Variations on a theme of Horowitz

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Abstract

Horowitz [Hor1] showed that for every $n \geq 2$, there exist elements $w_1, \ldots, w_n$ in $F_2 = \text{free}(a, b)$ which generate non-conjugate maximal cyclic subgroups of F_2 and which have the property that $\text{trace}(\rho(w_1)) = \cdots = \text{trace}(\rho(w_n))$ for all faithful representations ρ of F_2 into $\text{SL}_2(\mathbf{C})$. Randol [Ran] used this result to show that the length spectrum of a hyperbolic surface has unbounded multiplicity. Masters [Mas] has recently extended this unboundness of the length spectrum to hyperbolic 3-manifolds. The purpose of this note is to present a survey of what is known about characters of faithful representations of F_2 into $\text{SL}_2(\mathbf{C})$, to give a conjectural topological characterization of such n-tuples of elements of F_2, and to discuss the case of faithful representations of general surface groups and 3-manifold groups.

1 Introduction, history, and motivation

The purpose of this survey is to explore the following question, asked during the Special Session on Geometric Function Theory, held in Hartford, Connecticut, during the 898$^{\text{th}}$ meeting of the AMS in March, 1995 (for a list of all the questions asked during that session, we refer the reader to Basmajian [Bas1]):

> 15. According to a theorem of Horowitz (see Horowitz [Hor1], Randol [Ran]), there exist pairs of closed curves on a closed [orientable] surface S [with negative Euler characteristic] for which the lengths of the geodesics in the respective homotopy classes are equal for any hyperbolic structure on S. These constructions all involve writing down a pair of words in the fundamental group for S and then applying trace identities to show that the words have the same trace, independent of the representation into $\text{PSL}(2, \mathbf{R})$. Find a topological characterization of such a pair of curves.

We begin with some definitions, and make the observation that while we generally state the definitions in terms of the free group of rank two F_2, the definitions and many of the observations hold true for a general finitely generated group G. Also, while we generally restrict our attention to faithful representations of a group G into $\text{SL}_2(\mathbf{C})$, the assumption of faithfulness is not necessary. In fact, the results of Horowitz [Hor1] and Ginzburg and Rudnick [GR] hold for all representations of F_2 into $\mathrm{SL}_2(\mathbf{C})$.

Let $F_2 = \mathrm{free}(a, b)$ be the free group of rank 2, and let $\mathcal{F}(F_2)$ denote the space of all faithful representations of F_2 into $\mathrm{SL}_2(\mathbf{C})$. The topology on $\mathcal{F}(F_2)$ is given by realizing it as a subset of $\mathrm{SL}_2(\mathbf{C}) \times \mathrm{SL}_2(\mathbf{C}) = \mathrm{Hom}(F_2, \mathrm{SL}_2(\mathbf{C}))$, the space of all representations of F_2 into $\mathrm{SL}_2(\mathbf{C})$, by associating the representation $\rho \in \mathcal{F}(F_2)$ with the point $(\rho(a), \rho(b))$ in $\mathrm{SL}_2(\mathbf{C}) \times \mathrm{SL}_2(\mathbf{C})$. Note that $\mathcal{F}(F_2)$ is dense in $\mathrm{SL}_2(\mathbf{C}) \times \mathrm{SL}_2(\mathbf{C})$. We denote by $\mathcal{DF}(F_2)$ the subspace of $\mathcal{F}(F_2)$ consisting of all those faithful representations ρ of F_2 into $\mathrm{SL}_2(\mathbf{C})$ whose image $\rho(F_2)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbf{C})$.

For an element w of F_2, the *character associated to* w is the function $\chi[w] : \mathcal{F}(F_2) \to \mathbf{C}$ given by setting $\chi[w](\rho) = \mathrm{trace}(\rho(w))$, where $\mathrm{trace}(A)$ is the usual trace of the 2×2 matrix A. Note that by this definition, an element w of F_2 and its inverse w^{-1} determine equal characters $\chi[w] = \chi[w^{-1}]$, since $\mathrm{trace}(A) = \mathrm{trace}(A^{-1})$ for a 2×2 matrix A with determinant 1. Direct calculation also establishes the identities $\chi[g] = \chi[h \cdot g \cdot h^{-1}]$, since $\mathrm{trace}(A) = \mathrm{trace}(B \cdot A \cdot B^{-1})$ for 2×2 matrices A and B with determinant 1, and $\chi[g \cdot h] = \chi[g]\chi[h] - \chi[g \cdot h^{-1}]$, since $\mathrm{trace}(A \cdot B) = \mathrm{trace}(A)\mathrm{trace}(B) - \mathrm{trace}(A \cdot B^{-1})$ for 2×2 matrices A and B with determinant 1.

Hence, if we let $\mathcal{C}(F_2)$ denote the set of conjugacy classes of maximal cyclic subgroups of F_2, the first two of the three identities just described yield that there is a well-defined map from $\mathcal{C}(F_2)$ to the set of characters, by taking the character of a generator. In this language, the purpose of this note is to describe the extent to which this map is not injective, and to describe means of determining when different elements of $\mathcal{C}(F_2)$ give rise to the same character.

An element w of F_2 is *maximal* if it generates a maximal cyclic subgroup of F_2, and hence is not a proper power of another element of F_2. An element w of F_2 is *primitive* if there exists a free basis S for F_2 containing w. Note that primitive elements are necessarily maximal, though not conversely. For a general group G (admitting a faithful representation into $\mathrm{SL}_2(\mathbf{C})$), the notion of maximality of an element of G still holds, namely, that an element of G is maximal if it is not a proper power of another element of G, though the notion of primitivity is restricted to elements of free groups. We are able to restrict our attention to maximal elements, since for a maximal element w of G, the character $\chi[w^n]$ is a polynomial in $\chi[w]$, see Section 2; in fact, there exists a polynomial $\tau_n(x)$, independent of w, so that $\chi[w^n] = \tau_n(\chi[w])$.

A *character class* in F_2 is the collection of all maximal cyclic subgroups of F_2 which give rise to the same character; that is, two maximal cyclic subgroups $\langle w \rangle$ and $\langle u \rangle$ of F_2 belong to the same character class if and only if $\chi[w](\rho) = \chi[u](\rho)$ for all $\rho \in \mathcal{F}(F_2)$. The *stable multiplicity* $\mathrm{mult}(w)$ of a maximal element w of F_2 is the number of conjugacy classes in its character class.

It was shown by Horowitz, Theorem 8.1 of [Hor1], that the stable multiplicity of an element of F_2 is always finite.

The starting point for our discussion is the following result of Horowitz:

Theorem 1.1 (Example 8.2 of Horowitz [Hor1]) *Let F_2 be a free group on two generators. For each $m \geq 1$, there exist elements $w_1, \ldots, w_m$ of F_2 which generate pairwise non-conjugate maximal cyclic subgroups of F_2 and which satisfy $\chi[w_1] = \cdots = \chi[w_m]$. That is, the stable multiplicity of maximal elements of F_2 is unbounded.*

Note that this result of Horowitz does not apply directly to lengths of closed curves on surfaces or in 3-manifolds, since most surfaces and 3-manifolds have fundamental groups that are not free of rank two, but rather is a statement about characters of representations of F_2 into $\mathrm{SL}_2(\mathbf{C})$. This will be discussed in more detail in Section 5. For the time being, we focus our attention on algebraic properties of representations of F_2.

In the same paper, Horowitz also gives the following necessary condition for two elements of F_2 to have the same character.

Theorem 1.2 (Lemma 6.1 of Horowitz [Hor1]) *Let U, U^* be elements in the free group $F_2 = \mathrm{free}(a, b)$ on two generators a and b of the form*

$$U = a^{\alpha_1} \cdot b^{\beta_1} \cdot a^{\alpha_2} \cdot b^{\beta_2} \cdots a^{\alpha_s} \cdot b^{\beta_s},$$

$$U^* = a^{\alpha_1^*} \cdot b^{\beta_1^*} \cdot a^{\alpha_2^*} \cdot b^{\beta_2^*} \cdots a^{\alpha_t^*} \cdot b^{\beta_t^*},$$

where s, $t > 0$ and $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_s, \alpha_1^*, \ldots, \alpha_t^*, \beta_1^*, \ldots, \beta_t^*$ are non-zero integers. If $\chi[U] = \chi[U^*]$, then $s = t$. Also, the numbers $|\alpha_1^*|, \ldots, |\alpha_s^*|$ are a rearrangement of the numbers $|\alpha_1|, \ldots, |\alpha_s|$, and the numbers $|\beta_1^*|, \ldots, |\beta_s^*|$ are a rearrangement of the numbers $|\beta_1|, \ldots, |\beta_s|$.

We note here that given an element w of $F_2 = \mathrm{free}(a, b)$, Theorem 1.2 gives an inefficient algorithm for determining all elements u of F_2 for which $\chi[w] = \chi[u]$, namely by considering all elements u of F_2 constructed by permuting the exponents of a and b in w, as well as changing their signs. As it is known that Theorem 1.2 is not optimal, and as there is not obvious direct generalization of Theorem 1.2 to other groups, we continue the search for better necessary conditions for two elements of F_2 to determine the same character.

One corollary of Theorem 1.2 is the following result, which we highlight due to the role that it plays later.

Theorem 1.3 (Theorem 7.1 of Horowitz [Hor1]) *Let u be an element of a free group F (of any countable rank). If $\chi[u] = \chi[c^m]$, where c is a primitive element of F, then u is conjugate to $c^{\pm m}$.*

These results of Horowitz give necessary conditions for two elements of F_2 to give rise to the same character. Before stating what is known in terms of partial converses to Theorem 1.2, we need the following. There is an involution I on $F_2 = \text{free}(a,b)$, defined as follows. First consider the automorphism $J : F_2 \to F_2$ defined by setting $J(a) = a^{-1}$ and $J(b) = b^{-1}$ and then extending so that J is an automorphism of F_2.

Define the involution $I : F_2 \to F_2$ by $I(w) = J(w^{-1}) = (J(w))^{-1}$. We refer to I as the *canonical involution* for F_2 with respect to the generators a and b. Note that I is not an automorphism of F_2 but rather is an anti-automorphism, with $I(w \cdot u) = I(u) \cdot I(w)$ and $I(w^{-1}) = (I(w))^{-1}$. It is not difficult to see that I is character preserving, in the sense that for any element w of F_2, we always have that $\chi[w] = \chi[I(w)]$. This and other properties of the involution I are discussed in more detail in Section 3.

As a partial answer to the original question, we note the following Proposition, which is an easy exercise using the uniqueness of normal forms in F_2, see for example Lyndon and Schupp [LyS].

Proposition 1.4 *Let $w = a^{n_1} \cdot b^{m_1} \cdots a^{n_p} \cdot b^{m_p}$ be an element of $F_2 = \text{free}(a,b)$, with $n_1, \ldots, n_p$ and $m_1, \ldots, m_p$ all non-zero. Then, w is conjugate to $I(w)$ if and only if there exists c so that $n_k = n_l$ for $k + l \equiv c \pmod p$ and $m_k = m_l$ for $k + l \equiv c - 1 \pmod p$.*

In particular, if $p \geq 3$ and if either the n_k are distinct or the m_k are distinct, then w and $I(w)$ generate non-conjugate maximal cyclic subgroups of F_2. In the case $p = 2$, for $w = a^{n_1} \cdot b^{m_1} \cdot a^{n_2} \cdot b^{m_2}$ with n_1, m_1, n_2, and m_2 distinct integers, we have that w and $I(w)$ generate non-conjugate maximal cyclic subgroups of F_2.

