Hilbert Space compression and exactness of discrete groups.

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Abstract

We show that the Hilbert Space compression of any (unbounded) finite dimensional CAT(0) cube complex is 1 and deduce that any finitely generated group acting properly, co-compactly on a CAT(0) cube complex is exact, and hence has Yu's Property A. The class of groups covered by this theorem includes free groups, finitely generated Coxeter groups, finitely generated right angled Artin groups, finitely presented groups satisfying the B(4)-T(4) small cancellation condition and all those word-hyperbolic groups satisfying the B(6) condition. Another family of examples is provided by certain canonical surgeries defined by link diagrams.

Key words: Exactness, Hilbert Space compression, CAT(0) cube complex, Property A.

Introduction

We say that a group Γ is exact if the operation of taking the reduced crossed product with Γ preserves exactness of short exact sequences of Γ-C^*-algebras. In other words, Γ is exact if and only if for every exact sequence of Γ-C^*-algebras

$$0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$$

the sequence

$$0 \longrightarrow C_r^*(\Gamma, B) \longrightarrow C_r^*(\Gamma, C) \longrightarrow C_r^*(\Gamma, D) \longrightarrow 0$$

of crossed product algebras is exact. Kirchberg and Wassermann [9] proved that when Γ is discrete, it is exact if and only if its reduced C^*-algebra $C_r^*(\Gamma)$

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is exact. This means that the functor $B \mapsto C_r^*(\Gamma) \otimes_{\min} B$ is exact, i.e. preserves exactness of sequences of C^*-algebras.

In [19] Yu introduced Property A, analagous to Fölner's criterion for amenability, which for a finitely generated group is equivalent to exactness.

Definition 1 *A discrete metric space Γ is said to have Property A if for any $r > 0, \epsilon > 0$, there exist a family of finite subsets $\{A\gamma\}_{\gamma \in \Gamma}$ of $\Gamma \times \mathbb{N}$ such that*

- $(\gamma, 1) \in A_\gamma$ *for all* $\gamma \in \Gamma$;
- $\frac{|A_\gamma \backslash A_{\gamma'}| + |(A_{\gamma'} \backslash A_\gamma|}{|A_\gamma \cap A_{\gamma'}|} < \epsilon$ *for all γ and $\gamma' \in \Gamma$ satisfying $d(\gamma, \gamma') \leq r$, where, for each finite set A,$|A|$ is the number of elements in A;*
- $\exists R > 0$ *such that if $(x, m) \in A_\gamma$, $(y, n) \in A_\gamma$ for some $\gamma \in \Gamma$, then $d(x, y) \leq R$.*

The authors would like to thank Jacek Brodzki who explained to us the equivalence of Property A with exactness as follows: according to Higson and Roe, [8], a finitely generated group has Property A if and only if it acts amenably on its Stone-Cech compactification. By the theorem of Ozawa in [14] this is equivalent to exactness for the group. Exactness should be thought of as a weak form of amenability; the property was first made prominent by the work of Kirchberg and Wassermann [9], and studied by several authors [1,6,8,14,18,19]. Examples of exact groups include groups of finite asymptotic dimension, for example Gromovs word hyperbolic groups, discrete subgroups of connected Lie groups and amenable groups. The class is closed under the semi-direct product, [19].

By way of further motivation for the study of exactness we should point out that groups with property A admit a uniform embedding into Hilbert Space and satisfy the strong Novikov conjecture, and the coarse Baum Connes conjecture [19]. In [7] Guentner and Kaminker introduced a numerical quasi-isometry invariant of a finitely generated group, the values of which parametrize the difference between the group being uniformly embeddable in a Hilbert Space and the reduced C*-algebra of the group being exact.

Theorem (Guentner and Kaminker, see [6]) *Let G be a discrete group. If the Hilbert Space compression of G is strictly greater than $1/2$ then G is exact.*

We will define Hilbert Space compression later, but note here that it is a measure of the amount of distortion that is necessary when trying to embed the group in a Hilbert Space via a large scale Lipschitz map. Guentner and Kaminker illustrated their theorem by showing that the Hilbert Space compression of a finite rank free group is 1 thus giving a new proof of exactness for free groups. It should be noted that they did not construct an embedding of the free group in a Hilbert Space with (asymptotic) compression 1, but rather, thinking of the group as a tree via its Cayley graph, produced a family of large scale Lipschitz embeddings with asymptotic compression arbitrarily close to 1. Those familiar with CAT(0) complexes would recognize that the first of their embeddings (with asymptotic compression 1/2) can be used without change to embed the vertex set of a CAT(0) cube complex into a Hilbert Space with asymptotic compression 1/2 though this is not in itself enough to establish exactness for a group acting on the cube complex. Guentner and Kaminker showed that in the case of a tree the embedding can be modified to obtain new embeddings with asymptotic compression arbitrarily close to 1.

