Discontinuity geometry for an impact oscillator

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Abstract. We use methods of singularity theory to classify the local geometry of the discontinuity set, together with associated local dynamics, for a discrete dynamical system that represents a natural class of oscillator with one degree of freedom impacting against a fixed obstacle. We also include descriptions of the generic transitions that occur in the discontinuity set as the position of the obstacle is smoothly varied. The results can be applied to any choice of restitution law at impact. The analysis provides a general setting for the study of local and global dynamics of discontinuous systems of this type, for example giving a geometric basis for the possible construction of Markov partitions in certain cases.

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1. Introduction
An impact oscillator (sometimes called a ‘vibro-impact system’) here consists of a system of ordinary differential equations in which the ‘free’ dynamics are interrupted by an obstacle, together with a given ‘restitution rule’ for describing the dynamics immediately after impact with the obstacle. A typical system of this kind with one degree of freedom takes the form

\[ \ddot{x} + f(x, \dot{x}) = g(t) \]  

for \( x \in \mathbb{R} \), where we assume \( f \) and \( g \) are smooth functions (meaning \( C^\infty \) or analytic) and \( g \) is periodic with period \( T > 0 \), and where the obstacle is placed at \( x = c \) with restitution rule that \( \dot{x} \) is replaced by \(-r\dot{x}\) for constant \( 0 < r < 1 \) whenever a trajectory reaches \( x = c \). As usual the dot denotes differentiation with respect to \( t \). An important particular case is the linear system

\[ \ddot{x} + x = \cos \omega t \]  

with \( 0 < \omega \neq 1 \).

The geometric analysis that we give in this paper can be applied to quite general types of interaction of the obstruction with the ‘free’ dynamics of (2) such as, for example, the presence of a thin resistant layer at \( x = c \) which does not cause the velocity \( v \) to change sign but merely replaces \( v \) by \( rv \) with \( r \geq 0 \). The purpose of our analysis is to bring out the primary geometric role of the ‘obstacle’ at \( x = c \) in the overall phase space dynamics: after this, the imposition of any particular rule for interaction with the obstacle is a second step which determines how our geometric
information is to be used to help describe the dynamics. Indeed, taking \( r = 1 \) allows us to reconstruct most of the original free dynamics, an observation worth keeping in mind in what follows. Our results apply to fixed values of the ‘clearance’ \( c \) and also describe the transitions that take place as \( c \) is varied.

The main ideas of the paper are as follows. After straightening out the flow for (1) we describe in sections 2 and 3 a 2-manifold (almost) in \( \mathbb{R}^3 \) that represents the impacts of the trajectories of (1) with the obstacle \( x = c \); we call this the ‘impact surface’. The dynamics of the impact oscillator are then represented by a discrete dynamical system obtained by choosing initial data (velocity and phase) at a point where \( x = c \), proceeding parallel to the \( t \)-axis until meeting the impact surface, resetting the time to \( t = 0 \), applying the restitution rule and then repeating the process. This involves studying (sections 4 and 5) the geometry of two maps from the impact surface to the plane of initial data, namely the projection parallel to the \( t \)-axis (for which we need to understand its discontinuous inverse) and the time ‘re-set’ map. We use methods of singularity theory to analyse the geometry of both these maps under explicit generic assumptions on (1). The dynamics for the impact oscillator can then in principle be constructed (sections 6 and 7) from this information together with the restitution rule. A list of notation is given in section 8.

There is an extensive literature on impact oscillators which we do not attempt to survey here. Useful references include Nordmark (1991, 1997), Ivanov (1993, 1994), Bishop (1994), Budd and Dux (1994a,b), Chin et al. (1994), Casas et al. (1996), Babitsky (1998), Foale and Bishop (1992) with their own bibliographies to which the interested reader is referred. The first author to approach the geometry of discontinuities in a systematic way was Whiston (1987, 1992) whose pioneering approach was the inspiration for the present work. In this paper we outline the main geometric results; in a further paper (Chillingworth 2003) some examples and specific applications to dynamics are given.

2. The impact surface in phase space

The natural phase space for the study of the system (1) is \( \mathbb{R}^3 \) with coordinates \((x, y, u) = (x, \dot{x}, t)\) in which the system becomes a first-order system

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -f(x, y) + g(u) \\
\dot{u} &= 1
\end{align*}
\]

The standard way to investigate the dynamics is to study the time-\( T \) map \((x_0, y_0) \mapsto (x(T), y(T))\) where \((x(t), y(t)) = (x(t), \dot{x}(t))\) denotes the solution to (1) with initial data \((x(0), y(0)) = (x_0, y_0)\). However, as observed by Shaw and Holmes (1983), Whiston (1987), Foale and Bishop (1992), Bud and Dux (1994a) and others, in the case of an impact oscillator it is more useful instead to work in the plane

\[ \Sigma_c = \{(x, \dot{x}, t) \in \mathbb{R}^3 : x = c\} \]

and then to study the discrete dynamical system on (most of) \( \Sigma_c \) generated by the composition

\[ \tilde{I} = \tilde{R} \circ \tilde{G} : \Sigma_c \to \Sigma_c \]
where \( \tilde{G} : \Sigma_c \rightarrow \Sigma_c \) is given by following the dynamics of (3) with initial data on \( \Sigma_c \) until the trajectory hits \( \Sigma_c \) the next time (if at all), and \( \tilde{R} \) is the ‘restitution map’ taking \( y \) to \(-ry\) or other appropriate rule.

The map \( \tilde{G} \) is discontinuous at points of \( \tilde{S} = \tilde{G}^{-1}(\Sigma_c^0) \) where \( \Sigma_c^0 = \Sigma_c \cap \{ y = 0 \} \) : if a trajectory ‘grazes’ \( \Sigma_c \) at \( z \) (that is if it meets \( \Sigma_c \) at \( z \in \Sigma_c^0 \) where \( \dot{x} \neq 0 \) ) then nearby trajectories may miss \( \Sigma_c \) in a neighbourhood of \( z \) and not hit \( \Sigma_c \) until a considerable time later (if at all). Therefore, as emphasized by Whiston (1987), the dynamical behaviour of the system is crucially dependent on the nature and position of the ‘discontinuity set’ \( \tilde{S} \) and its inverse iterates \( I^{-n}(\tilde{S}) \) for \( n = 1, 2, 3, \ldots \). In what follows we give a local and general global geometric description of the discontinuity set \( \tilde{S} \) (in a slightly different setting) for the system (1) under generic assumptions on \( f \) and \( g \).

For the linear system (2) we give more specific information. We also describe the changes that the discontinuity structure undergoes as the clearance parameter \( c \) is varied. Some of the ways in which this geometric information gives insight into various aspects of dynamical behaviour are also analysed.

Our approach is based on viewing the dynamics as being generated not by families of curved trajectories meeting the plane \( \Sigma_c \) in \( \mathbb{R}^3 \) but rather by families of straight lines intersecting a corresponding curved surface. This allows us to use methods of singularity theory as applied to the study of apparent outlines (apparent contours). This line of thought is varied. Some of the ways in which this geometric information gives insight into various aspects of dynamical behaviour are also analysed.

Let \( x(c, v; \tau; t) \) denote the solution to (1) with initial data \( (x, \dot{x}) = (c, v) \) when \( t = \tau \). For fixed \( c \in \mathbb{R} \) define the ‘impact surface’ \( V_c \) as

\[
V_c = \{(v, \tau; t) \in \mathbb{R}^3 : x(c, v, \tau; \tau + t) = c \}.
\]

Observe that the plane

\[
\Pi = \{(v, \tau; t) \in \mathbb{R}^3 : t = 0 \}
\]

is automatically part of \( V_c \). The projection

\[
p : \mathbb{R}^3 \rightarrow \mathbb{R}^2 : (v, \tau; t) \mapsto (v, \tau)
\]

plays a major role in this paper, and we identify \( \Pi \) with its image under \( p \), that is the \((v, \tau)\)-plane \( \mathbb{R}^2 \). We write

\[
p_c = p|_{V_c} : V_c \rightarrow \Pi
\]

and in what follows we use known generic properties of maps from a 2-manifold to a plane in order to shed light on the structure of \( p_c \).

The straight line \( p^{-1}(v, \tau) \) intersects \( V_c \) at points whose \( t \)-coordinates are the times \( t \in \mathbb{R} \) at which the trajectory of (1) starting at \( x = c \) with initial velocity \( v \) at time \( \tau \) meets the obstacle at \( x = c \).

There is a natural map \( \Phi : \mathbb{R}^4 \rightarrow \mathbb{R}^4 \) corresponding to ‘re-setting the initial data’, namely

\[
\Phi(c, v; \tau; t) = (x(c, v, \tau; \tau + t), \dot{x}(c, v, \tau; \tau + t), \tau + t; 0).
\]

Then \( \Phi(\{c\} \times V_c) \subset \{c\} \times \mathbb{R}^2 \times \{0\} \) and therefore writing \( \Phi(c, z) = (c, \varphi_c(z); 0) \) for \( z \in V_c \) gives a map

\[
\varphi_c : V_c \rightarrow \mathbb{R}^2
\]

which we call the ‘re-set map’ as it represents re-setting \((\dot{x}, \tau + t)\) to the new initial data for velocity and phase at points of \( V_c \) (where \( x \) is automatically re-set to \( c \)). Thus \( \varphi_c = \Phi_c|_{V_c} \) where
\[ \Phi_c : \mathbb{R}^3 \to \mathbb{R}^2 : (v, \tau; t) \mapsto (\tilde{x}(v, \tau; t), \tau + t). \]

For \((v, \tau) \in \Pi\) let \(F_c(v, \tau)\) be the point of \(p^{-1}(v, \tau) \cap V_c\) with the smallest positive \(t\)-coordinate, provided this exists. We assume that \(p^{-1}(v, \tau) \cap V_c\) is a discrete set: we disregard systems (1) where \(f\) and \(g\) are bizarre enough to give an accumulation point of times \(t\) at which a solution \(x(v, \tau; t)\) takes the value \(c\). Thus \(F_c(v, \tau)\) is defined for \((v, \tau) \in \Pi_c\) where

\[ \Pi_c = p(V_c \setminus \Pi) \subset \Pi \]

and \(F_c\) is a right inverse for the projection \(p_c = p|V_c\). We call \(F_c\) the ‘first-hit map’. In many cases (and certainly in the case of (2) as we see below) \(\Pi_c\) will be almost all of \(\Pi\), as trajectories leaving \(x = c\) can be expected to return. Observe that the map \(G_c = \varphi_c \circ F_c : \Pi_c \to \Pi\) corresponds to the map \(\tilde{G} : \Sigma_c \to \Sigma_c\) previously described. Finally, write

\[ R : \Pi \to \Pi : (v, \tau) \mapsto (\rho(v), \tau) \]

where \(\rho(v) = -rv\) or some other appropriate formula. Using \(V_c\), we can now give the following geometrical characterization of the dynamics of the impact oscillator (1).

- Start at \(x = c\) at time \(\tau\) with initial velocity \(v\).
- Proceed along the \(t\)-axis in the positive direction until the first point of intersection with \(V_c\); that is apply the map \(F_c\).
- Re-set initial data by applying the map \(\varphi_c\).
- Replace the new \(v\) by \(\rho(v)\); that is apply the map \(R\).
- Repeat the process.

In other words our model for the dynamics of the impact oscillator is the (discontinuous) discrete system

\[ I_c = R \circ G_c : \Pi_c \to \Pi. \quad (4) \]

The points of discontinuity in the resulting discrete dynamical system are among the points \((v, \tau)\) for which the straight line \(p^{-1}(v, \tau)\) is tangent to (that is, fails to be transverse to) the impact surface \(V_c\). In visual terms, these are the points such that the line of sight along the \(t\)-direction is somewhere tangent to \(V_c\), that is, they form the ‘apparent outline’ or ‘apparent contour’ \(P_c\) of \(V_c\) viewed in the \(t\)-direction. Technically, they are the ‘singular values’ of the projection \(p_c = p|V_c : V_c \to \Pi\). The tangency points themselves are the ‘singular points’ of \(p_c\) (points where the derivative fails to have rank 2). We call the set \(H_c\) of singular points the ‘horizon’ of \(V_c\) viewed in the \(t\)-direction; thus \(P_c = p(H_c)\). See figure 1 for a schematic illustration.