So, in a loose sense, for most maximal elements w of F_2, there is another element u, necessarily maximal, so that w and u generate non-conjugate cyclic subgroups of F_2 and so that $\chi[w] = \chi[u]$.

We now refine our terminology. Say that a maximal element w of F_2 (or more precisely, the conjugacy class of maximal cyclic subgroups of F_2 generated by w) is *pseudo-simple* if for any element u with $\chi[u] = \chi[w]$, we have that u is conjugate either to $w^{\pm 1}$ or to $I(w)^{\pm 1}$. With this language, the stable multiplicity of a pseudo-simple element of F_2 is at most 2, since we allow the possibility that w and $I(w)$ are conjugate.

Further, say that a maximal element w of F_2 (or more precisely, the conjugacy class of maximal cyclic subgroups of F_2 generated by w) is *simple* if for any element u with $\chi[u] = \chi[w]$, we have that u is conjugate to $w^{\pm 1}$. For example, primitive elements in free groups are simple, by Theorem 1.3. In particular, if

w is simple, then $I(w)$ is conjugate to w. The stable multiplicity of a simple element of F_2 is 1.

An element w of F_2 is *strictly pseudo-simple* if it is pseudo-simple but not simple. In particular, if w is strictly pseudo-simple, then w and $I(w)$ are not conjugate. The stable multiplicity of a strictly pseudo-simple element w of F_2 is exactly 2, with the two conjugacy classes in its character class being represented by $\langle w \rangle$ and $\langle I(w) \rangle$.

Ginzburg and Rudnick [GR] prove the following. Given an element $w = a^{n_1} \cdot b^{m_1} \cdots a^{n_p} \cdot b^{m_p}$, consider the two p-tuples of exponents $\mathbf{n} = (n_1, \ldots, n_p)$ and $\mathbf{m} = (m_1, \ldots, m_p)$. Say that $\mathbf{n}$ is *non-singular* if $n_k \neq \sum_{j \in S} n_j$ for all $1 \leq k \leq p$ and for all subsets $S \subset \{1, \ldots, p\}$, $S \neq \{k\}$. (In particular, note that if $\mathbf{n}$ is non-singular, then all the n_j are distinct and also $\sum_{j \in S} n_j \neq 0$ if S is non-empty.) Say that the element w is *non-singular* if both p-tuples of its exponents $\mathbf{n}$ and $\mathbf{m}$ are non-singular.

Theorem 1.5 (Theorem 1.1 of Ginzburg and Rudnick [GR]) *If w is a non-singular element of F_2, then w is strictly pseudo-simple.*

(The terminology in this note differs slightly from Ginzburg and Rudnick [GR], who use simple where we use pseudo-simple.) Moreover, Ginzburg and Rudnick [GR] also refine the statement of Theorem 1.2 for a non-singular element w of F_2.

Theorem 1.6 (Corollary 3.1 of Ginzburg and Rudnick [GR]) *Let U, U^* be non-singular elements in the free group $F_2 = \mathrm{free}(a, b)$ on two generators a and b of the form*

$$U = a^{\alpha_1} \cdot b^{\beta_1} \cdot a^{\alpha_2} \cdot b^{\beta_2} \cdots a^{\alpha_s} \cdot b^{\beta_s},$$

$$U^* = a^{\alpha_1^*} \cdot b^{\beta_1^*} \cdot a^{\alpha_2^*} \cdot b^{\beta_2^*} \cdots a^{\alpha_t^*} \cdot b^{\beta_t^*},$$

where s, $t > 0$ and $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_s, \alpha_1^*, \ldots, \alpha_t^*, \beta_1^*, \ldots, \beta_t^*$ are non-zero integers. If $\chi[U] = \chi[U^*]$, then $s = t$. Moreover, either the numbers $\alpha_1^*, \ldots, \alpha_s^*$ are a rearrangement of the numbers $\alpha_1, \ldots, \alpha_s$, and the numbers $\beta_1^*, \ldots, \beta_s^*$ are a rearrangement of the numbers $\beta_1, \ldots, \beta_s$, or else the numbers $\alpha_1^*, \ldots, \alpha_s^*$ are a rearrangement of the numbers $-\alpha_1, \ldots, -\alpha_s$, and the numbers $\beta_1^*, \ldots, \beta_s^*$ are a rearrangement of the numbers $-\beta_1, \ldots, -\beta_s$.

Theorem 1.2 and Theorem 1.6 share a common approach to their proof. Namely, they are both proven starting from the observation that if w and u are elements of F_2 with $\chi[w] = \chi[u]$, then for any family $\mathcal{P}$ of representations in $\mathcal{F}(F_2)$, we have that $\mathrm{trace}(\rho(w)) = \mathrm{trace}(\rho(u))$ for all $\rho \in \mathcal{P}$. Any identities satisfied by all representations in $\mathcal{F}(F_2)$ must be satisfied by all the representations in $\mathcal{P}$, and so the identities arising from analyzing the representations in $\mathcal{P}$ give conditions that yield necessary conditions for all representations in $\mathcal{F}(F_2)$.

Horowitz [Hor1] considers the collection $\mathcal{P}$ of representations $\mathcal{P} = \{\rho\}$ defined by

$$\rho(a) = \begin{pmatrix} \lambda & t \\ 0 & \lambda^{-1} \end{pmatrix} \text{ and } \rho(b) = \begin{pmatrix} \mu & 0 \\ t & \mu^{-1} \end{pmatrix}$$

for complex numbers λ, μ, and t. Ginzburg and Rudnick [GR] consider the collection $\mathcal{R}$ of representations $\mathcal{R} = \{\rho\}$ defined by

$$\rho(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \text{ and } \rho(b) = \begin{pmatrix} 1 & x \\ 1 & x+1 \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 1 & x \\ 1 & x+1 \end{pmatrix}^{-1}$$

for complex numbers a, b, and x.

In fact, Horowitz analyzes the leading term of the Fricke polynomial of $\chi[w]$, described in Section 2, expressed as a polynomial in $\chi[a \cdot b]$, with coefficients in $\mathbf{Z}[\chi[a], \chi[b]]$, evaluated at the representations in $\mathcal{P}$. Ginzberg and Rudnick analyze the coefficient of $\chi[a \cdot b]$ in this expansion, evaluated at the representations in $\mathcal{R}$.

2 Fricke polynomials

One of the main results in Horowitz [Hor1] was to give a proof of the following claim of Fricke (see p. 338 and 366 of Fricke and Klein [FrK]).

Theorem 2.1 (Theorem 3.1 of Horowitz [Hor1]) *Let F_n be a free group on n generators $a_1, \ldots, a_n$. If u is an arbitrary element of F_n, then the character $\chi[u]$ of u can be expressed as a polynomial with integer coefficients in the $2^n - 1$ characters $\chi[a_{i_1} \cdot a_{i_2} \cdots a_{i_k}]$, where $1 \le k \le n$ and $1 \le i_1 < i_2 < \cdots < i_k \le n$.*

We refer to this polynomial as the *Fricke polynomial* for u. One of the keys to the proof of Theorem 2.1 is the identity $\chi[w \cdot u] = \chi[w]\chi[u] - \chi[w \cdot u^{-1}]$, which follows immediately from the analogous identity for traces of 2×2 matrices, as well as the other basic identities already mentioned, that $\chi[w] = \chi[w^{-1}]$ and that $\chi[w] = \chi[u \cdot w \cdot u^{-1}]$; see Section 1.

One consequence of Theorem 2.1 is the following construction of Fricke. Let w and u be any pair of elements of F_2 for which $\chi[w] = \chi[u]$, and let $p = p(a, b)$ be any element of $F_2 = \text{free}(a, b)$. Since $\chi[p]$ is expressible as a polynomial $\chi[p] = P(\chi[a], \chi[b], \chi[a \cdot b])$, we see that

$$\chi[p(w, u)] = P(\chi[w], \chi[u], \chi[w \cdot u]) = P(\chi[u], \chi[w], \chi[u \cdot w]) = \chi[p(u, w)],$$

where the middle equality is a consequence of the assumption that $\chi[w] = \chi[u]$ and the fact that $\chi[w \cdot u] = \chi[u \cdot w]$.

This leads to the following definition. Let w and u be elements of F_2 for which $\chi[w] = \chi[u]$, and let F be the subgroup of F_2 generated by w and u. There is an automorphism σ on F, the *switching automorphism*, defined by setting $\sigma(w) = u$ and $\sigma(u) = w$ and then extending σ to be an automorphism of F. The discussion above yields that σ is a character preserving automorphism on the subgroup F of F_2, which in general does not extend to an automorphism of all of F_2.

Theorem 2.1 can be thought of as the analogue for these characters of the result that the Teichmüller space of an orientable surface S of negative Euler characteristic and finite analytic type can be parametrized by the lengths of a fixed finite set of simple closed curves on the surface. For more information on this, we refer the interested reader to Abikoff [Ab], Schmutz Schaller [Sch2], and Hamenstädt [Ham].

Let $\mathcal{B}_n$ be the ring of polynomials with integer coefficients in the $2^n - 1$ indeterminates $x_{i_1 i_2 \cdots i_k}$, where $1 \le k \le n$ and $1 \le i_1 < i_2 < \cdots < i_k \le n$. Theorem 2.1 can also be interpreted as describing a map $\Theta : \mathcal{C}(F_n) \to \mathcal{B}_n$, by taking the character of a generator.

One question as yet unresolved is to determine the image $\Theta(\mathcal{C}(F_n))$ in $\mathcal{B}_n$. It is an easy observation that Θ is not surjective, even for $n = 1$. To see this, define a family $\tau_n(s)$, $n \ge 0$, of polynomials by setting $\tau_0(s) = 2$, $\tau_1(s) = s$, and $\tau_{n+1}(s) = s\tau_n(s) - \tau_{n-1}(s)$. The $\tau_n(s)$ are Chebychev polynomials of the second kind. Using the above identity for $\chi[w \cdot u]$, we see that $\chi[w^n] = \tau_n(\chi[w])$. This is discussed by Horowitz [Hor1], Section 2, and was exploited to great effect by Jørgensen [J4].

Let $\mathcal{I}_n$ be the ideal in $\mathcal{B}_n$ consisting of those polynomials which are identically 0 under the substitution

$$x_{i_1 i_2 \cdots i_k} = \chi[a_{i_1} \cdot a_{i_2} \cdots a_{i_k}].$$

(The polynomials in $\mathcal{I}_n$ are the obstruction to the uniqueness of the Fricke polynomial of a word in F_n.) Horowitz considered the question of determining the structure of $\mathcal{I}_n$. He showed, see Theorem 4.1 of [Hor1], that $\mathcal{I}_1$ and $\mathcal{I}_2$ are both the trivial ideal, so that the character of elements of $F_1 = \mathrm{free}(a_1)$ and of $F_2 = \mathrm{free}(a_1, a_2)$ are represented by unique polynomials.