The main purpose of this note is to show how to adapt the construction from [7] to the class of unbounded, finite dimensional CAT(0) cube complexes. In the case of a tree one uses the fact that there is a unique edge geodesic joining any two points in the tree; the same is of course not true for CAT(0) cube complexes of dimension at least 2 so the embedding and the argument need to be modified appropriately. In place of unique edge geodesics we will use the normal cube paths originally introduced in [11] to establish biautomaticity for groups acting freely and properly discontinuously on CAT(0) cube complexes.

Theorem 10 *Let X be an unbounded finite dimensional CAT(0) cube complex. The Hilbert space compression of X is* 1*.*

In [6] it is shown that Hilbert Space compression is a quasi-isometry invariant so if a discrete group G acts freely and co-compactly on an unbounded CAT(0) cube complex it follows that the group (regarded as a metric space via the word length metric) has Hilbert Space compression 1. Since $1 > 1/2$ we obtain:

Theorem 12 *If G is a group acting properly and co-compactly on a CAT(0) cube complex then G is exact and therefore has Yu's Property A.*

Note that if G acts properly on a bounded CAT(0) cube complex then G is finite and therefore exact, so the hypothesis that the cube complex should be unbounded (which is only inserted in Theorem 10 in order to ensure that asymptotic compression can be defined) is not needed in Theorem 12.

The paper is organised as follows: In section 1 we recall the definition of a CAT(0) cube complex and, stating the definitions, show how to construct a large scale Lipschitz embedding of such a complex in an associated Hilbert Space, with asymptotic compression 1/2. In section 2 we outline some preliminary results concerning the existence and properties of normal cube paths in a CAT(0) cube complex. The results in this section are taken from [11]. In section 3 we define a family of embeddings $\{f_\epsilon \mid 0 < \epsilon < 1/2\}$ of the vertices of a cube complex X into the Hilbert Space of square summable real valued functions on the set of hyperplanes of X. We also show that these embeddings are large-scale Lipschitz. In section 4 we show that the compression of each map f_ϵ is $1/2 + \epsilon$ and deduce that the Hilbert Space compression of the metric space $(X^{(0)}, d_1)$ is 1, where $X^{(0)}$ denotes the vertex set of X and d_1 is the edge metric. In section 5 we deduce the exactness of groups acting properly and co-compactly on a CAT(0) cube complex.

The class of groups covered by this theorem includes free groups, finitely generated Coxeter groups [13], and finitely generated right angled Artin groups (for which the Salvetti complex is a CAT(0) cube complex). A rich class of interesting examples is furnished by Wise, [17], in which it is shown that many small cancellation groups act properly and co-compactly on CAT(0) cube complexes. The examples include every finitely presented group satisfying the B(4)-T(4) small cancellation condition and all those word-hyperbolic groups satisfying the B(6) condition. Finally many 3-manifolds admit decompositions as CAT(0) cube complexes, so their fundamental groups are also covered by the theorem, a family of examples is provided by certain canonical surgeries defined by link diagrams (see [2] and [3]). Classical examples are furnished by groups acting simply transitively on buildings with the structure of a product of trees.

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1 CAT(0) cube complexes

A *cube complex* X is a metric polyhedral complex in which each cell is isometric to the Euclidean cube $[-1/2, 1/2]^n$, and the gluing maps are isometries. If there is a bound on the dimension of the cubes then the complex carries a complete geodesic metric, [4].

A cube complex is *non-positively curved* if for any cube C the following conditions on the link of C, $\mathrm{lk}C$, are satisfied:

(1) (*no bigons*) For each pair of vertices in $\mathrm{lk}C$ there is at most one edge containing them.
(2) (*no triangles*) Every edge cycle of length three in $\mathrm{lk}C$ is contained in a 2-simplex of $\mathrm{lk}C$.

The following theorem of Gromov relates the combinatorics and the geometry of the complex.

Lemma 2 (Gromov, [5]) *A cube complex X is locally $CAT(0)$ if and only if it is non-positively curved, and it is CAT(0) if and only if it is non-positively curved and simply connected.*

Any graph may be regarded as a 1-dimensional cube complex, and the curvature conditions on the links are trivially satisfied. The graph is CAT(0) if and only if it is a tree. Euclidean space also has the structure of a CAT(0) cube complex with its vertices at the integer lattice points.

A *midplane* of a cube $[-1/2, 1/2]^n$ is its intersection with a codimension 1 coordinate hyperplane. So every n-cube contains n midplanes each of which is an $(n-1)$-cube, and any m of which intersect in a $(n-m)$-cube. Given an edge in a non-positively curved cube complex, there is a unique codimension 1 hyperplane in the complex which cuts the edge transversely in its midpoint. This is obtained by developing the midplanes in the cubes containing the edge. In the case of a tree the hyperplane is the midpoint of the edge, and in the case of Euclidean space it is a geometric (codimension-1) hyperplane.