The points of \(H_c\) correspond to points in \((x, \dot{x}, t)\)-space where the trajectories of (1) ‘graze’ (that is, are tangent to) the plane \(\Sigma_c\), and these points are central to understanding the nonlinear behaviour of an impact oscillator. Many authors such as Foale and Bishop (1992), Chin et al. (1994) and Nordmark (1997) have studied aspects of the dynamical behaviour close to graze points. Our interpretation of these points in terms of singularities of smooth maps sheds new light on some existing results and extends these to situations with more complicated local phase space geometry. We study the structure of the horizon \(H_c \subset V_c\) and the outline \(P_c \subset \Pi\), and describe their role in the dynamics both for a given fixed clearance \(c\) and as \(c\) is allowed to vary. We give some key results for the general system (1), while for the basic linear system (2) we provide a fairly complete analysis.
3. Geometry of the impact surface

We have boldly called \( V_c \) a ‘surface’ but have yet to see whether it deserves the name: is \( V_c \) indeed a 2-manifold? By definition

\[
x_c(c, v, \tau; \tau + t) = x(c, v, \tau; \tau + t)
\]

and so \( V_c \) is a particular level set of the function \( x_c : \mathbb{R}^3 \rightarrow \mathbb{R} \). The Implicit Function Theorem implies that \( V_c \) will indeed be a smooth manifold in a neighbourhood of every point where \( \nabla x_c \) does not vanish, i.e. the regular points of the function \( x_c \). We call these points ‘regular points’ of \( V_c \). As we now see, all points of \( V_c \) where \( v \neq 0 \) are regular points.

Let \( V^*_c \) and \( V^0_c \) denote the subsets of \( V_c \) where \( v \neq 0 \) and \( v = 0 \), respectively.

**Lemma 1.** The set \( V^*_c \) is a smooth 2-manifold in \( \mathbb{R}^3 \).

**Proof.** Consider the flow \( \{ \Psi_t \} \) on \( \mathbb{R}^3 \) generated by the autonomous system of equations (3). Write \( q = (x, y, u) \in \mathbb{R}^3 \). Since \( \Psi_t \) takes trajectories to trajectories its derivative \( D\Psi_t(q_0) \) takes the vector \( \dot{q}_0 \) to \( \dot{q}_t \) (where \( q_t = \Psi_t(q_0) \)); thus if \( q_0 = (c, v, \tau) \) and \( \dot{q}_0 = (v, a, 1) \) we have

\[
D\Psi_t(q_0) = \begin{pmatrix}
v & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
\end{pmatrix} = \begin{pmatrix}
x \frac{\partial x}{\partial v} \frac{\partial x}{\partial \tau} + \dot{x} \\
\dot{x} \frac{\partial \dot{x}}{\partial v} \frac{\partial \dot{x}}{\partial \tau} + \ddot{x} \\
1 & 0 & 1 \\
\end{pmatrix}
\]

where \( x \) stands for \( x(c, v, \tau; \tau + t) \) and \( \partial / \partial \tau \) denotes partial differentiation with respect to the third variable. Since \( D\Psi_t(q_0) \) is invertible it follows that the second matrix is non-singular when \( v \neq 0 \) and so in particular

![Figure 1. Viewing the impact surface \( V_c \).](image-url)
\[
\text{grad } x_c(v, \tau; t) = \left( \frac{\partial x_c}{\partial v}, \frac{\partial x_c}{\partial \tau}, \frac{\partial x_c}{\partial t} \right) = \left( \frac{\partial x}{\partial v} + \dot{x}, \frac{\partial x}{\partial \tau} + \dot{\tau}, \frac{\partial x}{\partial t} + \dot{t} \right)
\]
is not zero. \hfill \Box

There is another consequence of (6) that we shall use later.

**Corollary 1.**

\[(v, a) = (0, 0) \iff \left( \frac{\partial x}{\partial \tau}, \frac{\partial \dot{x}}{\partial \tau} \right) = (0, 0).\]

**Proof.** Compare first and third columns of the matrices in (6).

In the above \(a = \ddot{x}_c(0, \tau; 0)\) which we denote more fully by \(a_c(\tau)\). From (1) we have

\[a_c(\tau) = -f(c, 0) + g(\tau). \tag{7}\]

Next we study the nature of \(V_c\) at points of \(V_c^0\). Denote the \(\tau\)-axis by \(L_0\); then \(L_0 \subset V_c^0\) since \(L_0 \subset \Pi \subset V_c\). Moreover, on \(\Pi\) we have \(\ddot{x}_c(v, \tau; 0) = v\) which vanishes on \(L_0\).

**Proposition 1.** Assume \(r \geq 2\). In a neighbourhood of \(L_0\) the impact surface \(V_c\) consists of the plane \(\Pi\) together with a 2-manifold \(V'_c\) that is the graph of a smooth function \(v = v_c(\tau, t)\) satisfying

\[v_c(\tau, 0) = 0, \quad \dot{v}_c(\tau, 0) = -\frac{1}{2} a_c(\tau).\]

**Proof.** Since \(x_c = c\) on \(\Pi\) we can write

\[x_c(v, \tau; t) = c + t y_c(v, \tau; t)\]
in a neighbourhood of \(\Pi\), where \(y_c = \ddot{x}_c = v\) and \(\dot{y}_c = \frac{1}{2} \dddot{x}_c\) on \(\Pi\) (i.e. \(t = 0\)). Solutions to \(x_c = c\) not on \(\Pi\) are given by \(y_c = 0\). Since \(\partial y_c/\partial v = 1 \neq 0\) we can (by the Implicit Function Theorem) express \(y_c = 0\) close to \(L_0\) as \(v = v_c(\tau, t)\) for a smooth function \(v_c\) with \(v_c(\tau, 0) = -\dot{y}_c(\tau, 0)\), which gives the result. \hfill \Box

**Corollary 2.** The contribution to the outline \(P_c\) from \(V'_c\) intersects \(L_0\) at the points \(\tau\) where \(a_c(\tau) = 0\).

We shall study the behaviour of \(P_c\) close to \(L_0\) in more detail in section 4 below.

The structure of \(V_c^0\) away from \(L_0\) seems harder to pin down. Instead, we make a conjecture on generic behaviour. Let \(M\) be a smooth manifold and \(h_c : M \to \mathbb{R}\) a smooth function varying smoothly with a real parameter \(c\). By a ‘Morse point’ we mean a non-degenerate critical point \(x_0 \in M\) of \(h_c\) for some particular \(c = c_0\), and we say that the level set \(h_c^{-1}(0)\) undergoes a ‘Morse transition’ at \(x_0\) as \(c\) passes through \(c_0\) if \((\partial/\partial c)h_c(x_0) \neq 0\). We apply this to \(h_c = x_c - c : \Pi \times \mathbb{R}^2 \to \mathbb{R}\) where \(V_c = h_c^{-1}(0)\).

**Conjecture 1.** It is a generic property of the pair of functions \(f, g\) in (1) that \(c \in \mathbb{R}\) is a regular value of the function \(x_c : \mathbb{R}^3 \setminus \Pi \to \mathbb{R}\) for all \(c \in \mathbb{R}\) except possibly for a discrete set \(\{c_i\}\). Each \(c_i\) is the value of \(x_{c_i}\) at one or more non-degenerate critical points of \(x_{c_i}\), with \(V_{c_i}\) undergoing one or more Morse transitions as \(c\) passes through \(c_i\).

The conjecture is supported by the correspondence between Morse points of \(V_c\) on \(V_c^0\) and points of quadratic tangency of separate branches of \(P_c\) which we discuss further below, and the fact that the latter occur as generic codimension-1 phenomena.
3.1. The linear system (2)

In this subsection we look at the precise structure of $V_c$ for the specific system (2). We can do this since we have an explicit general solution.

**Proposition 2.** The solution of (2) gives the following formula for $x_c$:

$$x_c(v, \tau; t) = A \cos t + B \sin t + \gamma \cos \omega(\tau + t)$$

where

$$A = c - \gamma \cos \omega \tau, \quad B = v + \omega \gamma \sin \omega \tau$$

and $\gamma = (1 - \omega^2)^{-1}$.

**Proof.** Immediate verification.

**Corollary 3.** The domain of definition $\Pi_c$ for the first hit map $F_c$ in the case of the linear system (2) is given by

$$\Pi_c = \begin{cases} \Pi, & \text{if } \omega \in \mathbb{Q} \text{ or } |c| < \gamma \\ \Pi \setminus \{(0, 2n\pi/\omega) : n \in \mathbb{Z}\}, & \text{if } \omega \notin \mathbb{Q} \text{ and } c > \gamma \\ \Pi \setminus \{(0, (2n + 1)\pi/\omega) : n \in \mathbb{Z}\}, & \text{if } \omega \notin \mathbb{Q} \text{ and } c < -\gamma \end{cases}$$

**Proof.** The only circumstances under which a trajectory $(x_c, \dot{x}_c)$ can leave $x = c$ at $t = 0$ never to return (although it will return arbitrarily closely) are when $\omega/2\pi$ is irrational and the initial point is an extreme point on the $x$-axis, that is $\sin \omega \tau = 0$ and $A$ has the same sign as $\cos \omega \tau$. These conditions characterize $\Pi_c$ as stated.

For (2) there are important symmetries in the system that manifest themselves as symmetries of $V_c$.

**Corollary 4.** For (2) the impact surface $V_c$ has the following symmetries:

(i) The surface $V_c$ is invariant under translation $\tau \mapsto \tau + 2\pi/\omega$. If $\omega = p/q \in \mathbb{Q}$ where $p, q \in \mathbb{N}$ then $V_c$ is also invariant under translation $t \mapsto t + 2q\pi$.

(ii) The surface $V_{-c}$ is obtained from $V_c$ by translating $\tau \mapsto \tau + \pi/\omega$ and changing the sign of $v$.

**Proof.** Clear from the expression (8), because (i) $2q\pi$ is an integer multiple of both $2\pi$ and $2\pi/\omega$, and (ii) $x_{-c}(-v, \tau + \pi/\omega; t) = -x_c(v, \tau; t)$.

**Corollary 5.** If $\omega = p/q$ where $p, q \in \mathbb{N}$ then the line segment in $\mathbb{R}^3$ parallel to the $t$-axis from $(v, \tau; 0)$ to $(v, \tau; 2\pi q)$ intersects $V_c$ an odd number of times between these points.

**Proof.** Any periodic trajectory meets $x = c$ an even number of times.

For (2) we find from Proposition 2 that
\[
\begin{align*}
\frac{\partial x_c}{\partial v} &= \sin t \\
\frac{\partial x_c}{\partial \tau} - \frac{\partial x_c}{\partial t} &= a_c(\tau) \sin t + v \cos t
\end{align*}
\] (10)

where
\[
a_c(\tau) = \cos \omega \tau - c = \ddot{x}_c(v, \tau; 0).
\] (11)

Therefore, \( \text{grad } x_c = 0 \) precisely where
\[
\sin t = a_c(\tau) \sin t + v \cos t = \dot{x}_c(v, \tau; t) = 0,
\]
that is, \( v = 0 \) (of course), \( t = k\pi, k \in \mathbb{Z} \) and \( \dot{x}_c(0, \tau; k\pi) = 0 \). Investigating the geometry of \( V_c \) in the neighbourhood of these singular points will occupy the remainder of this section.

For \( k \in \mathbb{Z} \) let \( L_k \) denote the line \( t = k\pi \) in the \((\tau, t)\)-plane, that is,
\[
L_k = \{(v, \tau; t) \in \mathbb{R}^3 : v = 0, t = k\pi\}.
\]

At points of \( L_k \) we have
\[
\begin{align*}
x_c(0, \tau; k\pi) &= (c - \gamma \cos \omega \tau)(-1)^k + \gamma \cos \omega(\tau + k\pi) \\
\dot{x}_c(0, \tau; k\pi) &= \alpha \sin \omega \tau (-1)^k - \alpha \sin \omega(\tau + k\pi)
\end{align*}
\] (12a) (12b)

where \( \alpha = \omega \gamma \), so the points \( (0, \tau; k\pi) \in V_c \cap L_k \) are points where the right-hand side of (12a) is equal to \( c \), and those where \( V_c \) may be singular are those where also the right-hand side of (12b) vanishes.

Recall from Proposition 1 that when \( k = 0 \) (that is \( t = 0 \)) we have \( L_0 \subset V_c \) and \( V_c \) fails to be a 2-manifold at all points of the line \( L_0 \) : there it is the transverse intersection of two 2-manifolds, namely the plane \( \Pi \) and the manifold \( V'_c \) that is a graph \( v = v_c(\tau, t) \). Therefore, we turn to cases where \( k \neq 0 \).

**Case 1.** \( k \) even.

With \((x_c, \dot{x}_c) = (c, 0)\) equations (12) become
\[
\begin{align*}
\cos \omega \tau &= \cos(\omega \tau + k\omega \pi) \\
\sin \omega \tau &= \sin(\omega \tau + k\omega \pi)
\end{align*}
\] (13a) (13b)

respectively, which imply \( k\omega \in 2\mathbb{Z} \). If \( \omega \notin \mathbb{Q} \) this can never be satisfied, while if \( \omega = p/q \in \mathbb{Q} \) where \( p, q \in \mathbb{N} \) then (13a), (13b) are satisfied simultaneously for all \( \tau \) when \( k \) is an integer multiple of \( 2q \). In view of Corollary 4 we summarize this as follows.