In the case $n = 3$, though, he shows that the ideal $\mathcal{I}_3$ is non-zero. Specifically, let $\mathbf{x} = (x_1, x_2, x_3, x_{12}, x_{13}, x_{23})$, and set

$$k_1(\mathbf{x}) = x_{12}x_3 + x_{13}x_2 + x_{23}x_1$$

and

$$k_0(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 + x_{12}^2 + x_{13}^2 + x_{23}^2 - x_1x_2x_{12} - x_1x_3x_{13} - x_2x_3x_{23} + x_{12}x_{13}x_{23} - 4.$$

Then, $\mathcal{I}_3$ is the principal ideal in $\mathcal{B}_3$ generated by $k(\mathbf{x}, x_{123}) = x_{123}^2 - k_1(\mathbf{x})x_{123} + k_0(\mathbf{x})$; this is derived from the character relation

$$\begin{aligned}\chi[a_1 \cdot a_2 \cdot a_3] = \chi[a_1]\chi[a_2 \cdot a_3] &+ \chi[a_2]\chi[a_1 \cdot a_3] + \chi[a_3]\chi[a_1 \cdot a_2] \\ &- \chi[a_1]\chi[a_2]\chi[a_3] - \chi[a_1 \cdot a_3 \cdot a_2],\end{aligned}$$

which is derivable from the basic identities for characters discussed at the beginning of this Section, and the consequent identity for $\chi[u \cdot v \cdot w]\chi[u \cdot w \cdot v]$. In contrast to this, Whittemore [W2] showed that $\mathcal{I}_n$ is not a principal ideal for $n \geq 4$.

Another reason for Whittemore's interest is the following question, as described in [W1]. Following Artin, define the *braid group* B_n to be the group of automorphisms of the free group $F_n = \mathrm{free}(a_1, \ldots, a_n)$ generated by the automorphisms β_k of F_n, $1 \leq k \leq n-1$, defined by: $\beta_k(a_k) = a_{k+1}$, $\beta_k(a_{k+1}) = a_{k+1} \cdot a_k \cdot a_{k+1}^{-1}$, and $\beta_k(a_j) = a_j$ for $j \neq k,\ k+1$. It is known that every knot group G (where a *knot group* is the fundamental group of $\mathbf{S}^3 - K$ for a knot K) can be obtained from F_n by identifying the generators of F_n with their images under an element β_G of B_n.

Using Theorem 2.1, we can realize the set $\mathrm{Hom}(F_n, \mathrm{SL}_2(\mathbf{C}))$ of all representations of F_n into $\mathrm{SL}_2(\mathbf{C})$ with a subset $\mathcal{T}_n$ of $\mathbf{C}^{2^n-1}$ by taking an element ρ of $\mathrm{Hom}(F_n, \mathrm{SL}_2(\mathbf{C}))$ to the point

$$(\chi[a_1](\rho), \chi[a_2](\rho), \ldots, \chi[a_{i_1} \cdot a_{i_2} \cdots a_{i_k}](\rho), \ldots, \chi[a_1 \cdot a_2 \cdots a_n](\rho))$$

of $\mathbf{C}^{2^n-1}$, for all $2^n - 1$ possible values of $i_1, \ldots, i_k$ satisfying $1 \leq k \leq n$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

Magnus conjectured that the points of $\mathcal{T}_n$ corresponding to a knot group G are exactly the fixed points in $\mathcal{T}_n$ of the automorphism of $\mathcal{T}_n$ induced by β_G. In Theorem 1 of [W1], Whittemore determined the points of $\mathcal{T}_2$ corresponding to the representations of the group G of Listing's knot, given by the presentation

$$G = \langle a, b \mid b^{-1} \cdot a^{-1} \cdot b \cdot a \cdot b^{-1} \cdot a \cdot b \cdot a^{-1} \cdot b^{-1} \cdot a = 1 \rangle.$$

Let $\mathcal{A}_n$ denote the group of automorphisms of the quotient ring $\mathcal{B}_n/\mathcal{I}_n$, and let $\mathrm{Out}(F_n)$ denote the group of outer automorphism classes of F_n. Each automorphism of F_n induces in a natural way an element of $\mathcal{A}_n$. Horowitz then argues that this induces a natural isomorphism between $\mathrm{Out}(F_n)$ and $\mathcal{A}_n$ for $n \geq 3$. (We refer the interested reader to the discussion in [Hor2] preceeding Corollary 1 for a more detailed treatment.)

Consequently, it was suggested $\mathrm{Out}(F_n)$ might be profitably studied by analyzing the structure of $\mathcal{A}_n$. However, it is unclear to what extent this programme was carried out, and it is unclear that it would significantly add to the current state of the knowledge of the structure of $\mathrm{Out}(F_n)$, though some further work on this general question has been done by Magnus [Mag2], González-Acuña and Montensinos-Amilibia [GM], and Humphries [Hum].

3 Properties of the involution I

The purpose of this Section is to explore some of the properties of the canonical involution I and the automorphism J on $F_2 = \mathrm{free}(a, b)$.

The first observation is that the property of an element w of F_2 being conjugate to $I(w)$ is independent of the choice of generating set for F_2. In fact, up to an inner automorphism of F_2, the two operations of changing generators and applying the canonical involution (with respect to the appropriate set of generators) commute. This is an easy application of Nielsen transformations. For a discussion of Nielsen transformations, see Lyndon and Schupp [LyS].

The second observation is that both the canonical involution I and the automorphism J of F_2 are character preserving. One proof of this begins with the following Lemma, due originally to Jørgensen [J1]. (There are other proofs, for instance the proof given by Ginzburg and Rudnick [GR].)

Lemma 3.1 (Section 4 of Jørgensen [J1]) *Let A and B be two elements of $\mathrm{SL}_2(\mathbf{C})$. Then, there exists an element E of $\mathrm{SL}_2(\mathbf{C})$ so that $E \cdot A \cdot E^{-1} = A^{-1}$ and $E \cdot B \cdot E^{-1} = B^{-1}$.*

Moreover, given the geometric description of E (for instance, in the case that A and B are hyperbolic elements of $\mathrm{SL}_2(\mathbf{C})$ with distinct fixed points, E is the half-turn whose axis is the common perpendicular to the axes of A and B), it is easy to see that E varies continuously with A and B. We note here that, as has been observed and exploited by Jørgensen and others, see in particular Jørgensen [J1], Jørgensen and Sandler [JS], and Pignataro and Sandler [PS], an element w of F_2 is equal to $I(w)$ if and only if w is a palindrome in a and b.

Combining Lemma 3.1 with the facts that conjugation and inversion are both trace preserving, we see that the automorphism J, and hence the canonical involution I, are both character preserving.

The third observation is that most of the examples and constructions found to date regarding elements of F_2 which generate non-conjugate maximal cyclic subgroups of F_2 and which give rise to the same character can largely be captured by the action of character preserving involutions, either the canonical involution I, the automorphism J, or the switching automorphism σ of some subgroup (as described in Section 2).

Consider, for example, the following elements of $F_2 = \mathrm{free}(a, b)$, due originally to Horowitz [Hor1]. Given an infinite-tuple $(\varepsilon_1, \varepsilon_1, \ldots, \varepsilon_n, \ldots)$, where each $\varepsilon_n = \pm 1$, we get a rooted binary tree T of elements of F_2. Namely, set $w_0 = a$

and for $m \geq 1$ set

$$\begin{aligned} w_m(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m) &= w_{m-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1})^{-\varepsilon_m} \cdot b^{2m} \cdot \\ & \quad w_{m-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1})^{\varepsilon_m} \cdot b^{2m-1} \cdot \\ & \quad w_{m-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1})^{-\varepsilon_m} \cdot b^{2m} \cdot \\ & \quad w_{m-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1})^{\varepsilon_m}. \end{aligned}$$

Note that for $m \geq 0$, there are 2^m elements of *depth* m in T, namely the elements $w_m(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$ for the 2^m choices of $\varepsilon_k = \pm 1$ for $1 \leq k \leq m$. Horowitz proves that the 2^m elements $w_1, \ldots, w_{2^m}$ generate pairwise non-conjugate maximal cyclic subgroups of F_2 and that $\chi[w_1] = \cdots = \chi[w_{2^m}]$. His proof of the former part of the statement is just an application of the existence of unique normal forms for elements in free groups. His proof of the latter part is a direct calculation, using the family of representations described at the end of Section 1. We give here an alternative proof of the second part of his statement, using a slightly difficult argument.

The basic fact we need is the following.

Lemma 3.2 *Let $F_2 = \mathrm{free}(a, b)$ be the free group on a and b, and let T be the tree described above. Then,*

$$I(w_m(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)) = w_m(-\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_m).$$

Proof The proof of the Lemma is by induction. We begin with the calculation of $I(w_1(\varepsilon_1))$. Note that

$$w_1(\varepsilon_1) = a^{-\varepsilon_1} \cdot b^2 \cdot a^{\varepsilon_1} \cdot b \cdot a^{-\varepsilon_1} \cdot b^2 \cdot a^{\varepsilon_1}$$

and that

$$\begin{aligned} I(w_1(\varepsilon_1)) &= I(a^{-\varepsilon_1} \cdot b^2 \cdot a^{\varepsilon_1} \cdot b \cdot a^{-\varepsilon_1} \cdot b^2 \cdot a^{\varepsilon_1}) \\ &= a^{\varepsilon_1} \cdot b^2 \cdot a^{-\varepsilon_1} \cdot b \cdot a^{\varepsilon_1} \cdot b^2 \cdot a^{-\varepsilon_1} = w_1(-\varepsilon_1), \end{aligned}$$

as desired.

Suppose now that

$$I(w_{m-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1})) = w_{m-1}(-\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_{m-1}),$$

and consider $I(w_m(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m))$. Using that I is an anti-automorphism and the inductive hypothesis, we see that

$$I(w_m(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)) \;=\; I(w_{m-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1}))^{\varepsilon_m} \cdot b^{2m} \cdot$$

$$\begin{aligned}
& I(w_{m-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1}))^{-\varepsilon_m} \cdot b^{2m-1} \cdot \\
& I(w_{m-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1}))^{\varepsilon_m} \cdot b^{2m} \cdot \\
& I(w_{m-1}(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1}))^{-\varepsilon_m} \\
= \; & w_{m-1}(-\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_{m-1})^{\varepsilon_m} \cdot b^{2m} \cdot \\
& w_{m-1}(-\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_{m-1})^{-\varepsilon_m} \cdot b^{2m-1} \cdot \\
& w_{m-1}(-\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_{m-1})^{\varepsilon_m} \cdot b^{2m} \cdot \\
& w_{m-1}(-\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_{m-1})^{-\varepsilon_m} \\
= \; & w_m(-\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_m),
\end{aligned}$$

as desired. **QED**

Note that the two elements $w_m(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m)$ and $w_m(-\varepsilon_1, -\varepsilon_2, \ldots, -\varepsilon_m)$ lie in different branches of the tree rooted at w_0. In fact, we can apply this Lemma to the subtree rooted at any element $w = w_k(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k)$, which corresponds in this construction to the subgroup of F_2 generated by w and b, with its canonical involution defined in terms of w and b. By considering all such subtrees and their relative canonical involutions, we see that all the elements in this tree below w_0 and of the same depth must have equal characters, as they are related by this collection of involutions.

