Minimal cubings

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Abstract

We combine ideas of Scott and Swarup on good position for almost invariant subsets of a group with ideas of Sageev on constructing cubings from such sets. We construct cubings which are more canonical than in Sageev's original construction. We also show that almost invariant sets can be chosen to be in very good position.

Let G be a finitely generated group, and let $H_1, \ldots, H_n$ be subgroups. For $i = 1, \ldots, n$, let X_i be a nontrivial H_i–almost invariant subset of G. In [6], Sageev gave a natural construction of a cubing $C(X_1, \ldots, X_n)$ with a G–action which reflects the way in which the translates of the X_i's intersect each other.

In order to give the reader a feel for this, we start by discussing a simple and closely related topological example. For other simple examples, the reader

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is referred to Sageev's paper [6]. Consider a finite family $\mathcal{F} = \{S_1, \ldots, S_n\}$ of compact curves in general position on an orientable surface M. There is a natural way to produce a 2–dimensional cubed complex $C(\mathcal{F})$ which reflects how the S_i's intersect each other. Let $\widetilde{M}$ denote the universal cover of M, let $\widetilde{\mathcal{F}}$ denote the pre-image of $\mathcal{F}$ in $\widetilde{M}$, and let D denote the collection of double points of the curves in $\widetilde{\mathcal{F}}$. Then $C(\mathcal{F})$ is the dual 2–complex to $\widetilde{\mathcal{F}}$ in $\widetilde{M}$. This means that $C(\mathcal{F})$ lies in $\widetilde{M}$, has one vertex in each component of $\widetilde{M} - \widetilde{\mathcal{F}}$, and for each segment of $\widetilde{\mathcal{F}} - D$ it has an edge which crosses this segment and no other and joins two vertices of $C(\mathcal{F})$. Further, for each point of the double set D, there is a square which contains that point and is a 2–cell of $C(\mathcal{F})$, and these are the only 2–cells of $C(\mathcal{F})$.

Now let G denote $\pi_1(M)$. If we assume that each S_i is essential in M, then S_i has an associated nontrivial H–almost invariant subset X_i of G, where H equals $\pi_1(S_i)$, so that H is trivial or infinite cyclic. There is a close connection between $C(\mathcal{F})$ and Sageev's cubing $C(X_1, \ldots, X_n)$, but in general these cubings are very different. Recall that $\widetilde{\mathcal{F}}$ consists of lines in $\widetilde{M}$. Both cubings encode information about how the lines of $\widetilde{\mathcal{F}}$ intersect. If one considers two lines of $\widetilde{\mathcal{F}}$, the cubing $C(\mathcal{F})$ encodes very detailed information about how they intersect, as it has a square for each double point, but the cubing $C(X_1, \ldots, X_n)$ encodes only the information about whether or not they intersect. On the other hand, if one has a family of k distinct lines in $\widetilde{\mathcal{F}}$, where $k \geq 3$, and if each line in the family meets all the others, then $C(X_1, \ldots, X_n)$ has a corresponding k–cube, but this information is not encoded in $C(\mathcal{F})$ at all. However, if we assume that each component of $M - \mathcal{F}$ is not simply connected, then the two cubings are equal. Note that this assumption implies that no component of $\widetilde{M} - \widetilde{\mathcal{F}}$ is compact, so that $\widetilde{\mathcal{F}}$ consists of embedded lines, and any pair of these lines meets transversely in at most one point. Further there is no triple of distinct lines such that each line meets the other two.

It is clear that $C(\mathcal{F})$ depends crucially on the precise configuration of the S_i's in M. For example, if the S_i's are disjoint, then $C(\mathcal{F})$ is 1–dimensional, but if we homotop the S_i's to meet each other, then $C(\mathcal{F})$ becomes 2–dimensional. Thus $C(\mathcal{F})$ is not an invariant of the homotopy classes of the curves in $\mathcal{F}$. A similar phenomenon occurs with $C(X_1, \ldots, X_n)$. Of course, one cannot talk of almost invariant sets being homotopic, but there is a natural idea of equivalence of almost invariant sets which corresponds to the idea of homotopy of the S_i's. For many groups G, it is easy to give examples where $C(X_1, \ldots, X_n)$ is 1–dimensional, but if we replace each X_i by an equivalent set Y_i, the cubing $C(Y_1, \ldots, Y_n)$ is at least 2–dimensional. Thus Sageev's cubing depends crucially on the precise choice of the X_i's, and is not an invariant of the equivalence classes of the X_i's.

In this paper, we consider the case when each of the H_i's is finitely generated and we show how to construct a cubing $L(X_1, \ldots, X_n)$ which in most cases depends only on the equivalence classes of the X_i's, i.e. replacing the X_i's by equivalent almost invariant sets yields the same cubing. The cubing we obtain is thus more canonical than $C(X_1, \ldots, X_n)$. We also show that it embeds naturally and equivariantly in $C(X_1, \ldots, X_n)$ and that it is minimal in a natural sense.

Sageev's original construction depended on the partial order on the X_i's given by inclusion. Our construction in this paper uses Sageev's ideas but replaces the partial order of inclusion by a partial order on the X_i's which is based on 'almost inclusion'. Such a partial order was introduced by Scott in [7] in a topological context, and it played a basic role in the purely algebraic work of Scott and Swarup in [9] and [11]. In order to define this partial order, the X_i's need to satisfy a technical condition which Scott and Swarup called "good position". In [11], they showed how to replace any finite family of almost invariant subsets of a group by a family of equivalent almost invariant subsets which are in good position. In this paper, we introduce an idea which we call "very good position" for almost invariant sets which is analogous to the properties possessed by shortest curves on surfaces or by least area surfaces in 3–manifolds. We discuss these analogies in section 4. We use our new cubing to show that any finite family of almost invariant subsets of a group can be replaced by a family of equivalent almost invariant subsets which are in very good position. We also show how to apply these ideas to strengthen some results of Niblo [3] and of Dunwoody and Roller [2].

1 Preliminaries

1.1 Almost invariant sets

In this section, we recall the definition of an almost invariant subset of a finitely generated group G, and we introduce some basic related ideas. Throughout this paper, we will always assume that G is finitely generated. We will need several definitions which we take from [9], but see [8] for a discussion.

Definition 1.1 *Two sets P and Q are almost equal if their symmetric difference $P - Q \cup Q - P$ is finite. We write $P \overset{a}{=} Q$.*

Definition 1.2 *If a group G acts on the right on a set Z, a subset P of Z is almost invariant if $Pg \overset{a}{=} P$ for all g in G. An almost invariant subset P of Z is nontrivial if P and its complement $Z - P$ are both infinite. The complement $Z - P$ will be denoted simply by P^*, when Z is clear from the context.*

This idea is connected with the theory of ends of groups via the Cayley graph Γ of G with respect to some finite generating set of G. (Note that in this paper groups act on the left on covering spaces and, in particular, G acts on its Cayley graph on the left.) Using $\mathbb{Z}_2$ as coefficients, we can identify 0–cochains and 1–cochains on Γ with sets of vertices or edges. A subset P of G represents a set of vertices of Γ which we also denote by P, and it is a beautiful fact, due to Cohen [1], that P is an almost invariant subset of G if and only if δP is finite, where δ is the coboundary operator in Γ. Thus G has a nontrivial almost invariant subset if and only if the number of ends $e(G)$ of G is at least 2. Further $e(G)$ can be identified with the number of nontrivial almost invariant subsets of G, when this count is made correctly. If H is a subgroup of G, we let $H\backslash G$ denote the set of cosets Hg of H in G, i.e. the quotient of G by the left action of H. Of course, G will no longer act on the left on this quotient, but it will still act on the right. Thus we have the idea of an almost invariant subset of $H\backslash G$. Further, P is an almost invariant subset of $H\backslash G$ if and only if δP is finite, where δ is the coboundary operator in the graph $H\backslash\Gamma$. Thus $H\backslash G$ has a nontrivial almost invariant subset if and only if the number of ends $e(G,H)$ of the pair (G,H) is at least 2. Considering the pre-image X in G of an almost invariant subset P of $H\backslash G$ leads to the following definitions.

