularity result:

Proposition 10. *The composition* $G_c : \Pi_c \to \Pi$ *is a local diffeomorphism at all points of* $\Pi_c \setminus P_c$.

Observe the analogous roles played on the one hand by H_c for p_c and on the other by V_c^0 for φ_c. More precisely, let $z \in H_c$: then $p_c(z) \in P_c$ and $\varphi_c(z) \in L_0$. If $w \in P_c$ and $w' \in L_0$ we say that w and w' 'correspond' if there exists $z \in H_c$ with $p_c(z) = w, \varphi_c(z) = w'$. Under this correspondence, points of interesting geometry on P_c correspond to points of interesting geometry on L_0. We pursue this principle in greater detail below. First we set up some general terminology to describe local behaviour of F_c and p_c.

Let $w_0 = (v, \tau) \in \Pi$; then $p_c^{-1}(w_0)$ is a discrete set of points

$$p_c^{-1}(w_0) = \{z_i : i \in N \subset \mathbf{Z}\} \in V_c, \qquad z_i = (w_0; t_i)$$

with $t_0 = 0$ and $t_i < t_j$ for $i < j \in N$, where N denotes a finite or infinite interval of integers. For the linear system (2) we have $N = \mathbf{Z}$ or (when $\omega \notin \mathbf{Q}$) exceptionally $N = \{0\}$ by Corollary 3. By definition $F_c(w_0) = z_1$. We occasionally blur the distinction between $w_0 \in \Pi$ and $z_0 = (w_0; 0) \in \Pi \times \mathbf{R}$.

Write $w_i = \varphi_c(z_i) \in \Pi$ and for each $i \in N$ let U_i be the connected component of $p^{-1}(U_0) \cap V_c$ containing z_i. Let $W_i = \varphi_c(U_i)$.

In the case of (2) if $\omega = p/q \in \mathbf{Q}$ then $\{z_i : i \in \mathbf{Z}\}$ is a finite set modulo $t = 2\pi q$ and $\{w_i : i \in \mathbf{Z}\}$ is a finite set modulo $\tau = 2\pi/\omega$ (see Proposition 3). In general for (1) where V_c is periodic in t with period T, in counting the points z_i we shall count modulo T, and write $z_j \equiv z_k$ when $t_j - t_k \in T\mathbf{Z}$, and also write $w_j \equiv w_k$ when $\tau_j - \tau_k \in (2\pi/\omega)\mathbf{Z}$ where ω is the frequency of the forcing function g. If $z_j \equiv z_k$ then $w_j \equiv w_k$. We write $\bar{z}$ (or $\bar{w}$) to denote the equivalence class of z modulo T (or w modulo $2\pi/\omega$).

The Correspondence Principle

We now list significant geometric features of P_c and their counterparts in L_0 under the Correspondence Principle. To simplify notation we write $F_c = F$, $H_c = H$ and $P_c = P$ with a fixed value of c understood, although we retain the suffix for V_c as a reminder.

We begin with 'codimension-0' features, that is, those which occur for an open set of values of $c \in \mathbf{R}$. First, the two features arising from local geometry of H:

(1) A 'single fold point' is a point $w_0 \in P^*$ such that $p^{-1}(w_0) \cap H = \{\bar{z}_j\}$ where z_j is a fold singularity of p_c, that is $z_j \in H$ and $\ddot{x}_c(z_j) \neq 0$. Then $w_j = (0, \tau_j; 0) \in L_0$ satisfies $\ddot{x}_c(w_j) \neq 0$, that is $a_c(\tau_j) \neq 0$. It is a point at which $\partial v_c/\partial t \neq 0$ so the sheet V_c' of V_c passes through L_0 with non-zero slope: recall Proposition 1. The only branch of P_c passing through w_j is L_0. We call any such point $(0, \tau; 0) \in L_0$ where $a_c(\tau) \neq 0$ a 'simple point' of L_0.

(2) A 'single cusp point' is a point $w_0 \in P^*$ such that $p^{-1}(w_0) \cap H = \{\bar{z}_j\}$ with $\ddot{x}_c(z_j) = 0$ but $x_c^{(3)}(x_j) \neq 0$; then $w_j \in L_0$ has $\ddot{x}_c(w_j) = 0$, $x_c^{(3)}(w_j) \neq 0$, that is $a_c(\tau_j) = 0$, $a_c'(\tau_j) \neq 0$ and so w_j is a point where a branch of P has quadratic tangency with L_0 by Proposition 4. We call $(0, \tau; 0) \in L_0$ where $a_c(\tau) = 0, a_c'(\tau) \neq 0$ a 'tangency point' of L_0.

Next the only codimension-0 feature arising from global geometry of H:

(3) A point $w_0 \in P^*$ is a 'transverse double point' if $p^{-1}(w_0) \cap H = \{\bar{z}_j, \bar{z}_k\}$ where z_j, z_k are both fold singularities of p_c with arcs H_j, H_k of H through z_j, z_k, respectively, such that the arcs $P_i = p_c(H_i)$ for $i = j, k$ intersect transversely at w_0. Then $w_j \in L_0$, and $p^{-1}(w_j) \cap H$ consists of one point mod $t = T$ apart from w_j at which $\ddot{x}_c \neq 0$. Likewise for w_k. Since by Proposition 6 the map φ_c is a local diffeomorphism at z_j there is a neighbourhood U_j of z_j in V_c such that $U_j \cap p^{-1}P_k$ is a smooth arc transverse to H_j taken by φ_c to an arc of P transverse to L_0 at w_j; likewise for w_k. We call w_j, w_k 'transverse points' of L_0.