We can recast this construction slightly as follows. Set $\nu_m(a,b) = a^{-1} \cdot b^{2m} \cdot a \cdot b^{2m-1} \cdot a^{-1} \cdot b^{2m} \cdot a$. Then, $w_1(\varepsilon_1) = \nu_1(a^{\varepsilon_1}, b)$ and $w_1(-\varepsilon_1) = \nu_1(a^{-\varepsilon_1}, b^{-1})^{-1} = I(\nu_1(a^{\varepsilon_1}, b))$. In general,

$$w_m(\varepsilon_1, \ldots, \varepsilon_m) = \nu_m(w_{m-1}(\varepsilon_1, \ldots, \varepsilon_{m-1})^{\varepsilon_m}, b)$$

and

$$\begin{aligned}
I(w_m(\varepsilon_1, \ldots, \varepsilon_m)) &= \nu_m(w_{m-1}(-\varepsilon_1, \ldots, -\varepsilon_{m-1})^{-\varepsilon_m}, b^{-1})^{-1} \\
&= w_m(-\varepsilon_1, \ldots, -\varepsilon_m).
\end{aligned}$$

Suppose however that we consider the action on $w_m(\varepsilon_1, \ldots, \varepsilon_m)$ of the involution I^*, where we consider $w_m(\varepsilon_1, \ldots, \varepsilon_m)$ as a word in $w_{m-1}(\varepsilon_1, \ldots, \varepsilon_{m-1})$ and b and I^* is the canonical involution for this generators. Then,

$$I^*(w_m(\varepsilon_1, \ldots, \varepsilon_m)) = w_m(\varepsilon_1, \ldots, \varepsilon_{m-1}, -\varepsilon_m).$$

There are several other constructions of n-tuples of words in $F_2 = \mathrm{free}(a,b)$ generating non-conjugate maximal cyclic subgroups of F_2 whose characters are equal.

One is due to Buser [Bus]. Set $\nu(a,b) = b \cdot a^{-1} \cdot b^{-1} \cdot a \cdot b$, set $W_1(\varepsilon_1)(a,b) = \nu(a^{\varepsilon_1}, b^{\varepsilon_1})$, and for $m \geq 2$ inductively define

$$W_m(\varepsilon_1, \ldots, \varepsilon_m)(a,b) = W_{m-1}(\varepsilon_1, \ldots, \varepsilon_{m-1})(a^{\varepsilon_m}, \nu(a^{\varepsilon_m}, b^{\varepsilon_m})),$$

where again each $\varepsilon_k = \pm 1$, so that there are 2^m words at the m^{th} step of the construction. As with Horowitz's construction, these 2^m words generate non-conjugate maximal cyclic subgroups of F_2 and give rise to the same character.

Since J is an automorphism, since $J(a^\varepsilon) = a^{-\varepsilon}$, and since $J(\nu(a^\varepsilon, b^\varepsilon)) = \nu(a^{-\varepsilon}, b^{-\varepsilon})$, we have that

$$\begin{aligned} J(W_m(\varepsilon_1, \ldots, \varepsilon_m)(a,b)) &= W_{m-1}(\varepsilon_1, \ldots, \varepsilon_{m-1})(J(a^{\varepsilon_m}), \nu(J(a^{\varepsilon_m}), J(b^{\varepsilon_m}))) \\ &= W_{m-1}(\varepsilon_1, \ldots, \varepsilon_{m-1})(a^{-\varepsilon_m}, \nu(a^{-\varepsilon_m}, b^{-\varepsilon_m})) \\ &= W_m(\varepsilon_1, \ldots, \varepsilon_{m-1}, -\varepsilon_m)(a,b). \end{aligned}$$

We note here that in his discussion, Buser also gives a very nice geometric description of this construction.

Masters [Mas] uses a slightly different approach. Let $\{p_n\}$, $\{q_n\}$, and $\{k_n\}$ be sequences of positive integers, for $n \geq 1$, and set

$$W_n(x,y) = (x^{p_n - 1 + q_n} y^{-q_n})^{k_n} \; x \; (x^{p_n - 1 + q_n} y^{-q_n}) \; x^{-1}$$

and

$$\overline{W}_n(x,y) = x \; (x^{p_n - 1 + q_n} y^{-q_n})^{k_n} \; x^{-1} \; (x^{p_n - 1 + q_n} y^{-q_n}).$$

Note that if I is the canonical involution for the free group generated by x and y, we have that

$$I(W_n(x,y)) = (x^{-1} \; y^{-q_n}) \; \overline{W}_n(x,y) \; (x^{-1} \; y^{-q_n})^{-1},$$

and so $\chi[W_n(x,y)] = \chi[\overline{W}_n(x,y)]$.

In this case, the nodes in the tree are ordered pairs of elements. Consider the words

$$w_{1,1} = W_1(a,b)$$

and

$$w_{1,2} = \overline{W}_1(a,b).$$

By the argument in the previous paragraph, $\chi[w_{1,1}] = \chi[w_{1,2}]$, and the root of the tree is the ordered pair $(w_{1,1}, w_{1,2})$. The left branch from the root corresponds to the ordered pair

$$(w_{2,1}, w_{2,2}) = (W_2(w_{1,1}, w_{1,2}), \overline{W}_2(w_{1,1}, w_{1,2})),$$

and the right branch from the root corresponds to the ordered pair

$$(w_{2,3}, w_{2,4}) = (W_2(w_{1,2}, w_{1,1}), \overline{W}_2(w_{1,2}, w_{1,1})).$$

The canonical involution yields that the two words in each ordered pair have the same character, and the switching automorphism relative to the generators for the root of the tree interchange the two branches.

To generate the binary tree, we iterate this construction: each node v in the tree is marked by an ordered pair of elements of F_2 of the same character; if the depth of v is m (where here the root has depth 1), one of the two branches of depth $m+1$ descending from v is marked by W_{m+1} and $\overline{W}_{m+1}$ applied to the pair of elements marking v, while the other branch descending from v is marked by first applying the switching automorphism to the pair of words marking v and then applying W_{m+1} and $\overline{W}_{m+1}$ to these words. Again, all the words of the same depth have the same character. (We note that Masters considers only a part of this tree, as he uses only m elements of depth m and not all 2^m.) The reason for choosing the sequences of exponents is to ensure that, when the free groups are realized inside a 3-manifold group, the m considered non-conjugate elements in the free groups remain non-conjugate in the ambient 3-manifold group.

Pignataro and Sandler [PS] also construct a binary tree in which each node is marked by an ordered pair of elements of $F_2 = \text{free}(a,b)$. The tree is rooted at F_2. Consider the word $W_0(a,b) = a \cdot b^2 \cdot a^{-1}$, and set $W(a,b) = W_0(a,b) \cdot J(W_0(a,b)) \cdot W_0(a,b)^{-1} \cdot J(W_0(a,b))^{-1}$. (Substituting in $W_0(a,b)$ into the expression for $W(a,b)$ gives that $W(a,b) = a \cdot b^2 \cdot a^{-2} \cdot b^{-2} \cdot a^2 \cdot b^{-2} \cdot a^{-2} \cdot b^2 \cdot a$, and it is important for their analysis that $W(a,b)$ is a palindrome in a and b.)

Suppose that a node v is marked by the ordered pair (U,V). The node on the left branch descending from v is then marked by the ordered pair $(W(U,V), W(U,V^{-1}))$; since

$$W(U,V) = W_0(U,V) \cdot J(W_0(U,V)) \cdot W_0(U,V)^{-1} \cdot J(W_0(U,V))^{-1}$$

and

$$\begin{aligned} &W(U,V^{-1}) \\ &\quad = W_0(U,V^{-1}) \cdot J(W_0(U,V^{-1})) \cdot W_0(U,V^{-1})^{-1} \cdot J(W_0(U,V^{-1}))^{-1} \\ &\quad = W_0(U,V)^{-1} \cdot J(W_0(U,V))^{-1} \cdot W_0(U,V) \cdot J(W_0(U,V)), \end{aligned}$$

we see that $W(U,V^{-1})$ is conjugate to $W(U,V)$ and hence the two elements marking the node v give rise to the same character, for all nodes except the root of the tree.

The node on the right branch descending from v is marked by the ordered pair $(W(V,U), W(V,U^{-1}))$, which is obtained from the ordered pair marking the node on the left branch by applying the switching automorphism relative to U, V to the first element in the pair and by applying the switching automorphism and the automorphism J, both relative to U, V, to the second element in the pair. However, since $W(U,V)$ is a palindrome, the action of J is the same as inversion.

4 The main structural conjecture

The following is an attempt to formulate a loose conjecture to describe when elements give rise to the same character in F_2:

Conjecture 4.1 *Let F_2 be the free group of rank two, and suppose that there are elements w and u of F_2 which generate non-conjugate maximal cyclic subgroups of F_2 and whose associated characters $\chi[w]$ and $\chi[u]$ are equal. Then, there exists a binary tree T of subgroups of F_2 with the following properties:*

1. *each node v of the tree is a free subgroup of F_2 of rank two;*
2. *the branches denote proper inclusion, so that if a branch descends from a node V to a node V', then V' is a proper subgroup of V, where we think of the tree as being arranged vertically, with the root at the top;*
3. *for each node V of T, there is a character preserving involution I_V on V which interchanges the two branches descending from V;*
4. *there are nodes V_w and V_u containing w and u, respectively, which have the same depth in T and which are related by the action of the character preserving involutions I_V for nodes V in the tree above V_w and V_u.*

Roughly speaking, the constructions of Horowitz, Buser, Masters, and Pignataro and Sandler all fall within the scope of the conjecture.

In the case of Horowitz's construction, the free subgroups in the tree are the subgroups generated by $w_m(\varepsilon_1, \ldots, \varepsilon_m)$ and b, and the involutions are the canonical involutions with respect to these generators.

In the case of Buser's construction, the free subgroups in the tree are the subgroups generated by the $W_m(\varepsilon_1, \ldots, \varepsilon_m)$ and a, and the involutions are the automorphisms J with respect to these generators.

In the case of Masters' construction, the free subgroups in the tree are generated by the pairs of elements marking the nodes in the tree, and the involutions are the switching automorphisms with respect to these generators. The main difficulty here is that the two elements marking the root may not generate a free group.

In the case of Pignataro and Sandlers' construction, the free subgroups are the tree are generated by the pairs of elements marking the nodes in the tree, and the involutions are the switching automorphisms on the ordered pairs marking the nodes. Here, though, for the conjecture to apply, we would need to take the tree in the conjecture to be the tree starting from one of the nodes of depth one in Pignataro and Sandler's construction as described in the previous Section.

5 Connections to lengths of curves

We now consider in more detail the connection between discrete, faithful representations of a group G into $\mathrm{SL}_2(\mathbf{C})$ and lengths of curves in hyperbolic 2- and 3-manifolds. We begin by resolving a slight ambiguity, as the fundamental groups of hyperbolic 2- and 3-manifolds are discrete subgroups of $\mathrm{PSL}_2(\mathbf{C})$ (or $\mathrm{PSL}_2(\mathbf{R})$, in the case of surfaces), and not of $\mathrm{SL}_2(\mathbf{C})$. Let $P : \mathrm{SL}_2(\mathbf{C}) \to \mathrm{PSL}_2(\mathbf{C})$ be the quotient map.