**Proposition 3.** There are no points on \( L_k \) for even \( k \neq 0 \) at which \( V_c \) fails to be regular, apart from (when \( \omega \in \mathbb{Q} \)) the images of \( L_0 \) under the translations in the \( t \)-direction which leave \( V_c \) invariant.

**Case 2.** \( k \) odd.

The equations to solve now are
\[
\cos \omega \tau + \cos(\omega \tau + k\omega \pi) = 2c/\alpha
\]  
(14a)
\[
\sin \omega \tau + \sin(\omega \tau + k\omega \pi) = 0.
\]  
(14b)

From (14b) we have one or other of the equalities

\[
\omega \tau + k\omega \pi = -\omega \tau + 2l\pi
\]  
(15a)
\[
\omega \tau + k\omega \pi = \omega \tau + (2l + 1)\pi
\]  
(15b)

for \(l \in \mathbb{Z}\). Now (15a) gives

\[
\omega \tau = l\pi - k\omega \pi/2
\]  
(16)

and then from (14a)

\[
\cos \omega \tau = c\omega/\alpha = c/\gamma.
\]  
(17)

On the other hand, (15b) gives

\[
k\omega \pi = (2l + 1)\pi
\]  
(15b)

and then from (14a)

\[c = 0.\]

Thus (14a) and (15b) can have simultaneous solutions \(\tau\) only when \(\omega\) is the ratio of two odd integers, and then when \(c = 0\). However, (14a) and (15a) will have solutions \(\tau\) for every \(\omega\) for certain choices of \(c\).

Therefore, we have two possible settings for solutions to (14) as follows:

**I:** \(k \in 2\mathbb{Z} + 1, \quad k\omega \notin 2\mathbb{Z} + 1;\)

**II:** \(k \in 2\mathbb{Z} + 1, \quad k\omega \in 2\mathbb{Z} + 1, \quad c = 0.\)

We consider these each in turn.

**Type I:** \(k \in 2\mathbb{Z} + 1, \quad k\omega \notin 2\mathbb{Z} + 1\)

From (15a), (17) we have \(\omega \tau = l\pi - k\omega(\pi/2)\) and \(c = c_{k,l} = \gamma \xi\) where \(\xi = \cos \omega \tau\).

We then find

\[
x_c(v, \tau; \sigma; k\pi + u) = -vu - \frac{1}{2} c\omega^2(2\sigma^2 + 4\sigma u + u^2) + O(3).
\]

The quadratic terms are non-degenerate in \((v, \sigma; u)\) so the Morse Lemma implies that (up to a local diffeomorphism which is the identity map to first order) the structure of \(V_c\) is locally that of a cone tangent to the \(v\)-axis. Notice moreover that for all \(\tau \in \mathbb{R}\) we have

\[
x_c(v, \tau; \sigma; k\pi) = \gamma(\cos \omega \tau + \cos \omega(\tau + k\pi)) - c
\]

which does not involve \(v\). Since \(|\cos \omega \tau + \cos \omega(\tau + k\pi)|\) has a local maximum value of \(2|\xi|\) at \(\tau = \tau_s\) it follows that the impact surface \(V_c\) contains a pair of lines intersecting \(L_k\) and parallel to the \(v\)-axis that are created as \(c\) passes through the value \(c_{k,l}\) from above if \(c_{k,l} > 0\) or from below if \(c_{k,l} < 0\). When \(c = c_{k,l}\) the line \(\tau = \tau_s, \ t = k\pi\) is a line of tangency of \(V_c\) with the plane \(t = k\pi\), so the local cone structure is not only tangent to the \(v\)-axis but contains it entirely. As \(c\) passes through \(c_{k,l}\) the structure of \(V_c\) undergoes a Morse transition at \((0, \tau_s; k\pi)\) between (locally) a hyperboloid of one sheet and a hyperboloid of two sheets. See figure 2.
**Type II:** $k \in 2\mathbb{Z} + 1$, $k\omega \in 2\mathbb{Z} + 1$, $c = 0$

Here we find

$$x_c(v, \tau; k\pi) = -c$$
$$\dot{x}_c(v, \tau; k\pi) = -v$$
$$\ddot{x}_c(v, \tau; k\pi) = c - \cos \omega \tau$$
$$\dddot{x}_c(v, \tau; k\pi) = v + \omega \sin \omega \tau$$

and so for all $(v, \tau) \in \mathbb{R}^2$ and $c \in \mathbb{R}$ we have

$$x_c(v, \tau; k\pi + u) = -c \cos u - v \sin u + u \left( -\frac{u}{2} \cos \omega \tau + \frac{u^2}{6} (\omega \sin \omega \tau) + O(u^3) \right).$$

Therefore in a neighbourhood of the plane $t = k\pi$ we may write $V_c = h^{-1}(c)$ where $h : \mathbb{R}^3 \to \mathbb{R}$ is given by

$$h(v, \tau; u) = (1 + \cos u)^{-1} \left( -v \sin u + u \left( -\frac{u}{2} \cos \omega \tau + \frac{u^2}{6} \omega \sin \omega \tau + O(u^3) \right) \right).$$

Thus $V_0$ consists of the plane $t = k\pi$ (that is $u = 0$) itself together with a 2-manifold $V_0''$ intersecting this plane along the line $L_k$ (that is $u = v = 0$) and locally the graph of an analytic function $v = v_0(\tau, u)$ given by

$$\psi(u)v_0(\tau, u) = -\frac{u}{2} \cos \omega \tau + \frac{u^2}{6} (\omega \sin \omega \tau) + O(u^3)$$

where again $\psi(u) = u^{-1} \sin u$ ($u \neq 0$), $\psi(0) = 1$. Clearly for $\omega \tau \neq \pm(\pi/2)$ (mod $2\pi$) we see $(\partial v_0/\partial u)(\tau, 0) \neq 0$ and the projection $p|V_0''$ is regular at $(\tau, 0)$. When $\omega \tau = \pm(\pi/2)$ (mod $2\pi$) we write $\tau = \pm(\pi/2\omega) + \sigma$ (mod $2\pi/\omega$) and find

$$\psi(u)v_0 \left( \pm \frac{2\pi}{\omega} + \sigma, u \right) = \pm \omega \left( \frac{1}{2} \sigma u + \frac{u^2}{6} + O(u^3) \right)$$

so $v_0$ has a saddle point at $(\sigma, u) = (0, 0)$, that is $(\tau, t) = (\pm\pi/2\omega, k\pi) \in L_k$. The singular points of $p|V_0''$ are given by $\partial v_0/\partial u = 0$, that is, (to first order) $3\sigma + 2u = 0$, giving the apparent outline $P_0$ locally as $v_0 = \mp 3/8\omega \sigma^2 + O(\sigma^3)$ and showing that $P_0$ locally is a curve with quadratic tangency to $v = 0$ from the $v < 0$ side at $\tau = \pi/2\omega$ and from the $v > 0$ side at $\tau = -\pi/2\omega$. 

![Figure 2. Morse transition for (2) at $c = c_k$; note that the vertical line lies in the surface $V_c$.](image_url)
As \( c \) varies away from \( c = 0 \) observe that \( V_c \) no longer contains the plane \( t = k\pi \); contrast this with \( k = 0 \) where the plane II remains automatically part of \( V_c \). See figure 3.

Closer analysis shows that \( P_c \) unfolds into the disjoint union (locally) of a regular curve and a cusped curve as \( c \) moves away from zero in either sense: see figure 4.

An analogous description applies in the neighbourhood of \( \tau = -\pi/2\omega \), related to the above by the symmetry described in Corollary 4.

The full set of local transitions of \( V_c \) that occur as \( c \) varies can therefore be summarized as follows:

**Theorem 1.** There are no transitions for \(|c| > |\gamma|\).

(i) As \( c \) descends through positive values \( c_{k,l} = \gamma \cos(k\omega(\pi/2) - l\pi) \), where \( k \in 2\mathbb{Z} + 1 \) and \( k\omega \notin 2\mathbb{Z} + 1 \) and \( l \in \mathbb{Z} \), there are simultaneous Morse (Type I) transitions from two sheets to one sheet at corresponding points \((v, \tau; t) = (0, -k(\pi/2) + l(\pi/\omega); k\pi)\).

(ii) If \( \omega = p/q \) where \( p, q \in 2\mathbb{N} + 1 \) then as \( c \) descends through the value 0 there is a Type II transition at \((v, \tau; t) = (0, \pi/2\omega; q\pi) \mod t = 2q\pi\).
(iii) As $c$ descends through negative values $c_{k,l} = \gamma \cos(k\omega(\pi/2) - l\pi)$ as in (ii) there are simultaneous Morse (Type I) transitions from one sheet to two sheets at corresponding points $(v, \tau; t) = (0, -k(\pi/2) + l(\pi/\omega); k\pi)$.

4. Geometry of the projection $p_c = p|V_c$

In section 2 we analyzed the local geometry of the impact surface $V_c$ for the general system (1), in particular at points where it fails to be a smooth 2-manifold, and in Theorem 1 we classified the local changes in topology for the system (2) as $c$ passes through certain special values.

We now turn to look closely at the set of singular points and the set of singular values of the projection $p_c = p|V_c : V_c \to \Pi$, at points where $v$ may or may not be zero. We give some results for the general system (1), and a more specific analysis for (2).

First, some facts about apparent outlines in general. See for example Bruce (1984a, b).

Theorem 2. Typically the only singularities exhibited by the projection into a plane of a smooth 2-manifold in $\mathbb{R}^3$ are curves of ‘folds’ with isolated ‘cusps’. See figure 5.

Typically the only transitions that occur in the projection of a 1-parameter family of surfaces are ‘lips’, ‘beaks’ and ‘swallowtails’. See figure 6.

The word ‘typically’ can be expressed precisely in terms of openness and density of relevant properties in certain function spaces: we omit a more formal statement. The interpretation is that these phenomena are structurally stable (robust) under sufficiently small perturbations, and that any other singular phenomena that may arise can be perturbed into collections of these alone.

If the manifold is compact and the above conditions prevail then the singular set will be compact and will in particular contain only finitely many cusps. In our case
the manifold \( V_c \) is not compact. However, for the system (2) with \( \omega \in \mathbb{Q} \) it is periodic in \( t \) and hence compact modulo this period. The apparent outline \( P_c \) is thus compact. For (1) in general and for (2) with \( \omega \notin \mathbb{Q} \) the apparent outline will not be compact and may contain infinitely many cusps.

To obtain more than the most basic results for (1) we need to make some generic assumptions about the functions \( f \) and \( g \).

**Definition 1.** The pair of functions \( f, g \) in (1) is ‘generic’ if the following holds: \( f(c,0) \) is a regular value of the function \( g \) except possibly for a finitely many values \( c_i \) of \( c \), these being regular points of \( f(\cdot,0) \). Moreover, all critical points of \( g \) with value \( c_i \) are non-degenerate. Thus

\[
g(t) = f(c,0), \quad c \neq c_i \quad \Rightarrow \quad g'(t) \neq 0 \tag{18}
\]

and

\[
g(t) = f(c_i,0), \quad g'(t) = 0 \quad \Rightarrow \quad g''(t) \neq 0, \quad \frac{\partial f}{\partial c}(c_i,0) \neq 0. \tag{19}
\]

If the pair \( f, g \) is generic then as \( c \) varies the set of solutions \( t \) to \( g(t) = f(c,0) \) undergoes no worse than quadratic (Morse) transitions.
Recall that for (1) the set \( V^*_c \) is a smooth 2-manifold. We begin by studying singularities of \( p_c|V^*_c \).

4.1. Behaviour of \( p_c \) at points of \( V^*_c \)

Let \( H^*_c = H_c \cap V^*_c \) be the set of singular points of the map \( p_c|V^*_c \), thus
\[
H^*_c = \{ (v, \tau; t) \in \Pi \times \mathbb{R} : v \neq 0, x_c(v, \tau; t) - c = \dot{x}_c(v, \tau; t) = 0 \}.
\]

**Theorem 3.**

(1) For the system (1) the horizon \( H^*_c \) is a smooth 1-manifold.

(2) For the system (1) with the pair \( f, g \) generic, for \( c \neq c_i \) there is a fold singularity at every point of \( H^*_c \) except possibly for isolated cusp singularities. There are swallowtail transitions as \( c \) passes through \( c_i \).