Note that for (2) when $c = 0$ and $\omega = p/q \in \mathbf{Q}$ the reduction of the t-periodicity from $2\pi q$ to πq means in particular that all single fold points and all single cusp points of P^* become double points (mod $2\pi q$): this is reflected in the doubling of the tangency points of L_0 at $\pm\pi/\omega$ (Type II transition) as c passes through zero.

The three codimension-0 features are nicely illustrated in figures 6 and 7: at a swallowtail transition an arc of single fold points acquires two cusp points and a pair of transverse double points. Correspondingly, at a swan transition an arc of simple points of L_0 acquires two tangency points and a pair of transverse points.

Now we turn to 'codimension-1' features, meaning those which typically occur at isolated values of c. There is only one such local feature.

(4) A 'swallowtail point' $w_0 \in P^*$ (arising for (2) when $c = 1$) has $p^{-1}(w_0) \cap H = \{\bar{z}_j\}$ where $\ddot{x}_c(z_j) = x_c^{(3)}(z_j) = 0$ but $x^{(4)}(z_j) \neq 0$; then $w_j \in L_0$ has $\ddot{x}_c(w_j) = x_c^{(3)}(w_j) = 0$ but $x^{(4)}(w_j) \neq 0$, that is $a_c(\tau_j) = a_c'(\tau_j) = 0$, $a_c'(\tau_j) \neq 0$. By Proposition 4 this means V_c' has cubic tangency with $v = 0$ and generically a lips (swan) transition takes place at w_j. We call w_j a 'swan point' of L_0.

There are two global codimension-1 features:

(5) A 'tangency point' $w_0 \in P^*$ is such that $p^{-1}(w_0) \cap H = \{\bar{z}_j, \bar{z}_k\}$ with $\ddot{x}_c(z_j), \ddot{x}_c(z_k) \neq 0$ and with arcs H_j, H_k of H through z_j, z_k, respectively, such that $P_i = p(H_i)$ for $i = j, k$ are arcs of P^* mutually quadratically tangent at w_0. Then $L_j = \varphi_c(H_j)$ is an interval of L_0 containing w_j. The quadratic tangency implies that near z_j the set $p_c^{-1}(P_k)$ is either:

(a) a pair of arcs through z_j transverse to each other and to H, or
(b) the single point z_j.

In case (a) since φ_c is a local diffeomorphism at z_j it follows from Proposition 9 that P near w_j consists of a pair of smooth arcs crossing each other and L_0 transversely at w_j. If Conjecture 1 holds this must correspond to a Morse point of V_c at the point $(w_j, t_k - t_j) \in H^0 = H \cap V_c^0$. For the linear system (1) (recall Theorem 1) this means $c = \pm c_m$ with $c_m = \gamma \cos \omega m\pi/2$ for some $m \in \mathbf{Z}$, and $|t_j - t_k| = |m|\pi$. We call w_j a

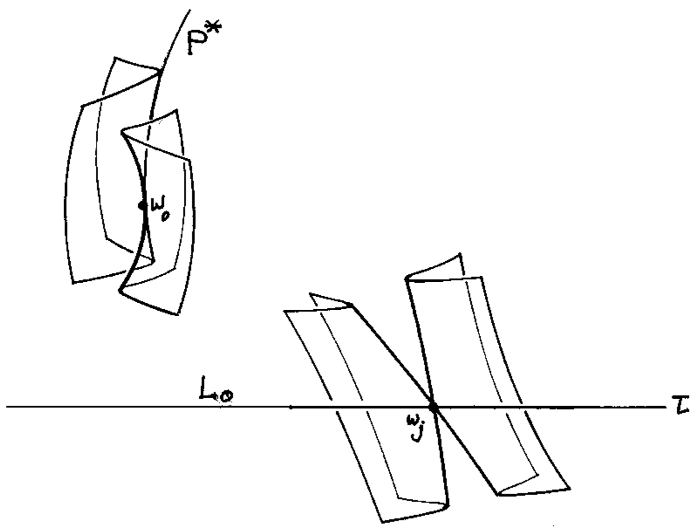

Figure 8. Tangency point $w_0 \in P^*$ and corresponding Morse point $w_j \in L_0$.

'Morse point' of L_0. An analogous discussion applies to w_k. See figure 8, in which points of V_c are included to emphasize the local geometry.

In general for apparent outlines of surfaces in $\mathbf{R}^3$ we expect either (a) or (b) to apply to z_j and z_k independently. In the context of (2) it is only (a) that occurs. It is unclear whether (b) can occur for (1); if so it would be necessary to include further types of Morse point w_j on L_0 including those which are the image under p of an isolated point of H.

Remark 2. The fact that for (2) all double points of P on L_0 where $a_c(\tau) \neq 0$ are transverse points or Morse points shows that for this system all tangencies of P^* with itself must indeed be quadratic.

(6) A 'cusp/fold' point w_0 of P^* is a coincidence of a cusp point and a fold point: thus $p^{-1}(w_0) \cap H = \{\bar{z}_j, \bar{z}_k\}$ with $\ddot{x}_c(z_j) \neq 0$ but with $\ddot{x}_c(z_k) = 0$ and $x_c^{(3)}(z_k) \neq 0$. Such points occur generically for apparent outlines and presumably also for (1) but do not occur for (2) by Proposition A1 in Whiston (1992). As c varies the generic local behaviour of P^* is again a transition between zero and two intersections, with corresponding transitions at the 'double tangency' point w_j and 'lips' or 'beaks' point w_k of L_0. We omit the details.

In place of cups/fold points there is a codimension-1 phenomenon special to the particular system (2) that would be expected to occur in the general system (1) only in codimension 2, that is at isolated values of c for certain discrete choices of a further parameter in f or g.