Definition 1.3 *If G is a finitely generated group and H is a subgroup, then a subset X of G is H–*almost invariant *if X is invariant under the left action of H, and simultaneously $H\backslash X$ is an almost invariant subset of $H\backslash G$. We may also say that X is* almost invariant *over H. In addition, X is a* nontrivial *H–almost invariant subset of G, if the quotient sets $H\backslash X$ and $H\backslash X^*$ are both infinite.*

Remark 1.4 *Note that if X is a nontrivial H–almost invariant subset of G, then $e(G,H)$ is at least 2, as $H\backslash X$ is a nontrivial almost invariant subset of $H\backslash G$. In fact $e(G,H)$ can be identified with the number of nontrivial H–almost invariant subsets of G, when this count is made correctly. See [12] for details.*

Definition 1.5 *If G is a group and H is a subgroup, then a subset W of G is H–*finite *if it is contained in the union of finitely many left cosets Hg of H in G.*

Definition 1.6 *If G is a group and H is a subgroup, then two subsets V and W of G are H–*almost equal *if their symmetric difference is H–finite.*

It will also be convenient to avoid this rather clumsy terminology sometimes, particularly when the group H is not fixed, so we make the following definition.

Definition 1.7 *If X is a H–almost invariant subset of G and Y is a K–almost invariant subset of G, and if X and Y are H–almost equal, then we will say that X and Y are* equivalent *and write $X \sim Y$.*

Remark 1.8 *Note that H and K must be commensurable, so that X and Y are also K–almost equal and $(H \cap K)$–almost equal.*

A more elegant and equivalent formulation is that X is equivalent to Y if and only if each is contained in a bounded neighbourhood of the other. In the context of the study of quasi-isometries, two such sets are called coarsely equivalent.

Equivalence is important because usually one is interested in an equivalence class of almost invariant subsets of a group rather than a specific such subset.

The next definitions makes precise the notion of crossing of almost invariant sets. This is an algebraic analogue of crossing of codimension–1 manifolds, but it ignores "inessential" crossings.

Definition 1.9 *Let X be a H–almost invariant subset of G and let Y be a K–almost invariant subset of G. The four sets $X \cap Y$, $X^* \cap Y$, $X \cap Y^*$ and $X^* \cap Y^*$ are called the* corners *of the pair (X, Y).*

Definition 1.10 *Let X be a H–almost invariant subset of G and let Y be a K–almost invariant subset of G. We will say that Y* crosses *X if each of the four corners of the pair (X, Y) is not H–finite. Thus each of the four corners projects to an infinite subset of $H \backslash G$.*

The motivation for the above definition is that when one of the four corners is empty, we clearly have no crossing, and if one of the four corners is "small", then we have "inessential crossing". Note that Y may be a translate of X in which case such crossing corresponds to the self-intersection of a single immersion.

Remark 1.11 *It is shown in [8] that if X and Y are nontrivial, then $X \cap Y$ is H–finite if and only if it is K–finite. It follows that crossing of nontrivial almost invariant subsets of G is symmetric, i.e. that X crosses Y if and only if Y crosses X.*

Definition 1.12 *Let U be a nontrivial H–almost invariant subset of G and let V be a nontrivial K–almost invariant subset of G. We will say that $U \cap V$ is* small *if it is H–finite.*

Remark 1.13 *This terminology will be extremely convenient, particularly when we want to discuss translates U and V of X and Y, as we do not need to mention the stabilisers of U or of V. However, the terminology is symmetric in U and V and makes no reference to H or K, whereas the definition is not symmetric and does refer to H, so some justification is required. If U is also H'–almost invariant for a subgroup H' of G, then H' must be commensurable with H. Thus $U \cap V$ is H–finite if and only if it is H'–finite. In addition, Remark 1.11 tells us that $U \cap V$ is H–finite if and only if it is K–finite. This provides the needed justification of our terminology.*

In the context of the study of quasi-isometries, the terminology "deep" is used for a subset of a metric space which contains balls of arbitrarily large radius. One can show that $U \cap V$ is H–infinite if and only if it is deep in this sense.

1.2 Cubings

We review here the construction in [6], to which the reader is referred for details (see also [4]).

A *cubing* C is a $CAT(0)$ cubical complex. If σ is an n–dimensional cube in C, viewed as a standard unit cube in $\mathbb{R}^n$ and $\hat{\sigma}$ denotes the barycentre of σ, then a *dual cube* in σ is the intersection with σ of an $(n-1)$–dimensional plane running through $\hat{\sigma}$ and parallel to one of the $(n-1)$–dimensional faces of σ. Given a cubing, one may consider the equivalence relation on edges generated by the relation which declares two edges to be equivalent if they are opposite sides of a square in C. Now given an equivalence class of edges, the *hyperplane* associated to this equivalence class is the collection of dual cubes whose vertices lie on edges in the equivalence class. It is not hard to show that hyperplanes are totally geodesic subspaces. Moreover, in [6] it is shown that hyperplanes do not self-intersect (i.e. a hyperplane meets a cube in a single dual cube) and that a hyperplane separates a cubing into precisely two components, which we call the *half-spaces* associated to the hyperplane.

Consider a finitely generated group G with subgroups $H_1, \ldots, H_n$. For $i = 1, \ldots, n$, let X_i be a nontrivial H_i–almost invariant subset of G, and let $E = \{gX_i, gX_i^* : g \in G, 1 \leq i \leq n\}$. In [6], Sageev gave a construction of a cubing from the set E equipped with the partial order given by inclusion. We need the following definition.

Definition 1.14 *Let E be a partially ordered set, equipped with an involution $A \to A^*$ such that $A \neq A^*$, and if $A \leq B$ then $B^* \leq A^*$. An ultrafilter V on E is a subset of E satisfying*

1. *For every $A \in E$, we have $A \in V$ or $A^* \in V$ but not both.*
2. *If $A \in V$ and $A \leq B$ then $B \in V$.*

Sageev constructs a cubed complex K whose vertex set $K^{(0)}$ is the collection of all ultrafilters on E. There is a natural action of G on K, and Sageev shows that a certain component C of K is G–invariant and a cubing.

Let $K^{(0)}$ denote the collection of all ultrafilters on E. Construct $K^{(1)}$ by attaching an edge to two vertices $V, V' \in K^{(0)}$ if and only if they differ by replacing a single element by its complement, i.e. there exists $A \in V$ such that $V' = (V - \{A\}) \cup \{A^*\}$. Note that the fact that V and V' are both ultrafilters implies that A must be a minimal element of V. Also if A is a minimal element of V, then the set $V' = (V - \{A\}) \cup \{A^*\}$ must be an ultrafilter on E. Now attach 2–dimensional cubes to $K^{(1)}$ to form $K^{(2)}$, and inductively attach n–cubes to $K^{(n-1)}$ to form $K^{(n)}$. All such cubes are attached by an isomorphism of their boundaries and, for each $n \geq 2$, one n–cube is attached to $K^{(n-1)}$ for each occurrence of the boundary of an n–cube appearing in $K^{(n-1)}$. The complex K constructed in this way will not be connected, but one special component can be picked out in the following way. For each element g of G, define the ultrafilter $V_g = \{A \in E : g \in A\}$. These special vertices of K are called *basic*. Two basic vertices V and V' of K differ on only finitely many complementary pairs of elements of E, so that there exist elements $A_1, \ldots, A_n$ of E which lie in V such that V' can be obtained from V by replacing each A_i by A_i^*. By re-ordering the A_i's if needed, we can arrange that A_1 is a minimal element among the A_i's. It follows that $V_1 = (V - \{A_1\}) \cup \{A_1^*\}$ is also an ultrafilter on E, and so is joined to V by an edge of K. By repeating this argument, we will find an edge path in K of length n which joins V and V'. It follows that the basic vertices of K all lie in a single component C. As the collection of all basic vertices is preserved by the action of G on K, it follows that this action preserves C. Finally, Sageev shows in [6] that C is $CAT(0)$ and hence is a cubing.

At first sight, one might think that C should equal K. To show that this will not be the case, here are two examples.

Example 1.15 *Let E be the family of subsets of the integers $\mathbb{Z}$ of the form $\{x \in \mathbb{Z} : x \leq a\}$ or $\{x \in \mathbb{Z} : x \geq b\}$, with the partial order given by inclusion and the involution given by reflection in the endpoint. Let K and C be constructed as above. Let V denote the ultrafilter on E which consists of all element of E of the form $\{x \in \mathbb{Z} : x \leq a\}$. Then V is not basic. In fact, V differs from any basic ultrafilter V_g on infinitely many elements, so that V is not a vertex of C. Further, as V has no minimal elements, it constitutes an entire component of K.*

The second example is closely related to the first, but may seem more interesting to topologists.

