A flatness property of acts over monoids

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Abstract. A lot has been written lately about various flatness concepts of acts over monoids and in particular a property that has come to be known as *condition* (P). We introduce in this paper a new property which is related to this condition and develop connections with weak flatness and in particular absolute weak flatness.

1. Introduction and Preliminaries

Throughout this paper S will denote a monoid. We refer the reader to [6] for basic definitions and terminology relating to semigroups and acts over monoids and to [4] for definitions and results relating to the various flatness properties referred to in this paper.

A right S-act A is said to satisfy *condition* (P) if whenever $a, a' \in A, u, v \in S$ and $au = a'v$, there exist $a'' \in A, s, t \in S$ such that $a = a''s, a' = a''t, su = tv$. We sometimes deal in this paper with cyclic acts (that is acts of the form S/ρ where ρ is a right congruence on S) and so we give characterizations of the various flatness concepts for such acts which will be used most often in this paper.

Lemma 1.1. [4, Lemma 2.1] *Let S be a monoid and let ρ be a right congruence on S.*

(a) S/ρ satisfies condition (P) if and only if for all $u, v \in S$ with $u \; \rho \; v$ there exist $s, t \in S$ such that $su = tv$ and $s \; \rho \; 1 \; \rho \; t$.

(b) S/ρ is weakly flat if and only if for all $u, v \in S$ with $u \; \rho \; v$ there exist $s, t \in S$ such that $su = tv, s(\rho \vee \Delta u)1$ and $t(\rho \vee \Delta v)1$

Note that in *(b)* above Δ denotes the equality relation on S.

Lemma 1.2. ([4, Lemma 3.1]) *Let S be any monoid such that every flat cyclic right $S-$act satisfies condition (P). Then every $e \in E(S)\backslash\{1\}$ is a right zero element of S.*

A monoid S is called *left PP* if every principal left ideal of S is projective. It can be shown that S is left PP if and only if for every $x \in S$, there exists $e^2 = e \in S$ such that $ex = x$ and $ux = vx$ implies $ue = ve$. Every regular and every right cancellative monoid is left PP.

2. Condition (P_E)

We introduce here a new condition, related to condition (P), and establish some connections with flatness. First notice that a right S-act A satisfies condition (P) if and only if for $a, a' \in A, u, v \in S$ and $au = a'v$ there exist $a'' \in A, s, t, e = e^2, f = f^2 \in S$ such that $a = a''se,\ a' = a''tf,\ eu = u,\ fv = v$ and $su = tv$.

Definition Let S be a monoid and A a right $S-$act. We shall say that A *satisfies condition* (P_E) if whenever $a, a' \in A, u, v \in S$ and $au = a'v$, there exist $a'' \in A, s, t \in S$ and $e = e^2, f = f^2 \in S$ such that

$$ae = a''se, a'f = a''tf, eu = u, fv = v \text{ and } su = tv.$$

That there is an abundance of acts over monoids which possess condition (P_E) follows from :

Theorem 2.1. *Let S be a left inverse monoid (i.e. S is regular and $fef = fe$ for all $e, f \in E(S)$). Then every right $S-$act satisfies condition (P_E).*

Proof. Let A be a right $S-$act and suppose that $au = a'v$ with $a, a' \in A, u, v \in S$. Let $u' \in V(u), v' \in V(v)$ and put $e = uu', f = vv'$. Then clearly $u = eu, v = fv$. Let $s = u'uv'vu', t = u'uv'$ and $a'' = a'v$. Then

$$ae = auu'e = a'vv'vu'e = auv'vu'e = auu'uv'vu'e = ause = a''se$$

and

$$a'f = a'vv'vv' = (au)(v'v)v' = (au)(u'u)(v'v)v' = (a'v)tf = a''tf.$$

Finally, $su = (u'u)(v'v)(u'u) = (u'u)(v'v) = tv$ as required. ∎

It is well known that $(P) \Rightarrow$ flat $\Rightarrow$ weakly flat and clearly $(P) \Rightarrow (P_E)$. We show now that condition (P_E) is incomparable with flatness.