(7) A 'cusp coincidence point' $w_0 \in P^*$ has $p^{-1}(w_0) \cap H = \{\bar{z}_j, \bar{z}_k\}$ with $\ddot{x}_c(z_j) = \ddot{x}_c(x_k) = 0$ and $x_c^{(3)}(z_j), x_c^{(3)}(x_k) \neq 0$: then w_j and w_k are points of L_0 corresponding to a double degenerate graze as described at the end of section 4. We call these 'cusp contact points' of L_0. At these points the outline P_c has two branches quadratically tangent to L_0 and tangent to each other with order 9/2. We refer to Chillingworth (2003) for further details.

6.1. *Some local geometry of G_c*

We now turn to study the behaviour of the map G_c itself near its points of singularity and/or discontinuity. These are the points of L_0 together with the points $w_0 \in P^*$ for which $F_c(w_0) = z_1 \in H$. The local geometry of G_c needs to be described on a case-by-case basis, and for reasons of space we shall consider only single fold/simple points and single cusp/tangency points, as well as transverse double points of P^* and transverse points of L_0. Further cases are studied in Chillingworth (2003).

For definiteness we focus on $w_0 \in P^* \cap \Pi^+$; the description for $P^* \cap \Pi^-$ is analogous, with the sign of v reversed.

The following results are useful in keeping track of local geometry.

Proposition 11. *Let $z = (v, \tau; t) \in H$ be a fold point for p_c with $w = (0, \tau + t) = \varphi_c(z) \in L_0$. Then the image under $D\varphi_c(z)$ of* $\ker Dp_c(z)$ *is spanned by the vector $(a, 1)$ where $a = \ddot{x}_c(w) = a_c(\tau + t)$.*

Proof. Since $\ker Dp_c(z)$ is spanned by $(0, 0; 1)$ its image under $D\varphi_c(z)$ is spanned by $(\ddot{x}_c(z), 1)$ by Proposition 6, and $\ddot{x}_c(z) = \ddot{x}_c(w)$ as φ_c simply re-sets the clock.

Definition 2. For $w = (0, \tau) \in L_0$ the 'distinguished direction' at w is that spanned by the vector $(a_c(\tau), 0)$.

Proposition 12. *If $z = (0, \tau; t) \in H$ with $a_c(\tau) \neq 0$ then* $\ker D\varphi_c(z) = T_z H$.

Proof. The tangent space to H is spanned by a vector orthogonal to both $\operatorname{grad} x_c$ and $\operatorname{grad} \dot{x}_c$ with $x_c = c$ and $\dot{x}_c = 0$. Such a vector is

$$\left(\frac{\partial x}{\partial \tau}, -\frac{\partial x}{\partial v}, \frac{\partial x}{\partial v}\right) + (0, 0, \det W)$$

with W given by (31). At singular points of φ_c we have $\det W = 0$, and the first vector above spans $\ker D\varphi_c$.

As with all the figures in this paper, those which illustrate key features of G_c below are qualitative and not intended to be numerically accurate.

(1a) *Single fold points*

Let $w_0 \in P^* \cap \Pi^+$ be a single fold point, with $H \cap p^{-1}(w_0) = \{\bar{z}_1\}$. Choose a sufficiently small disc neighbourhood U_0 of w_0 so that the connected component U_1 of $p_c^{-1} U_0$ containing z_1 intersects H in a smooth arc H_1; thus U_1 is folded by p along H_1 which is mapped by p diffeomorphically to a smooth arc P_1 in Π^+. We may assume P_1 separates U_0 into two connected components $U_0^+ = p(U_1)$ and U_0^-. We say that U_0^+ lies on the 'shadow side' of P_1 while U_0^- lies on the 'free side'.

As $z_1 \in H$ we have $w_1 \in L_0$ and F is discontinuous along P_1. Assuming $2 \in N$ there is a neighbourhood U_2 of z_2 such that $p|U_2 : U_2 \to U_0$ is a diffeomorphism, and $K_2 = U_2 \cap p^{-1} P_1$ is an arc through z_2 separating U_2 into two open sets $U_2^{\pm}$ with $p(U_2^{\pm}) = U_0^{\pm}$. Thus $F|U_0^-$ is the restriction to U_0^- of the diffeomorphism $(p|U_2)^{-1} : U_0 \to U_2$. If $2 \notin N$ then $F|U_0^-$ is undefined.

In contrast $p|U_1 : U_1 \to U_0$ is a fold. We have $G_c(U_0) = G_c(U_0^+ \cup P_1 \cup U_0^-) = \varphi_c(U_1^+ \cup H_1 \cup U_2^-) = W_1^+ \cup J_1 \cup W_2^-$ with J_1 an open interval of L_0 containing the simple point w_1, with $W_1^+ \cup J_1$ an open neighbourhood of w_1 in the closed half-plane $\Pi^- \cup L_0$, and with W_2^- one of the two components into which $Q_2 = \varphi_c(K_2)$ separates W_2. See figure 9.