Example 1.16 *Let E be the family of all closed half-spaces in the hyperbolic plane $\mathbb{H}^2$, with the partial order given by inclusion and the involution given by reflection of a half-space in its boundary line. Let K and C be constructed as above. Let w denote a point on the circle at infinity of $\mathbb{H}^2$, and let V_w denote the elements of E whose closure contains w. Then V_w is not basic, and as V_w differs from any basic ultrafilter V_g on infinitely many elements, it follows that V_w is not a vertex of C. Further, as V has no minimal elements, it constitutes an entire component of K.*

As noted in Roller's survey article [5], one can characterise the vertices of C as being those ultrafilters on E which satisfy the descending chain condition. Note that the ultrafilters V and V_w in the above two examples obviously do not satisfy the descending chain condition.

An important aspect of Sageev's construction is that one can recover the elements of E from the action of G on the cubing C. Recall that an edge f of C joins two vertices V and V' if and only if there exists $A \in V$ such that $V' = (V - \{A\}) \cup \{A^*\}$. If f is oriented towards V', we will say that f *exits* A. We let $\mathcal{H}_A$ denote the hyperplane associated to the equivalence class of f. This equivalence class consists of all those edges of C which, when suitably oriented, exit A. Now let X denote an H–almost invariant subset of G which is an element of E, such that X contains the identity e of G. Thus X lies in the basic vertex $V_e = \{A \in E : e \in A\}$. As X^* is non-empty, it contains some element k and so lies in the basic vertex V_k. Now any path joining V_e to V_k must contain an edge which exits X, so we can define the hyperplane $\mathcal{H}_X$ as above. Let $\mathcal{H}_X^+$ denote the half-space determined by $\mathcal{H}_X$ which contains the basic vertex V_e. Recall that an edge of C lies in the equivalence class which determines $\mathcal{H}_X$ if and only if it exits X when suitably oriented. It follows that a vertex V of C lies in $\mathcal{H}_X^+$ if and only if $X \in V$. Now we claim that the subset $\{g \in G : gV_e \in \mathcal{H}_X^+\}$ of G equals X. For

$$\begin{aligned}\{g \in G : gV_e \in \mathcal{H}_X^+\} &= \{g \in G : X \in gV_e\} = \{g \in G : g^{-1}X \in V_e\} \\ &= \{g \in G : e \in g^{-1}X\} = \{g \in G : g \in X\} = X.\end{aligned}$$

The following result implies that if we consider a vertex V of C other than V_e, then the subset $\{g \in G : gV \in \mathcal{H}_X^+\}$ of G is still H–almost invariant, and although it need not be equal to X, it is still equivalent to X.

Lemma 1.17 *Suppose that G is a finitely generated group which acts on a cubing C. Let $\mathcal{H}$ be a hyperplane in C with stabilizer H, let $\mathcal{H}^+$ and $\mathcal{H}^-$ denote the two half-spaces defined by $\mathcal{H}$, and suppose that H preserves each of $\mathcal{H}^+$ and $\mathcal{H}^-$. Then, for any vertex v, the set $X_v = \{g \in G | gv \in \mathcal{H}^+\}$ is almost invariant over H and all these subsets of G are equivalent.*

Proof. We need to show that $hX_v = X_v$, for all h in H, and that $X_v a$ and X_v are H–almost equal for all a in G.

As H stabilises $\mathcal{H}^+$, it follows immediately that $hX_v = X_v$, for all h in H.

Next consider $X_v - X_v a$. From the definition of X_v, we have that

$$X_v a = \{ga \in G | gv \in \mathcal{H}^+\} = \{g' \in G | g' a^{-1} v \in \mathcal{H}^+\}.$$

Hence

$$X_v - X_v a = \{g \in G | gv \in \mathcal{H}^+ \text{ and } ga^{-1}v \notin \mathcal{H}^+\}.$$

Thus $g \in X_v - X_v a$ if and only if $\mathcal{H}$ separates gv from $ga^{-1}v$. Now there are only finitely many hyperplanes in C which separate v from $a^{-1}v$. We denote these hyperplanes by $\mathcal{H}_1, \ldots, \mathcal{H}_n$. It follows that if $g \in X_v - X_v a$, then $\mathcal{H} = g\mathcal{H}_i$, for some i. For any two elements g and g' such that $\mathcal{H} = g\mathcal{H}_i$ and $\mathcal{H} = g'\mathcal{H}_i$, we have that $g'g^{-1}\mathcal{H} = \mathcal{H}$, so that $g'g^{-1} \in H$ and $Hg = Hg'$. It follows that $X_v - X_v a$ is contained in HF for some finite set F, and so is H–finite. Similarly, $X_v a - X_v$ is H–finite. As this holds for any element a of G, it follows that X_v is almost invariant over H, as required.

Now let v and w denote two vertices of C, and let k be an element of $X_v - X_w$. Thus $kv \in \mathcal{H}^+$ and $kw \notin \mathcal{H}^+$. Hence $\mathcal{H}$ separates kv from kw, so that $k^{-1}\mathcal{H}$ separates v and w. As in the above argument, it follows that $X_v - X_w$ is H–finite. Similarly, $X_w - X_v$ is H–finite. It follows that X_v and X_w are equivalent, which completes the proof of the lemma. ∎

2 The new partial order

In this section, we recall some of the ideas of Scott and Swarup in [11] and [9].

Consider a finitely generated group G with finitely generated subgroups $H_1, \ldots, H_n$. For $i = 1, \ldots, n$, let X_i be a nontrivial H_i–almost invariant subset of G, and let $E = \{gX_i, gX_i^* : g \in G, 1 \leq i \leq n\}$. As E is a collection of subsets of G, it has a natural partial order by inclusion. But one can sometimes define a more interesting partial order. The idea is to define $U \leq V$ when U is "nearly" contained in V. Precisely, we want $U \leq V$ if $U \cap V^*$ is small. However, an obvious difficulty arises when two of the corners $U^{(*)} \cap V^{(*)}$ are small, as we have no way of deciding between two possible inequalities. It turns out that we can avoid this difficulty if we know that whenever two of the corners of U and V are small, then one of them is empty. Thus we consider the following condition on E:

Condition (*): If U and V are in E, and two of their corners are small, then one of their corners is empty.

If E satisfies Condition (*), we will say that the family $X_1 \ldots, X_n$ is in *good position*.

Assuming that this condition holds, we can define a relation $\leq$ on E by saying that $U \leq V$ if and only if $U \cap V^*$ is empty or is the only small set among the four corners of U and V. Despite the seemingly artificial nature of this definition, one can show that $\leq$ is a partial order on E. This is not entirely trivial, but the proof is in Lemma 1.14 of [9]. Condition (*) plays a key role in the proof. If $U \leq V$ and $V \leq U$, it is easy to see that we must have $U = V$, using the fact that E satisfies Condition (*). Most of the proof of Lemma 1.14 of [9] is devoted to showing that $\leq$ is transitive.

We will need the following fact, which follows immediately from Lemma 2.31 of [11]. Note that the number D is independent of the element g of G.

Lemma 2.1 *Let G be a finitely generated group with finitely generated subgroups H and K, a nontrivial H–almost invariant subset A and a nontrivial K–almost invariant subset U. Let Γ denote the Cayley graph of G with respect to some finite generating set. Then there is $D > 0$, such that if $gU \leq A$, then gU is contained in the D–neighbourhood of A in Γ.*

Remark 2.2 *This result will play a key role in our construction of a cubing in section 3. This explains why we need to restrict our attention to almost invariant subsets of G which are over finitely generated subgroups.*

In general, the family $X_1, \ldots, X_n$ need not be in good position, but we will use the results in [9] to show that we can find almost invariant sets $Y_1, \ldots, Y_n$ such that Y_i is equivalent to X_i and the Y_i's are in good position. We will also show that the partial order obtained is unique in most cases. We did not state such results in [9], as we were concentrating on almost invariant sets associated to splittings, but all the arguments needed are essentially there.

It turns out that the case when $n = 1$ contains almost all of the difficulties, so we will start by discussing that case. Let H be a finitely generated subgroup of G, and let X be a H–almost invariant subset of G. If X is not in good position, there must be two translates U and V of X such that two of their corners are small, and neither is empty. If $U \cap V$ is one of the two small corners, the other must be $U^* \cap V^*$, as otherwise U or V would be small which contradicts the fact that X is nontrivial. Similarly, if $U \cap V^*$ is one of the two small corners, the other must be $U^* \cap V$. It follows that U is equivalent to V or to V^*. This naturally leads one to consider the subgroup $\mathcal{K}(X)$ of G defined by $\mathcal{K}(X) = \{g \in G : gX \sim X \text{ or } X^*\}$. It will also be convenient to consider the subgroup $\mathcal{K}_0(X) = \{g \in G : gX \sim X\}$ of $\mathcal{K}$, so that the index of $\mathcal{K}_0$ in $\mathcal{K}$ is at most 2. We will say that the collection $E(X)$ of all translates of X and X^* is *nested with respect to* $\mathcal{K}$, if for any $k \in \mathcal{K}$, one of the four corners of X and kX is empty. It is clear that X is in good position if and only if $E(X)$ is nested with respect to $\mathcal{K}$. The following lemma summarises results proved by Scott and Swarup in the proof of Proposition 2.14 of [9].