**Proof.** Let \( z = (v, \tau; t) \in \mathbb{R}^3 \) and write
\[
X_c(z) = (x_c(z), \dot{x}_c(z)).
\]
Then \( H_c = X_c^{-1}(c, 0) \), and
\[
DX_c(z) = \begin{pmatrix}
\frac{\partial x_c}{\partial v} & \frac{\partial x_c}{\partial \tau} & \frac{\partial x_c}{\partial t} \\
\frac{\partial \dot{x}_c}{\partial v} & \frac{\partial \dot{x}_c}{\partial \tau} & \frac{\partial \dot{x}_c}{\partial t}
\end{pmatrix}
\]
(20)

\[
= \begin{pmatrix}
\frac{\partial x}{\partial v} & \frac{\partial x}{\partial \tau} + \dot{x} & \ddot{x} \\
\frac{\partial \dot{x}}{\partial v} & \frac{\partial \dot{x}}{\partial \tau} + \ddot{x} & \dddot{x}
\end{pmatrix}
\]
(21)

where here \( \dot{x} = 0 \). From the proof of Lemma 1 we see that this matrix has rank 2 (regardless of \( \dot{x} \)) provided \( v \neq 0 \). Hence \( H^*_c \) is a smooth 1-manifold by the Implicit Function Theorem.

The projection \( p^*_c \) has a fold singularity at \( z \in H^*_c \) precisely when \( \dot{x}_c(z) \neq 0 \) and a cusp point at \( z \) when \( \ddot{x}_c(z) = 0 \) but \( \dddot{x}_c(z) \neq 0 \). From (1) we have
\[
\dddot{x}_c(z) = 0 \iff f(c, 0) = g(t)
\]
(22)
and differentiating (1) with respect to \( t \) at \( z \) we find that if \( \dddot{x}_c(z) = 0 \) then
\[
\dddot{x}_c = \dddot{g}(t).
\]
(23)

Thus the genericity of \( f, g \) guarantees finitely many cusp points (modulo \( t = T \)) for \( c \neq c_i \), and also guarantees \( x^{(4)}_c \neq 0 \) when \( \dot{x}_c = \ddot{x}_c = \dddot{x}_c = 0 \) which can occur when \( c = c_i \). The fact that swallowtail transitions occur comes from the Morse transitions in solutions of \( g(t) = f(c, 0) \) as \( c \) passes through \( c_i \); we omit the details which can be found in Chillingworth (2003). \( \square \)

A fold point \( (v, \tau; t) \in V^*_c \) for \( p_c \) corresponds to a non-degenerate ‘graze’ (quadratic contact) in the trajectory for (1) with initial data \( (v, \tau) \); a cusp point corresponds to a ‘degenerate graze’ with cubic contact. A swallowtail transition is the coalescence of two (cubic) degenerate grazes into one with quartic contact. Our expression of these
phenomena in terms of singularity theory is a way of describing how they respond to (are ‘unfolded by’) the initial data \(v\) and \(\tau\).

Next we look at the behaviour of \(p_c\) at points where \(V_c\) intersects the plane \(v = 0\). From Theorem 3 we know that it is only here that \(H_c\) can fail to be a smooth 1-manifold. We first consider points of \(L_0\).

### 4.2. Behaviour of \(p_c\) at points of \(L_0\)

Recall from (7) that along \(L_0\) the impact surface \(V_c\) consists of \(\Pi\) intersected transversally along \(L_0\) by a sheet \(V'_c\) of \(V_c\) that is the graph of a smooth function \(v = v_c(\tau, t)\). Thus for small \(|t|\) the horizon \(H_c\) consists of \(L_0\) together with points of \(V'_c\) where \(\dot{v}_c = 0\). By Corollary (2) these occur on \(L_0\) where \(a_c(\tau) = 0\).

**Proposition 4.** Let \(g(\tau_0) = f(c_0, 0)\) so \(a_{c_0}(\tau_0) = 0\), and suppose \(a_{c_0}'(\tau_0) \neq 0\), that is, \(g(\tau_0) = k \neq 0\). Then \(v_{c_0}\) has a non-degenerate saddle point at \((\tau_0, 0)\). The horizon \(H_c\) is parametrized locally as

\[
(v, \tau; t) = \left(\frac{\tau}{3} k \sigma^2 + O(\sigma^4), \tau_0 + \sigma; -\frac{\sigma}{2} + O(\sigma^2)\right)
\]  
(24)

so the outline \(P_c\) has quadratic contact with \(L_0\) at \(\tau_0\) from the side \(v > 0\) or \(v < 0\) accordingly as \(g(\tau_0)\) is negative or positive, respectively.

**Proof.** Since \(v_c(\tau, 0) = 0\) we have \((\partial^2 / \partial \tau^2)v_c(\tau, 0) = 0\). Implicit differentiation of

\[
y_c(v_c(\tau, t), \tau; t) = 0
\]  
(25)

and use of (1) gives

\[
\dot{v}_c(\tau, 0) = -\dot{y}_c(0, \tau; 0) = -\frac{1}{2} a_c(\tau),
\]

\[
\ddot{v}_c(\tau, 0) = -\ddot{y}_c(0, \tau; 0) = \frac{1}{4} \chi_c(0, \tau; 0) - \frac{\partial \ddot{x}_c}{\partial v}(0, \tau; 0) \ddot{v}_c(\tau, 0)
\]

\[
= -\frac{2}{3} \ddot{f}_c(c_0, 0) + \frac{\partial \ddot{x}_c}{\partial v}(0, \tau; 0) \ddot{v}_c(\tau, 0) - \frac{1}{3} \dot{g}(\tau)
\]

from (1), so if \(g(\tau_0) = f(c_0, 0)\) then \(a_{c_0}(\tau_0) = 0\) and \(\ddot{v}_{c_0}(\tau_0, 0) = -\frac{1}{2} k\). Also

\[
\frac{\partial \ddot{v}_{c_0}}{\partial \tau}(\tau_0, 0) = -\frac{1}{2} a_{c_0}'(\tau_0) = -\frac{1}{2} k
\]

and so the Hessian matrix for \(v_{c_0}\) at \((\tau_0, 0)\) is

\[
\frac{k}{6} \left(\begin{array}{cc}
0 & 3 \\
3 & 2
\end{array}\right).
\]

Thus if \(\tau = \tau_0 + \sigma\) we have

\[
v_c(\tau_0 + \sigma, t) = -\frac{k}{6} (3 \sigma t + \sigma^2) + O(3)
\]  
(26)

so \(V'_c\) cuts the plane \(v = 0\) in a curve through \((\tau_0, 0)\) with tangent direction \((1, -3)\). The horizon \(H_{c_0}\) given by \(\dot{v}_c = 0\) satisfies \(t = -\frac{\sigma}{2} + O(\sigma^2)\) which yields (24). □

If \(g(\tau_0) = f(c_0, 0)\) and \(g(\tau, 0) = 0\) then the 2-jet of \(v_c\) at \((\tau_0, 0)\) vanishes. If the critical point \(\tau_0\) of \(g\) is non-degenerate and \((\partial f / \partial c)(c_0, 0) \neq 0\) we expect the coalescence or creation of two saddle points as in Proposition 4. This cannot happen
Proposition 5. Let \((\tau_0, c_0)\) satisfy \(g(\tau_0) = f(c_0, 0)\). Suppose
\[
\dot{g}(\tau_0) = 0, \quad \ddot{g}(\tau_0) = m \neq 0, \quad \frac{\partial f}{\partial x}(c_0, 0) \neq 0
\]
so that \(f(c, 0)\) passes with non-zero speed through the non-degenerate critical value \(g(\tau_0)\) of \(g\) as \(c\) passes through \(c_0\). Then \(p_c\mid V'_c : V'_c \to \Pi\) undergoes a lips creation or annihilation as \(c\) passes through \(c_0\).

Proof. Further implicit differentiation of (25) yields
\[
v_c^{(3)}(\tau, 0) = -v_c^{(3)}(0, \tau; 0) = -\frac{1}{4}v_c^{(4)}(0, \tau; 0)
\]
\[
= -\frac{1}{4}\dot{g}(\tau) = -\frac{1}{4}m \neq 0
\]
at \((c, \tau) = (c_0, \tau_0)\), so the 3-jet of \(v_c(\tau_0, \cdot)\) does not vanish. We now invoke techniques from singularity theory. After reparametrizing the \(t\)-axis (depending on \((c, \tau)\)) by \(t = \psi(c, \tau; u)\) we may assume \(v_c^{(3)}(\tau_0, u) = u^3\) and moreover (since a versal unfolding of \(u^3\) has the form \(\alpha + \beta u + u^3\) and we have \(v_c(\tau, 0) = 0\)) that
\[
v_c(\tau, u) = \beta(c, \tau)u + u^3
\]
for \((c, \tau)\) near \((c_0, \tau_0)\) and small \(u\), where \(\beta\) is a \(C^\infty\) function with \(\beta(c_0, \tau_0) = 0\). We find \(\beta(c, \tau) = (\partial v_c / \partial u)(\tau, 0) = -\frac{1}{2}a_c(\tau)\). Now consider
\[
h(v, c, \tau; u) = -v + \beta(c, \tau)u + u^3.
\]
The condition for a lips transition in the outline of the graph \(v = v_c(\tau, t)\) in the \((v, \tau)\)-plane as \(c\) passes through \(c_0\) is (see Arnold 1986, Corollary 4.5) that the function \(c\) has a non-degenerate critical point when restricted to the ‘spine’ of the discriminant of \(h\), that is the set \(\Psi^{-1}(0, 0)\) where \(\Psi : (v, c, \tau) \mapsto (v, \beta(c, \tau))\). This condition is thus that the curve \(\beta(c, \tau) = 0\) has quadratic tangency with the \(\tau\)-axis; this is precisely what the conditions (27) guarantee.

The geometry of the lips here has a particular form more complicated than the ‘standard’ picture from Arnold (1976), Bruce (1984b) and Bruce and Giblin (1985), for example. Inspection of the graph of \(v_c(\tau, \cdot)\) for \((c, \tau)\) near \((c_0, \tau_0)\) shows that the cusp points of the lips occur with opposite signs of \(v\), and the lips themselves have two points of quadratic tangency with \(v = 0\) from opposite sides as in Proposition 4. See figure 7. Compare Bruce (1989) where the geometry of outlines rather than just

Figure 7. The swan configuration.
their differential topology is explored in some detail. To reflect the slender elegance of this form of the lips we call this a ‘swan’ configuration.

4.3. Behaviour of $p_c$ at other points of $V^0_c$

Assuming Conjecture (1) the only points away from $L_0$ that are not regular points of $V^0_c$ are the Morse points for particular values of $c$. This still leaves room for points of $V^0_c$ that are regular points of $V_c$ yet such that $p_c$ has a singularity more degenerate than a fold. We conjecture that for most systems (1) this does not occur. However, it turns out that for (2) this does happen as a result of the special symmetries in (2).

In the terminology of Theorem 3, at a singularity $z$ of $p_c$ on $V^0_c$ we have

$$DX_c(z) = \begin{pmatrix} \sin t & -a \sin t & 0 \\ \cos t & -a \cos t & x_c \end{pmatrix}$$

and so for the system (2) the point $z$ can fail to be a fold for $p_c$ only if

$$\sin t = 0,$$

that is, $z \in L_k$ (28)

or

$$\ddot{x}_c(z) = 0 \quad \text{with} \quad z \notin L_k.$$ (29)

The first case (28) has already been studied in our earlier analysis of $V_c$ itself at points of $L_k$, therefore it remains to study $H_c$ at the points $z$ which satisfy (29), corresponding to degenerate graze points. It turns out that the geometry and symmetry of the solutions to (2) imply that if the trajectory of (3) for (2) has a degenerate graze then any other graze in the same trajectory must also be degenerate: this can be seen in Whiston (1992) as a special case of Proposition A1. The consequence is that such $z = (0, \tau; t)$ must satisfy $a_c(\tau) = 0$ and the local geometry of $p_c$ at $z$ and of $P_c$ at $p_c(z)$ become particularly degenerate. This corresponds to the fact that a cusp of $P^*_c$ cannot cross a fold curve of $P^*_c$ but can meet $P^*_c$ only at another cusp, in contrast to the typical behaviour of apparent outlines. See the Correspondence Principle below.

5. Geometry of the re-set map $\varphi_c = \Phi_c|V_c$

The singularity structure of $\varphi_c$ naturally has close analogies with that of $p_c$.

**Proposition 6.** The map $\varphi_c : V_c \to \Pi$ is a local diffeomorphism at all points of $V^*_c$ and is singular at all (regular) points of $V^0_c$. If $z = (0, \tau; t) \in V^0_c$ and $a_c(\tau) \neq 0$ then $\varphi_c$ has a fold singularity at $z$.

**Proof.** At $z = (v, \tau; t) \in \Pi \times \mathbb{R}$ we have

$$D\Phi_c(z) = \begin{pmatrix} \frac{\partial \dot{x}_c}{\partial v} & \frac{\partial \dot{x}_c}{\partial \tau} & \frac{\partial \dot{x}_c}{\partial t} \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial \dot{x}}{\partial v} & \frac{\partial \dot{x}}{\partial \tau} + \dot{x} & \ddot{x} \\ 0 & 1 & 1 \end{pmatrix}$$ (30)

where $x = x(c, v, \tau; t)$, and the matrix has kernel dimension 1 or 2 according to whether $D\Phi_c(z)$ has rank 2 or 1, respectively. We consider the two cases separately.