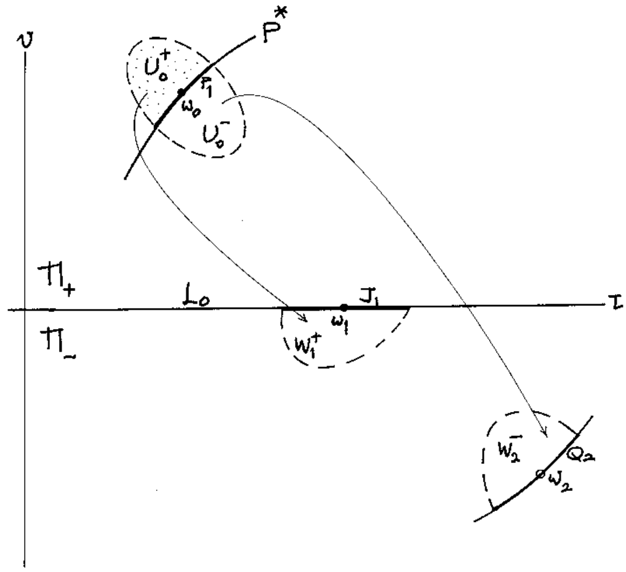

Figure 9. Action of G_c near a single fold point $w_0 \in P^*$.

In the linear case with $\omega = p/q \in \mathbf{Q}$ and $z_2 \equiv z_0$ the above still holds although now $G_c : U_0^- \to W_2^- \equiv W_0^- = U_0^-$ is the identity map.

Note that $G_c : U_0^- \to W_2^-$ is a diffeomorphism while $G_c : U_0^+ \cup P_1 \to W_1^+ \cup J_1$ has the geometry of what we may call an 'inverse fold' at all points of P_1. In particular, any path in $U_0^+ \cup P_1$ transverse to P_1 at $w \in P_1$ is taken by G_c to a path with tangent in the distinguished direction at $G_c(w)$.

(1b) *Simple points of* L_0

At a simple point $w_0 = (0, \tau_0) \in L_0$ we have $(\partial/\partial t)v_c(\tau_0; 0) = -\frac{1}{2}a_c(\tau_0) \neq 0$. To fix matters we take $a_c(\tau_0) < 0$, the other case being completely analogous. The implicit function theorem allows V_c' to be expressed locally as a graph $t = u_c(v, \tau)$, where $u_c(v, \tau)$ here has the same sign as v. If $U_0 = U_0^+ \cup J_0 \cup U_0^-$ is a sufficiently small connected open neighbourhood of w_0 in Π with $U_0 \cap \Pi^\pm = U_0^\pm$ and $U_0 \cap L_0 = J_0$ then $F|U_0^+$ is just the restriction to U_0^+ of the diffeomorphism $F^0 = (p|U_0')^{-1} : U_0 \to U_0' : (v, \tau) \mapsto (v, \tau; u_c(v, \tau))$ where U_0' is a neighbourhood of z_0 in V_c'. However, if $1 \in N$ then F is discontinuous along J_0 and $F|U_0^- \cup J_0$ is (for U_0 sufficiently small) the restriction to $U_0^- \cup J_0$ of the diffeomorphism $F_1 = (p|U_1)^{-1} : U_0 \to U_1$.

In the linear case (2) if $z_1 \equiv z_0$ then $F = F^0$ on all of U_0.

We have described the action of F on U_0; we now describe the action of φ_c on $F(U_0)$. Since $F(U_0^+) \subset U_0'$ we need in particular to understand the action of φ_c on U_0'. Let $W_0' = \varphi_c(U_0')$.

Lemma 2. *The restriction* $\varphi_c|U_0' : U_0' \to W_0'$ *is a diffeomorphism.*

Proof. The tangent space to V_c' at z_0 is spanned by

$$\left\{(0, 1; 0), \left(-\frac{1}{2}a_c(\tau_0), 0; 1\right)\right\}$$

since $(\partial v_c/\partial \tau)(w_0) = 0$ as $L_0 \subset V_c'$, and

$$\ker D\Phi_c(z_0) = \operatorname{span}\{(0, 1; -1)\}$$

as $(\partial x_c/\partial \tau)(w_0) = 0$ since $L_0 \subset V_c$. Therefore, $\varphi_c|V_c'$ is a local diffeomorphism at z_0 if and only if

$$\det \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2}a_0 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \neq 0,$$

where $a_0 = a_c(\tau_0)$. The value of the determinant is $\frac{1}{2}a_0$ which is non-zero as w_0 is a simple point of L_0. □

Corollary 6. *The map* $G_c|U_0^+$ *is the restriction to* U_0^+ *of the diffeomorphism* $\varphi_c \circ F^0 : U_0 \to W_0'$.

Note that $\varphi_c \circ F^0$ is the identity on J_0. Therefore, to first order the effect of G_c on U_0^+ is a shear in the direction of increasing τ composed with a reflection in L_0; we call this a 'shear-reflection'. From Corollaries 6 and 5.1 we therefore see that $G_c|U_0$ is discontinuous along J_0, being the restriction of a shear-reflection on U_0^+ and a fold on $U_0^- \cup J_0$. See figure 10.

For (2) when $z_1 \equiv z_0$ we have $F|U_0 = F^0|U_0$ and $G_c|U_0 \to W_0'$ is a shear-reflection, fixed along J_0; then $(G_c)^2 = \mathrm{id} : U_0 \to U_0$ because $p^{-1}(U_0) = U_0 \cup U_0'$ (modulo $2\pi q$-periodicity in t): trajectories leaving $x = c$ with initial state $(v, \tau) = w$ close to $(v_0, \tau_0) = w_0$ and with $v \neq 0$ return to w after passing through $x = c$ once with non-zero speed.