It is well known (see for instance Kra [Kra] and the references contained therein) that a discrete, faithful representation $\widehat{\rho}$ of a finitely generated group G into $\mathrm{PSL}_2(\mathbf{C})$ lifts to a discrete, faithful representation ρ of G into $\mathrm{SL}_2(\mathbf{C})$ (by which we mean that $\widehat{\rho} = P \circ \rho$) if G contains no 2-torsion. Conversely, if G is a finitely generated group containing no 2-torsion and if ρ is a faithful representation of G into $\mathrm{SL}_2(\mathbf{C})$, then the composition $\widehat{\rho} = P \circ \rho$ is necessarily a faithful representation of G into $\mathrm{PSL}_2(\mathbf{C})$, as the image $\rho(G)$ of G in $\mathrm{SL}_2(\mathbf{C})$ cannot contain the non-trivial element of the kernel of P, namely $-\mathrm{id}$. (However, *a priori* there still may be a several-to-one correspondence between representations into $\mathrm{SL}_2(\mathbf{C})$ and representations into $\mathrm{PSL}_2(\mathbf{C})$, as there may be distinct representations ρ_1 and ρ_2 of G into $\mathrm{SL}_2(\mathbf{C})$ for which $P \circ \rho_1 = P \circ \rho_2$.)

So, given a discrete, faithful representation ρ of a finitely generated group G with no 2-torsion into $\mathrm{SL}_2(\mathbf{C})$, we can compose with P to obtain a discrete, faithful representation $\widehat{\rho} = P \circ \rho$ of G into $\mathrm{PSL}_2(\mathbf{C})$, which then gives rise to an orientable hyperbolic 3-manifold, namely the quotient $\mathbf{H}^3/\widehat{\rho}(G)$. (We make here the convention that when G is the fundamental group of a surface, we consider discrete, faithful representations ρ of G into $\mathrm{SL}_2(\mathbf{R})$, with quotient surface $\mathbf{H}^2/\widehat{\rho}(G)$, unless explicitly stated otherwise.) (In the cases of interest to us here, the group G will be the fundamental group of an orientable surface of negative Euler characteristic or of a compact hyperbolizable 3-manifold, and will in fact be torsion-free.)

Let A be a loxodromic (or hyperbolic) element of $\mathrm{PSL}_2(\mathbf{C})$, so that A is conjugate to $z \mapsto \lambda^2 z$ for some λ^2 in $\mathbf{C}$ with $|\lambda^2| > 1$. The number λ^2 is the *multiplier* of the loxodromic element A. Note that the multiplier of a loxodromic element of $\mathrm{PSL}_2(\mathbf{C})$ determines the trace of its lift to $\mathrm{SL}_2(\mathbf{C})$ up to sign, as there are two possible lifts of A to $\mathrm{SL}_2(\mathbf{C})$, with traces $\pm(\lambda+\lambda^{-1})$. The *axis* $\mathrm{axis}(A)$ of A is the hyperbolic line in $\mathbf{H}^3$ joining its two fixed points; A acts as translation along its axis. The *translation distance* of A along $\mathrm{axis}(A)$, defined to be the hyperbolic distance between x and $A(x)$ for any point x on $\mathrm{axis}(A)$, is $\ln(|\lambda^2|)$.

Let Γ be a discrete torsion-free subgroup of $\mathrm{PSL}_2(\mathbf{C})$. There is a one-to-one correpondence between free homotopy classes of closed curves in $\mathbf{H}^3/\Gamma$ (or in $\mathbf{H}^2/\Gamma$, in the case that Γ lies in $\mathrm{PSL}_2(\mathbf{R})$) and conjugacy classes of maximal cyclic subgroups of Γ. For a maximal loxodromic element A of Γ, the axis of A projects to a closed geodesic of length $\ln(|\lambda^2|)$ in the quotient manifold $\mathbf{H}^3/\Gamma$. Among all closed curves in the free homotopy class determined by A, the projection of the axis of A has minimal length. We define the length of the free homotopy class of curves determined by A, or equivalently of the conjugacy class of maximal cyclic subgroups of Γ determined by A, to be the length of this geodesic.

For a maximal parabolic element A of Γ, the axis of A is not defined, and there are closed curves in the free homotopy class of A whose lengths go to 0. We define the length of the free homotopy class of curves determined by A, or equivalently of the conjugacy class of maximal cyclic subgroups of Γ determined by A, to be 0. (There are no elliptic elements of Γ, by assumption.)

For a finitely generated group G and an element ρ of $\mathcal{F}(G)$ with discrete image, the *length spectrum* of $\widehat{\rho}(G)$ (where $\widehat{\rho} = P \circ \rho$), or of its quotient manifold $\mathbf{H}^3/\widehat{\rho}(G)$, is the set of lengths of closed geodesics in $\mathbf{H}^3/\widehat{\rho}(G)$, counted with multiplicity. (Actually, in the case of interest to us here, since we have a representation of G into $\mathrm{PSL}_2(\mathbf{C})$, we have the *marked length spectrum*, which we can think of as the map from G into $\mathbf{R}$ obtained by composing $\widehat{\rho}$ with the function from $\widehat{\rho}(G)$ giving the length of a conjugacy class of maximal cyclic subgroups of $\widehat{\rho}(G)$, using the correspondence described in the previous paragraphs. For closed orientable surfaces equipped with a metric of constant negative curvature, the marked length spectrum contains sufficient information to completely determine the geometry of the surface. The marked length spectrum has been studied by a number of authors; we refer the interested reader to Croke [Cr] or Otal [Ot] for more information about the behavior of the length spectra of surfaces.)

We pause here to note the following. In recent years, there has been a great deal of interest in determining the exact behavior of the number $\mathcal{N}(\ell)$ of closed geodesics of length at most ℓ in a hyperbolic n-manifold, or n-orbifold, which is known to be asymptotically $\mathcal{N}(\ell) \sim \frac{1}{(n-1)\ell} e^{(n-1)\ell}$, as well as the statistics of their distribution. We will not explore this connection here, other than to say that arithmetic and non-arithmetic hyperbolic n-manifolds behave differently when viewed by $\mathcal{N}(\ell)$. For further information, we refer the interested reader to Schmutz [Sch1], Luo and Sarnak [LuS], Marklof [Mar], and Bolte [Bol], and to the references contained therein.

Let ρ be an element of $\mathcal{F}(G)$ with discrete image. If two maximal elements w and u of G satisfy $\mathrm{trace}(\rho(w)) = \mathrm{trace}(\rho(u))$ with $\rho(w)$ (and hence $\rho(u)$ loxodromic), then $\widehat{\rho}(w)$ and $\widehat{\rho}(u)$ correspond to closed geodesics of equal length in the quotient manifold $\mathbf{H}^3/\widehat{\rho}(G)$, where $\widehat{\rho} = P \circ \rho$. This follows immediately, since the trace of an element in $\mathrm{SL}_2(\mathbf{C})$ determines the multiplier of the corresponding element in $\mathrm{PSL}_2(\mathbf{C})$, which in turn determines the length of the closed geodesic in the quotient manifold. Specifically, if $\widehat{\rho}(w)$ is loxodromic with multiplier λ^2, then $c = \text{trace}(\widehat{\rho}(w)) = \pm(\lambda + \lambda^{-1})$, and so $\lambda^2 = \frac{1}{2}(c^2 - 2 \pm c\sqrt{c^2 - 4})$, where the sign of the $\pm$ is chosen so that $|\lambda^2| > 1$.

In particular, if G is any finitely generated group and if w and u are two elements of G which generate non-conjugate maximal cyclic subgroups and which satisfy $\chi[w] = \chi[u]$, then $\text{trace}(\rho(w)) = \text{trace}(\rho(u))$ for all $\rho \in \mathcal{F}(G)$, and so the lengths of the free homotopy classes determined by w and u are equal in $\mathbf{H}^3/\widehat{\rho}(G)$ (where $\widehat{\rho} = P \circ \rho$) (or in $\mathbf{H}^2/\widehat{\rho}(G)$, in the case that ρ is a representation into $\mathrm{SL}_2(\mathbf{R})$) for all representations ρ in $\mathcal{F}(G)$ with discrete image. So, finding pairs of closed curves on S whose geodesic representatives have the same hyperbolic length over all hyperbolic structures on S is equivalent to the problem of finding pairs of elements in G that generate non-conjugate maximal cyclic subgroups of G and that give rise to the same character over the space of faithful representations of G into $\mathrm{SL}_2(\mathbf{C})$. We refer the interested reader to Leininger [Lei], particularly Section 3, for a more detailed discussion of this point.

Randol proved the following result for the length spectrum of a surface.

Theorem 5.1 (Main result of Randol [Ran]) *Let S be an orientable surface of negative Euler characteristic. Then, the length spectrum of S has unbounded multiplicity.*

We pause here to make the following aside. Randol's theorem, Theorem 5.1, arose out of his interest in earlier work of Guillemin and Kazhdan [GK], who prove the following. Let M be a closed surface with a metric of negative curvature and simple length spectrum; here, by *simple length spectrum*, we mean that there do not exist closed geodesics on M such that the ratio of their lengths is a rational number. Let Δ be the Laplace-Beltrami operator on $C^\infty(M)$. If there are functions q_1 and q_2 in $C^\infty(M)$ for which the operators $\Delta + q_1$ and $\Delta + q_2$ have coincident spectra, then $q_1 \equiv q_2$. (We note that this result has been generalized to compact negatively curved Riemannian manifolds by Croke and Sharafutdinov [CS], to whom we refer the interested reader for more information.) In this language, Theorem 5.1 implies that surfaces with a constant negative curvature metric never satisfy this condition of simple length spectrum.

Of course, when discussing the spectrum of the Laplace-Beltrami operator on a hyperbolic surface, it would be remiss to not mention the Selberg trace formula. We refer the interested reader to the paper of McKean [McK] and the books of Hejhal [Hej] for a more detailed discussion of the trace formula.

Masters proved the following result for the length spectrum of a hyperbolic 3-manifold.

Theorem 5.2 (Theorem 1.2 of Masters [Mas]) *Let N be a hyperbolic 3-manifold with non-elementary fundamental group. Then, the length spectrum of N has unbounded multiplicity.*

Both Randol and Masters used the earlier work of Horowitz in their proofs. The main difficulty in both cases, more pronounced for 3-manifolds than for surfaces, is not the construction a free subgroup F_2 of the fundamental group G, but rather is to control the problem of elements in F_2 being non-conjugate in F_2 but becoming conjugate in G. For surfaces, the easiest way to get around this difficulty is to make use of the fact that the fundamental group of an orientable surface of negative Euler characteristic contains a large number of nicely behaved free subgroups of rank two. The nicest behaved such subgroups are the malnormal free subgroups. Recall that a subgroup H of a group G is *malnormal* if $gHg^{-1} \cap H = \{1\}$ for all $g \in G - H$. In particular, if F_2 is a malnormal subgroup of G, then elements of F_2 are conjugate in G if and only if they are conjugate in F_2. Hence, one approach to handling the case of a general group G is to construct malnormal free subgroups of G of rank two, and then apply the results from the preceeding Sections.