(a) rank $D\Phi(z) = 2$.

Here $(\partial \dot{x}/\partial v, \partial \dot{x}/\partial \tau) \neq (0, 0)$ and ker $D\Phi_c(z)$ is spanned by the non-zero vector
\[ u = \left( \frac{\partial \hat{x}}{\partial \tau}, \frac{\partial \hat{x}}{\partial v}, \frac{\partial \hat{x}}{\partial \tau} \right) \]

so \( \varphi_c = \Phi_c | V_c \) is singular precisely when \( u \) is orthogonal to \( \operatorname{grad} x_c(z) \), that is

\[ \frac{\partial x}{\partial \tau} \frac{\partial x_c}{\partial v} - \frac{\partial x}{\partial \tau} \frac{\partial x_c}{\partial \tau} + \frac{\partial x}{\partial \tau} \frac{\partial x_c}{\partial \tau} = 0 \]

which reduces to \( \det W = 0 \) where

\[ W = W(z) = \begin{pmatrix} \frac{\partial x}{\partial v} & \frac{\partial x}{\partial \tau} \\ \frac{\partial x}{\partial v} & \frac{\partial x}{\partial \tau} \\ \frac{\partial x}{\partial \tau} & \frac{\partial x}{\partial \tau} \end{pmatrix}, \quad (31) \]

evaluated at \( z \). From (6) we see this determinant vanishes if and only if the matrix on the left-hand side of (6) is also singular, that is \( v = 0 \). We have \( \dim \ker D\varphi_c(z) = \dim \ker \Phi_c(z) = 1 \), and the conditions for a fold are that \( V_c \) be transverse to the plane \( \{ v = 0 \} \) and that \( \ker D\varphi_c(z) \) not lie in that plane. Now transversality fails only if \( \frac{\partial x_c}{\partial \tau} = \hat{x}_c = 0 \), i.e. \( \frac{\partial x}{\partial \tau} = \hat{x} = 0 \). At a regular point of \( V_c \), we then must have \( \frac{\partial x}{\partial \tau} \neq 0 \), so \( \det W \) vanishes only if \( \frac{\partial \hat{x}}{\partial \tau} = 0 \); by Corollary (1) this happens only if \( a_c(\tau) = 0 \). Likewise \( \ker D\varphi_c(z) \) lies in \( \{ v = 0 \} \) only if \( \frac{\partial \hat{x}}{\partial \tau} = a_c(\tau) = 0 \).

(b) rank \( D\Phi(z) = 1 \).

Here \( (\partial x/\partial v, \partial x/\partial \tau) = (0,0) \) and \( \ker D\Phi_c(z) \) is spanned by \( \{(1,0,0), (0,1,-1)\} \). Clearly \( \det W = 0 \) so \( v = 0 \). Now \( \dim \ker D\varphi_c(z) = 1 \) if and only if \( \ker D\Phi_c(z) \) is the tangent space to \( V_c \) at \( z \), that is \( (1,0,0) \) and \( (0,0,1) \) are both orthogonal to \( \operatorname{grad} x_c \), that is

\[ \frac{\partial x_c}{\partial v} = 0 = \frac{\partial x_c}{\partial \tau} - \frac{\partial x_c}{\partial t} \]

so \( (\partial x/\partial v, \partial x/\partial \tau) = (0,0) \). This cannot occur (in particular, \( \partial x/\partial v \neq 0 \)) as the second columns of both matrices in (6) are non-zero. Thus \( \ker D\varphi_c(z) = 1 \) has dimension 1 and is spanned by \( \hat{u} = (\partial x/\partial \tau, -\partial x/\partial v, \partial x/\partial v) \). It fails to be a fold only if \( \partial x/\partial \tau = 0 \) which (since \( \partial \hat{x}/\partial \tau = 0 \)) occurs only if \( a_c(\tau) = 0 \). \( \square \)

**Remark 1.** At singular points \( z \) of \( \varphi_c \) the vectors \( \{u, \hat{u}\} \) are linearly dependent since \( \det W(z) = 0 \); either vector (if non-zero) can be taken as a basis for \( \ker D\varphi_c(z) \).

Next we identify the cusp singularities of \( \varphi_c \).

**Proposition 7.** At regular points \( z = (v, \tau, t) \) of \( V_c \setminus H \) where \( v = 0 \), \( a_c(\tau) = 0 \) and \( a_c(\tau) \neq 0 \) the map \( \varphi_c : V_c \to \Pi \) has cusp singularities.

**Proof.** From the proof of Proposition 6 we know that \( \varphi_c \) has a singularity at \( z \) that is not a fold since \( (v, a_c(\tau)) = (0,0) \) implies \( (\partial x/\partial \tau, \partial \hat{x}/\partial \tau) = (0,0) \).

The condition for a cusp is that the tangency of \( \ker D\varphi_c(z) \) with \( \{ v = 0 \} \) be as non-degenerate as possible, meaning here that the \( v \)-component of \( \ker D\varphi_c(z) \) have non-zero derivative along \( V^0_c \) at \( z \). As \( \operatorname{grad} x_c(z) = (\partial x/\partial v, \hat{x}, \hat{x}) \) with \( \hat{x} \neq 0 \) the tangent space to \( V^0_c \) at \( z \) is spanned by \( (0,1,-1) \), so the cusp condition is \( \partial m/\partial \tau - \hat{m} \neq 0 \) where \( m \) denotes \( (\partial x/\partial \tau)(c, 0; \tau; \tau + t) \) in case (a) or \( (\partial x/\partial \tau)(c, 0; \tau; \tau + t) \) in case (b). These reduce to \( \partial^2 \hat{x}/\partial \tau^2 \neq 0 \) (case (a)) or \( \partial^2 \hat{x}/\partial \tau^2 \neq 0 \) (case (b)). Now differentiating (6) with respect to \( \tau \) at \( z \) yields
$D_{\psi_1}(q) = -\begin{pmatrix} 0 & \partial^2 x / \partial \tau^2 + \partial^2 x / \partial \tau \partial v \\ a'_c(\tau) & \partial^2 x / \partial \tau^2 + \partial^2 x / \partial \tau \partial v \end{pmatrix}.$

We have $\partial \dot{x} / \partial \tau = 0$ as $a_c(\tau) = 0$, and also $\partial \dot{x} / \partial \tau = 0$ as follows from differentiating (1) since $(\partial x / \partial \tau, \partial \dot{x} / \partial \tau) = (0, 0)$; thus $a'_c(\tau) \neq 0$ implies

$$\left( \frac{\partial^2 x}{\partial \tau^2}, \frac{\partial^2 x}{\partial \tau \partial v} \right) \neq (0, 0).$$

Also, differentiating $\det W = 0$ along $V^0_c$ gives

$$\det \left( \begin{array}{ccc} \partial^2 \dot{x} / \partial \tau^2 & \partial \dot{x} / \partial v \\ \partial^2 x / \partial \tau^2 & \partial x / \partial v \end{array} \right) = 0.$$

In case (a) we have $\partial \dot{x} / \partial v \neq 0$, so if the cusp condition fails so that $\partial^2 \dot{x} / \partial \tau^2 = 0$ then $\partial^2 x / \partial \tau^2 = 0$, contradicting (32). In case (b) we have $\partial \dot{x} / \partial v = 0$ and $\partial x / \partial v \neq 0$ so $\partial^2 \dot{x} / \partial \tau^2 = 0$ which by (32) implies $\partial^2 x / \partial \tau^2 \neq 0$.

Finally, we can identify swallowtail transitions, using the generic assumptions (18), (19) on $f$ and $g$. We omit the proof; see Chillingworth (2003).

**Proposition 8.** Assume the pair $f, g$ is generic. Then $a_c(\tau) = a'_c(\tau) = 0$ precisely when $c = c_i$, and as $c$ passes through $c_i$ there are creations or annihilations of pairs of cusps of $\varphi_c$ at swallowtail transitions.

Using $\varphi_c$ we can show that swallowtails are the only transitions that occur where $v \neq 0$, in other words there are no lips or beaks transitions in this context.

**Theorem 4.** As $c$ varies there are no lips or beaks transitions occurring for $p_c$ on $V^*_c$.

**Proof.** On $V^*_c$ the map $\varphi_c$ is everywhere a local diffeomorphism, and takes $H_c$ to the line $L_c$. At a lips transition $H_c$ would be a single point, while at a beaks transition it would be a pair of mutually tangent arcs: in neither case is it a 1-manifold.

**6. The composition $G_c = \varphi_c \circ F_c$**

Recall that $P_c = p(H_c)$ is the apparent outline of $V_c$ viewed in the $t$-direction, including the $\tau$-axis $L_0$, and corresponds to initial data $(c, v, \tau)$ that give rise to grazing trajectories. Clearly $P_c \subset \Pi_c$ except possibly for some points of $L_0$. Write $\Pi_c \setminus P_c = \Pi_c \setminus (\Pi_c \cap P_c)$. The first result is a straightforward consequence of the definitions.

**Proposition 9.** For the general system (1) the map $G_c = \varphi_c \circ F_c : \Pi_c \to \Pi$ has the following properties:

(i) $G_c$ is injective;

(ii) $G_c(\Pi_c \cap P_c) \subset P_c$;

(iii) $G_c(\Pi_c \setminus P_c) \subset \Pi \setminus P_c$. 
**Proof.** Statement (i) follows from uniqueness of solutions to (1): two solutions with \( x = c \) and different \((v, \tau)\) cannot next pass through \( x = c \) at the same time and with the same velocity. Statements (ii), (iii) record the fact that a given trajectory of (1) either is or is not a grazing trajectory regardless of the choice of initial point. \( \square \)

Since \( p_c \) is a local diffeomorphism away from \( H_c \) (which includes \( L_0 \)), and in the previous section we have seen that \( \varphi_c \) is a local diffeomorphism away from \( \{v = 0\} \), we immediately have the following regularity result:

**Proposition 10.** The composition \( G_c : \Pi \to \Pi \) is a local diffeomorphism at all points of \( \Pi \setminus P_c \).

Observe the analogous roles played on the one hand by \( H_c \) for \( p_c \) and on the other by \( V_c^0 \) for \( \varphi_c \). More precisely, let \( z \in H_c : \) then \( p_c(z) \in P_c \) and \( \varphi_c(z) \in L_0 \). If \( w \in P_c \) and \( w' \in L_0 \) we say that \( w \) and \( w' \) ‘correspond’ if there exists \( z \in H_c \) with \( p_c(z) = w, \varphi_c(z) = w' \). Under this correspondence, points of interesting geometry on \( P_c \) correspond to points of interesting geometry on \( L_0 \). We pursue this principle in greater detail below. First we set up some general terminology to describe local behaviour of \( F_c \) and \( p_c \).

Let \( w_0 = (v, \tau) \in \Pi \); then \( p_c^{-1}(w_0) \) is a discrete set of points

\[
p_c^{-1}(w_0) = \{z_i : i \in N \subset \mathbb{Z}\} \in V_c, \quad z_i = (w_0; t_i)
\]

with \( t_0 = 0 \) and \( t_i < t_j \) for \( i < j \in N \), where \( N \) denotes a finite or infinite interval of integers. For the linear system (2) we have \( N = \mathbb{Z} \) or (when \( \omega \notin \mathbb{Q} \)) exceptionally \( N = \{0\} \) by Corollary 3. By definition \( F_c(w_0) = z_1 \). We occasionally blur the distinction between \( w_0 \in \Pi \) and \( z_0 = (w_0; 0) \in \Pi \times \mathbb{R} \).

Write \( w_i = \varphi_c(z_i) \in \Pi \) and for each \( i \in N \) let \( U_i \) be the connected component of \( p^{-1}(U_0) \cap V_c \) containing \( z_i \). Let \( W_i = \varphi_c(U_i) \).

In the case of (2) if \( \omega = p/q \in \mathbb{Q} \) then \( \{z_i : i \in \mathbb{Z}\} \) is a finite set modulo \( t = 2\pi q \) and \( \{w_i : i \in \mathbb{Z}\} \) is a finite set modulo \( \tau = 2\pi / \omega \) (see Proposition 3). In general for (1) where \( V_c \) is periodic in \( t \) with period \( T \), in counting the points \( z_i \) we shall count modulo \( T \), and write \( z_j \equiv z_k \) when \( t_j - t_k \in T\mathbb{Z} \), and also write \( w_j \equiv w_k \) when \( \tau_j - \tau_k \in (2\pi / \omega)\mathbb{Z} \) where \( \omega \) is the frequency of the forcing function \( g \). If \( z_j \equiv z_k \) then \( w_j \equiv w_k \). We write \( \bar{z} \) (or \( \bar{w} \)) to denote the equivalence class of \( z \) modulo \( T \) (or \( w \) modulo \( 2\pi / \omega \)).