Lemma 2.3 *(Scott-Swarup) Let G be a finitely generated group with a finitely generated subgroup H, and let X be a nontrivial H–almost invariant subset of G.*

1. *If H has finite index in $\mathcal{K}$, there is an almost invariant subset W of G with stabiliser $\mathcal{K}_0$ which is equivalent to X, such that $E(W)$ is nested with respect to $\mathcal{K}$.*

2. *If H has infinite index in $\mathcal{K}$, then $\mathcal{K}$ has finite index in G, and there is a subgroup H'' of $\mathcal{K}$ which is commensurable with H and normal in $\mathcal{K}$. Further, $H''\backslash\mathcal{K}$ is isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}_2 * \mathbb{Z}_2$. In the first case $\mathcal{K} = \mathcal{K}_0$, and in the second case $\mathcal{K}_0$ has index 2 in $\mathcal{K}$. There is an almost invariant subset W of G with stabiliser H'' which is equivalent to X, such that $E(W)$ is nested with respect to $\mathcal{K}$.*

Now we can prove the following result.

Lemma 2.4 *Let G be a finitely generated group with a finitely generated subgroup H, and let X be a nontrivial H–almost invariant subset of G. Then X is equivalent to a K–almost invariant subset W of G which is in good position. Thus the set $E(W)$ of all translates of W and W^* has the partial order $\leq$ described above.*

Proof. Lemma 2.3 shows that in all cases, there is an almost invariant subset W of G which is equivalent to X such that $E(W)$ is nested with respect to $\mathcal{K}(X)$. As X and W are equivalent, the subgroups $\mathcal{K}(X)$ and $\mathcal{K}(W)$ are equal, so that $E(W)$ is nested with respect to $\mathcal{K}(W)$. As remarked just before the statement of Lemma 2.3, this implies that W is in good position, which completes the proof. ■

We would like to show that the partial order obtained by applying the above result is unique. More precisely, if Y and Z are equivalent to X and in good position, we want to show that there is a G–equivariant bijection between $E(Y)$ and $E(Z)$ which preserves complementation and the partial orders. It is natural to attempt to define such a map $\varphi : E(Y) \to E(Z)$, by sending Y to Z, and extending appropriately. If it is to be G–equivariant, it must send gY to gZ for every g in G. This immediately raises a potential problem, which is that it seems possible that $gY = Y$, but $gZ \neq Z$. However the following result shows that this cannot occur.

Lemma 2.5 *Let G be a finitely generated group, let Y and Z be equivalent almost invariant subsets of G each of which is in good position. Then the stabilisers of Y and Z are equal.*

Proof. Let K and L denote the stabilisers of Y and Z respectively, so that K and L must be commensurable subgroups of G. Let k denote an element of K, so that $kY = Y$. As Z is equivalent to Y, it follows that kZ is equivalent to Z. As Z is in good position, we must have $kZ = Z$, or $Z \subset kZ$ or $kZ \subset Z$.

As K and L are commensurable, some power k^n of K must lie in L, so that $k^n Z = Z$. It follows that in all cases we must have $kZ = Z$, so that k lies in L. Thus K is contained in L. Similarly L is contained in K, so that $K = L$ as required. ∎

Now we return to the question of the uniqueness of the partial order on $E(W)$ obtained by applying Lemma 2.4. Suppose that Y and Z are equivalent to X and in good position. We want to define a bijection $\varphi : E(Y) \to E(Z)$, which is G–equivariant and preserves complementation. If φ sends Y to Z it must also send gY to gZ and gY^* to gZ^*, for every g in G. The fact that the stabilisers of Y and Z are equal implies that this gives a well defined map on the translates of Y. There is still a potential problem, which is that it seems possible that $gY = Y^*$, but $gZ \neq Z^*$. If this does not occur, it is clear that we do have a well defined map from $E(Y)$ to $E(Z)$ which is G–equivariant and preserves complementation. In order to discuss the general situation, we will use the following piece of terminology which Scott and Swarup introduced in [11].

Definition 2.6 *If X is an H–almost invariant subset of a group G, then X is invertible if there is an element g in G such that $gX = X^*$.*

Note that in [11], Scott and Swarup only used this term when X was associated to a splitting, but in this paper, we will not make that restriction.

Our previous discussion shows that if Y is not invertible, then we have a well defined map $\varphi : E(Y) \to E(Z)$, described by sending gY to gZ and gY^* to gZ^*, for every g in G. If, in addition, Z is not invertible, then the same comment applies to the inverse map showing that φ must be a bijection, which is G–equivariant and preserves complementation. It is also clear that $\varphi(U)$ is equivalent to U for every U in $E(Y)$. We will say that φ *preserves equivalence classes.*

Now we can prove our first uniqueness result for partial orders.

Lemma 2.7 *Let G be a finitely generated group with a finitely generated subgroup H. Let X be a nontrivial H–almost invariant subset of G, and suppose that X is equivalent to Y and to Z such that each of Y and Z is in good position. In addition, suppose that Y and Z are both not invertible. Then there is a G–equivariant bijection $\varphi : E(Y) \to E(Z)$ which is order preserving and preserves complementation and equivalence classes.*

Proof. As discussed above, we can define a G–equivariant bijection $\varphi : E(Y) \to E(Z)$, by sending gY to gZ and gY^* to gZ^* for every g in G, and φ also preserves complementation and equivalence classes. In many cases, φ is already order preserving, but if it is not we will describe a simple modification of φ which will arrange this.

Let U and V denote elements of $E(Y)$. As U is equivalent to $\varphi(U)$ and V is equivalent to $\varphi(V)$, it follows that a corner of U and V is small if and only if the corresponding corner of $\varphi(U)$ and $\varphi(V)$ is small. Hence U and V are comparable in $E(Y)$ if and only if $\varphi(U)$ and $\varphi(V)$ are comparable in $E(Z)$.

Further, it follows that φ is order preserving, except possibly when there are U and V such that two of the four corners of U and V are small. If this happens, then U and V must be equivalent, and we again consider the group $\mathcal{K}(X) = \{g \in G : gX \sim X \text{ or } X^*\}$. Note that as X, Y and Z are equivalent, the groups $\mathcal{K}(X)$, $\mathcal{K}(Y)$ and $\mathcal{K}(Z)$ are all equal. We denote this group by $\mathcal{K}$. We also have the subgroup $\mathcal{K}_0 = \{g \in G : gX \sim X\}$ of $\mathcal{K}$, whose index in $\mathcal{K}$ is at most 2.

Suppose that H has finite index in $\mathcal{K}$. Then part 1) of Lemma 2.3 implies that there is an almost invariant subset W of G with stabiliser $\mathcal{K}_0$ which is equivalent to X and in good position. The fact that W is in good position combined with Lemma 2.5 implies that the stabilisers of Y and Z also equal $\mathcal{K}_0$. If $\mathcal{K} = \mathcal{K}_0$, it follows that φ is order preserving, because there are no distinct equivalent elements of $E(Y)$. If $\mathcal{K}_0$ has index 2 in $\mathcal{K}$, it is possible that φ is not order preserving, so we need some special arguments. If k denotes an element of $\mathcal{K} - \mathcal{K}_0$, then kY^* is equivalent to Y. As Y is in good position, we must have $kY^* \subset Y$ or $Y \subset kY^*$. Note that as we are assuming that Y is not invertible, we cannot have $Y = kY^*$. We can suppose that $kY^* \subset Y$, by replacing k by k^{-1} and Y by Y^*, if necessary. Thus either φ is order preserving, or this fails to hold only in that $kY^* \subset Y$ but $Z \subset kZ^*$, for all $k \in \mathcal{K} - \mathcal{K}_0$. If φ is not order preserving, we replace Z by $Z' = kZ$ and we replace Y by $Y' = Y^*$. As Y' and Z' are each in good position, and equivalent to each other, there is a natural G–equivariant bijection $\varphi' : E(Y') \to E(Z')$ sending Y' to Z' which must be order preserving, except possibly when one compares Y', kY' and Z', kZ', where $k \in \mathcal{K} - \mathcal{K}_0$. Now the inclusion $kY^* \subset Y$ tells us that $kY' \subset (Y')^*$, and the inclusion $Z \subset kZ^*$ tells us that $kZ' = k^2 Z = Z \subset kZ^* = (Z')^*$. We conclude that φ' is order preserving, and preserves complementation and equivalence classes.