The fundamental group of an orientable surface S of negative Euler characteristic contains a large number of non-conjugate malnormal free subgroups of rank 2. Some can be constructed geometrically. For example, every pair of pants decomposition of S, of which there are infinitely many (if S is not itself a pair of pants) gives a number of embedded copies of a pair of pants in S, and the fundamental group of each such pair of pants is a malnormal free subgroup of rank 2 of $\pi_1(S)$. (Here, a *pair of pants* is topologically a thrice-punctured sphere, though conformally there are four types: a sphere with 3 points removed, with 2 points and 1 disc removed, with 1 point and 2 discs removed, and with 3 discs removed.) There are also malnormal subgroups of $\pi_1(S)$ corresponding to each embedded torus with one point or disc removed in S. This means that in order to characterize elements of $\pi_1(S)$ with the same character, it becomes necessary to characterize all malnormal free subgroups of $\pi_1(S)$, and even then there are elements with the same character that arise from other constructions, as will be described below.

We may also take a larger embedded subsurface of S whose fundamental group injects into $\pi_1(S)$. For example, if S is closed and we take the standard presentation

$$G = \langle a_1, b_1, \ldots, a_p, b_p \mid [a_1, b_1] \cdots [a_p, b_p] = 1 \rangle,$$

for $G = \pi_1(S)$, then the subgroup $\langle a_1, \ldots, a_p \rangle$ is malnormal and free. This subgroup is the fundamental group of the subsurface constructed by taking a regular neighborhood of $a_1 \cup \cdots \cup a_p$ in S. In fact, this is the subgroup used by Randol [Ran].

Note that Theorem 5.1 can also be extended to surfaces of infinite type, as such surfaces contain many malnormal free subgroups, again arising from embedded copies of a pair of pants or a torus with one puncture or hole.

In attempting to generalize this method to the fundamental group of a 3-manifold M, we run into the difficulty that the construction of malnormal free subgroups of 3-manifold groups is much more difficult than the construction of such subgroups for surface groups.

Masters resolves this difficulty in the proof of Theorem 5.2 by choosing the elements in the free subgroup carefully and showing directly that they are not conjugate in $\pi_1(M)$, using number theory and a careful choice of the exponent p_n, q_n, and k_n, as described in Section 3.

It is possible to obtain a separate proof of part of Theorem 5.2 in the case of convex co-compact hyperbolic 3-manifolds, a class which includes closed hyperbolic 3-manifolds, using the following Theorem of I. Kapovich, avoiding number theory. (This approach does use different machinery, namely the fact that convex co-compact Kleinian groups are word hyperbolic in the sense of Gromov.)

Theorem 5.3 (Theorem C of Kapovich [Kap]) *Let G be a torsion-free word hyperbolic group and let Γ be a non-elementary (i.e. not cyclic) subgroup of G. Then there exists a subgroup H of Γ such that H is free of rank 2 which is quasiconvex and malnormal in G.*

Malnormality is a strong condition to impose on a free subgroup F of a group G. There is a less exact but nonetheless still effective method, due to Pignataro and Sandler, which addresses the issue of when non-conjugate elements of F become conjugate in G, which avoids malnormal subgroups. The following Lemma is adapted from an argument given in the proof of Theorem 1 of Pignataro and Sandler [PS].

Lemma 5.4 *Let G be a finitely generated group without torsion and without $\mathbf{Z}\oplus\mathbf{Z}$ subgroups, and suppose that there exists a discrete faithful representation ρ_0 of G into $\mathrm{SL}_2(\mathbf{C})$. Then, there exists a constant $K > 0$ so that the following holds: for any faithful (but not necessarily discrete) representation ρ of G into $\mathrm{SL}_2(\mathbf{C})$ and for any free subgroup F of rank 2 in G, the inclusion map from the collection $\mathcal{C}(\rho(F))$ of conjugacy classes of maximal cyclic subgroups of $\rho(F)$ to the collection $\mathcal{C}(\rho(G))$ of conjugacy classes of maximal cyclic subgroups of $\rho(G)$ is at most K-to-1.*

Proof First, we can assume without loss of generality that $P \circ \rho_0(G)$ is a purely loxodromic, geometrically finite subgroup of $\mathrm{PSL}_2(\mathbf{C})$. [If $P \circ \rho_0(G)$ is not geometrically finite, then let M be a *compact core* for $\mathbf{H}^3/(P \circ \rho_0(G))$. We can uniformize M as $(\mathbf{H}^3 \cup \Omega(\Gamma))/\Gamma$ for a purely loxodromic, geometrically finite subgroup of $\mathrm{PSL}_2(\mathbf{C})$. Since Γ is necessarily isomorphic to $P \circ \rho_0(G)$, we can write $\Gamma = P \circ \rho_1(G)$ for a discrete faithful representation ρ_1 of G into $\mathrm{SL}_2(\mathbf{C})$, and then replace ρ_0 with ρ_1.] Set $\Gamma = P \circ \rho_0(G)$.

Since there are no cusps by assumption, there is a one-to-one correspondence between the collection $\mathcal{C}(\rho_0(G))$ of conjugacy classes of maximal cyclic subgroups of $\rho_0(G)$ and the collection of closed geodesics in the hyperbolic 3-manifold $\mathbf{H}^3/\Gamma$. Let $\mathrm{CC}(\mathbf{H}^3/\Gamma)$ be the *convex core* of the hyperbolic 3-manifold $\mathbf{H}^3/\Gamma$, and note that $\mathrm{CC}(\mathbf{H}^3/\Gamma)$ contains all of the closed geodesics in $\mathbf{H}^3/\Gamma$.

Note that $P \circ \rho_0(F)$ is a purely loxodromic, geometrically finite subgroup of Γ. Let $\pi : \mathbf{H}^3/(P \circ \rho_0(F)) \to \mathbf{H}^3/\Gamma$ be the covering map. Since $\mathbf{H}^3/(P \circ \rho_0(F))$ is compact, its image under π is compact as well. Since there is a positive lower bound on the injectivity radius of $\mathbf{H}^3/\Gamma$, there is some $K > 0$ so that π is at most K-to-1.

We can reinterpret this geometric fact as saying that the map from the collection $\mathcal{C}(F)$ of maximal cyclic subgroups of F to the collection $\mathcal{C}(G)$ of conjugacy classes of maximal cyclic subgroups of G is at most K-to-1. Hence, any faithful representation ρ of G into $\mathrm{SL}_2(\mathbf{C})$ has the same property. **QED**

Note that this argument can be made to work for a group G containing $\mathbf{Z} \oplus \mathbf{Z}$ subgroups, by carefully analyzing the behavior of the covering map at the cusps.

Underlying all of this discussion is the fact that free subgroups of rank 2 are very common in any group that admits a faithful representation into $\mathrm{SL}_2(\mathbf{C})$, as such groups satisfy the Tits alternative: for any two elements A and B of $\mathrm{SL}_2(\mathbf{C})$ of infinite order and with disjoint fixed point sets, there are integers n and m so that $\langle A^n, B^m \rangle$ is free of rank 2. We may then apply Lemma 5.4 to these subgroups. In particular, this implies that the characterization of pairs of elements of G with equal characters is extremely complicated.

We note that more is known about 2-generator subgroups of Kleinian groups. For instance, Ratcliffe [Rat] shows that for a torsion-free, two generator, discrete subgroup Γ of either $\mathrm{SL}_2(\mathbf{C})$ or of $\mathrm{PSL}_2(\mathbf{C})$, either Γ is free abelian of rank two, $\mathbf{H}^3/\Gamma$ has finite volume, or Γ is free of rank two. Reid [Rei] has shown that there are infinitely many closed 2-generator hyperbolic 3-manifolds which have a proper finite sheeted cover which is also 2-generator, which is behavior that is very unlike the surface case.

We now expand our horizons. Let S be an orientable surface of negative Euler characteristic, and let $\mathcal{T}(S)$ denote the Teichmüller space of hyperbolic structures on S. Let $\mathcal{C}(S)$ denote the set of free homotopy classes of homotopically non-trivial closed curves on S. There is a natural map $\mathcal{L} : \mathcal{C}(S) \times \mathcal{T}(S) \to [0, \infty)$, given by setting

$$\mathcal{L}([c], g) = \text{length}_g([c]),$$

where $\text{length}_g([c])$ is defined to be the infimum of the lengths of the closed curves on S in the free homotopy class $[c]$ determined by c, measured using the hyperbolic structure g on S.

As has already been noted, since an element of $\pi_1(S)$ and its inverse correspond to the same curve on S with opposite orientations, there is a one-to-one correspondence between the collection $\mathcal{C}(\pi_1(S))$ of conjugacy classes of maximal cyclic subgroups of $\pi_1(S)$ and the collection $\mathcal{C}(S)$ of free homotopy classes of closed curves on S. Theorem 1.5 and Proposition 1.4 can be thought of as evidence for the view that for fixed $g \in \mathcal{T}(S)$, the function $\mathcal{L}(\cdot, g) : \mathcal{C}(S) \to [0, \infty)$ often has multiplicity at least two, with two representative conjugacy classes generated by w and $I(w)$.

As noted by Randol [Ran], the Bumpy Metric theorem (see Abraham [Abr], Anosov [An]) implies that, if we expand the second factor of the domain to be the space $\mathcal{R}(S)$ of all Riemannian metrics on S, then the function

$$\mathcal{L}(\cdot, g) : \mathcal{C}(S) \to [0, \infty)$$

for fixed $g \in \mathcal{R}(S)$ is generically injective.

Hence, there is something non-generic about the hyperbolic metrics on a surface, and it would be nice to have a conjecture that captures this non-genericity. Note that it cannot be as simple as saying that hyperbolic metrics are exactly the metrics g for which the function $\mathcal{L}(\cdot, g)$ on $\mathcal{C}(S)$ has unbounded multiplicity, by the following example due to Buser [private communication]. Let S be a closed orientable surface of genus 2, let c be a simple closed separating curve on S, let U be an open regular neighborhood of c, and consider a metric g on S that is hyperbolic on one component of $S - U$ and not hyperbolic on the other component. The hyperbolic component of the surface then contributes to the unboundedness of the multiplicity of the length spectrum of S, and the metric on the other component can be chosen to be anything.

So, consider the action of $\text{Diff}(S)$ on $\mathcal{R}(S)$ by pullback. Let

$$G_S = \{f \in \text{Diff}(S) \mid f^*(\mathcal{T}(S)) = \mathcal{T}(S)\}$$

be the collection of all diffeomorphisms of S that pull hyperbolic metrics back to hyperbolic metrics. It is immediate that G_S is a subgroup of $\text{Diff}(S)$, by elementary properties of pullback.