In general a hyperplane is analogous to an immersed codimension 1 submanifold in a Riemannian manifold and in a CAT(0) cube complex one can show that the immersion is a local isometry. An application of the Cartan-Hadamard theorem then shows that the hyperplane is isometrically embedded. Furthermore any hyperplane in a CAT(0) cube complex separates it into two components referred to as the half spaces associated with the hyperplane. This is a consequence of the fact that the complex is simply connected. The hyperplane gives rise to 1-cocycle which is necessarily trivial, and hence the hyperplane separates the space.

The set of vertices of a CAT(0) cube complex X can be viewed as a discrete metric space, where the metric $d_1(u, v)$ is given by the length of a shortest edge path between the vertices u and v. We will refer to this as the ℓ^1 metric on the vertices. Alternatively we can measure the distance by restricting the path metric on X to obtain the ℓ^2 metric on the vertices. If X is finite dimensional these metrics are quasi-isometric, and we have $d(u, v) \leq d_1(u, v) \leq \sqrt{n} d(u, v)$ where d denotes the CAT(0) (geodesic) metric on X and n is the dimension of the complex.

Sageev [16] observed that the shortest path in the 1-skeleton crosses any hyperplane at most once, and since every edge crosses exactly one hyperplane, the ℓ^1 distance between two vertices is the number of hyperplanes separating them.

Finally we will need the concept of a median. In any CAT(0) cube complex there is a well defined notion of an interval; given any two vertices u, v the interval between them, denoted $[u, v]$ consists of all the vertices which lie on an edge geodesic from u to v. Given any three vertices u, v, w there are three intervals $[u, v], [v, w], [w, u]$ and the intersection of these three intervals is always a single point m known as the median of the triple u, v, w (see [15] for details). It has the following important property: If we consider the hyperplanes which separate the pair u, v and those which separate the pair u, w the intersection of these two families consists of precisely the hyperplanes which separate u and the median m. Furthermore the hyperplanes which separate v from w are precisely those hyperplanes which separate m from v together with those which separate m from w so we have $d_1(v, w) = d_1(v, m) + d_1(m, w) = d_1(v, u) + d_1(w, u) - 2d_1(m, u)$. We will use this fact in section 4.

In [10] it was shown how to use the hyperplane structure of a CAT(0) cube complex X to obtain an ℓ^1 embedding of the cube complex in the Hilbert Space $\ell^2(H, \mathbb{R})$ of square summable (real valued) functions on the set H of hyperplanes in X. An alternative description of the embedding, based on the one used in [7] in the context of a tree, is as follows:

Choose a basepoint v in $X^{(0)}$ and for each vertex $w \in X^{(0)}$ set $H_w = \{h \in H \mid h \text{ separates } v \text{ and } w\ \}$. Define $f_w : H \longrightarrow \mathbb{R}$ by $f_w = \sum_{h \in H_w} \delta_h$ where δ_h denotes the characteristic function of the singleton $\{h\} \subset H$.

It is easy to see that the function f_w is ℓ^1 and therefore ℓ^2 and since the Hilbert Space is contractible (in fact uniquely geodesic) the map extends to an embedding of X in $\ell^2(H, \mathbb{R})$. If X is a cube then this embedding is isometric, however in the case of a tree (consisting of more than a single edge) then it is not. For example let T be the tree consisting of two edges e_s, e_t both adjacent to a vertex v, and with the other two vertices labelled s, t. The tree has two hyperplanes, corresponding to the midpoints of the two edges, so that $\ell^2(T, \mathbb{R}) \sim \mathbb{R}e_s \bigoplus \mathbb{R}e_t$. The vertex v is not separated from itself by either of the hyperplanes so we have $f_v = 0$. The vertex s is only separated from v by the hyperplane s so we have $f_s = \delta_{e_s}$ and similarly $f_t = \delta_{e_t}$. Now in the tree we have $d_1(s, t) = d_2(s, t) = 2$ however in the Hilbert Space we have $d_1(f_s, f_t) = 2 \neq \sqrt{2} = d(f_s, f_t)$, where we have used d_1 to denote the ℓ^1 metric and d to denote the Hilbert metric.

Although the embedding defined above is not necessarily an isometry it is relatively easy to show that it is a large scale Lipshcitz map, and we can measure the distortion of such a map in terms of its compression:

Definition 3 *A function $f : X \to Y$ is large-scale Lipschitz if there exist $C > 0$ and $D \geq 0$ such that $d_Y(f(x), f(y)) \leq Cd_X(x, y) + D$. Following Gromov, the compression $\rho(f)$ of f is given by $\rho_f(r) = \inf_{d_X(x,y) \geq r} d_Y(f(x), f(y))$. Assuming that X is unbounded the asymptotic compression R_f is given by*

$$R_f = \liminf_{r \to \infty} \frac{\log \rho_f^*(r)}{\log r}$$

where $\rho_f^*(r) = \max\{\rho_f(r), 1\}$.

In the case of the embedding of the vertices described above the map is large scale Lipschitz with $C = 1, D = 0$. The argument used by Guentner and Kaminker [7] to compute the asymptotic compression of the embedding of a tree goes through without change to our more general context to show that the asymptotic compression is $1/2$. (It should be noted here that we are regarding the cube complex as a metric space via the ℓ^1 metric not the (geodesic) ℓ^2 metric.)