**The Correspondence Principle**

We now list significant geometric features of \( P_c \) and their counterparts in \( L_0 \) under the Correspondence Principle. To simplify notation we write \( F_c = F, H_c = H \) and \( P_c = P \) with a fixed value of \( c \) understood, although we retain the suffix for \( V_c \) as a reminder.

We begin with ‘codimension-0’ features, that is, those which occur for an open set of values of \( c \in \mathbb{R} \). First, the two features arising from local geometry of \( H \):

(1) A ‘single fold point’ is a point \( w_0 \in P \) such that \( p_c^{-1}(w_0) \cap H = \{z_j\} \) where \( z_j \) is a fold singularity of \( p_c \), that is, \( z_j \in H \) and \( \bar{x}_c(z_j) \neq 0 \). Then \( w_j = (0, \tau_j; 0) \in L_0 \) satisfies \( \bar{x}_c(w_j) \neq 0 \), that is \( a_c(\tau_j) \neq 0 \). It is a point at which \( \partial V_c / \partial \tau \neq 0 \) so the sheet \( V_c' \) of \( V_c \) passes through \( L_0 \) with non-zero slope: recall Proposition 1. The only branch of \( P_c \) passing through \( w_j \) is \( L_0 \). We call any such point \( (0, \tau, 0) \in L_0 \) where \( a_c(\tau) \neq 0 \) a ‘simple point’ of \( L_0 \).
(2) A ‘single cusp point’ is a point \( w_0 \in P^\ast \) such that \( p^{-1}(w_0) \cap H = \{z_j\} \) with \( \tilde{x}_c(z_j) = 0 \) but \( x_c^{(3)}(z_j) \neq 0 \); then \( w_j \in L_0 \) has \( \tilde{x}_c(w_j) = 0, x_c^{(3)}(w_j) \neq 0 \), that is \( a_c(\tau_j) = 0, a_c'(\tau_j) \neq 0 \) and so \( w_j \) is a point where a branch of \( P \) has quadratic tangency with \( L_0 \) by Proposition 4. We call \((0, \tau; 0) \in L_0\) where \( a_c(\tau) = 0, a_c'(\tau) \neq 0 \) a ‘tangency point’ of \( L_0 \).

Next the only codimension-0 feature arising from global geometry of \( H \):

(3) A point \( w_0 \in P^\ast \) is a ‘transverse double point’ if \( p^{-1}(w_0) \cap H = \{z_j, z_k\} \) where \( z_j, z_k \) are both fold singularities of \( p_c \) with arcs \( H_j, H_k \) of \( H \) through \( z_j, z_k \), respectively, such that the arcs \( P_i = p_c(H_i) \) for \( i = j, k \) intersect transversely at \( w_0 \). Then \( w_j \in L_0 \), and \( p^{-1}(w_j) \cap H \) consists of one point \( t = T \) apart from \( w_j \) at which \( \tilde{x}_c \neq 0 \). Likewise for \( w_k \). Since by Proposition 6 the map \( \varphi_c \) is a local diffeomorphism at \( z_j \) there is a neighbourhood \( U_j \) of \( z_j \) in \( V_c \) such that \( U_j \cap p^{-1}(P_k) \) is a smooth arc transverse to \( H_j \) taken by \( \varphi_c \) to an arc of \( P \) transverse to \( L_0 \) at \( w_j \); likewise for \( w_k \). We call \( w_j, w_k \) ‘transverse points’ of \( L_0 \).

Note that for (2) when \( c = 0 \) and \( \omega = p/q \in \mathbb{Q} \) the reduction of the \( t \)-periodicity from \( 2\pi q \) to \( \pi q \) means in particular that all single fold points and all single cusp points of \( P^\ast \) become double points (mod \( 2\pi q \)): this is reflected in the doubling of the tangency points of \( L_0 \) at \( \pm \pi/\omega \) (Type II transition) as \( c \) passes through zero.

The three codimension-0 features are nicely illustrated in figures 6 and 7: at a swallowtail transition an arc of single fold points acquires two cusp points and a pair of transverse double points. Correspondingly, at a swallow transition an arc of simple points of \( L_0 \) acquires two tangency points and a pair of transverse points.

Now we turn to ‘codimension-1’ features, meaning those which typically occur at isolated values of \( c \). There is only one such local feature.

(4) A ‘swallowtail point’ \( w_0 \in P^\ast \) (arising for (2) when \( c = 1 \)) has \( p^{-1}(w_0) \cap H = \{z_j, z_k\} \) where \( \tilde{x}_c(z_j) = x_c^{(3)}(z_j) = 0 \) but \( x_c^{(4)}(z_j) \neq 0 \); then \( w_j \in L_0 \) has \( \tilde{x}_c(w_j) = x_c^{(3)}(w_j) = 0 \) but \( x_c^{(4)}(w_j) \neq 0 \), that is \( a_c(\tau_j) = a_c'(\tau_j) = 0, a_c''(\tau_j) \neq 0 \). By Proposition 4 this means \( V_{c'} \) has cubic tangency with \( v = 0 \) and generically a lips (swan) transition takes place at \( w_j \). We call \( w_j \) a ‘swan point’ of \( L_0 \).

There are two global codimension-1 features:

(5) A ‘tangency point’ \( w_0 \in P^\ast \) is such that \( p^{-1}(w_0) \cap H = \{z_j, z_k\} \) with \( \tilde{x}_c(z_j), \tilde{x}_c(z_k) \neq 0 \) and with arcs \( H_j, H_k \) of \( H \) through \( z_j, z_k \), respectively, such that \( P_i = p_c(H_i) \) for \( i = j, k \) are arcs of \( P^\ast \) mutually quadratically tangent at \( w_0 \). Then \( L_j = \varphi_c(H_j) \) is an interval of \( L_0 \) containing \( w_j \). The quadratic tangency implies that near \( z_j \) the set \( p_{c'}^{-1}(P_k) \) is either:

(a) a pair of arcs through \( z_j \) transverse to each other and to \( H \), or
(b) the single point \( z_j \).

In case (a) since \( \varphi_c \) is a local diffeomorphism at \( z_j \) it follows from Proposition 9 that \( P \) near \( w_j \) consists of a pair of smooth arcs crossing each other and \( L_0 \) transversely at \( w_j \). If Conjecture 1 holds this must correspond to a Morse point of \( V_c \) at the point \((w_j, t_k - t_j) \in H^0 = H \cap V_c^0 \). For the linear system (1) (recall Theorem 1) this means \( c = \pm e_m \) with \( e_m = \gamma \cos \omega m \pi / 2 \) for some \( m \in \mathbb{Z} \), and \( |t_j - t_k| = |m| \pi \). We call \( w_j \) a
Morse point’ of \( L_0 \). An analogous discussion applies to \( w_k \). See figure 8, in which points of \( V_c \) are included to emphasize the local geometry.

In general for apparent outlines of surfaces in \( \mathbb{R}^3 \) we expect either (a) or (b) to apply to \( z_j \) and \( z_k \) independently. In the context of (2) it is only (a) that occurs. It is unclear whether (b) can occur for (1); if so it would be necessary to include further types of Morse point \( w_j \) on \( L_0 \) including those which are the image under \( p \) of an isolated point of \( H \).

**Remark 2.** The fact that for (2) all double points of \( P \) on \( L_0 \) where \( a_c(\tau) \neq 0 \) are transverse points or Morse points shows that for this system all tangencies of \( P^* \) with itself must indeed be quadratic.

(6) A ‘cusp/fold’ point \( w_0 \) of \( P^* \) is a coincidence of a cusp point and a fold point: thus \( p^{-1}(w_0) \cap H = \{ z_j, z_k \} \) with \( \dot{x}_c(z_j) \neq 0 \) but with \( \dot{x}_c(z_k) = 0 \) and \( x_c^{(3)}(z_k) \neq 0 \). Such points occur generically for apparent outlines and presumably also for (1) but do not occur for (2) by Proposition A1 in Whiston (1992). As \( c \) varies the generic local behaviour of \( P^* \) is again a transition between zero and two intersections, with corresponding transitions at the ‘double tangency’ point \( w_j \) and ‘lips’ or ‘beaks’ point \( w_k \) of \( L_0 \). We omit the details.

In place of cups/fold points there is a codimension-1 phenomenon special to the particular system (2) that would be expected to occur in the general system (1) only in codimension 2, that is at isolated values of \( c \) for certain discrete choices of a further parameter in \( f \) or \( g \).

(7) A ‘cusp coincidence point’ \( w_0 \in P^* \) has \( p^{-1}(w_0) \cap H = \{ z_j, z_k \} \) with \( \dot{x}_c(z_j) = \dot{x}_c(x_k) = 0 \) and \( x_c^{(3)}(z_j), x_c^{(3)}(x_k) \neq 0 \) : then \( w_j \) and \( w_k \) are points of \( L_0 \) corresponding to a double degenerate graze as described at the end of section 4. We call these ‘cusp contact points’ of \( L_0 \). At these points the outline \( P_c \) has two branches quadratically tangent to \( L_0 \) and tangent to each other with order 9/2. We refer to Chillingworth (2003) for further details.

Figure 8. Tangency point \( w_0 \in P^* \) and corresponding Morse point \( w_j \in L_0 \).
6.1. Some local geometry of \( G_c \)
We now turn to study the behaviour of the map \( G_c \) itself near its points of singularity and/or discontinuity. These are the points of \( L_0 \) together with the points \( w_0 \in P^s \) for which \( F_c(w_0) = z_1 \in H \). The local geometry of \( G_c \) needs to be described on a case-by-case basis, and for reasons of space we shall consider only single fold/simple points and single cusp/tangency points, as well as transverse double points of \( P^s \) and transverse points of \( L_0 \). Further cases are studied in Chillingworth (2003).

For definiteness we focus on \( w_0 \in P^s \cap \Pi^+ \); the description for \( P^s \cap \Pi^- \) is analogous, with the sign of \( v \) reversed.

The following results are useful in keeping track of local geometry.

**Proposition 11.** Let \( z = (v, \tau; t) \in H \) be a fold point for \( p_c \) with \( w = (0, \tau + t) = \varphi_c(z) \in L_0 \). Then the image under \( D\varphi_c(z) \) of \( \ker Dp_c(z) \) is spanned by the vector \((a, 1)\) where \( a = \dot{x}_c(w) = a_c(\tau + t) \).

**Proof.** Since \( \ker Dp_c(z) \) is spanned by \((0, 0; 1)\) its image under \( D\varphi_c(z) \) is spanned by \((\dot{x}_c(z), 1)\) by Proposition 6, and \( \dot{x}_c(z) = \dot{x}_c(w) \) as \( \varphi_c \) simply re-sets the clock.

**Definition 2.** For \( w = (0, \tau) \in L_0 \) the ‘distinguished direction’ at \( w \) is that spanned by the vector \((a_c(\tau), 0)\).

**Proposition 12.** If \( z = (0, \tau; t) \in H \) with \( a_c(\tau) \neq 0 \) then \( \ker D\varphi_c(z) = T_zH \).

**Proof.** The tangent space to \( H \) is spanned by a vector orthogonal to both \( \text{grad} x_c \) and \( \text{grad} \dot{x}_c \) with \( x_c = c \) and \( \dot{x}_c = 0 \). Such a vector is

\[
\left( \frac{\partial x}{\partial \tau}, -\frac{\partial x}{\partial \nu}, \frac{\partial x}{\partial v} \right) + (0, 0, \det W)
\]

with \( W \) given by (31). At singular points of \( \varphi_c \) we have \( \det W = 0 \), and the first vector above spans \( \ker D\varphi_c \).

As with all the figures in this paper, those which illustrate key features of \( G_c \) below are qualitative and not intended to be numerically accurate.

(1a) Single fold points
Let \( w_0 \in P^s \cap \Pi^+ \) be a single fold point, with \( H \cap p^{-1}(w_0) = \{z_1\} \). Choose a sufficiently small disc neighbourhood \( U_0 \) of \( w_0 \) so that the connected component \( U_1 \) of \( p_c^{-1}U_0 \) containing \( z_1 \) intersects \( H \) in a smooth arc \( H_1 \); thus \( U_1 \) is folded by \( p \) along \( H_1 \) which is mapped by \( p \) diffeomorphically to a smooth arc \( P_1 \) in \( \Pi^+ \). We may assume \( P_1 \) separates \( U_0 \) into two connected components \( U_0^+ = p(U_1) \) and \( U_0^- \). We say that \( U_0^+ \) lies on the ‘shadow side’ of \( P_1 \) while \( U_0^- \) lies on the ‘free side’.