Now suppose that H has infinite index in $\mathcal{K}$. Then part 2) of Lemma 2.3 tells us that $\mathcal{K}$ has finite index in G, and there is a subgroup H'' of $\mathcal{K}$ which is commensurable with H and normal in $\mathcal{K}$. Further, $H''\backslash\mathcal{K}$ is isomorphic to $\mathbb{Z}$ or to $\mathbb{Z}_2 * \mathbb{Z}_2$. In the first case $\mathcal{K} = \mathcal{K}_0$, and in the second case $\mathcal{K}_0$ has index 2 in $\mathcal{K}$. It also implies that there is an almost invariant subset W of G with stabiliser H'' which is equivalent to X and in good position. As before, it follows that the stabilisers of Y and Z must also equal H''. The facts that H'' is normal in $\mathcal{K}$ with quotient a group with two ends, and that $\mathcal{K}$ has finite index in G, imply that $e(G, H'') = 2$. If $\mathcal{K} = \mathcal{K}_0$, we let λ denote an element of $\mathcal{K}$ which maps to a generator of $H''\backslash\mathcal{K}$, and we choose λ so that $Y \subset \lambda Y$. Then either $Z \subset \lambda Z$ or $\lambda Z \subset Z$. As Y and Z are equivalent, there is a number D such that Y and Z each lie in the D–neighbourhood of the other. Hence the unions $\bigcup_{n\geq 1} \lambda^n Y$ and $\bigcup_{n\geq 1} \lambda^n Z$ each lie in the D–neighbourhood of the other. As $Y \subset \lambda Y$, and $e(G, H'') = 2$, the union $\bigcup_{n\geq 1} \lambda^n Y$ equals G. It follows that the union $\bigcup_{n\geq 1} \lambda^n Z$ also equals G, so that the inclusion $\lambda Z \subset Z$ is impossible. Thus $Z \subset \lambda Z$, which implies that φ is order preserving.

If $\mathcal{K} \neq \mathcal{K}_0$, so that $H''\backslash\mathcal{K}$ is $\mathbb{Z}_2 * \mathbb{Z}_2$, the situation is more complicated. Fix an element k of $\mathcal{K} - \mathcal{K}_0$, so that kY is equivalent to Y^*. As Y is in good position, we must have either $Y \subset kY^*$ or $Y^* \subset kY$. (Again the assumption that Y is not invertible implies that we cannot have $Y = kY^*$.) Similarly, for each integer n, we must have $\lambda^n Y \subset k\lambda^n Y^*$ or $\lambda^n Y^* \subset k\lambda^n Y$. Suppose that $\lambda^n Y \subset k\lambda^n Y^*$, for some n. As $Y \subset \lambda Y$, it follows that $\lambda^n Y \subset \lambda^{m+n} Y$, for every $m \geq 1$, and so $k\lambda^n Y \subset k\lambda^{m+n} Y$, for every $m \geq 1$. As the union of the $\lambda^{m+n} Y$, for $m \geq 1$, equals G, so does the union of the $k\lambda^{m+n} Y$, for $m \geq 1$. It follows that we cannot have $\lambda^{m+n} Y \subset k\lambda^{m+n} Y^*$, for every $m \geq 1$. In particular, the inclusion $\lambda^n Y \subset k\lambda^n Y^*$ cannot hold for all values of n. Similarly, the inclusion $\lambda^n Y^* \subset k\lambda^n Y$ cannot hold for all values of n. If $\lambda^N Y \subset k\lambda^N Y^*$ for some integer N, then $\lambda^n Y \subset k\lambda^n Y^*$ whenever $n \leq N$. It follows that there is an integer $N(Y)$ such that $\lambda^n Y \subset k\lambda^n Y^*$ whenever $n \leq N(Y)$, and $\lambda^n Y^* \subset k\lambda^n Y$ whenever $n > N(Y)$. A similar discussion for Z yields an integer $N(Z)$ such that $\lambda^n Z \subset k\lambda^n Z^*$ whenever $n \leq N(Z)$, and $\lambda^n Z^* \subset k\lambda^n Z$ whenever $n > N(Z)$. If $N(Y) = N(Z)$, it is now easy to see that φ is order preserving. Otherwise, we let d denote $N(Z) - N(Y)$ and let Z' denote $\lambda^d Z$, so that Z' is equivalent to Z, and let $\varphi' : E(Y) \to E(Z')$ be the equivariant bijection which sends Y to Z'. As $N(Z') = N(Y)$, it follows that φ' is order preserving, and so is the required order preserving bijection from $E(Y)$ to $E(Z)$. ■

The above result shows that when one replaces X by an almost invariant set in good position, one obtains a unique partial order if we do not allow invertible almost invariant sets. We now discuss the general situation. Clearly if Y and Z are equivalent to X and one is invertible and the other is not, we do not obtain exactly the same partial order, so we now restrict attention to the case where both Y and Z are invertible.

Lemma 2.8 *Let G be a finitely generated group with a finitely generated subgroup H. Let X be a nontrivial H–almost invariant subset of G, and suppose that X is equivalent to Y, Z and V such that each of Y, Z and V is in good position. In addition, suppose that Y, Z and V are each invertible. Then one of the following holds:*

1. *There are G–equivariant bijections between $E(Y)$, $E(Z)$ and $E(V)$ which preserve complementation, ordering and equivalence classes.*

2. *H has infinite index in $\mathcal{K}$ and there is a G–equivariant bijection between two of $E(Y)$, $E(Z)$ and $E(V)$ which preserves complementation, ordering and equivalence classes.*

Remark 2.9 *This means that in case 1) there is only one partially ordered set as in Lemma 2.7, and in case 2) there are at most two possible partially ordered sets. The case of two distinct partial orders can occur. The simplest example occurs when G is $\mathbb{Z}_2 * \mathbb{Z}_2$ and H is trivial.*

Proof. For simplicity, we start by considering Y and Z only. The assumption that Y and Z are both invertible implies that $\mathcal{K}_0$ has index 2 in $\mathcal{K}$. It is no longer obvious that we can define a G–equivariant map $\varphi : E(Y) \to E(Z)$, by sending gY to gZ and gY^* to gZ^* for every g in G, because it is possible that there is g in G such that $gY = Y^*$ but $gZ \neq Z^*$.

If H has finite index in $\mathcal{K}$, then as in the proof of Lemma 2.12 the stabilisers of Y and Z must both equal $\mathcal{K}_0$. As each of Y and Z is invertible, it follows that $kY = Y^*$ and $kZ = Z^*$ for every k in $K - K_0$. Hence φ can be defined as above, and it is a G–equivariant bijection. It is also order preserving because there are no distinct equivalent elements of $E(Y)$.

Now suppose that H has infinite index in $\mathcal{K}$. Then as in the proof of Lemma 2.12 the stabilisers of Y and Z must equal H''. In this case, it is possible that φ cannot be defined as above, because the elements which invert Y and Z need not be the same. As in the case when Y and Z were not invertible, we let λ denote an element of $\mathcal{K}$ which maps to a generator of $H''\backslash\mathcal{K}$, and we choose λ so that $Y \subset \lambda Y$. As in that case, it follows that $Z \subset \lambda Z$. Now let k denote an element of $\mathcal{K} - \mathcal{K}_0$ such that $kY = Y^*$. As $Y \subset \lambda Y$ and so $\lambda Y^* \subset Y^*$, it is clear that k cannot invert $\lambda^n Y$, for any $n \neq 0$. If $kZ = Z^*$, then φ can be defined as above and is a G–equivariant bijection. Further it is easy to see that φ is order preserving. If $kZ \neq Z^*$, the fact that Z is invertible means that there is an integer $n \neq 0$ such that $k\lambda^n Z = Z^*$. If n is even, say $n = 2m$, this is equivalent to the equation $k\lambda^m Z = \lambda^m Z^*$, and we let $Z' = \lambda^m Z$. We can now define $\varphi' : E(Y) \to E(Z')$ to send gY to gZ' and gY^* to gZ'^*, and φ' is a G–equivariant bijection which preserves complementation and is order preserving. As $E(Z') = E(Z)$, this is the required bijection. However, if n is odd, this cannot be done.