In order to obtain large scale Lipschitz embeddings with asymptotic compression close to 1 we need to adapt the embedding described above. The idea, taken from [7] is to weight the functions δ_h according to how far the hyperplane is from the basepoint. Whereas in the case of a tree the hyperplanes which separate two vertices are linearly ordered in a higher dimensional cube complex they are not and there are several partial orders one could use in modifying the argument. It turns out that the appropriate ordering is furnished by the normal cube paths introduced in [11] and we describe these next.

2 Normal cube paths

Definition 4 *A cube path is a sequence of cubes $\mathcal{C} = \{C_0, \dots C_n\}$, each of dimension at least 1, such that each cube meets its successor in a single vertex, $v_i = C_{i-1} \cap C_i$ and such that for $1 \leq i \leq n-1$, C_i is the (unique) cube of minimal dimension containing v_i and v_{i+1}. Note that v_i and v_{i+1} are diagonally opposite vertices of C_i. We define v_0 to be the vertex of C_0 which is diagonally opposite v_1, and v_n to be the vertex of C_n diagonally opposite v_{n-1}. We call the v_i, vertices of the cube-path, with v_0 the initial vertex and v_n the terminal vertex. Given a cube path from u to v we can construct edge paths from u to v which travel via the edges of the cubes C_i so every hyperplane separating u from v must intersect at least one of the cubes C_i. We say the cube path is normal if $C_{i+1} \cap st(C_i) = v_i$ for each i, where $st(C_i)$ is the union of all cubes which contain C_i as a face (including C_i itself).*

In [11] it was shown that given any two vertices u, v there is a unique normal cube path $\mathcal{C} = \{C_0, \dots, C_n\}$ from u to v. We will need the following key facts about normal cube paths all of which may be found in [11].

Lemma 5 *Let s, t, v_0 be vertices of a CAT(0) cube complex with s and t diagonally adjacent across some cube E_0. Let $s = s_0, s_1, \dots, s_m = v$, $t = t_0, t_1, \dots, t_n = u$ be the vertices of the (unique) normal cube paths from s to v_0 and from t to v_0 respectively. Let $\{C_i \mid i = 1, \dots m\}$ be the cubes on the normal cube path from s to v_0 and $\{D_j \mid j = 1, \dots n\}$ be the cubes on the normal cube path from t to v_0. Then:*

(1) Each hyperplane separating s from v_0 intersects exactly one of the cubes

C_i and each hyperplane separating t from v_0 intersects exactly one of the cubes D_j.

(2) For each $i \le \min\{m,n\}$ there is a cube E_i such that s_i is diagonally adjacent to t_i across E_i .

We will need the following technical lemma:

Lemma 6 *Let s,t,v be vertices of the CAT(0) cube complex X with s and t diagonally opposite across some cube E_0. Let $s = s_0, s_1, \dots, s_m = v$, $t = t_0, t_1, \dots, t_n = v$ be the vertices of the (unique) normal cube paths from s to v and from t to v respectively. Let $\{C_i \mid i = 1, \dots m\}$ be the cubes on the normal cube path from s to v and $\{D_j \mid j = 1, \dots n\}$ be the cubes on the normal cube path from t to v. If h is a hyperplane in X which separates both s and t from v and which intersects the cube C_i then h also intersects one of the cubes D_{i-1}, D_i, D_{i+1}.*

PROOF. By lemma 5 the hyperplane h can only (and must) intersect the normal cube path from t to v in one of the cubes D_j, and the hypothesis that $s = s_0$ and $t = t_0$ are diagonally opposite across the cube E_0 ensures that for each $i \le \min\{m,n\}$ s_i is diagonally opposite to t_i across some cube E_i.

Now h separates s_{i-1}, s_i and also separates t_{j-1}, t_j. let $k = \min\{i,j\}$. Assume first that h separates s_{k-1} and s_k so $i = k \le j$; if h also separates t_{k-1} and t_k then h crosses $D_k = D_i$ as required. If on the other hand h does not separate t_{k-1} from t_k then, since it does not separate t from t_{k-1} by the minimality of k, but does separate t from v, it must also separate t_k from v. Now we construct an edge path from t_k to v as follows. First cross over the cube E_k from s_k to t_k then follow a path through the cubes $C_{k+1}, C_{k+2}, \dots, C_m$ to v. This gives an edge path from t_k to v so it must cross h. However none of the cubes $C_{k+1}, \dots, C_m$ intersect h so h must cross E_k and hence h is adjacent to t_k. But as h separates t_k from v and is adjacent to t_k it must cross the first cube (D_{k+1}) on the normal cube path from t_{k+1} to v as required. The case when h separates t_{k-1} and t_k so $i = k$ but does not separate s_{k-1} and s_k is argued in exactly the same way reversing the roles of s and t, C and D and so on.