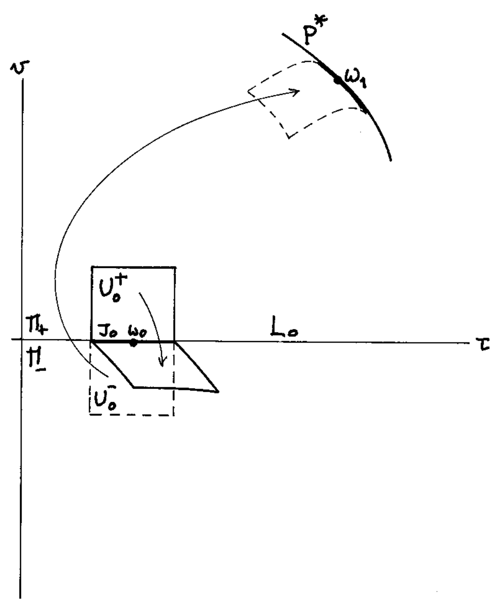

Figure 10. Action of G_c near a simple point $w_0 \in L_0$.

This completes the local description of G_c at single fold points of P^* and simple points of L_0. Next we turn to cusp points of P^* and corresponding tangency points of L_0.

(2a) *Single cusp points*
Let $w_0 \in P^* \cap \Pi^+$ be a single cusp point, with $p^{-1}(w_0) \cap H = \{\bar{z}_1\}$. Choose U_0 sufficiently small so that the connected component H_1 of $p^{-1}(U_0) \cap H$ containing z_1 is a smooth arc of fold points of p_c apart from the cusp point at z_1.

The projection p_c has a cusp singularity at z_1, so F is discontinuous along one branch of the cusp of P^* at w_0 (although not at w_0 itself). The connected component U_1 of $p_c^{-1}(U_0)$ containing z_1 is a neighbourhood of z_1, and if U_0 is small enough the set $U_1 \cap p^{-1}(P)$ is the union of a pair of smooth arcs $H_1 = U_1 \cap H$ and K_1 with quadratic tangency at z_1. Now φ_c is a local diffeomorphism at z_1 by Proposition 6, and $\varphi_c(H_1) = J_1 \subset L_0$; hence $\varphi_c(K_1) = J_1'$ is an arc in $\Pi^- \cup L_0$ quadratically tangent to L_0 at w_1. From the geometry of the cusp we see that F takes the two branches of $U_0 \cap P^*$ at w_0 to two arcs of a C^1 (but not C^2) curve in V_c passing through z_1: one is an arc of H_1 while the other is an arc of K_1. Thus G_c takes U_0 to a subset of an open neighbourhood of w_1 bounded by two arcs of a C^1 (but not C^2) curve in Π passing through $w_1 \in L_0$: one arc is a subinterval of J_1 (included) while the other is an arc of J_1' (not included). See figure 11.

(2b) Tangency points of L_0
Let $w_0 \in L_0$ be a tangency point. At w_0 we have $\dot{x}_c = 0$, $\ddot{x}_c = 0$ and from the correspondence principle we know that w_1 is a cusp point (unless $w_1 = w_0$) and that φ_c has a 'cusp' singularity at $z_1 \notin H$. We now put together this information on F at w_0 and φ_c at $z_1 \notin H$ to describe the action of G_c on a neighbourhood of the tangency point $w_0 = (0, \tau_0; 0)$. For definiteness we take $x_c^{(3)}(w_0) > 0$ (so $v_c(\tau_0, t)$ has a maximum at $t = 0$) and $a_c'(\tau_0) > 0$ (so $(\partial/\partial\tau)v_c(\tau_0, 0) < 0$), although the geometrical description would be analogous in other cases.

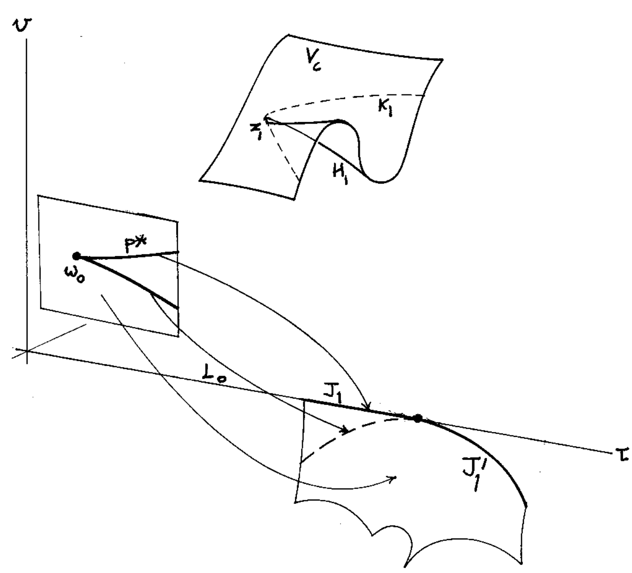

Figure 11. Action of G_c near a cusp point $w_0 \in P^*$.

Choose U_0 small enough so that the connected component of $p^{-1}(U_0) \cap H$ containing z_0 is the union of an interval J_0 of L_0 and a smooth arc H_0 of H meeting only at z_0 ; write $P_0 = p(H_0)$ which is a smooth arc of P quadratically tangent to L_0 at w_0 from the side $v > 0$. Let $P_0^{\pm}$ denote the subarcs of P_0 with τ greater than or less than τ_0, respectively.

From the geometry of V_c near z_0 (recall section 2) we see that F is discontinuous along L_0 and along P_0^-, so we shall consider individually the three connected components of the complement of $L_0 \cup P_0^-$ in U_0. We have already noted:

(1) $G_c(w_0) = w_1$ is a cusp point of $P^* \cap \Pi^-$

and we also have

(2) $G_c(P_0^-) = J_0^+$, an open interval (τ_0, τ_+) of L_0 for some $\tau_+ > \tau_0$.