As \( z_1 \in H \) we have \( w_1 \in L_0 \) and \( F \) is discontinuous along \( P_1 \). Assuming \( 2 \in N \) there is a neighbourhood \( U_2 \) of \( z_2 \) such that \( p|_{U_2} : U_2 \rightarrow U_0 \) is a diffeomorphism, and \( K_2 = U_2 \cap p^{-1}P_1 \) is an arc through \( z_2 \) separating \( U_2 \) into two open sets \( U_2^\pm \) with \( p(U_2^+ \cap P_1) = U_0^+ \). Thus \( F|_{U_0^-} \) is the restriction to \( U_0 \) of the diffeomorphism \( (p|_{U_2})^{-1} : U_0 \rightarrow U_2 \). If \( 2 \notin N \) then \( F|_{U_0^-} \) is undefined.

In contrast \( p|_{U_1} : U_1 \rightarrow U_0 \) is a fold. We have \( G_c(U_0) = G_c(U_0^+ \cup P_1 \cup U_0^-) = \varphi_c(U_1^+ \cup H_1 \cup U_2^-) = W_1^+ \cup J_1 \cup W_2^- \) with \( J_1 \) an open interval of \( L_0 \) containing the simple point \( w_1 \), and with \( W_1^+ \cup J_1 \) an open neighbourhood of \( w_1 \) in the closed half-plane \( \Pi^- \cup L_0 \), and with \( W_2^- \) one of the two components into which \( Q_2 = \varphi_c(K_2) \) separates \( W_2 \). See figure 9.
In the linear case with \( \omega = p/q \in \mathbb{Q} \) and \( z_2 \equiv z_0 \) the above still holds although now \( G_c : U^- \rightarrow W^- \equiv W_0^- = U_0^- \) is the identity map.

Note that \( G_c : U_0^-. \rightarrow W_0^- \) is a diffeomorphism while \( G_c : U_0^+ \cup P_1 \rightarrow W_1^+ \cup J_1 \) has the geometry of what we may call an ‘inverse fold’ at all points of \( P_1 \). In particular, any path in \( U_0^+ \cup P_1 \) transverse to \( P_1 \) at \( w \in P_1 \) is taken by \( G_c \) to a path with tangent in the distinguished direction at \( G_c(w) \).

(1b) Simple points of \( L_0 \)

At a simple point \( w_0 = (0, \tau_0) \in L_0 \) we have \( (\partial/\partial t) \nu_c(\tau_0; 0) = -\frac{1}{2} a_c(\tau_0) \neq 0 \). To fix matters we take \( a_c(\tau_0) < 0 \), the other case being completely analogous. The implicit function theorem allows \( V_c' \) to be expressed locally as a graph \( t = u_c(v, \tau) \), where \( u_c(v, \tau) \) here has the same sign as \( v \). If \( U_0 = U_0^+ \cup J_0 \cup U_0^- \) is a sufficiently small connected open neighbourhood of \( w_0 \) in \( \Pi \) with \( U_0 \cap \Pi^2 = U_0^2 \) and \( U_0 \cap L_0 = J_0 \) then \( F|U_0^+ \) is just the restriction to \( U_0^+ \) of the diffeomorphism \( F^0 = (p|U_0')^{-1} : U_0 \rightarrow U_0' \) where \( U_0' \) is a neighbourhood of \( z_0 \) in \( V_c' \). However, if \( 1 \in \mathbb{N} \) then \( F \) is discontinuous along \( J_0 \) and \( F|U_0^- \cup J_0 \) is (for \( U_0 \) sufficiently small) the restriction to \( U_0^- \cup J_0 \) of the diffeomorphism \( F_1 = (p|U_1')^{-1} : U_0 \rightarrow U_1 \).

In the linear case (2) if \( z_1 \equiv z_0 \) then \( F = F^0 \) on all of \( U_0 \).

We have described the action of \( F \) on \( U_0 \); we now describe the action of \( \varphi_c \) on \( F(U_0) \). Since \( F(U_0^+) \subset U_0' \) we need in particular to understand the action of \( \varphi_c \) on \( U_0' \). Let \( W_0' = \varphi_c(U_0') \).

Lemma 2. The restriction \( \varphi_c|U_0' : U_0' \rightarrow W_0' \) is a diffeomorphism.

Proof. The tangent space to \( V_c' \) at \( z_0 \) is spanned by

\[
\left\{ (0, 1; 0), \left( -\frac{1}{2} a_c(\tau_0), 0; 1 \right) \right\}
\]
since \((\partial v_c/\partial \tau)(w_0) = 0\) as \(L_0 \subset V'_c\), and
\[
\ker D\Phi_c(z_0) = \text{span}\{(0, 1; -1)\}
\]
as \((\partial x_c/\partial \tau)(w_0) = 0\) since \(L_0 \subset V_c\). Therefore, \(\varphi_c|V'_c\) is a local diffeomorphism at \(z_0\) if and only if
\[
\begin{vmatrix}
0 & 1 & 0 \\
-\frac{1}{2}a_0 & 0 & 1 \\
0 & 1 & -1
\end{vmatrix} \neq 0,
\]
where \(a_0 = a_c(\tau_0)\). The value of the determinant is \(\frac{1}{2}a_0\) which is non-zero as \(w_0\) is a simple point of \(L_0\).

**Corollary 6.** The map \(G_c|U_0^+\) is the restriction to \(U_0^+\) of the diffeomorphism \(\varphi_c \circ F^0 : U_0 \rightarrow W_0\).

Note that \(\varphi_c \circ F^0\) is the identity on \(J_0\). Therefore, to first order the effect of \(G_c\) on \(U_0^+\) is a shear in the direction of increasing \(\tau\) composed with a reflection in \(L_0\); we call this a ‘shear-reflection’. From Corollaries 6 and 5.1 we therefore see that \(G_c|U_0\) is discontinuous along \(J_0\), being the restriction of a shear-reflection on \(U_0^+\) and a fold on \(U_0^- \cup J_0\). See figure 10.

For (2) when \(z_1 \equiv z_0\) we have \(F|U_0 = F^0|U_0\) and \(G_c|U_0 \rightarrow W_0'\) is a shear-reflection, fixed along \(J_0\); then \((G_c)^2 = \text{id} : U_0 \rightarrow U_0\) because \(p^{-1}(U_0) = U_0 \cup U_0'\) (modulo \(2\pi q\)-periodicity in \(t\)): trajectories leaving \(x = c\) with initial state \((v, \tau) = w\) close to \((v_0, \tau_0) = w_0\) and with \(v \neq 0\) return to \(w\) after passing through \(x = c\) once with non-zero speed.

**Figure 10.** Action of \(G_c\) near a simple point \(w_0 \in L_0\).
This completes the local description of $G_c$ at single fold points of $P^*$ and simple points of $L_0$. Next we turn to cusp points of $P^*$ and corresponding tangency points of $L_0$.

(2a) Single cusp points
Let $w_0 \in P^* \cap \Pi^+$ be a single cusp point, with $p^{-1}(w_0) \cap H = \{z_1\}$. Choose $U_0$ sufficiently small so that the connected component $H_1$ of $p^{-1}(U_0) \cap H$ containing $z_1$ is a smooth arc of fold points of $p_c$ apart from the cusp point at $z_1$.

The projection $p_c$ has a cusp singularity at $z_1$, so $F$ is discontinuous along one branch of the cusp of $P^*/C_3$ at $w_0$ (although not at $w_0$ itself). The connected component $U_1$ of $p_c^{-1}(U_0)$ containing $z_1$ is a neighbourhood of $z_1$, and if $U_0$ is small enough the set $U_1 \cap p^{-1}(P)$ is the union of a pair of smooth arcs $H_1 = U_1 \cap H$ and $K_1$ with quadratic tangency at $z_1$. Now $\varphi_c$ is a local diffeomorphism at $z_1$ by Proposition 6, and $\varphi_c(H_1) = J_1 \subset L_0$; hence $\varphi_c(K_1) = J_1'$ is an arc in $\Pi^- \cup L_0$ quadratically tangent to $L_0$ at $w_1$. From the geometry of the cusp we see that $F$ takes the two branches of $U_0 \cap P^*$ at $w_0$ to two arcs of a $C^1$ (but not $C^2$) curve in $V_c$ passing through $z_1$: one is an arc of $H_1$ while the other is an arc of $K_1$. Thus $G_c$ takes $U_0$ to a subset of an open $\omega_0 \subset H_c / C_2$ bounded by two arcs of a $C^1$ (but not $C^2$) curve in $\Pi$ passing through $w_1 \in L_0$: one arc is a subinterval of $J_1$ (included) while the other is an arc of $J_1'$ (not included). See figure 11.

(2b) Tangency points of $L_0$
Let $w_0 \in L_0$ be a tangency point. At $w_0$ we have $\dot{x}_c = 0$, $\ddot{x}_c = 0$ and from the correspondence principle we know that $w_1$ is a cusp point (unless $w_1 = w_0$) and that $\varphi_c$ has a ‘cusp’ singularity at $z_1 \notin H$. We now put together this information on $F$ at $w_0$ and $\varphi_c$ at $z_1 \notin H$ to describe the action of $G_c$ on a neighbourhood of the tangency point $w_0 = (0, \tau_0; 0)$. For definiteness we take $x_c^{(3)}(w_0) > 0$ (so $v_c(\tau_0, t)$ has a maximum at $t = 0$) and $a_c(\tau_0) > 0$ (so $(\partial/v_c)(\tau_0, 0) < 0$), although the geometrical description would be analogous in other cases.

Figure 11. Action of $G_c$ near a cusp point $w_0 \in P^*$. 
Choose $U_0$ small enough so that the connected component of $p^{-1}(U_0) \cap H$ containing $z_0$ is the union of an interval $J_0$ of $L_0$ and a smooth arc $H_0$ of $H$ meeting only at $z_0$; write $P_0 = p(H_0)$ which is a smooth arc of $P$ quadratically tangent to $L_0$ at $w_0$ from the side $v > 0$. Let $P_0^\pm$ denote the subarcs of $P_0$ with $\tau$ greater than or less than $\tau_0$, respectively.

From the geometry of $V_c$ near $z_0$ (recall section 2) we see that $F$ is discontinuous along $L_0$ and along $P_0^\pm$, so we shall consider individually the three connected components of the complement of $L_0 \cup P_0^\pm$ in $U_0$. We have already noted:

1. $G_c(w_0) = w_1$ is a cusp point of $P^* \cap \Pi^-$

and we also have

2. $G_c(P_0^-) = J_0^+$, an open interval $(\tau_0, \tau_+)$ of $L_0$ for some $\tau_+ > \tau_0$.

Let $D$ denote the open subset of $U_0 \cap \Pi^+$ bounded by $P_0^-$ and the interval $J_0^- = \{ \tau \in J_0 : \tau < \tau_0 \}$. Then

3. $G_c|D \cup P_0^-$ has an inverse fold along $P_0^-$, and

4. close to $J_0^-$ the diffeomorphism $G_c|D$ is the restriction of a local shear-reflection that is the identity on $J_0^-$, although $G_c$ itself is discontinuous along $J_0^-$. See figure 12 which indicates the contours $\tau = \text{const.}$ and $\dot{x}_c = \text{const.}$ in $D$, as well as their images under $G_c$. The latter images are of course the straight lines $v = \text{const.}$, while the former images are curves transverse to $J_0^+$ along the distinguished direction.

Next we consider the open set $E = U_0 \cap \Pi^-$, which we may take as $G_c(D)$.

5. $G_c|E$ is the restriction of a local shear-reflection along $J_0^+$, although $G_c$ itself is discontinuous along $J_0^-$. In contrast, $G_c$ is continuous on $E \cup J_0^-$ and is a fold along $J_0^-$ (compare (1b) above) with $G_c(J_0^-) = P_0^+$. 

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![Diagram](image-url)

Figure 12. Action of $G_c$ near a tangency point $w_0 \in L_0$. 

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*Impact oscillator*
Finally, it remains to consider the action of $G_c$ on the subsets $C^+, C^-$ of $U_0$ bounded by $P_0^+$ together with $J_0^+, P_0^-$, respectively. We may take $C^+ = G_c(E)$. Let $C = C^+ \cup C^-$. 

(6) $G_c(C)$ is an open neighbourhood of the cusp point $w_1$ with one branch $Q_1$ of the cusp of $P^*$ at $w_1$ deleted; here $\hat{C}^+ = G_c(C^+) \cap H$ lies ‘inside’ the cusp while $\hat{C}^- = G_c(C^-)$ lies ‘outside’.