From now on we fix a vertex v as a basepoint and for each vertex s we define an integer-valued weight function w_s on the set of hyperplanes as follows. Let $\mathcal{C} = \{C_0, \dots, C_n\}$ be the unique normal cube path from s to v. If the hyperplane h separates s and v then set $w_s(h) = i + 1$ where h intersects the cube C_i, otherwise set $w(h) = 0$. Hence w_s has finite support. From Lemma 6 we get:

Corollary 7 *If s and t are adjacent in X and h separates both s and t from v then $|w_t(h) - w_s(h)| \leq 1$.*

PROOF. If the normal cube path from s to v is denoted by the cubes C_i as above and the normal cube path from t to v is denoted by D_j then h intersects precisely the cubes $C_{w_s(h)}$ and $D_{w_t(h)}$ so by the lemma $D_{w_t(h)} = D_{w_s(h)\pm 1}$, and $w_t(h) = w_s(h)$ or $w_t(h) = w_s(h) \pm 1$ as required.

Note that in the statement of the corollary "adjacent" may be taken to mean adjacent across the diagonal of any cube, however in our application we will only need it to mean that s and t are vertices of a common edge.

3 The large scale Lipschitz embeddings

As in the last section we fix a CAT(0) cube complex X (not necessarily finite dimensional) and a base vertex v. We will show how to construct a family (indexed by the interval $(0, 1/2)$) of large scale Lipschitz embeddings of the vertex set $X^{(0)}$ into the Hilbert Space of ℓ^2 functions from the set H of hyperplanes in X to $\mathbb{R}$.

For each $\epsilon \in (0, 1/2)$ define $f_\epsilon(s) = \sum_{h\in h} w_s(h)^\epsilon \delta_h$. As noted before since the s-weight of a hyperplane is 0 unless the hyperplane is one of the finitely many separating s from the basepoint v, this sum is always finite and therefore is an element of $\ell^2(H, \mathbb{R})$.

In order to show that f_ϵ is a large scale Lipschitz map it suffices to show that there is a constant C such that whenever $d_1(s,t) = 1$, $\| f_\epsilon(s) - f_\epsilon(t) \|^2 \leq C$.

Lemma 8 *For each $\epsilon \in (0, 1/2)$ there is a constant C such that for any vertices $s, t \in X^{(0)}$ with $d_1(s,t) = 1$ we have $\| f_\epsilon(s) - f_\epsilon(t) \|^2 \leq C$.*

PROOF. Let h_0 be the hyperplane cutting the edge joining s, t. Assume, without loss of generality that h separates t from v but not s from v so that $d_1(s,v) + 1 = d_1(t,v)$ and the set of hyperplanes separating t from v is the union of the set $\{h_1, \ldots, h_m\}$ of the hyperplanes separating s from v together with h.

We need to compute

$$\| f_\epsilon(s) - f_\epsilon(t) \|^2 = \sum_{i=0}^{m} [w_s(h_i)^\epsilon - w_t(h_i)^\epsilon]^2 = 1^{2\epsilon} + \sum_{i=1}^{m} [w_s(h_i)^\epsilon - w_t(h_i)^\epsilon]^2 \quad (1)$$

Now according to corollary 7 we have $|w_t(h_i) - w_s(h_i)| \leq 1$. Suppose that for a particular hyperplane h_i we have $w_s(h_i) = k$ so that $w_t(h_i)$ takes one of the values $k-1, k, k+1$ and $[w_s(h_i)^\epsilon - w_t(h_i)^\epsilon]^2$ takes one of the values $[k^\epsilon - (k+1)^\epsilon]^2, [k^\epsilon - k^\epsilon]^2, [k^\epsilon - (k-1)^\epsilon]^2$

An elementary calculation of the first derivative shows that the function $X \mapsto [X^\epsilon - (X+1)^\epsilon]^2$. is strictly increasing so we have $[k^\epsilon - (k+1)^\epsilon]^2 > [(k-1)^\epsilon - k^\epsilon]^2 = [k^\epsilon - (k-1)^\epsilon]^2 > 0 = [k^\epsilon - k^\epsilon]^2$ hence we have $[w_s(h_i)^\epsilon - w_t(h_i)^\epsilon]^2 \leq [w_s(h_i)^\epsilon - (w_s(h_i)+1)^\epsilon]^2$ and so

$$\sum_{i=1}^{m} [w_s(h_i)^\epsilon - w_t(h_i)^\epsilon]^2 \leq \sum_{i=1}^{m} [w_s(h_i)^\epsilon - (w_s(h_i)+1)^\epsilon]^2. \quad (2)$$

We can split the final sum as a double sum taken over all hyperplanes with a given s-weight. Let J denote the set of all s-weights.