Let D denote the open subset of $U_0 \cap \Pi^+$ bounded by P_0^- and the interval $J_0^- = \{\tau \in J_0 : \tau < \tau_0\}$. Then

(3) $G_c|D \cup P_0^-$ has an inverse fold along P_0^-, and
(4) close to J_0^- the diffeomorphism $G_c|D$ is the restriction of a local shear-reflection that is the identity on J_0^-, although G_c itself is discontinuous along J_0^-.

See figure 12 which indicates the contours $\tau =$ const. and $\dot{x}_c =$ const. in D, as well as their images under G_c. The latter images are of course the straight lines $v =$ const., while the former images are curves transverse to J_0^+ along the distinguished direction.

Next we consider the open set $E = U_0 \cap \Pi^-$, which we may take as $G_c(D)$.

(5) $G_c|E$ is the restriction of a local shear-reflection along J_0^+, although G_c itself is discontinuous along J_0^+. In contrast, G_c is continuous on $E \cup J_0^-$ and is a fold along J_0^- (compare (1b) above) with $G_c(J_0^-) = P_0^+$.

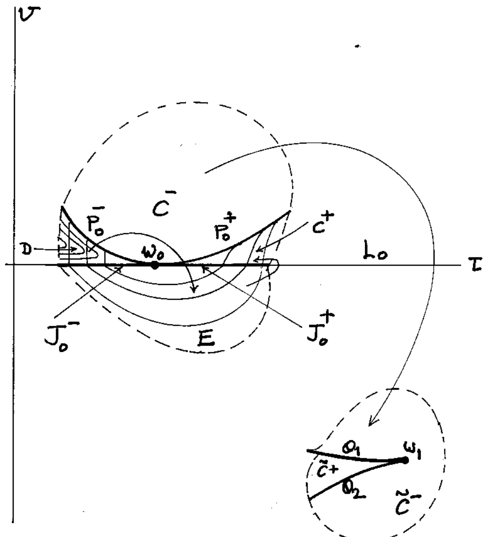

Figure 12. Action of G_c near a tangency point $w_0 \in L_0$.

Finally, it remains to consider the action of G_c on the subsets C^+, C^- of U_0 bounded by P_0^+ together with J_0^+, P_0^-, respectively. We may take $C^+ = G_c(E)$. Let $C = C^+ \cup C^-$.

(6) $G_c(C)$ is an open neighbourhood of the cusp point w_1 with one branch Q_1 of the cusp of P^* at w_1 deleted; here $\tilde{C}^+ = G_c(C^+)$ lies 'inside' the cusp while $\tilde{C}^- = G_c(C^-)$ lies 'outside'.

(7) $G_c|C \cup J_0^+$ has a fold along J_0^+ with $G_c(J_0^+) = Q_1$, and is the restriction of the diffeomorphism $\varphi_c \circ (p|U_1)^{-1}$ on a neighbourhood of P_0^-, where U_1 is a neighbourhood of z_1 in V_c. The second branch of the cusp at w_1 is $Q_2 = G_c(P_0^+)$. See figure 12.

(3a) *Transverse double points*

Let w_0 be a transverse double point with $p^{-1}(w_0) \cap H = \{\bar{z}_1, \bar{z}_k\}$ with $1 < k$. Choose a neighbourhood U_0 of w_0 small enough so that for $i = 1, k$ the connected components of $p^{-1}(U_0) \cap H$ containing z_i are smooth arcs H_i that project by p to smooth arcs P_i of $P^* \cap \Pi^+$ intersecting transversely at w_0.

Since $j = 1$ we have $w_1 \in L_0$ and (as in (1a)(ii) above) the map $G_c|U_0$ is discontinuous along the arc P_1 through w_0. Writing $U_0 = U_0^+ \cup P_1 \cup U_0^-$ as in (1a), the restriction $G_c|U_0^+ \cup P_1$ is an inverse fold along P_1, with $J_1 = G_c(P_1)$ an open interval of L_0. The image of $P_k \cap (U_0^+ \cup P_1)$ under G_c is $\tilde{P}_k \cap (\Pi_- \cup L_0)$ where $\tilde{P}_k$ is an arc of P through w_1 transverse to L_0; the shadow side of P_k is taken to the shadow side of $\tilde{P}_k$.

The effect of G_c on U_0^- will differ according as $k = 2$ or $k > 2$.

If $k > 2$ then $F|U_0^-$ is the restriction of the diffeomorphism $(p|U_2)^{-1} : U_0 \to U_2$, so $G_c|U_0^-$ is the restriction of a diffeomorphism $U_0 \to W_2 = \varphi_c(U_2)$. Let $K_2^l = U_2 \cap p^{-1}(P_l)$ for $l = 1, k$. Then $G_c(P_k \cap U_0^-)$ is the free side W_2^- of $J_2^1 = \varphi_c(K_2^1)$ in W_2, and $G_c(P_k \cap U_0^-) = \tilde{P}_k' \cap W_2^-$ where $\tilde{P}_k'$ is an arc of P^* intersecting J_2^1 transversely at $w_2 \in \Pi^-$. See figure 13(i).

If $k = 2$ then $G_c|U_0^-$ is discontinuous along the arc $P_2 \cap U_0^-$, which separates U_0^- into two open sets U_0^{-+} and U_0^{--} on the shadow and free sides of P_2, respectively. Now $F|U_0^{--}$ is the restriction to U_0^{--} of $(p|U_3)^{-1} : U_0 \to U_3$, and G_c takes U_0^{--} to an open subset W_3^{--} of $W_3 = \varphi_c(U_3)$ bounded by two arcs of P^* through $w_3 \in \Pi^-$ and on the free side of both arcs. (If $z_3 \equiv z_0$ so $w_3 \equiv w_0$ then $G_c|U_0^{--}$ is the identity map.) On the other hand $U_0^{-+} \cup (P_2 \cap U_0^-)$ is taken to an open subset of $W_2 \cap (\Pi^- \cup L_0)$ bounded by an arc of P through w_2 (transverse to L_0) and an interval of L_0, the latter being included. An inverse fold for G_c occurs along $(P_2 \cap U_0^-)$. See figure 13(ii).