(7) $G_c(C \cup J_0^+)$ has a fold along $J_0^+$ with $G_c(J_0^+) = Q_1$, and is the restriction of the diffeomorphism $\varphi_c \circ (p(U_1))^{-1}$ on a neighbourhood of $P_0^+$, where $U_1$ is a neighbourhood of $z_1$ in $V_c$. The second branch of the cusp at $w_1$ is $Q_2 = G_c(P_0^+)$. See figure 12.

(3a) Transverse double points

Let $w_0$ be a transverse double point with $p^{-1}(w_0) \cap H = \{ \bar{z}_1, \bar{z}_k \}$ with $1 < k$. Choose a neighbourhood $U_0$ of $w_0$ small enough so that for $i = 1, k$ the connected components of $p^{-1}(U_0) \cap H$ containing $\bar{z}_i$ are smooth arcs $H_i$ that project by $p$ to smooth arcs $P_i$ of $P^* \cap \Pi^+$ intersecting transversely at $w_0$.

Since $j = 1$ we have $w_1 \in L_0$ and (as in (1a)(ii) above) the map $G_c|U_0$ is discontinuous along the arc $P_1$ through $w_0$. Writing $U_0 = U_0^+ \cup P_1 \cup U_0^-$ as in (1a), the restriction $G_c|U_0^+ \cup P_1$ is an inverse fold along $P_1$, with $J_1 = G_c(P_1)$ an open interval of $L_0$. The image of $P_k \cap (U_0^+ \cup P_1)$ under $G_c$ is $P_k \cap (\Pi_+ \cup L_0)$ where $P_k$ is an arc of $P$ through $w_1$ transverse to $L_0$; the shadow side of $P_k$ is taken to the shadow side of $P_k$.

The effect of $G_c$ on $U_0^+$ will differ according as $k = 2$ or $k > 2$.

If $k > 2$ then $F|U_0^- \cap H$ is the restriction of the diffeomorphism $(p|U_2)^{-1} : U_0 \to U_2$, so $G_c|U_0^-$ is the restriction of a diffeomorphism $U_0 \to W_2 = \varphi_c(U_2)$. Let $K_2 = U_2 \cap p^{-1}(P_l)$ for $l = 1, k$. Then $G_c(P_k \cap U_0^-)$ is the free side $W_2^1$ of $J_2^1 = \varphi_c(K_2)$ in $W_2$, and $G_c(P_k \cap U_0^+) = P_k^1 \cap W_2$ where $P_k^1$ is an arc of $P^+$ intersecting $J_2^1$ transversely at $w_2 \in \Pi^-$. See figure 13(i).

If $k = 2$ then $G_c|U_0^-$ is discontinuous along the arc $P_2 \cap U_0^-$, which separates $U_0^-$ into two open sets $U_0^-$ and $U_0^-$ on the shadow and free sides of $P_2$, respectively. Now $F|U_0^- \cap H$ is the restriction of $(p|U_3)^{-1} : U_0 \to U_3$, and $G_c$ takes $U_0^-$ to an open subset $W_3$ of $W_3 = \varphi_c(U_3)$ bounded by two arcs of $P^+$ through $w_3 \in \Pi^-$ and on the free side of both arcs. (If $z_3 \equiv z_0$ so $w_3 \equiv w_0$ then $G_c|U_0^-$ is the identity map.) On the other hand $U_0^+ \cup (P_2 \cap U_0^-) \cap H$ is an open subset of $W_2 \cap (\Pi_+ \cup L_0)$ bounded by an arc of $P$ through $w_2$ (transverse to $L_0$) and an interval of $L_0$, the latter being included. An inverse fold for $G_c$ occurs along $(P_2 \cap U_0^-) \cap H$. See figure 13(ii).

(3b) Transverse points of $L_0$

Let $w_0 \in L_0$ be a transverse point with $p^{-1}(w_0) \cap H = \{ \bar{z}_0, \bar{z}_j \}$. Assume $U_0$ chosen small enough so that the connected component $U_j$ of $p^{-1}(U_0)$ containing $\bar{z}_j$ meets $H$ in a smooth arc $H_j$ that projects diffeomorphically by $p$ to a smooth arc $P_j$ of $P$ intersecting $J_0$ transversely at $w_0$.

As in case (1b) above, we assume $a_c(\tau_0) < 0$. Then $G_c|U_0^+$ is the restriction of a diffeomorphism that is to first order a shear-reflection fixed along $J_0$. Hence $G_c(P_j \cap U_0^+)$ is the intersection of $W_0 \cap \Pi^-$ (which we can take to be $U_0^-$) with an arc of $P$ through $w_0$ and transverse to $L_0$; this arc must therefore be $P_j \cap U_0^-$ as $G_c$ preserves $P$ by Proposition 9.

To see the effect of $G_c$ on $U_0^-$ we consider separately the cases $j > 1$ and $j = 1$. 

The restriction of $G_c$ to $U_0^- \cup J_0$ is the restriction of a fold map with singular set $J_0$. A typical arc through $w_0$ and transverse to $J_0$ is taken by $G_c$ to an arc tangent to $P^*$ at $w_1$. The exceptional arcs are those whose tangent at $w_0$ is in the direction $p \ker D\varphi_c(z_i) = \ker W(z_i)$, these being taken to arcs transverse to $P^*$ at $w_1$ (that is, transverse as 1-manifolds: as parametrized curves their speed becomes zero at $w_1$). Since from the Correspondence Principle (see (5)) self-intersections of $P^*$ at fold points are always transverse unless they correspond to Morse points on $L_0$ as in (2b) in which case they are quadratic, it follows that $P_j$ must be just such an exceptional arc. See figure 14. This reflects the geometry already seen in (3a) above, as $P_j$ can be seen as the image under an inverse fold of an arc transverse to $P^*$ at $w_1$. A similar argument applies at every $z_i \in p^{-1}(w_0)$ with $z_i \notin H$, replacing $F_c$ by a local right inverse to $p|V_c$ near $z_j$. Thus we have the following result.

**Proposition 13.** If $w_0$ is a transverse point of $L_0$ then the branch of $P$ through $w_0$ transverse to $L_0$ is tangent at $w_0$ to $p \ker D\varphi_c(z_i) = \ker W(z_i)$, this direction being the same for every $z_i \in p^{-1}(w_0)$ with $z_i \notin H$.

(i) $j > 1$

The restriction of $G_c$ to $U_0^- \cup J_0$ is the restriction of a fold map with singular set $J_0$. A typical arc through $w_0$ and transverse to $J_0$ is taken by $G_c$ to an arc tangent to $P^*$ at $w_1$. The exceptional arcs are those whose tangent at $w_0$ is in the direction $p \ker D\varphi_c(z_i) = \ker W(z_i)$, these being taken to arcs transverse to $P^*$ at $w_1$ (that is, transverse as 1-manifolds: as parametrized curves their speed becomes zero at $w_1$). Since from the Correspondence Principle (see (5)) self-intersections of $P^*$ at fold points are always transverse unless they correspond to Morse points on $L_0$ as in (2b) in which case they are quadratic, it follows that $P_j$ must be just such an exceptional arc. See figure 14. This reflects the geometry already seen in (3a) above, as $P_j$ can be seen as the image under an inverse fold of an arc transverse to $P^*$ at $w_1$. A similar argument applies at every $z_i \in p^{-1}(w_0)$ with $z_i \notin H$, replacing $F_c$ by a local right inverse to $p|V_c$ near $z_j$. Thus we have the following result.

**Proposition 13.** If $w_0$ is a transverse point of $L_0$ then the branch of $P$ through $w_0$ transverse to $L_0$ is tangent at $w_0$ to $p \ker D\varphi_c(z_i) = \ker W(z_i)$, this direction being the same for every $z_i \in p^{-1}(w_0)$ with $z_i \notin H$. 

Figure 13. Action of $G_c$ near a transverse double point $w_0 \in P^*$: (i) $k > 2$; (ii) $k = 2$. 
Both the maps $p_c$ and $\varphi_c$ exhibit fold singularities at $z_1$ with their singular sets mutually transverse. Thus $G_c$ near $w_0$ is the result of composing an inverse fold with a fold. We have that $G_c|U_0^-$ is discontinuous along $P_1 \cap U_0^-$, which separates $U_0^-$ into two open subsets $U_0^-_1$ and $U_0^+_1$ on the shadow side of $P_1$. Then $G_c$ takes $U_0^-$ to an open subset of $W_0^+$ bounded by two transverse arcs of $P^+$ through the transverse double point $w_2 \in P^+ \cap \Pi^+$. If $z_2 \in N$ and $z_2 \neq z_0$ then $F|U_0^-$ is the restriction to $U_0^-$ of the diffeomorphism $(p|U_2)^{-1} : U_0 \rightarrow U_2$; if $z_2 \equiv z_0$ so $w_2 \equiv w_0$ then $G_c$ is the identity map on $U_0^-$. Compare case (1b) above.

The arc $P_1^- = P_1 \cap U_0^-$ is taken by $G_c$ to an arc $J_1^-$ of $J_1 \subset L_0$ with an endpoint at $w_1$; an arc $J_0^+$ of $J_0 \subset L_0$ with endpoint at $w_0$ and on the shadow side of $P_1$ is taken to an arc $J_0^+$ of $P \cap \Pi^+$ with endpoint at $w_1$ which by Proposition 13 is transverse to $L_0$. The map $G_c$ takes the open set $U_0^+$ of $U_0^-$ bounded by $P_1^-, J_0^+$ to an open set in $W_0^+ = W_1^+ \cap \Pi^+$ bounded by the arcs $J_0^-$ and $J_1^+$, exhibiting a fold along $J_0^+ \subset L_0$ and an inverse fold along $P_1^-$. See figure 14.

**Remark 3.** A local model for a pair of folds with transverse singular sets is the pair of maps $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by $(x,y) \mapsto (x^2, y)$ and $(x,y) \mapsto (x, y^2)$, and in this case the inverse of one composed with the other takes the form $(u,v) \mapsto (\sqrt{u}, v^2)$ in the positive quadrant. We cannot, however, assume that $G_c$ has exactly this form near
$w_0$ as there may be no coordinate change on $V_c$ near $z_1$ that takes both folds to their standard forms simultaneously (Teixeira 1982, Kurokawa 2003).

7. Global considerations
For a system such as (2) with $\omega \in Q$ and for a given choice of clearance $c$ the apparent outline $P_c$ (consisting of the part $P_c^r$ where $v > 0$ together with the $\tau$-axis $L_0$) separates the plane $\Pi$ into a finite number of complementary regions. In other cases this description is still adequate if we bound the time that would be allowed to elapse under the ‘free’ dynamics of (1). The dynamics of $G_c$ can then be regarded as a re-arrangement of these regions with a diffeomorphism on each open region, but with discontinuities and singularities along $L_0$ and along some of the boundary arcs that comprise $P_c^r$. In this paper we have discussed some of the geometry of this decomposition of $G_c$. To construct the dynamics of the impact oscillator it is then necessary to compose $G_c$ with the restitution map $R$. Certain local features of this picture have been previously used by other authors (Budd and Dux 1994a,b) to show mechanisms for interesting dynamical behaviour such as ‘chatter’, and global features are used by Dippnall (2003) to detect horseshoes. We propose this overall geometric description, in which the complementary regions to $P_c$ form a kind of Markov partition (Adler 1998) for the dynamics, with local dynamics near discontinuities as partially described in this paper, as a versatile setting for future study of impact oscillators (1) and their generalizations.

8. List of notation
- $c$: $x$-coordinate of the obstacle
- $F_c$: first-hit map $\Pi_c \rightarrow V_c$
- $G_c \varphi_c \circ F_c$
- $H_c \{ (v, \tau; t) \in V_c : \dot{x}(c, v, \tau; t) = 0 \}$
- $I_c R \circ G_c$ : the dynamical system to be studied
- $L_K \{ (v, \tau; t) \in \mathbb{R}^3 : (v, t) = (0, k\pi) \}$
- $L_0$ $\tau$-axis
- $p$: projection $(v, \tau; t) \mapsto (v, \tau)$
- $p_c$: restriction of $p$ to $V_c$
- $P_c$: $p(H_c)$ = apparent outline of $V_c$ in $t$-direction
- $R$: restitution map $\Pi \rightarrow \Pi$
- $v$: initial velocity
- $v_c$: function of $(\tau, t)$ whose graph is $V'_c$ near $L_0$
- $V_c$: impact surface $\{ (v, \tau; t) \in \mathbb{R}^3 : x(c, v, \tau; t) = c \}$
- $V'_c$: sheet of $V_c$ passing through $L_0$ other than $\Pi$
- $x(c, v, \tau; t)$: solution to (1) with initial data $(c, v, \tau)$
- $x_c(v, \tau; t)$: $x(c, v, \tau; t)$
- $\varphi_c$: re-set map $V_c \rightarrow \Pi$
- $\Pi$: $(v, \tau)$-plane
- $\Pi_c$ subset of $\Pi$ for which future impact exists
- $\tau$: initial phase.
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