$$\begin{aligned} \sum_{i=1}^{m} [w_s(h_i)^\epsilon - (w_s(h_i)+1)^\epsilon]^2 &= \sum_{j \in J} \sum_{w_s(h_i)=j} [w_s(h_i)^\epsilon - (w_s(h_i)+1)^\epsilon]^2 \\ &= \sum_{j \in J} \sum_{w_s(h_i)=j} [j^\epsilon - (j+1)^\epsilon]^2. \end{aligned} \quad (3)$$

Since the cube complex has dimension n we can cross at most n hyperplanes in any given cube so the number of hyperplanes with $w_s(h) = j$ is at most n for any j and

$$\sum_{j} \sum_{w_s(h_i)=j} [j^\epsilon - (j+1)^\epsilon]^2 \leq \sum_{j} n [j^\epsilon - (j+1)^\epsilon]^2.$$

Putting $w_j = j^\epsilon$ and adding additional positive terms we see that

$$\| f_\epsilon(s) - f_\epsilon(t) \|^2 \leq n \sum_{j=0}^{\infty} [w_j - w_{j+1}]^2.$$

The series $\sum_{j=0}^{\infty} [w_j - w_{j+1}]^2$ converges so putting $C = n \sum_{j=0}^{\infty} [w_j - w_{j+1}]^2$ we get $\| f_\epsilon(s) - f_\epsilon(t) \|^2 \leq C$ and f_ϵ is large scale Lipschitz as required.

4 Hilbert Space compression

While establishing that the map is large scale Lipschitz required us to show that $\| f_\epsilon(s) - f_\epsilon(t) \|^2$ is small for nearby vertices, to establish that the embedding has large asymptotic compression requires us to show that $\| f_\epsilon(s) - f_\epsilon(t) \|^2$ is relatively large for points s, t which are sufficiently far apart.

Specifically we will prove:

Lemma 9 *For any positive r and any $\epsilon \in (0, 1/2)$ there is a constant C_ϵ such that $\| f_\epsilon(s) - f_\epsilon(t) \|^2 \geq C_\epsilon r^{1+2\epsilon}$. Hence $\rho_{f_\epsilon}(r) \geq \sqrt{C_\epsilon} r^{1/2+\epsilon}$*

PROOF. Let $D = d_1(s,t) \geq r$ and assume $d(1,s) \leq d(1,t)$ so that, letting m denote the median of the triple $1, s, t$, we have $d(m,t) \geq d(m,s)$. It follows that $d(m,t) \geq \sharp(\frac{D}{2}) \geq \sharp(\frac{r}{2})$ where $\sharp(n)$ denotes the smallest integer greater than n. Hence there are at least $\sharp(\frac{r}{2})$ hyperplanes which separate t from 1 but which do not separate s from 1. We will denote these hyperplanes $h_1, h_2, \ldots, h_{\sharp(\frac{r}{2})}$. Now consider the normal cube path $C_0, C_1, \ldots C_n$ from t to 1. As noted in lemma 5 each of the hyperplanes h_i must intersect exactly one of the cubes C_j, and by definition $w_t(h_i) = (j+1)$. By relabelling if necessary we may assume that the t-weight increases (not necessarily strictly) with the index i of the hyperplane, and given that the cube complex has dimension n at most n of the hyperplanes can have the same t-weight, i.e., at most n of the hyperplanes have weight 1^ϵ and the others have weight at least 2^ϵ; at most n of the remaining hyperplanes can have weight 2^ϵ and the others have to have weight at least 3^ϵ and so on. Recall that for each of these hyperplanes $w_s(h_i) = 0$ by hypothesis so, writing $\sharp(\frac{r}{2}) = kn + m$ for some integer $0 \leq m < n$ we have

$$\| f_\epsilon(s) - f_\epsilon(t) \|^2 \geq w_t(h_1)^{2\epsilon} + \ldots + w_t(h_{\sharp(\frac{r}{2})})^{2\epsilon} \geq n(1^{2\epsilon} + 2^{2\epsilon} + \ldots + k^{2\epsilon}) + m(k+1)^{2\epsilon}.$$

We will now show that the RHS of this equation is greater than the expression

$$\begin{aligned}
&\frac{1}{n}\left(1^{2\epsilon} + 2^{2\epsilon} + \ldots + \sharp(\frac{r}{2})^{2\epsilon}\right) \\
= &\frac{1}{n}(1^{2\epsilon} + \ldots + n^{2\epsilon} + (n+1)^{2\epsilon} + \ldots + (2n)^{2\epsilon} + (2n+1)^{2\epsilon} + \ldots \\
&+ (kn)^{2\epsilon} + (kn+1)^{2\epsilon} + \ldots + (kn+m))^{2\epsilon})
\end{aligned}$$

Claim: For any $i \geq 1$,

$$ni^{2\epsilon} > \frac{1}{n}[((i-1)n+1)^{2\epsilon} + \ldots + (in)^{2\epsilon}]$$

Since $\epsilon < \frac{1}{2}$ and $n \geq 1$ we have $ni^{2\epsilon} > n^{2\epsilon}i^{2\epsilon} = (in)^{2\epsilon}$