Example 1. Let J be a proper right ideal of a monoid S. Let x, y, z be symbols not belonging to S and define $A(J) = (\{x, y\} \times (S \backslash J)) \cup (\{z\} \times J)$ with a right S-action defined by

$$(x,u)s = \begin{cases} (x, us) & \text{if } us \notin J \\ (z, us) & \text{if } us \in J \end{cases} \qquad (y,u)s = \begin{cases} (y, us) & \text{if } us \notin J \\ (z, us) & \text{if } us \in J \end{cases}$$

$$(z,u)s = (z, us).$$

Then $A(J)$ is a right S-act (in fact it is the pushout of two copies of S amalgamating J). It is shown in [1] that $A(J)$ is flat if and only if $j \in Jj$ for every $j \in J$. Suppose now that there exist $u, s \in S$ with $us \in J$ and such that $[\forall f \in E(S), fs = s \Rightarrow f = 1]$. Then it follows that $A(J)$ does not satisfy condition (P_E). To see this suppose that it does and note that $(x, u)s = (y, u)s$. Hence there exist $w \in \{x, y, z\}, v \in S, e, f \in E(S), p, q \in S$ with

$$(x, u)e = (w, v)pe, (y, u)f = (w, v)qf, es = s, fs = s, ps = qs.$$

But then $e = f = 1$ and so $w = x = y$. Consider then the monoid S with multiplication table

	0	1	e	a
0	0	0	0	0
1	0	1	e	a
e	0	e	e	0
a	0	a	a	0

and let $J = eS = \{0, e\}$, with $u = s = a$. It is easy to check from the above that $A(J)$ is flat but does not satisfy condition (P_E).

Example 2. Let $U = \{a, b\}, V = \{c, d\}$ be left zero semigroups and let $S = U \dot{\cup} V$. Extend the multiplications in U and V to S by letting a and b be left zero elements for S and $cU = \{a\}, dU = \{b\}$. It is straightforward to check that S is a left normal band (i.e. S is regular and $efg = egf$ for all $e, f, g \in S$). From Theorem 2.1 we see that all right S^1-acts satisfy property P_E but by [2, Corollary 2.7], not all right S^1-acts are flat.

Lemma 2.2. ([3]) *Let S be a monoid and A a right $S-$act. Then A is weakly flat if and only if for every $x, y \in S, a, a' \in A$, $ax = a'y$ implies the existence of $a_1, \ldots, a_n \in A, x_1, \ldots, x_{n-1} \in \{x, y\}$ and $u_1, v_1, \ldots, u_n, v_n \in S$ such that*

$$\begin{aligned} a &= a_1u_1 & u_1x &= v_1x_1 \\ a_1v_1 &= a_2u_2 & u_2x_1 &= v_2x_2 \\ &\cdots & &\cdots \\ a_{n-1}v_{n-1} &= a_nu_n & u_nx_{n-1} &= v_ny \\ a_nv_n &= a' \end{aligned}$$

Theorem 2.3. *Let S be a monoid and A a right $S-$act satisfying condition (P_E). Then A is weakly flat.*

Proof. Suppose $ax = a'y$ in A. Then, by assumption, there exist $s, t, e, f \in S$ and $a'' \in A$ such that $ae = a''se, a'f = a''tf, ex = x, fy = y$ and $sx = ty$. Notice that $(se)x = sx = ty = (tf)y$ and that

$$\begin{aligned} a &= a.1 & 1.x &= ex \\ ae &= a''(se) & (se)x &= (tf)y \\ a''(tf) &= a'f & fy &= 1.y \\ a'.1 &= a' \end{aligned}$$

So by Lemma 2.2, A is weakly flat. ■

That the converse of the preceding theorem is not true follows from Example 1.

Lemma 2.4. ([3, Proposition 1.2 and Lemma 2.1]) *Let S be a left PP monoid. Then a right $S-act$, A, is principally weakly flat if and only if for every $a, a' \in A$ and $x \in S, ax = a'x$ implies there exists $e \in E(S)$ such that $ex = x$ and $ae = a'e$.*
A is weakly flat if and only if A is principally weakly flat and for all left ideals I and J of S, $AI \cap AJ = A(I \cap J)$.

Theorem 2.5. *Let S be a left PP monoid and A a right $S-$act. Then A is weakly flat if and only if A satisfies condition (P_E).*

Proof. If A satisfies condition (P_E) then we know that A is weakly flat. Suppose then that A is weakly flat and that $ax = a'y$ in A. Then by the previous lemma, there exist $a'' \in A, z \in Sx \cap Sy$ with $ax = a'y = a''z$. Now $z = sx = ty$ for some $s, t \in S$ and so $ax = (a''s)x, a'y = (a''t)y$. Hence by the previous lemma, there exist $e, f \in E(S)$ such that $ex = x, ae = a''se$ and $fy = y, a'f = a''tf$. ■

Notice that this result does not extend to flat acts by Example 2. Since right absolutely weakly flat monoids must be regular and hence left-PP, then all their right $S-$acts satisfy condition (P_E). Conversely, if all right S-acts satisfy condition (P_E) then S must be right absolutely weakly flat. Hence we have proved

Theorem 2.6. *S is a right absolutely weakly flat monoid if and only if all right S-acts satisfy condition (P_E).*

V. Fleischer ([5, Theorem 4]) gave a description of such monoids : S is regular and for every $x, y \in S$ there exists $z \in Sx \cap Sy$ such that $(x, z) \in \rho(x, y)$ (where $\rho(x, y)$ is the smallest right S-congruence containing (x, y)).