(3b) *Transverse points of L_0*

Let $w_0 \in L_0$ be a transverse point with $p^{-1}(w_0) \cap H = \{\bar{z}_0, \bar{z}_j\}$. Assume U_0 chosen small enough so that the connected component U_j of $p^{-1}(U_0)$ containing z_j meets H in a smooth arc H_j that projects diffeomorphically by p to a smooth arc P_j of P intersecting J_0 transversely at w_0.

As in case (1b) above, we assume $a_c(\tau_0) < 0$. Then $G_c|U_0^+$ is the restriction of a diffeomorphism that is to first order a shear-reflection fixed along J_0. Hence $G_c(P_j \cap U_0^+)$ is the intersection of $W_0' \cap \Pi^-$ (which we can take to be U_0^-) with an arc of P through w_0 and transverse to L_0; this arc must therefore be $P_j \cap U_0^-$ as G_c preserves P by Proposition 9.

To see the effect of G_c on U_0^- we consider separately the cases $j > 1$ and $j = 1$.

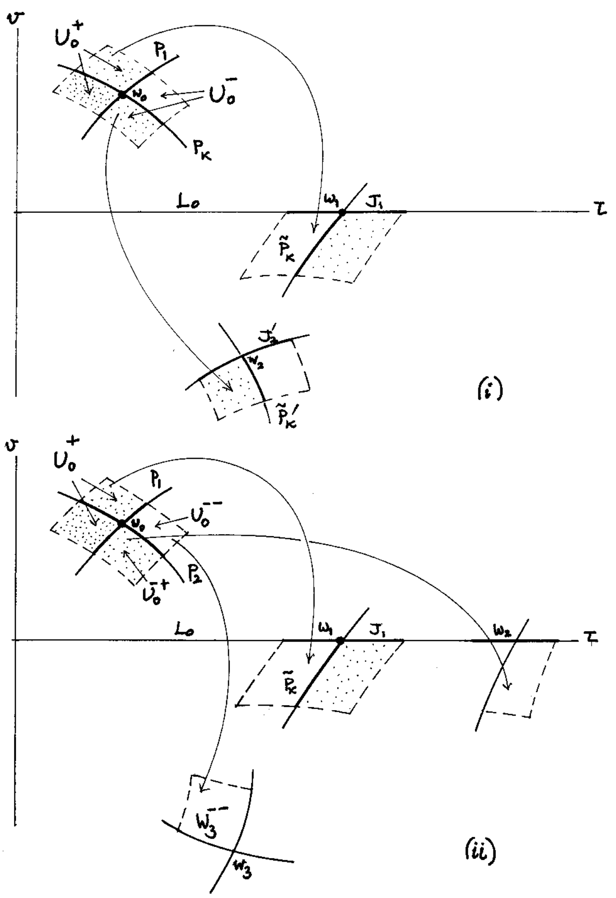

Figure 13. Action of G_c near a transverse double point $w_0 \in P^*$: (i) $k > 2$; (ii) $k = 2$.

(i) $j > 1$

The restriction of G_c to $U_0^- \cup J_0$ is the restriction of a fold map with singular set J_0. A typical arc through w_0 and transverse to J_0 is taken by G_c to an arc tangent to P^* at w_1. The exceptional arcs are those whose tangent at w_0 is in the direction $p \ker D\varphi_c(z_1) = \ker W(z_1)$, these being taken to arcs transverse to P^* at w_1 (that is, transverse as 1-manifolds: as parametrized curves their speed becomes zero at w_1). Since from the Correspondence Principle (see (5)) self-intersections of P^* at fold points are always transverse unless they correspond to Morse points on L_0 as in (2b) in which case they are quadratic, it follows that P_j must be just such an exceptional arc. See figure 14. This reflects the geometry already seen in (3a) above, as P_j can be seen as the image under an inverse fold of an arc transverse to P^* at w_{-1}. A similar argument applies at every $z_i \in p^{-1}(w_0)$ with $z_i \notin H$, replacing F_c by a local right inverse to $p|V_c$ near z_i. Thus we have the following result.

Proposition 13. *If w_0 is a transverse point of L_0 then the branch of P through w_0 transverse to L_0 is tangent at w_0 to $p \ker D\varphi_c(z_i) = \ker W(z_i)$, this direction being the same for every $z_i \in p^{-1}(w_0)$ with $z_i \notin H$.*