On the other hand, since $\epsilon > 0$ and $in > ik$ for all $k < n$ we have

$$\frac{1}{n}[((i-1)n+1)^{2\epsilon} + \ldots + (in)^{2\epsilon}] < \frac{1}{n}(n(in)^{2\epsilon}) = (in)^{2\epsilon}$$

So

$$ni^{2\epsilon} > (in)^{2\epsilon} > \frac{1}{n}[((i-1)n+1)^{2\epsilon} + \ldots + in^{2\epsilon}]$$

Claim:

$$m(k+1)^{2\epsilon} > \frac{1}{n}\left((kn+1)^{2\epsilon} + \ldots + (kn+m)^{2\epsilon}\right)$$

We have that

$$m(k+1)^{2\epsilon} \geq m^{2\epsilon}(k+1)^{2\epsilon} = (mk+m)^{2\epsilon}$$

Looking at the RHS of the statement of the claim we have:

$$\begin{aligned}
\frac{1}{n}\left((kn+1)^{2\epsilon} + \ldots + (kn+m)^{2\epsilon}\right) &< \frac{m}{n}(kn+m)^{2\epsilon} \quad \text{(since (kn + m) is the biggest term)} \\
&< \left(\frac{m}{n}\right)^{2\epsilon}(kn+m)^{2\epsilon} \quad (\text{since} \frac{m}{n} < 1) \\
&= (mk + \frac{m^2}{n})^{2\epsilon} \\
&< (mk+m)^{2\epsilon} \quad (\text{since} \frac{m}{n} < 1)
\end{aligned}$$

And so

$$m(k+1)^{2\epsilon} \geq (mk+m)^{2\epsilon} > \frac{1}{n}\left((kn+1)^{2\epsilon} + \ldots + (kn+m)^{2\epsilon}\right)$$

Putting both claims together, we have that:

$$
\begin{aligned}
n * 1^{2\epsilon} &> \frac{1}{n}\left(1^{2\epsilon} + \ldots + n^{2\epsilon}\right) \\
n * 2^{2\epsilon} &> \frac{1}{n}\left((n+1)^{2\epsilon} + \ldots + (2n)^{2\epsilon}\right) \\
&\vdots \\
n * k^{2\epsilon} &> \frac{1}{n}\left(((k-1)n+1)^{2\epsilon} + \ldots + (kn)^{2\epsilon}\right) \\
m * (k+1)^{2\epsilon} &\geq \frac{1}{n}\left((kn+1)^{2\epsilon} + \ldots + (kn+m)^{2\epsilon}\right)
\end{aligned}
$$

And so

$$
n * 1^{2\epsilon} + n * 2^{2\epsilon} + \ldots + n * k^{2\epsilon} + m(k+1)^{2\epsilon} > \frac{1}{n}\left(1^{2\epsilon} + 2^{2\epsilon} + \ldots + \sharp(\frac{r}{2})^{2\epsilon}\right)
$$

Hence,

$$
\begin{aligned}
\| f_\epsilon(s) - f_\epsilon(t) \|^2 &\geq w_t(h_1)^{2\epsilon} + \ldots + w_t(h_{\sharp(\frac{r}{2})})^{2\epsilon} \\
&\geq n * 1^{2\epsilon} + n * 2^{2\epsilon} + \ldots + n * k^{2\epsilon} + m(k+1)^{2\epsilon} \\
&\geq \frac{1}{n}(1^{2\epsilon} + 2^{2\epsilon} + \ldots + \sharp(\frac{r}{2})^{2\epsilon})
\end{aligned}
$$

In [6] Guentner and Kaminker showed that $(1^{2\epsilon}+2^{2\epsilon}+\ldots+\sharp(\frac{r}{2})^{2\epsilon}) \geq \frac{r^{2\epsilon+1}}{(2^{2\epsilon+1})(2\epsilon+1)}$ so putting $C_\epsilon = \frac{1}{n(2^{2\epsilon+1})(2\epsilon+1)}$ we obtain, as required,

$$
\| f_\epsilon(s) - f_\epsilon(t) \|^2 \geq C_\epsilon r^{2\epsilon+1}
$$

Now we obtain:

Lemma 10 *For each ϵ the asymptotic compression of the map f_ϵ is at least $1/2+\epsilon$.*

PROOF. We have

$$
R_{f_\epsilon} = \liminf_{r\to\infty} \frac{\log \rho_{f_\epsilon}(r)}{\log r} \geq \liminf_{r\to\infty} \frac{\log \sqrt{C_\epsilon} r^{1/2+\epsilon}}{\log r} = 1/2 + \epsilon
$$

5 Exactness for groups acting properly and co-compactly on a CAT(0) cube complex

The Hilbert Space compression of an unbounded metric space is defined to be the supremum of the asymptotic compression of all possible large scale Lipschitz maps from the metric space to a Hilbert Space so putting together the results of sections 3 and 4 we get

Theorem 11 *The Hilbert Space compression of an unbounded finite dimensional CAT(0) cube complex* (X, d) *is* 1.