Question 5.5 *Does there exist a diffeomorphism f of S so that $f^*(\mathcal{T}(S))$ is a proper subset of $\mathcal{T}(S)$?*

Say that a metric $g \in \mathcal{R}(S)$ is *wacky* if the map $\mathcal{L}(\cdot, g) : \mathcal{C}(S) \to [0, \infty)$ has unbounded multiplicity. For example, every hyperbolic metric is wacky, while a generic metric is not wacky. Let $\mathcal{W}(S)$ be the collection of all wacky metrics on S, and consider the group

$$H_S = \{f \in \mathrm{Diff}(S) \mid f^*(\mathcal{W}(S)) = \mathcal{W}(S)\}.$$

The following conjecture attempts to capture what is special about hyperbolic metrics in this context.

Conjecture 5.6 *Let S be an orientable surface of negative Euler characteristic. Then, G_S is a maximal connected subgroup of H_S.*

6 Character preserving automorphisms

Let $\mathrm{Aut}(G)$ denote the group of all automorphisms $\varphi : G \to G$ of G, and let $\mathrm{Inn}(G)$ denote the subgroup of $\mathrm{Aut}(G)$ consisting of the inner automorphisms $\varphi_g : G \to G$, given by $\varphi_g(h) = g \cdot h \cdot g^{-1}$ for $g \in G$. Let

$$\mathrm{Aut}_\chi(G) = \{\varphi \in \mathrm{Aut}(G) \mid \chi[g] = \chi[\varphi(g)] \text{ for all } g \in G\}$$

be the group of *character preserving automorphisms*. Note that $\mathrm{Inn}(G) \subset \mathrm{Aut}_\chi(G)$, by the basic properties of trace. As the constructions described in Section 3 and the conjecture given in Section 4 rely on the fact that J is a character preserving automorphism of F_2, and in some sense is the only one defined on all of F_2, we need to understand the group $\mathrm{Aut}_\chi(G)$.

This group has been completely determined for free groups by Horowitz.

Theorem 6.1 (Theorem 1 of Horowitz [Hor2]) *Let F_n be the free group of rank n. If $n \geq 3$, we have that $\mathrm{Aut}_\chi(F_n) = \mathrm{Inn}(F_n)$. If $n = 2$, we have that $\mathrm{Aut}_\chi(F_2) = \langle \mathrm{Inn}(F_2), J \rangle$, where J is the automorphism defined in Section 1.*

It is known that automorphisms of the free group F_2 of rank two are all geometric, in that if we realize F_2 as the fundamental group of a punctured torus T, then every automorphism is induced by the action of a homeomorphism of T. (However, this is no longer true if we realize F_2 as the fundamental group of a thrice-punctured sphere.) Moreover, given any two elements w and u of $F_2 = \mathrm{free}(a, b)$, the homomorphism $\varphi : F_2 \to F_2$ defined by $\varphi(a) = w$ and $\varphi(b) = u$ is an automorphism if and only if the commutator $[w, u]$ is conjugate to $[a, b]$. However, very few automorphisms of F_p for $p \geq 3$ are geometric, see Gersten [Ger].

Let $G_p = \langle a_1, b_1, \ldots, a_p, b_p | [a_1, b_1] \cdots [a_p, b_p] = 1\rangle$ be the standard presentation of the fundamental group of the closed orientable surface S_p of genus $p \geq 2$. We consider the question of determining $\mathrm{Aut}_\chi(G_p)$. It is a result of Nielsen [Nie] that all of the automorphisms of G_p are geometric.

In the case $p = 2$, there is an analogue on S_2 of the involution J on F_2, namely the *hyperelliptic involution*. This is a conformal involution of S_2. For more information about the hyperelliptic involution, we refer the interested reader to Farkas and Kra [FK].

By work of Haas and Susskind [HS], the hyperelliptic involution has the following characterization. For $p \geq 2$, let S_p be the closed orientable surface of genus p, and let f be an orientation-preserving homeomorphism of S_p with the property that for every simple closed curve α on S_p, $f(\alpha)$ is freely homotopic to either α or $-\alpha$ (where $-\alpha$ is the curve α with the opposite orientation). Then, either f is homotopic to the identity, or $p = 2$ and f is homotopic to the hyperelliptic involution. Conversely, on a closed orientable surface S_2 of genus two, the hyperelliptic involution J preserves the free homotopy class of every simple closed curve, and reverses the orientation of the curve if and only if the curve is non-separating. So, in terms of the standard presentation for G_2 given above, we see that $J(a_k) = a_k^{-1}$ and $J(b_k) = b_k^{-1}$. In particular, since J preserves the length of every simple closed curve on S_2, we have that J is character preserving on G_2.

We begin with the following Lemma, which is the analogue for G_p of Theorem 1.3. We note that a different proof of this Lemma is given by McShane [McS1].

Lemma 6.2 *Let G_p be the fundamental group of the closed orientable surface S_p of genus $p \geq 2$. Let $g \in G_p$ be a maximal element that represents a simple closed curve on S_p. Then, g determines $\chi[g]$; that is, if there exists a maximal element $h \in G_p$ with $\chi[h] = \chi[g]$, then h is conjugate to $g^{\pm 1}$.*

Proof First, we restrict attention to the discrete, faithful representations of G_p into $\mathrm{SL}_2(\mathbf{R})$, so that we get hyperbolic structures on S_p by taking the quotient $\mathbf{H}^2$ by $P \circ \rho(G_p)$. (Here, we are using the fact that since G_p has no 2-torsion, $\rho(G_p)$ in $\mathrm{SL}_2(\mathbf{R})$ is isomorphic to $P \circ \rho(G_p)$ in $\mathrm{PSL}_2(\mathbf{R})$.)

For each hyperbolic structure on S_p, the length of a closed geodesic on S_p determines the character of the corresponding element of G_p, and vice versa, by the discussion in Section 5. In particular, equal characters for two elements of G_p imply that the corresponding closed geodesics on S_p have equal lengths, independent of the hyperbolic structure on S_p.

If c is a homotopically non-trivial non-simple closed curve on S_p, there is a uniform positive lower bound of $2\ln(1 + \sqrt{2})$ for the length for the closed geodesic homotopic to c over all hyperbolic structures on S_p, see Hempel

[Hem]. However, if c' is a homotopically non-trivial simple closed curve, there is no positive minimum length for the closed geodesic homotopic to c' over all hyperbolic structures on S_p. In fact, there exist hyperbolic structures on S_p for which the length of the closed geodesic homotopic to c' goes to 0. Hence, since g represents a simple closed curve on S_p, h must also represent a simple closed curve on S_p.

Now, we are reduced to considering two simple closed curves on S_p so that the lengths of their corresponding closed geodesics are equal, independent of the hyperbolic structure on S_p. If they intersect, then the Collar Lemma, see for instance Buser [Bus], implies that the length of one goes to infinity as the length of the other goes to 0. If they are disjoint, we may use Fenchel-Nielsen coordinates, see for instance Abikoff [Ab], to see that the length of one can be made to go to 0 without changing the length of the other. Hence, we see that the curves must coincide, which is equivalent to saying that g and h are conjugate up to inverse, as desired. **QED**

Theorem 6.3 *Let G_p be the fundamental group of the closed orientable surface S_p of genus $p \geq 2$. For $p \geq 3$, we have that $\mathrm{Aut}_\chi(G_p) = \mathrm{Inn}(G_p)$. For $p = 2$, we have that $\mathrm{Aut}_\chi(G_2) = \langle \mathrm{Inn}(G_2), J \rangle$, where the involution J of G_2 arises from the hyperelliptic involution on S_2.*

Proof As in the case of free groups, the basic properties of trace yield immediately that $\mathrm{Inn}(G_p) \subset \mathrm{Aut}_\chi(G_p)$. For surfaces of genus 2, the hyperelliptic involution is an isometry for every hyperbolic structure on S, and so preserves the lengths of closed geodesics and hence also preserves characters. This shows that $\langle \mathrm{Inn}(G_2), J \rangle \subset \mathrm{Aut}_\chi(G_2)$

Now, let φ be an element of $\mathrm{Aut}_\chi(G_p)$. Let $g \in G_p$ be any element that represents a homotopically non-trivial simple closed curve on S_p. By Lemma 6.2, we see that $\varphi(g)$ must be conjugate to g. In particular, the automorphism φ of G_p corresponds to a homeomorphism f_φ of S_p that takes each simple closed geodesic to itself, possibly reversing the orientation of the geodesic.

We apply Theorem 1 of McShane [McS1] to see that this homeomorphism f_φ of S_p must be homotopic to an isometric map. (If we knew that φ was induced by an orientation-preserving homeomorphism of S_p, then we could apply the result of Haas and Susskind described above.) In the case $p = 2$, the only self-maps of S_p that are isometries of every hyperbolic structure are the identity and the hyperellliptic involution J. In the case $p \geq 3$, the only self-map of S_p that is an isometry of every hyperbolic structure on S_p is the identity, as desired. **QED**

7 Variants

There has been a wide variety of work in related areas by a number of authors.

Jørgensen [J4], [J3], [J2] has studied various aspects, properties, and applications of trace identities in $\mathrm{SL}_2(\mathbf{C})$ and $\mathrm{PSL}_2(\mathbf{C})$.

Sandler [San] extended results of the sort discussed in this survey to certain families of faithful representations of F_2 into $\mathrm{SU}(2,1)$, with similar applications to the length spectra of certain complex hyperbolic manifolds.

Thompson [Th] showed for each $n \geq 2$, there exists a field k and a subgroup G_n of $\mathrm{SL}_2(k)$ which contains a free group F_n of rank n, so that two elements of F_n give rise to the same character if and only if they are conjugate in G_n. Moreover, the field k is explicitly constructed as the algebraic closure of a finitely generated extension field of the rationals $\mathbf{Q}$.

Traina [Tr] gives an explicit though complicated expression for the Fricke polynomial for an element of $F_2 = \mathrm{free}(a, b)$.

Baribaud [Bar] studied the lengths of closed geodesics on a pair of pants. She defined a parameter, the *number of strings*, and gave a complete description of those closed geodesics which have the shortest length given their number of strings, for those geodesics with an odd numbers of strings.

Magnus [Mag1] considered this question for other groups. For example, consider the group G with the presentation

$$G = \langle a, b \mid W^k = 1 \rangle,$$

where W is a freely reduced word in a and b, and $k > 1$. Then, a necessary condition for G to have a faithful representation into $\mathrm{PSL}_2(\mathbf{C})$ is that if U is an element of G with the same Fricke polynomial as W, then U is conjugate to $W^{\pm 1}$.

McShane [McS2] (see also Bowditch [Bow1]) showed that for any hyperbolic structure on a punctured torus T, the equality

$$\sum_{\gamma} \frac{1}{1 + \exp(|\gamma|)} = \frac{1}{2}$$

holds, where the sum ranges over all simple closed geodesics γ on T and where $|\gamma|$ is the length of the closed geodesic γ. Bowditch [Bow2] has generalized this equality to hyperbolic once-punctured torus bundles.

This series can also be generalized as follows, see McShane [McS2]. Let M be a convex surface without boundary and with a hyperbolic structure of finite area and a cusp x. Then, the equality

$$\sum \frac{1}{1 + \exp(\frac{1}{2}(|\alpha| + |\beta|))} = \frac{1}{2}$$

holds, where the sum is over all pairs α and β of simple closed geodesics which bound a pair of pants containing the cusp x.