Lemma 2.7. *Let S be a monoid and ρ a right congruence on S. If S/ρ satisfies condition (P_E) then for all $u, v \in S$ with $u\ \rho\ v$, there exist $s, t, e^2 = e, f^2 = f \in S$ such that $se\ \rho\ e, tf\ \rho\ f, eu = u, fv = v$ and $su = tv$.*

Proof. Let $u\ \rho\ v$ and let $a = 1\rho$. Then $au = av$ and so there exist $a'' \in S/\rho, s', t', e^2 = e, f^2 = f \in S$ such that $ae = a''s'e, af = a''t'f, eu = u, fv = v$ and $s'u = t'v$. Now $a'' = x\rho$ for some $x \in S$ and so letting $s = xs', t = xt'$ we have $se = xs'e\ \rho\ e, tf = xt'f\ \rho\ f$ and $su = xs'u = xt'v = tv$ as required. ■

Consequently, using Theorem 2.5 we can deduce the following characterisation of weakly flat cyclic right S-acts.

Lemma 2.8. (cf. [3, Proposition 2.5]) *Let S be a left PP monoid and ρ a right congruence on S. Then S/ρ is weakly flat if and only if for all $u, v \in S$ with $u \; \rho \; v$, there exist $s, t, e^2 = e, f^2 = f \in S$ such that $se \; \rho \; e, tf \; \rho \; f, eu = u, fv = v$ and $su = tv$.*

Proof. One way round follows from Lemma 2.7. Suppose then that $(a\rho)u = (b\rho)v$ for $a, b, u, v \in S$, so that there exist $s, t, e^2 = e, f^2 = f \in S$ such that $se \; \rho \; e, tf \; \rho \; f, eau = au, fbv = bv$ and $sau = tbv$. Since S is left PP then there exists $g = g^2 \in S$ such that $gu = u$ and $eau = au$ implies $eag = ag$. Thus $(ag)\rho = (eag)\rho$ and so $(ag)\rho = (seag)\rho = 1\rho.(seag)$. In a similar way, there exists $h = h^2 \in S$ with $(bh)\rho = 1\rho.(tfbh)$ and $fbh = bh$. Now let $s' = sea$ and $t' = tfb$ so that $(a\rho)g = (1\rho)s'g, (b\rho)h = (1\rho)t'h, gu = u, hv = v$ and $s'u = t'v$. Hence S/ρ satisfies condition (P_E) as required. ■

Theorem 2.9. *Let S be a monoid such that every $e \in E(S)\backslash\{1\}$ is right zero and let ρ be a right congruence on S. Then S/ρ satisfies condition (P) if and only if S/ρ satisfies condition (P_E).*

Proof. One way round is clear. Suppose then that S/ρ satisfies condition (P_E) and that $u \; \rho \; v$. Then by Lemma 2.7, there exist $s, t, e^2 = e, f^2 = f \in S$ with $se \; \rho \; e, tf \; \rho \; f$, $eu = u, fv = v$ and $su = tv$. If $e = f = 1$ then the result is clear. If $e = 1, f \neq 1$ then f, and hence v, is right zero and we have $su = 1.v$ and $s \; \rho \; 1 \; \rho \; 1$. If $e \neq 1, f \neq 1$ then e and f are both right zero and hence so are u and v. Consequently, $1.u = 1.v$ and S/ρ satisfies condition (P) as required. ■

From Lemma 1.2, Theorem 2.9 and Theorem 2.5 we can deduce

Corollary 2.10. *Let S be a left-PP monoid. Then all weakly flat cyclic right S-acts satisfy condition (P) if and only if every $e \in E(S)\backslash\{1\}$ is right zero.*

Lemma 2.11. [3, Theorem 2.6] *Let S be a left PP-monoid such that for all $u, v \in E(S)$ there exists $z \in uS \cap vS$ such that $(z, u) \in \rho(u, v)$. Then every weakly flat right $S-$act is flat.*

From Theorem 2.5, it now follows that:

Corollary 2.12. *Let S be a left PP-monoid such that for all $u, v \in E(S)$ there exists $z \in uS \cap vS$ such that $(z, u) \in \rho(u, v)$. Then a right $S-$act A is flat if and only if A satisfies condition (P_E).*
Consequently, if S is a monoid such that for all $u, v \in E(S)$ there exists $z \in uS \cap vS$ such that $(z, u) \in \rho(u, v)$, then S is right absolutely flat if and only if all right S-acts satisfy condition (P_E).

It seems to us that in view of Theorem 2.5, the new condition given in this paper may be useful in the study of absolutely (weakly) flat monoids.

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