To complete the proof of the lemma, we consider all three of Y, Z and V. If H has finite index in $\mathcal{K}$, the above proof applies to each pair to show that the required G–equivariant bijections exist. If H has infinite index in $\mathcal{K}$, we consider the preceding paragraph. Choose λ and k as described there. There is an integer n such that $k\lambda^n Z = Z^*$. Similarly, there is an integer r such that $k\lambda^r V = V^*$. If either of n or r is even, the preceding paragraph provides a G–equivariant bijection between $E(Y)$ and one of $E(Z)$ or $E(V)$. If both n and r are odd, we let k' denote $k\lambda$, so that we have the equations $k'\lambda^{n-1}Z = Z^*$ and $k'\lambda^{r-1}Z = Z^*$. As $n-1$ and $r-1$ are both even, say $n-1 = 2m$ and $r-1 = 2s$, we let $Z' = \lambda^m Z$ and $V' = \lambda^s V$. Thus k' inverts Z' and inverts V'. Now we can define $\varphi' : E(Z') \to E(V')$ to send gZ' to gV' and gZ'^* to gV'^*, and φ' is the required G–equivariant bijection $E(Z) = E(V)$. ■

The above discussion shows that if one considers all possible ways of replacing X by an almost invariant set in good position, only one partially ordered set can be obtained in this way, unless X is equivalent to an invertible almost invariant set. In this case, at most two partially ordered sets can be obtained with Y invertible. Thus in all cases, at most three partially ordered sets can be obtained by replacing X by an almost invariant set in good position.

This completes our discussion of good position when one starts with a single almost invariant subset of G. It is now easy to extend this to the general case.

Lemma 2.10 *Let G be a finitely generated group with finitely generated subgroups $H_1, \ldots, H_n$. For $i = 1, \ldots, n$, let X_i be a nontrivial H_i–almost invariant subset of G. Then each X_i is equivalent to a K_i–almost invariant subset Y_i of G such that the Y_i's are in good position. Thus the set $E(Y_1, \ldots, Y_n)$ of all* *translates of all the Y_i's and their complements has the partial order $\leq$ described above.*

Proof. By Lemma 2.4, we can replace each X_i by an equivalent almost invariant set Y_i, such that each Y_i is in good position. Thus for each i, the set $E(Y_i)$ of all translates of Y_i and Y_i^* satisfies Condition (*). Suppose that the set $E(Y_1, \ldots, Y_n)$ of all translates of all the Y_i's and Y_i^*'s does not satisfy Condition (*). Then there exist distinct i and j and translates U and V of Y_i and Y_j respectively such that two of their corners are small, and neither is empty. As before, this implies that U is equivalent to V or to V^*, so that Y_i is equivalent to some translate of Y_j or Y_j^*. In this case we simply replace Y_i by the same translate of Y_j or Y_j^*. By repeating this process, we will be able to arrange that the collection $Y_1, \ldots, Y_n$ is also in good position, as required. ∎

In the preceding proof, it may seem that we took the easy way out by simply replacing Y_i by a translate of Y_j or Y_j^*. However the following simple example shows that there are cases when there is no other way to arrange that the Y_i's are in good position.

Example 2.11 *Let G denote the integers under addition and let H denote the trivial subgroup of G. As G has two ends, it has nontrivial almost invariant subsets over H. The natural examples are sets of the form $L_a = \{n \in G : n \leq a\}$ or $R_a = \{n \in G : n \geq a\}$ for some integer a. If X is an almost invariant subset of G over H which is in good position, it is easy to see that X must be one of the sets L_a or R_a, for some a. Thus the set $E(X)$ of all translates of X and X^* consists of all the sets L_a and R_a. It follows that it is impossible to have two almost invariant subsets X_1 and X_2 of G such that $E(X_1, X_2)$ satisfies Condition (*) unless X_2 is some translate of X_1 or X_1^*. Thus in this group there is simply not room for more than one almost invariant set to be in good position.*

The above example suggests that if we want the Y_i's we choose in Lemma 2.10 to be in good position and to reflect the properties of the X_i's, then we should exclude the possibility that there are X_i and X_j, with $i \neq j$, such that some translate of X_i is equivalent to X_j or X_j^*. If this occurs, we will say that the G–orbits of X_i and X_j are *parallel*. We use this word because we are thinking of parallel G–orbits as corresponding to homotopic curves on a surface. The following simple uniqueness result covers most situations. However, if one allows some of the Y_i's to be invertible, then it is possible to get more than one partially ordered set, but clearly the number is finite and is bounded above by 3^n.

Lemma 2.12 *Let G be a finitely generated group with finitely generated subgroups $H_1, \ldots, H_n$. For $i = 1, \ldots, n$, let X_i be a nontrivial H_i–almost invariant subset of G, and suppose that, for distinct i and j, the G–orbits of X_i and X_j are not parallel. Suppose that X_i is equivalent to Y_i and to Z_i such that the Y_i's are in good position and the Z_i's are in good position. Further suppose that, for each i, Y_i and Z_i are not invertible. Then there is a G–equivariant bijection $\varphi : E(Y_1, \ldots, Y_n) \to E(Z_1, \ldots, Z_n)$ which is order preserving and preserves equivalence classes.*

Proof. Consider the G–equivariant bijection $\varphi : E(Y_1, \ldots, Y_n) \to E(Z_1, \ldots, Z_n)$ described above. The proof of Lemma 2.7 shows how to modify φ to be order preserving when restricted to each $E(Y_i)$. Now either φ is itself order preserving or there are distinct i and j and a translate U_i of Y_i or Y_i^* and a translate U_j of Y_j or Y_j^* such that $U_i \subset U_j$ but $\varphi U_j \subset \varphi U_i$. In particular, this implies that U_i and U_j are equivalent. But this contradicts our hypothesis that the G–orbits of X_i and Y_i are not parallel, so that φ must be order preserving. ∎

3 Constructing cubings from almost invariant sets in good position

As in the previous section, we consider a finitely generated group G with finitely generated subgroups $H_1, \ldots, H_n$. For $i = 1, \ldots, n$, let X_i be a nontrivial H_i–almost invariant subset of G, and let $E = \{gX_i, gX_i^* : g \in G, 1 \leq i \leq n\}$. In [6], Sageev gave a construction of a cubing from E, which we outlined in section 1.2. A key ingredient of his construction was the use of the partial order induced by inclusion on E. In the previous section, we established that given a finite family of nontrivial almost invariant sets, there exists an equivalent family in good position, and, if the X_i's are in good position, we described a new partial order on E. In this section, we describe a variant of Sageev's construction which uses this new partial order. We will see that this gives a cubing which is minimal in a natural sense and in most cases is canonically associated to the equivalence classes of the X_i's.

Now suppose that the X_i's are in good position and consider E with the partial order of almost inclusion discussed in the previous section. As in section 1.2, let $\Lambda^{(0)}$ denote the collection of all ultrafilters on E, defined using the new partial order. Exactly as in section 1.2, we can inductively construct a cubed complex Λ whose vertex set is $\Lambda^{(0)}$. Again Λ will not be connected, but we wish to pick out a component L which corresponds in a natural way to the component C picked out in the previous case. In fact the vertices of L, like the vertices of C, will be characterised as ultrafilters on E which satisfy the descending chain condition. We cannot proceed exactly as before because the set $V_g = \{A \in E : g \in A\}$ need not be an ultrafilter with respect to the new partial order. For example, it is quite possible that $g \in A \leq B$, but that $g \notin B$. We will thus need to adjust the construction of basic vertices.

We will need the following technical lemma, which will allow us to start by constructing an ultrafilter for all but a finite number of elements of E.

Lemma 3.1 *There exists $R > 0$ such that if $A, B \in E$ and $A \leq B$ and if $g \in A$ such that $N_R(g) \subset A$, then $g \in B$.*

Proof. As $A \leq B$, we also have $B^* \leq A^*$. Now Lemma 2.1 tells us that there is $D > 0$ such that $B^* \subset N_D(A^*) = A^* \cup N_D(\delta A)$. If g lies in A but not in B, it follows that g lies in $N_D(\delta A)$. This implies there is a point h of A^* such that $d(g, h) \leq D + 1$, so that $N_{D+1}(g)$ is not contained in A. Thus the lemma holds with $R = D + 1$. ∎

We are now ready to describe the special ultrafilters which will pick out the component L of Λ which corresponds to C. Given $g \in G$, we want to describe an ultrafilter W_g which will be almost the same as the set $V_g = \{A \in E : g \in A\}$. Consider first the ball $N = N_R(g)$ of radius R about g in the Cayley graph of G, where R is as in Lemma 3.1 above. We let

$$E_R = \{A \in E | \delta A \cap N \neq \emptyset\}.$$

We then denote $E - E_R$ by E_R^*. As E consists of the translates of a finite family of X_i's and their complements, it follows that E_R is finite.

