Some free-by-cyclic groups

Ian J. Leary [*] Graham A. Niblo Daniel T. Wise [†]

January 8, 2008

A group is said to be locally free if every finitely generated subgroup of it is free. One example is the additive group of the rationals. We exhibit a finitely generated group G that is free-by-cyclic and contains a non-free, locally free subgroup. The smallest such example that we have found is of the form $G \cong F_n \rtimes \mathbb{Z}$ for $n = 3$. We also construct word-hyperbolic examples for larger values of n, and show that the groups are not subgroup separable. We used Bestvina and Brady's 'Morse theory for cube complexes' in the construction of these groups. The authors thank Jim Anderson, who posed a question concerning 3-manifolds that led to these examples. This work was started at a conference at Southampton, immediately before Groups St. Andrews, which was funded by EPSRC visitor grants GR/L06928 and GR/L31135, and by a grant from the LMS.

Throughout this note, F_n denotes a free group of rank n, $\bar{x}$ denotes x^{-1}, and $x^y = \bar{y}xy$.

Proposition 1. *The group G given by the presentation*

$$G = \left\langle a, b, t : a^t = b,\ b^t = ab\bar{a} \right\rangle$$

contains a non-free, locally free subgroup, and is isomorphic to a split extension $F_3 \rtimes \mathbb{Z}$.

Proof The given presentation expresses G as an ascending HNN extension, with base group freely generated by a and b and with stable letter t. Define $\phi\colon G \to \mathbb{Z}$ by $\phi(t) = 1$, $\phi(a) = \phi(b) = 0$, and let K be the kernel of ϕ. Then K is a strictly ascending union of 2-generator free groups

$$\langle a, b\rangle \subseteq \langle a, b\rangle^{\bar{t}} \subseteq \langle a, b\rangle^{\bar{t}^2} \subseteq \cdots \subseteq K.$$

Any such group is locally free and not free, since it is not finitely generated and the rank of its abelianization is at most two. (In fact, the abelianization of K is infinite cyclic.)

[*]Partially supported by EPSRC grant no. GR/L69398

[†]Supported by NSF grant no. DMS-9627506

It remains to show that G is free-by-cyclic. Define $\psi: G \to \mathbb{Z}$ by $\psi(t) = \psi(a) = \psi(b) = 1$. It will be shown that the kernel of ψ is free of rank 3. A presentation 2- complex Y for G may be constructed by attaching two 2-cells to a rose with edges a, b, and t according to the maps given in figure 1. Since Y is obtained from the presentation of G as an HNN extension with free base group, it follows that Y is an Eilenberg-Mac Lane space for G. Represent the three 1-cells of Y as unit intervals, and represent the two 2-cells of Y as a unit square in and a 2×1 rectangle, as indicated in figure 1. This makes Y into an affine cell complex in the sense of Bestvina and Brady ([1], Def. 2.1).

Figure 1.

Now take $S^1 = \mathbb{R}/\mathbb{Z}$, viewed as a cell complex with one vertex and one edge of length 1, as an Eilenberg-Mac Lane space for the integers. A cellular map $g: Y \to S^1$ may be defined that induces the homomorphism $\psi: G \to \mathbb{Z}$ on fundamental groups and is affine on each cell—in figure 1 this map is represented by 'height'. The inverse image of the vertex of S^1 is a rose consisting of one vertex and three 1-cells (the dotted lines on figure 1). Now let X be the cover of Y corresponding to the subgroup $H = \ker(\psi)$. The map g lifts to a map $f: X \to \mathbb{R}$. X is an affine cell complex, and f is a Morse function in the sense of [1], Def. 2.2. By construction, X is an Eilenberg-Mac Lane space for H, and for any integer t, $X_t = f^{-1}(t)$ consists of a disjoint union of copies of a 3- petalled rose.

Bestvina and Brady's Morse theory allows one to compare X and X_t: by lemma 2.5 of [1], a space homotopy equivalent to X may be obtained from X_t by coning off a subspace homeomorphic to a copy of the descending link (resp. ascending link) at v for each vertex v of X such that $f(v) > t$ (resp. $f(v) < t$). All vertices of X have isomorphic links, since Y has only one vertex, and the ascending and descending links at each vertex are as shown in figure 2.

The descending link The ascending link

Figure 2.

Both the ascending and descending link are contractible. Since coning off a contractible subspace does not change the homotopy type of a space, it follows that X is homotopy equivalent to X_t. But it is already known that X is an Eilenberg-Mac Lane space for H, and that X_t is a disjoint union of 3-petalled roses. It follows that X_t is connected, and that H is free of rank three. ∎

With the benefit of hindsight, a shorter proof that G as above is free-by-cyclic may be given—see Proposition 2 below. Such a proof gives no indication as to how G was discovered however. Moreover, the techniques of Proposition 1 generalize easily to more complicated presentations such as those given in Proposition 3.

Proposition 2. *Let H' be freely generated by x, y and z, and define an automorphism θ of H' by*

$$\theta(x) = y, \quad \theta(y) = z, \quad \theta(z) = y^2\bar{x}.$$

The group G of Proposition 1 is isomorphic to $H' \rtimes \langle t \rangle$, where the conjugation action of t on H' is given by θ.