Pignataro and Sandler [PS] use techniques similar to those described in this note to generalize earlier work of Jørgensen and Sandler [JS]. Let S be an orientable surface of negative Euler characteristic and let c, c' be two closed curves on S, neither homotopic to a peripheral curve, that intersect essentially. For each hyperbolic structure on S, let c and c' also refer to the closed geodesics on S with the given hyperbolic structure that lie in the free homotopy classes determined by c and c'. Then, for any hyperbolic structure on S, let x be a point of intersection of two closed geodesics c and c'. (Note that, even though c and c' will vary on S as the hyperbolic structure varies, there is always a point of intersection corresponding to x for the corresponding closed geodesics with the new hyperbolic structure.) Then, for any $m \geq 2$, there are closed curves $w_1, \ldots, w_m$ passing through x with $\text{length}_g(w_1) = \cdots = \text{length}_g(w_m)$ for every hyperbolic structure g on S.

There is a necessary condition in terms of homology for two elements of a surface group to have the same character.

Proposition 7.1 (Corollary 3.4 of Leininger [Lei]) *Let G be the fundamental group of an orientable surface S. Let w and u be elements of G with $\chi[w] = \chi[u]$. Then, w and u may be oriented so that they represent the same class in $G/[G,G] = H_1(S, \mathbf{Z})$.*

To close this Section, there is a folklore conjecture, that two closed curves on a surface S of equal length over the Teichmüller space of S can be characterized by their intersection numbers with simple closed curves. Specifically, given two closed curves w and u on an orientable surface S of negative Euler characteristic, let $\mathrm{i}(w,u)$ denote their geometric intersection number, which is equal to the minimum number of intersection points of w' and u', where w' is freely homotopic to w and u' is freely homotopic to u. Equivalently, define $\mathrm{i}(w,u)$ to be the number of (necessarily transversal) intersection points of the geodesic representatives of w and u for any hyperbolic structure on S.

The strong form of this conjecture has recently been resolved in the negative by Leininger [Lei]. The following Proposition is essentially a consequence of the Collar Lemma; for a complete proof, see [Lei]. Also see [Lei] for an explicit example of two elements w and u of $\pi_1(S)$ for which $\mathrm{i}(w,c) = \mathrm{i}(u,c)$ for all simple closed curves c on S but $\chi[w] \neq \chi[u]$, and for a more detailed discussion of this question.

Proposition 7.2 (Corollary 5.4 of Leininger [Lei]) *Given an orientable surface S of negative Euler characteristic, let w and u be closed curves on S for which* $\text{length}_g(w) = \text{length}_g(u)$ *for all hyperbolic structures g on S. Then,* $\mathrm{i}(w,c) = \mathrm{i}(u,c)$ *for all simple closed curves c on S.*

8 Questions and conjectures

There are a number of other questions that can be asked. We present a few of them here.

- The first concerns the relationship between elements of F_2 that are simple in the sense of having stable multiplicity 1 and those that are simple in the sense that they correspond to a simple closed curve on the punctured torus S, when we realize F_2 as the fundamental group of S.

  There is an algorithm, see Series [Ser], for determining when a closed curve on a punctured torus S, given as a word $w = a^{n_1} \cdot b^{m_1} \cdots a^{n_k} \cdot b^{m_k}$ in $F_2 = \text{free}(a, b) = \pi_1(S)$, is a simple curve. This algorithm involves constructing a finite collection of nested free subgroups of F_2. However, it seems that the character preserving automorphisms do not shed any light on this question of determining simplicity of curves. Even though w is conjugate to $I(w)$ for every simple closed curve w, there are also non-simple closed curves for which w is conjugate to $I(w)$.

  Consider the element $w = a^p \cdot b^y \cdot a^p \cdot b^q \cdot a^x \cdot b^q$ in $F_2 = \text{free}(a, b)$, where p, q, x, and y are arbitrary distinct non-zero integers. For most values of p, q, x, and y, w does not represent a simple curve on S, by the algorithm [Ser]. However, for any choice of p, q, x, and y, $I(w) = b^q \cdot a^x \cdot b^q \cdot a^p \cdot b^y \cdot a^p$ is conjugate (by $a^p \cdot b^y \cdot a^p$) to w; preliminary computer calculations in this case support the conjecture that w is then a simple element of F_2, and hence has $\text{mult}(w) = 1$. The following Conjecture attempts to make this link precise.

  Conjecture 8.1 *Let w be an element of F_2. If w conjugate to $I(w)$, then* $\text{mult}(w) = 1$.

  It is a difficult question to characterize those elements of F_2 with stable multiplicity 1. One possibility is the following. Let $w = w(a, b)$ be an element of $F_2 = \text{free}(a, b)$. Say that w is *prime* if w does not admit a non-trivial decomposition as $w(a, b) = w'(u, v)$, where $u = u(a, b)$ and $v = v(a, b)$ are elements of F_2. (We require that at least one of u and v be non-trivial in F_2, that is, not a primitive element of F_2.) If w is not prime, then say that w is *composite*.

  Conjecture 8.2 (Ginzburg and Rudnick [GR]) *Let w be a prime element of F_2. Then, w is pseudo-simple.*

  Conjecture 8.3 *Let w be a prime element of F_2, and suppose there exists an element u of F_2 for which $\chi[w] = \chi[u]$. Then, u is prime, and u is conjugate to $w^{\pm 1}$.*

The difficulty now becomes characterizing which elements of F_2 are prime.

One small piece of evidence for Conjecture 8.2 is the following construction. Let w be a composite element of F_2, so that we may write w as a word $w = w'(u, v)$, where $u = u(a, b)$ and $v = v(a, b)$ are non-trivial words in F_2. Let I be the canonical involution on F_2, and let I' be the canonical involution on the subgroup $F = \text{free}(u, v)$ of F_2. Then, $\chi[w'] = \chi[I'(w')]$, but in general, one expects that w' and $I'(w')$ are not conjugate in F, and hence not in F_2, depending on the specifics of the expressions of w, u, and v. Moreover, in this case one expects that, when expressed in terms of a and b, $I'(w'(u(a, b), v(a, b)))$ is not conjugate to either w or $I(w)$, and so the stable multiplicity of w is then at least three.

There is a related question, due to Riven:

Question 8.4 *Let u and w be elements of F_2 so that $\chi[w] = \chi[u]$. Does there exist a generating set $\{x, y\}$ for F_2 so that $u^{\pm 1}$ is conjugate to $I(w)$, where I is the canonical involution with respect to the generating set $\{x, y\}$?*

It could also be asked whether the question of the existence of such a generating set is or is not decidable. I would like to thank the first referee for bringing this question to my attention.

There is a topological interpretation of Conjecture 4.1. Let S be the punctured torus, and consider F_2 as $\pi_1(S)$. Let g_1 and g_2 be two elements of F_2 corresponding to closed curves on S not homotopic to the puncture on S. Then, $H = \langle g_1, g_2 \rangle$ is free of rank two and of infinite index in $\pi_1(S)$, unless g_1 and g_2 generate $\pi_1(S)$. As all automorphisms of F_2 are realized by homeomorphisms of S, we can phrase Conjecture 4.1 in terms of a tree of covers of S, where each node is a torus with a hole or a pair of pants, and of homeomorphisms of the node surfaces that realize the respective automorphisms.

- (Asked by U. Hamenstädt at the Workshop on Kleinian Groups and Hyperbolic 3-Manifolds, held at the University of Warwick, September 2001) Is there a connection between the stable multiplicity of a closed curve on a surface and the number of its self-intersections?

Let S be an orientable surface of negative Euler characteristic. Given a closed curve c on S, let $\text{mult}(c)$ denote its stable multiplicity and let $\text{self\_int}(c)$ denote the number of its self intersections, defined to be the minimum of the number of self intersections of any closed curve freely homotopic to c. Note that self_int(c) is independent of the hyperbolic structure on S.

Basmajian [Bas2] showed, see Corollary 1.2 of [Bas2], that for each $k \geq 1$, there exists a constant M_k, depending only on k and satisfying $\lim_{m\to\infty} M_k = \infty$, so that if self_int$(c) = k$, then $\text{length}_g(c) \geq M_k$ for every hyperbolic structure g on S.

So, for a closed curve c on S with self_int$(c) = k$, choose a hyperbolic structure z on S which minimizes $\text{length}_g(c)$ as g ranges over $\mathcal{T}(S)$. Since $\lim_{k\to\infty} M_k = \infty$, there exists K so that $M_k > \text{length}_z(c)$ for all $k \geq K$. Hence, if c' is another closed curve on S and self_int$(c') \geq K$, then c and c' must determine distinct character classes, since there is a hyperbolic structure on S, namely z, for which the closed geodesics freely homotopic to c and c' must have different lengths. However, this argument has the flaw that it is not uniform in the self intersection number of c, but relies on first determining the minimal length of c over all hyperbolic structures on S.

On a punctured torus, Conjecture 8.1 and the algorithm for simplicity described in Series [Ser] imply that there exist closed curves c_n on S for which self_int$(c_n) \to \infty$ but mult$(c_n) = 1$ for all n. So, the following question remains unresolved: does a bound on self_int(c) give a bound on mult(c)?

- This whole paper has been concerned with determining when there are elements w and u of F_2, or of a finitely generated group G, for which $\chi[w] - \chi[u] = 0$. Are there other functions, perhaps variants of McShane's identity, as discussed in Section 7, that hold for characters?

  In general, there cannot exist w and u for which $\chi[w] + \chi[u] = 0$. Such pairs of elements would correspond to closed curves of equal length on the quotient manifold but would not be detected by the methods that have been discussed in this note, as their characters are not equal. Let G be a finitely generated group with the property that every point in $\text{Hom}(G, \text{SL}_2(\mathbf{C}))$ is an accumulation point of $\mathcal{F}(G)$; in particular, $\mathcal{F}(G)$ is dense in $\text{Hom}(G, \text{SL}_2(\mathbf{C}))$. Free groups of finite rank and fundamental groups of closed orientable surfaces are examples of such groups. If there were elements w and u of G for which $\chi[w] = -\chi[u]$, then for every odd $m \geq 1$ we would have that $\chi[w^m] = -\chi[u^m]$. We could choose m large enough so that $\langle w^m, u^m \rangle$ is a free group of rank two. By the assumption on G, there would exist a sequence of representations $\{\rho_n\}$ in $\mathcal{F}(G)$ converging to the element of $\text{Hom}(G, \text{SL}_2(\mathbf{C}))$ taking every element of G to the identity. In particular, both $\{\rho_n(w^m)\}$ and $\{\rho_n(u^m)\}$ would converge to the identity, at which point $\chi[w^m] = \chi[u^m] = 2$, as both would be equal to $\chi[\mathrm{id}]$, a contradiction. (This argument is adapted from an argument due to Horowitz [Hor2].)

- Are there analogous results for the length spectra of more general classes of spaces?

  In this note, we have discussed this question for hyperbolic 2- and 3-manifolds. Leininger [Lei] discusses and answers this question for certain classes of path metrics on surfaces, specifically the singular Euclidean metrics. However, the question of whether analogous results hold, for instance, for pleated surfaces, or singular hyperbolic surfaces, or for 3-dimensional hyperbolic cone manifolds, is still open.

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