Now for each pair $\{A, A^*\}$ of elements of E we need to decide whether or not A or A^* is in W_g, consistent with condition 2) of Definition 1.14. We will make this decision first for pairs (A, A^*) in E_R^*. As in the definition of V_g, we do this by taking those elements that contain g. That is, let

$$U_g = \{A \in E_R^* | g \in A\}.$$

Note that if $A \in U_g$, then $N_R(g) \subset A$.

Lemma 3.2 *U_g is an ultrafilter on E_R^*.*

Proof. For each pair $\{A, A^*\} \in E_R^*$, we either have $g \in A$ or $g \in A^*$, so that condition 1) of Definition 1.14 holds. Now suppose that $A \in U_g$, $B \in E_R^*$ and $A \leq B$. Then $N_R(g) \subset A$, so that Lemma 3.1 tells us that $g \in B$. Hence $B \in U_g$, and we have shown that condition 2) of Definition 1.14 holds. ■

We now wish to complete U_g to an ultrafilter W_g on all of E. There are only finitely many pairs $\{A, A^*\}$ about which we need to make a decision as to whether A or A^* is in W_g.

First of all, for each $B \in E_R$ for which there exists $A \in U_g$, with $A \leq B$, we add B to U_g. That is, set

$$U_1 = U_g \cup \{B \in E_R \mid \exists\, A \in U_g, A \leq B\}.$$

Lemma 3.3 *U_1 is an ultrafilter on the set $U_1 \cup U_1^*$, where U_1^* denotes the set $\{X^* : X \in U_1\}$.*

Proof. By construction U_1 satisfies condition 2) of Definition 1.14, namely that if $A \in U_1$ and $A \leq B$ then $B \in U_1$. We claim that U_1 also satisfies condition 1) of Definition 1.14, namely that we do not have $B \in U_1$ and $B^* \in U_1$. For if this occurs, we have A_1 and A_2 in U_g, with $A_1 \leq B$ and $A_2 \leq B^*$. Thus we have $A_1 \leq B \leq A_2^*$. As $N_R(g) \subset A_1$, Lemma 3.1 tells us that $g \in A_2^*$, which contradicts the fact that $g \in A_2$. It follows that U_1 is an ultrafilter on $U_1 \cup U_1^*$, as required. ■

Now let V_1 denote the collection of the remaining elements of E, so that $V_1 = E - (U_1 \cup U_1^*)$, and let A_1 denote a minimal element of V_1. We form U_2 by adding A_1 to U_1 and then adding every $B \in V_1$ such that $A_1 \leq B$.

Lemma 3.4 *U_2 is an ultrafilter on the set $U_2 \cup U_2^*$.*

Proof. Clearly U_2 does not contain B and B^*, for any B in $U_2 \cup U_2^*$, and so U_2 satisfies condition 1) of Definition 1.14. We will show that it also satisfies condition 2). For suppose $C \in U_2$ and $C \leq D$, where $D \in U_2 \cup U_2^*$. If $C \in U_1$, then the definition of U_1 implies that $D \in U_1$ also and hence $D \in U_2$. If $C \notin U_1$, and $D \notin U_1 \cup U_1^*$, then $D \in U_2$ by our construction. If $C \notin U_1$ and $D \in U_1^*$, then $D^* \leq C^*$ and $D^* \in U_1$, which implies that $C^* \in U_1$. Thus $C^* \in U_2$ which contradicts our assumption that $C \in U_2$. ∎

Next let V_2 denote the collection of the remaining elements of E, so that $V_2 = E - (U_2 \cup U_2^*)$, and let A_2 denote a minimal element of V_2. We form U_3 by adding A_2 to U_2 and then adding every $B \in V_2$ such that $A_2 \leq B$. As above, U_3 is an ultrafilter on the set $U_3 \cup U_3^*$.

We continue in this way until all the elements of E have been exhausted. The resulting subset W_g of E is then an ultrafilter on E.

Note that W_g is not determined by g. The construction of U_2 and its successors involves making choices of minimal elements. Thus, for each g in G, the above construction will yield finitely many such ultrafilters W_g. A vertex W_g of Λ constructed in this way is called a *basic vertex*. As one sees from the construction, it agrees with the notion of a basic vertex in the original construction of the cubing in [6] except on a finite subset of E. The natural action of G on E preserves the partial order of almost inclusion, and so induces an action of G on Λ.

Next we need to show that the basic vertices of Λ all lie in a single component L. Recall that any two basic vertices of the cubed complex K constructed by Sageev in [6] agree except on a finite number of pairs of elements of E. Now each basic vertex W_g of Λ associated to an element g of G by the above construction agrees with the basic vertex V_g of K except on a finite number of pairs of elements of E. It follows that any two basic vertices of Λ are ultrafilters on $(E, \leq)$ which agree except on a finite number of pairs of elements of E. Suppose that v and v' disagree on k pairs of elements of E. Then, as discussed in section 1.2, there is a path of length k in Λ which joins v to v'. It follows that the basic vertices of Λ all lie in a single component L, as required.

Finally one needs to show that L is $CAT(0)$. The argument here is essentially the same as in [6] and will be left to the reader.

Having constructed L, we are now ready to compare it with the cubing C constructed by Sageev in [6].

Theorem 3.5 *Let G be a finitely generated group with finitely generated subgroups $H_1, \ldots, H_n$. For $i = 1, \ldots, n$, let X_i be a nontrivial H_i–almost invariant subset of G, and let $E = \{gX_i, gX_i^* : g \in G, 1 \leq i \leq n\}$. Suppose that the X_i's are in good position. Let $(E, \subset)$ denote the set E with the partial order given by inclusion, and let $(E, \leq)$ denote the set E with the partial order given by almost inclusion, as described in section 3. Let C denote the cubing constructed from the poset $(E, \subset)$ as in Sageev's original construction in [6], and let L be the cubing constructed from the poset $(E, \leq)$ as in the previous section. Then there is a natural G–equivariant embedding $L \to C$.*

Proof. Let K denote the cubed complex constructed from $(E, \subset)$, and let Λ denote the cubed complex constructed from $(E, \leq)$, so that C is a component of K and L is a component of Λ. We claim first that a vertex of Λ is a vertex of K. For if V is an ultrafilter on $(E, \leq)$, then V is a subset of E which satisfies the following conditions,

- For any $A \in E$ either $A \in V$ or $A^* \in V$, but not both.
- If $A \in V$ and $A \leq B$, then $B \in V$.

Now if $A \subset B$, then certainly $A \leq B$, so it follows immediately that V is also an ultrafilter on $(E, \subset)$. Thus $\Lambda^{(0)} \subset K^{(0)}$. The description of the construction of the cubed complexes K and Λ from their vertices shows that this inclusion naturally extends to an embedding of Λ in K, and that this embedding is G–equivariant. As any basic vertex of L differs from some basic vertex of C by only finitely many elements, it follows that they can be joined by a path in K. Thus the embedding of Λ in K induces an embedding of L in C, as required. ∎

Now given two collections of good position almost invariant sets, $Y_1, \ldots, Y_n$ and $Z_1, \ldots, Z_n$ with $Y_i \sim Z_i$, such that no Y_i or Z_i is invertible, Lemma 2.12 provides a G–equivariant, order preserving bijection from $E(Y_1, \ldots, Y_n) \to E(Z_1, \ldots, Z_n)$, which provides a G–equivariant isomorphism from L_Y to L_Z. Thus the cubing constructed from the poset $(E, \leq)$ is determined solely by the equivalence classes of the almost invariant sets $X_1, \ldots, X_n$.

Remark 3.6 *If we define distance functions on L and C, by assigning length 1 to each edge, then the inclusion of the cubing L into the cubing C is isometric. For if v and w are two vertices in L, then the number of edges in any C–geodesic between v and w equals the number of hyperplanes of C which separate v from w. Similarly the number of edges in any L–geodesic between v and w equals the number of hyperplanes of L which separate v from w. These numbers are equal because the vertices are ultrafilters and, in both cases, the number of hyperplanes separating the vertices measures the number of sets in E which need to be replaced by their complements.*

4 Applications

We saw in section 2 that given a family of almost invariant sets $X_1 \ldots, X_n$, there is a family of almost invariant sets $Y_1 \ldots, Y_n$, such that Y_i is equivalent to X_i, and the Y_i's are in good position. This means that if two elements of $E(Y) = E(Y_1, \ldots, Y_n)$ have two of their four corners small, then one is empty. Thus two elements of $E(Y)$ must cross, be nested or have only one small corner. In this section, we will show that the third possibility can be removed. Precisely, we say that the X_i's are in *very good position* if given two elements of $E(X)$, either they cross or they are nested. We will show that we can always arrange this situation by replacing each X_i by an equivalent almost invariant set Z_i.

