

Commensurability of Fuchsian groups and their axes

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The purpose of this note is to prove the following Theorem. This answers a question raised during the problem session at the Special Session on Geometric Function Theory of the 898th meeting of the AMS in Hartford, CT, the weekend of the fourth of March, 1995, and is related to a question in Mess [4] on whether equality of axis sets for Fuchsian groups implies commensurability.

Theorem *For each arithmetic Fuchsian group Γ , there exists an infinite order elliptic element e such that $e(\text{ax}(\Gamma)) = \text{ax}(\Gamma)$.*

Recall that a *Fuchsian* group is a discrete subgroup of $\text{PSL}_2(\mathbf{R}) \cong \text{Isom}^+(\mathbf{H}^2)$. We denote by $\text{ax}(\Gamma)$ the set of axes of hyperbolic elements of the Fuchsian group Γ .

The proof follows easily from known properties of arithmetic Fuchsian groups. Recall that an *arithmetic* Fuchsian group is constructed as follows, see [5]. Let k be a totally real number field, and A a quaternion algebra over k which is ramified at all places of k except one, which we may take to be the identity. Let ρ be the embedding of A into $M_2(\mathbf{R})$ induced by splitting A at the identity place of k . Let \mathcal{O} be an order of A , and \mathcal{O}^1 the elements of reduced norm one in \mathcal{O} . Then $\rho(\mathcal{O}^1)$ is a discrete subgroup of $\text{SL}_2(\mathbf{R})$ and its projection to $\text{PSL}_2(\mathbf{R})$ via P is a finitely generated Fuchsian group of the first kind. A Fuchsian group Γ is arithmetic if it is commensurable with some such group $P\rho(\mathcal{O}^1)$.

Subgroups Γ_1 and Γ_2 of $\text{PSL}_2(\mathbf{R})$ are *commensurable* if $\Gamma_1 \cap \Gamma_2$ has finite index in both Γ_1 and Γ_2 . Define the *commensurator* $\text{Comm}(\Gamma)$ of a subgroup Γ of $\text{PSL}_2(\mathbf{R})$ by

$$\text{Comm}(\Gamma) = \{\varphi \in \text{PSL}_2(\mathbf{R}) \mid \varphi\Gamma\varphi^{-1} \text{ is commensurable to } \Gamma\}.$$

It is a theorem of Margulis [3] that $\text{Comm}(\Gamma)$ is non-discrete if and only if Γ arithmetic. In particular, since it is non-elementary and non-discrete, $\text{Comm}(\Gamma)$ must contain an infinite order elliptic e [2]. It is easy to verify that, if Γ_1 and Γ_2 are commensurable, then $\text{ax}(\Gamma_1) = \text{ax}(\Gamma_2)$. Since $e \in \text{Comm}(\Gamma)$, $e\Gamma e^{-1}$ and Γ are commensurable, and so $\text{ax}(\Gamma) = \text{ax}(e\Gamma e^{-1}) = e(\text{ax}(\Gamma))$.

In fact, the Theorem holds true for any arithmetic subgroup Γ of $\text{Isom}^+(\mathbf{H}^{2n})$. Briefly, for such groups, Margulis' result still holds, and Abikoff and Haas [1] have shown that a non-elementary subgroup of $\text{Isom}^+(\mathbf{H}^{2n})$ which does not keep a proper subspace invariant is discrete if and only if

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it contains no infinite order elliptic elements. Since Γ arithmetic implies \mathbf{H}^{2n}/Γ has finite volume, Γ cannot keep invariant a proper totally geodesic subspace of \mathbf{H}^{2n} , and therefore the proof follows as before.

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References

- [1] Wm. Abikoff and A. Haas, ‘Nondiscrete groups of hyperbolic motions’, *Bull. L. M. S.* **22** (1990), 233–238.
- [2] T. Jørgensen, ‘A note on subgroups of $\mathrm{SL}_2(\mathbf{C})$ ’, *Quart. J. Math. Oxford* **28** (1977), 209–212.
- [3] G. A. Margulis, *Discrete subgroups of semi-simple Lie groups*, Springer–Verlag, 1991.
- [4] G. Mess, ‘Fuchsian groups with the same simple axes’, preprint, 1990.
- [5] M-F. Vignéras, *Arithmétique des algéras de Quaternions*, Lecture Notes in Mathematics, vol. 800, Springer-Verlag, 1980.