PROOF. For each $\epsilon \in (0, 1/2)$ we have constructed a large scale Lipschitz embedding f_ϵ of the metric space $(X^{(0)}, d_1)$ into the Hilbert Space $\ell^2(H, \mathbb{R})$ with compression at least $1/2 + \epsilon$. Hence the Hilbert Space compression of $(X^{(0)}, d_1)$ is 1. Since X is finite dimensional, of dimension n say, we have $d(s,t) \leq d_1(s,t) \leq \sqrt{n}d(s,t)$ so $(X^{(0)}, d_1)$ is quasi-isometric to (X, d), and since Hilbert Space compression is a quasi-isometry invariant we obtain the result.

Now suppose G s a group acting properly and co-compactly on an unbounded CAT(0) cube complex X. Choose a finite generating set for G and regard G as a metric space via the edge metric on the Cayley graph. Then G is quasi-isometric to (X, d). Again by quasi-isometry invariance we obtain

Corollary 12 *Let G be a finitely generated group regarded as a metric space via the word metric with respect to some finite generating set. If G acts properly and co-compactly on an unbounded CAT(0) cube complex then G has Hilbert Space compression* 1.

Finally since Guentner and Kaminker showed that a discrete group with Hilbert Space compression strictly greater than $1/2$ is exact we obtain:

Theorem 13 *If G is a group acting properly and co-compactly on a CAT(0) cube complex then G is exact and therefore has Yu's Property A.*

As noted in the introduction, if G acts properly on a bounded CAT(0) cube complex then G is finite and therefore exact, so the hypothesis that the cube complex should be unbounded (which was only inserted in the supporting results in order to ensure that asymptotic compression can be defined) would be superfluous here.

References

[1] C. Anantharaman-Delaroche, J. Renault, Monographies de L'Enseignement Mathématique, 36. L'Enseignement Mathmatique, Geneva, 2000.

[2] I. R. Aitchison, J. H. Rubinstein, An introduction to polyhedral metrics of non-positive curvature on 3-manifolds in "Geometry of Low-dimensional Manifolds: 2 Symplectic Manifolds and Jones-Witten Theory", London Mathematical Society Lecture Notes Series 151, Cambridge University Press (1990) 127–161.

[3] I. R. Aitchison, J. H. Rubinstein, Canonical surgery on alternating link diagrams in Knots 90, (1992) 543–558.

[4] M.R. Bridson and A. Haefliger, Metric spaces of non-positive curvature, Grundlehren der mathematischen Wissenschaften Volume 319, Springer, 1999.

[5] M. Gromov, Hyperbolic groups in S.M. Gersten (ed.) Essays on Group Theory (Mathematical Sciences Research Institute Publications) Springer-Verlag 1987.

[6] E. Guentner, J. Kaminker, Exactness and the Novikov conjecture. Topology 41 (2002), no. 2, 411–418; Addendum: Topology 41 (2002) no. 2, 419–420.

[7] E. Guentner, J. Kaminker, Exactness and Uniform Embeddability of Discrete Groups, ArXiv preprint math.OA/0309166.

[8] N. Higson, J. Roe, Amenable group actions and the Novikov conjecture. J. Reine Angew. Math. 519 (2000) 143–153.

[9] Kirchberg, Wasserman, Exact Groups and Continuous Bundles of C^*-algebras, Mathematische Annalen, 1999.

[10] G. Niblo, M. Roller, Groups acting on cubes and Kazhdan's property (T). Proc. Amer. Math. Soc. **126** (1998), no. 3, 693–699.

[11] G.A .Niblo, L.D. Reeves. The geometry of cube complexes and the complexity of their fundamental groups,Topology, Vol. 37,No 3 (1998) 621–633.

[12] G.A.Niblo, L.D.Reeves. Groups acting on CAT(0) cube complexes, Geometry and Topology, Vol. 1 (1997) 1–7.

[13] G.A.Niblo, L.D.Reeves. Coxeter groups act on CAT(0) cube complexes, Journal of Group Theory, 6, (2003) 309–413.

[14] N. Ozawa, Amenable actions and exactness for discrete groups, *Comptes Rendus Acad. Sci. Paris* **330** (2000) 691–695.

[15] M. A. Roller, Poc Sets, Median Algebras and Group Actions. An extended study of Dunwoody's construction and Sageev's theorem *Habilitationeschrift, Regensberg* 1998.

[16] M. Sageev, Ends of group pairs and non-positively curved cube complexes. Proc. London Math. Soc. (3), 71(3) (1995) 585–617.

[17] D.T. Wise, Cubulating Small Cancellation Groups, Preprint, http://www.gidon.com/dani/tl.cgi?athe=pspapers/SmallCanCube.ps.

[18] G. Yu, The Novikov conjecture for groups with finite asymptotic dimension. Ann. of Math. (2) 147 (1998), no. 2, 325–355.

[19] G. Yu, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert Space. Invent. Math. 139 (2000), no. 1, 201–240.