As we stated in the introduction, very good position for almost invariant sets is closely analogous to the properties enjoyed by shortest curves on surfaces or by least area surfaces in 3–manifolds. For simplicity, we will discuss only curves on surfaces. In order to explain the analogy, we first need to recall how curves on a surface are related to almost invariant sets. Let F denote a closed orientable surface and let S denote a simple closed curve on F. Let H denote the infinite cyclic subgroup of $G = \pi_1(F)$ carried by S, and let F_H denote the cover of F whose fundamental group is H. Thus S lifts to a circle in F_H which we also denote by S. Pick a generating set for G and represent it by a bouquet of circles embedded in F. We will assume that the base point of the bouquet does not lie on S. The pre-image of this bouquet in the universal cover $\widetilde{F}$ of F will be a copy of the Cayley graph Γ of G with respect to the chosen generating set. The pre-image in F_H of the bouquet will be a copy of the graph $H\backslash\Gamma$, the quotient of Γ by the action of H on the left. Consider the closed curve S on F_H. Let P denote the set of all vertices of $H\backslash\Gamma$ which lie on one side of S. Then P has finite coboundary, as δP equals exactly the edges of $H\backslash\Gamma$ which cross S. Hence P is an almost invariant subset of $H\backslash G$. Let X denote the pre-image of P in Γ, so that X equals the set of vertices of Γ which lie on one side of the line l. Then X is a H–almost invariant subset of G. If S is not simple, but we choose it to be shortest in its homotopy class, its lift to F_H will still be simple, so that the same construction can be made. Now the fact that S is shortest implies that, for each $g \in G$, the translate gl of the line l in $\widetilde{F}$ must equal l, be disjoint from l or meet l transversely in a single point. If $gl = l$, it follows that the translate gX of X must equal X (it cannot equal X^* as F is orientable). If gl is disjoint from l, it follows that gX and X are nested. If gl meets l transversely in a single point, it follows that X and gX cross each other. We conclude that if S is shortest in its homotopy class, then X is in very good position.

Lemma 4.1 *Let G be a finitely generated group with finitely generated subgroups $H_1, \ldots, H_n$. For $i = 1, \ldots, n$, and let X_i be a nontrivial H_i–almost invariant subset of G. Then, for each i, there exists a K_i–almost invariant subset Z_i of G which is equivalent to X_i, such that the Z_i's are in very good position.*

Proof. For simplicity we will consider the case when $n = 1$, and will denote X_1 by X and H_1 by H. The general case is essentially the same. Start with Y in good position such that Y is equivalent to X. Then construct the cubing C given by Y and using the poset $(E(Y), \subset)$. As discussed just before Lemma 1.17, there is a hyperplane $\mathcal{H}$ of C and a half-space $\mathcal{H}^+$ determined by $\mathcal{H}$ such that a vertex of C lies in $\mathcal{H}^+$ if and only if, when regarded as an ultrafilter on E, it contains Y. Further, for any vertex v of C, the set $Y_v = \{g \in G \mid g(v) \in \mathcal{H}^+\}$ is H–almost invariant and equivalent to Y.

Next consider the cubing L given by Y and using the poset $(E(Y), \leq)$, as constructed in section 3. Recall that $\mathcal{H}$ is associated to an equivalence class F of edges of C given by the equivalence relation generated by saying that two edges are equivalent if they are opposite edges of a square in C. Now two edges of L are opposite edges of a square in L if and only if they are opposite edges of a square in C. It follows that if f is an edge of F which also lies in L, then the equivalence class of f in L is precisely $F \cap L$. Let $\mathcal{K}$ denote the hyperplane in L associated to this equivalence class. Then it follows that $\mathcal{H}^+ \cap L$ equals one of the two half-spaces in L determined by $\mathcal{K}$. We denote this half-space by $\mathcal{K}^+$. Pick a vertex w of L, and apply Lemma 1.17 to obtain a new almost invariant set Z over H equal to $Y_w = \{g \in G \mid g(w) \in \mathcal{K}^+\}$. Since the inclusion of L in C is G–equivariant, Z may also be viewed as the set $\{g \in G \mid g(w) \in \mathcal{H}^+\}$. Now Lemma 1.17 tells us that Y_v and Y_w are equivalent. As Y_v is equivalent to Y, and $Z = Y_w$, it follows that Z is equivalent to Y.

In particular, two translates of Y in G are almost nested if and only if the corresponding translates of Z are almost nested. However, we claim that if two translates of Y in G are almost nested then the corresponding translates of Z are actually nested. This means exactly that Z is in very good position. To prove our claim, suppose, for example, that $aY \leq Y$. We need to show that $aZ \subset Z$. Recall that Z can be viewed as $\{g \in G \mid g(w) \in \mathcal{H}^+\}$. The description of the vertices of $\mathcal{H}^+$ given in the first paragraph of this proof shows that $Z = \{g \in G \mid Y \in g(w)\}$. Thus

$$\begin{aligned} aZ &= \{ag \in G \mid g(w) \in \mathcal{H}^+\} = \{g \in G \mid a^{-1}g(w) \in \mathcal{H}^+\} \\ &= \{g \in G \mid Y \in a^{-1}g(w)\} = \{g \in G \mid aY \in g(w)\}. \end{aligned}$$

As w is an ultrafilter on $(E(Y), \leq)$, so is $g(w)$. As $aY \leq Y$, it follows that if $aY \in g(w)$, then $Y \in g(w)$. Thus $aZ \subset Z$ as claimed. As this holds for all a, and analogous arguments apply if $aY \leq Y^*$, $aY^* \leq Y$ or $aY^* \leq Y^*$, it follows that Z is in very good position, completing the proof of the lemma. ■

We next consider applications which strengthen results of Niblo in [3] on the existence of splittings of a given group. Let H be a finitely generated subgroup of a finitely generated group G, and let X be a nontrivial H–almost invariant subset of G. In [3], Niblo defined a group $T(X)$ which is the subgroup of G generated by H and $\{g \in G : gX \text{ and } X \text{ are not nested}\}$. He proved, using Sageev's construction of cubings, that if $T(X) \neq G$, then G splits over a subgroup of $T(X)$. One can also define $S(X)$ to be the subgroup of G generated by H and $\{g \in G : gX \text{ crosses } X\}$. Clearly $S(X)$ is contained in $T(X)$. Further they are equal if X is in very good position. Thus the fact that X is equivalent to an almost invariant set in very good position yields a strengthening of Niblo's result in which one can replace the condition of not being nested by the condition of crossing. This strengthening was obtained previously by Scott and Swarup in [11] using their theory of regular neighbourhoods, but the present argument is more elementary.

In [3], Niblo proved an analogous result for two almost invariant subsets of a finitely generated group G. Again he used Sageev's construction of cubings. Let K is another finitely generated subgroup of G, and let Y be a nontrivial K–almost invariant subset of G. Suppose that any translate of X and any translate of Y are nested. Then G splits over a subgroup of $T(X)$ and over a subgroup of $T(Y)$. More precisely G is the fundamental group of a graph of groups with two edges such that the edge groups are conjugate into $T(X)$ and $T(Y)$ respectively.

As above, the fact that X and Y can be replaced by equivalent almost invariant sets in very good position means that the assumption that any translate of X and any translate of Y are nested can be replaced by the assumption that any translate of X and any translate of Y do not cross. This strengthening was also obtained previously by Scott and Swarup in [11] using their theory of regular neighbourhoods.

Finally, we state a result which generalises a result of Dunwoody and Roller in [2] and strengthens a result of Niblo in [3].

Theorem 4.2 *Let G be a finitely generated group with a finitely generated subgroup H and a nontrivial H–almost invariant subset X. If $\{g \in G : gX$ crosses $X\}$ lies in $Comm_G(H)$, the commensuriser of H in G, then G splits over a subgroup commensurable with H.*

In [3], Niblo proved this result on the stronger assumption that $\{g \in G : gX$ and X are not nested$\}$ lies in $Comm_G(H)$. In [2], Dunwoody and Roller proved the special case of this result when G commensurises H. One way to prove the result stated above is simply to apply Niblo's result using the fact that X is equivalent to an almost invariant subset of G in very good position. Alternatively, as Niblo's argument used Sageev's construction of cubings, one could obtain the strengthened result more directly by using our new cubing in place of Sageev's in Niblo's argument.

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