Proof First, check that the endomorphism θ is an automorphism of H' by exhibiting an inverse:

$$\theta^{-1}(z) = y, \quad \theta^{-1}(y) = x, \quad \theta^{-1}(x) = \bar{z}x^2.$$

Now, eliminate b from the given presentation for G to obtain

$$G = \langle a, t : \bar{t}\bar{t}att a\bar{t}\bar{a}t\bar{a} \rangle.$$

Substitute $a = xt$, and eliminate a, obtaining

$$G = \langle x, t : \bar{t}\bar{t}(xt)tt(xt)\bar{t}(\bar{t}\bar{x})t(\bar{t}\bar{x}) \rangle = \langle x, t : \bar{t}^2xt^3x\bar{t}\bar{x}^2 \rangle.$$

Add new generators $y = x^t$ and $z = x^{t^2}$, obtaining

$$\begin{aligned} G &= \langle x, y, z, t : x^t = y,\ y^t = z,\ ztx\bar{y}^2\bar{t} \rangle \\ &= \langle x, y, z, t : x^t = y,\ y^t = z,\ z^t = y^2\bar{x} \rangle. \end{aligned}$$

Thus G is seen to be isomorphic to $H' \rtimes \langle t \rangle$ as claimed. ∎

Next, we show how to construct a word-hyperbolic group having similar properties to the group G.

Proposition 3. *For $s \geq 3$ define a word $W(x,y)$ by*

$$W(x,y) = xy^4xy^5x\cdots xy^{4+s}x.$$

The group G_s with presentation

$$G_s = \langle a, b, t : a^t = b,\, b^t = W(b,a)b(W(a,b))^{-1}\rangle$$

is free-by-cyclic and contains a non-free, locally free subgroup. For s sufficiently large, G_s is word-hyperbolic.

Proof As in Proposition 1, G_s is a strictly ascending HNN-extension with base group freely generated by a and b, so contains a non-free, locally free subgroup. As in Proposition 1, an Eilenberg-Mac Lane space for G_s with an affine cell structure can be made by attaching a unit square and an $m \times 1$ rectangle to a rose with three petals of length 1. (Here m is one more than the length of the word W, as shown in figure 3.) The argument given in Proposition 1 shows that G_s is expressible as $F_n \rtimes \mathbb{Z}$, where n is the total area of the two 2-cells in figure 3, i.e., $n = m + 1 = s + 4 + (8+s)(s+1)/2$.

Figure 3.

It remains to show that G_s is word-hyperbolic for s sufficiently large. For this, it suffices to show that some presentation for G_s satisfies the $C'(1/7)$ small cancellation condition (see [3]). Eliminate b from the presentation for G_s. This leaves a 1-relator group, with relator

$$btab^4ab^5a\cdots ab^{4+s}a\bar{b}^2\bar{a}^{4+s}\bar{b}\cdots\bar{b}\bar{a}^5\bar{b}\bar{a}^4\bar{b}\bar{t}$$
$$=\bar{t}at^2a(\bar{t}a^4ta)(\bar{t}a^5ta)\cdots(\bar{t}a^{4+s}ta)\bar{t}\bar{a}^2(t\bar{a}^{4+s}\bar{t}\bar{a})(t\bar{a}^{3+s}\bar{t}\bar{a})\cdots(t\bar{a}^5\bar{t}\bar{a})(t\bar{a}^4\bar{t}\bar{a}).$$

The total length of this relator as a word in a and t may be seen to be $8+(14+s)(s+1)$. (The bracketing of the word is intended to facilitate this check.) For any $4\leq r\leq 4+s$, the word $\bar{t}a^rt$ occurs exactly once as a subword of either the relator or its inverse (but not both). Any subword of the relator of length at least $2s+15$ contains a subword of the form $(\bar{t}a^rt)^{\pm 1}$ for some such r. (The worst case is the subword $a^{4+s}ta\bar{t}\bar{a}^2t\bar{a}^{4+s}$, of length $2s+14$.) Hence any subword of the relator or its inverse of length $2s+15$ occurs in a unique place. It follows that G_s is word-hyperbolic whenever $2s+15\leq 1/7\,(8+(14+s)(s+1))$. This inequality is satisfied for all sufficiently large s. (In fact, $s\geq 9$ suffices.) ∎

P. Scott asked if free-by-cyclic groups are necessarily subgroup separable. An example due to Burns, Karass and Solitar showed that this is not the case (see [2]). The groups constructed above give another, simpler, argument to show this.

Proposition 4. *The groups G and G_s, constructed in Propositions 1 and 3, are not subgroup separable.*

Proof In each case, let L_1 be the subgroup generated by a and b, and let $L_2=tL_1\bar{t}$. Then L_1 and L_2 are free of rank two and are conjugate in G (resp. in G_s). Moreover, L_1 is a proper subgroup of L_2. It follows that L_1 cannot be closed, since it cannot be separated from any element of $L_2\setminus L_1$: in any finite quotient, the images of L_1 and L_2 have the same order, since they are conjugate, and so must be equal since the image of L_1 is a subgroup of the image of L_2. ∎

References.

[1] M. Bestvina and N. Brady, Morse theory and finiteness properties of groups, to appear in *Inventiones Math.*

[2] R. G. Burns, A. Karrass,and D. Solitar, A note on groups with separable finitely generated subgroups. *Bull. Austral. Math. Soc.* 36 (1987), no. 1, 153–160.

[3] R. Strebel, Small cancellation groups, Appendix to: *Sur les groupes hyperboliques d'aprés Mikhael Gromov*, 227–273, Progr. Math., 83, Birkhäuser Boston, Boston, MA, 1990.

Authors' addresses:
Faculty of Mathematical Studies,
University of Southampton,
Southampton,
SO17 1BJ.

Department of Mathematics,
Cornell University,
Ithaca, NY
14853