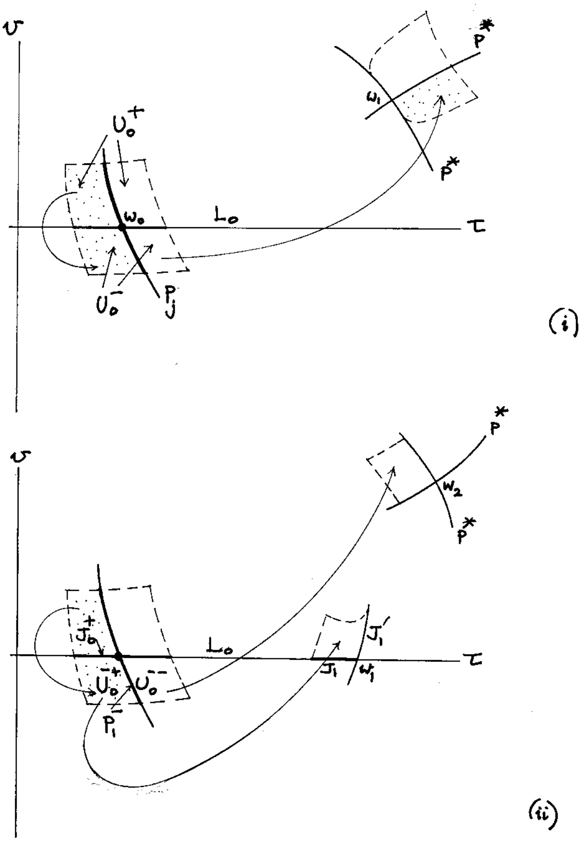

Figure 14. Action of G_c near a transverse point $w_0 \in L_0$: (i) $j > 1$; (ii) $j = 1$.

(ii) $j = 1$
Both the maps p_c and φ_c exhibit fold singularities at z_1 with their singular sets mutually transverse. Thus G_c near w_0 is the result of composing an inverse fold with a fold. We have that $G_c|U_0^-$ is discontinuous along $P_1 \cap U_0^-$, which separates U_0^- into two open subsets U_0^{--} and U_0^{-+} with U_0^{-+} on the shadow side of P_1. Then G_c takes U_0^{--} to an open subset of W_2 bounded by two transverse arcs of P^* through the transverse double point $w_2 \in P^* \cap \Pi^+$. If $z_2 \in N$ and $z_2 \not\equiv z_0$ then $F|U_0^{--}$ is the restriction to U_0^{--} of the diffeomorphism $(p|U_2)^{-1} : U_0 \to U_2$; if $z_2 \equiv z_0$ so $w_2 \equiv w_0$ then G_c is the identity map on U_0^{--}. Compare case (1b) above.

The arc $P_1^- = P_1 \cap U_0^-$ is taken by G_c to an arc J_1^- of $J_1 \subset L_0$ with an endpoint at w_1; an arc J_0^+ of $J_0 \subset L_0$ with endpoint at w_0 and on the shadow side of P_1 is taken to an arc J_1' of $P \cap \Pi^+$ with endpoint at w_1 which by Proposition 13 is transverse to L_0. The map G_c takes the open set U_0^{-+} of U_0^- bounded by P_1^-, J_0^+ to an open set in $W_1^+ = W_1 \cap \Pi^+$ bounded by the arcs J_1^- and J_1', exhibiting a fold along $J_0^+ \subset L_0$ and an inverse fold along P_1^-. See figure 14.

Remark 3. A local model for a pair of folds with transverse singular sets is the pair of maps $\mathbf{R}^2 \to \mathbf{R}^2$ given by $(x, y) \mapsto (x^2, y)$ and $(x, y) \mapsto (x, y^2)$, and in this case the inverse of one composed with the other takes the form $(u, v) \mapsto (\sqrt{u}, v^2)$ in the positive quadrant. We cannot, however, assume that G_c has exactly this form near

w_0 as there may be no coordinate change on V_c near z_1 that takes both folds to their standard forms simultaneously (Teixeira 1982, Kurokawa 2003).

7. Global considerations

For a system such as (2) with $\omega \in \mathbf{Q}$ and for a given choice of clearance c the apparent outline P_c (consisting of the part P_c^* where $v > 0$ together with the τ-axis L_0) separates the plane Π into a finite number of complementary regions. In other cases this description is still adequate if we bound the time that would be allowed to elapse under the 'free' dynamics of (1). The dynamics of G_c can then be regarded as a re-arrangement of these regions with a diffeomorphism on each open region, but with discontinuities and singularities along L_0, and along some of the boundary arcs that comprise P_c^*. In this paper we have discussed some of the geometry of this decomposition of G_c. To construct the dynamics of the impact oscillator it is then necessary to compose G_c with the restitution map R. Certain local features of this picture have been previously used by other authors (Budd and Dux 1994a,b) to show mechanisms for interesting dynamical behaviour such as 'chatter', and global features are used by Dippnall (2003) to detect horseshoes. We propose this overall geometric description, in which the complementary regions to P_c form a kind of Markov partition (Adler 1998) for the dynamics, with local dynamics near discontinuities as partially described in this paper, as a versatile setting for future study of impact oscillators (1) and their generalizations.

8. List of notation

c	x-coordinate of the obstacle
F_c	first-hit map $\Pi_c \to V_c$
G_c	$\varphi_c \circ F_c$
H_c	$\{(v,\tau;t) \in V_c : \dot{x}(c,v,\tau;t) = 0\}$
I_c	$R \circ G_c$: the dynamical system to be studied
L_k	$\{(v,\tau;t) \in \mathbf{R}^3 : (v,t) = (0,k\pi)\}$
L_0	τ-axis
p	projection $(v,\tau;t) \mapsto (v,\tau)$
p_c	restriction of p to V_c
P_c	$p(H_c) =$ apparent outline of V_c in t-direction
R	restitution map $\Pi \to \Pi$
v	initial velocity
v_c	function of (τ,t) whose graph is V_c' near L_0
V_c	impact surface $\{(v,\tau;t) \in \mathbf{R}^3 : x(c,v,\tau;t) = c\}$
V_c'	sheet of V_c passing through L_0 other than Π
$x(c,v,\tau;t)$	solution to (1) with intial data $(c,v;\tau)$
$x_c(v,\tau;t)$	$x(c,v,\tau;\tau+t)$
φ_c	re-set map $V_c \to \Pi$
Π	(v,τ)-plane
Π_c	subset of Π for which future impact exists
τ	initial phase.

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