THE MINIMAL ENTROPY PROBLEM FOR 3-MANIFOLDS WITH ZERO SIMPLICIAL VOLUME

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Dedicated to Jacob Palis on his sixtieth birthday

ABSTRACT. In this note, we consider the *minimal entropy problem*, namely the question of whether there exists a smooth metric of minimal (topological) entropy, for certain classes of closed 3-manifolds. Specifically, we prove the following two results.

Theorem A. *Let M be a closed orientable irreducible 3-manifold whose fundamental group contains a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup. The following are equivalent:*

(1) *the simplicial volume $\|M\|$ of M is zero and the minimal entropy problem for M can be solved;*
(2) *M admits a geometric structure modelled on $\mathbb{E}^3$ or* Nil*;*
(3) *M admits a smooth metric g with $\mathrm{h_{top}}(g) = 0$.*

Theorem B. *Let M be a closed orientable geometrizable 3-manifold. The following are equivalent:*

(1) *the simplicial volume $\|M\|$ of M is zero and the minimal entropy problem for M can be solved;*
(2) *M admits a geometric structure modelled on $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{E}^3$, or* Nil*;*
(3) *M admits a smooth metric g with $\mathrm{h_{top}}(g) = 0$.*

1. Introduction and statement of results

Let M^n be a closed orientable n-dimensional manifold. For a smooth Riemannian metric g on M, let $\mathrm{Vol}(M, g)$ denote the volume of M calculated with respect to g.

Let $\mathrm{h_{top}}(g)$ be the *topological entropy* of the geodesic flow of g, as defined in Section 2.6. Set the *minimal entropy* of M to be

$$\mathrm{h}(M) := \inf\{\mathrm{h_{top}}(g) \mid g \text{ is a smooth metric on } M \text{ with } \mathrm{Vol}(M, g) = 1\}.$$

A smooth metric g_0 with $\mathrm{Vol}(M, g_0) = 1$ is *entropy minimizing* if

$$\mathrm{h_{top}}(g_0) = \mathrm{h}(M).$$

The *minimal entropy problem* for M is whether or not there exists an entropy minimizing metric on M. Say that the *minimal entropy problem can be solved* for M if there exists an entropy minimizing metric on M. Smooth manifolds are hence naturally divided into two classes: those for which the minimal entropy problem can be solved and those for which it cannot.

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There are a number of classes of manifolds for which the minimal entropy problem can be solved. For instance, the minimal entropy problem can always be solved for a closed orientable surface M. For the 2-sphere and the 2-torus, this follows from the fact that both a metric with constant positive curvature and a flat metric have zero topological entropy. For surfaces of higher genus, A. Katok [11] proved that each metric of constant negative curvature minimizes topological entropy, and conversely that any metric that minimizes topological entropy has constant negative curvature.

This result of Katok has been generalized to higher dimensions by Besson, Courtois, and Gallot [1], as follows. Suppose that M^n ($n \geq 3$) admits a locally symmetric metric g_0 of negative curvature, normalized so that $\mathrm{Vol}(M, g_0) = 1$. Then g_0 is the unique entropy minimizing metric up to isometry. Unlike the case of a surface, a locally symmetric metric of negative curvature on a closed orientable n-manifold ($n \geq 3$) is unique up to isometry, by the rigidity theorem of Mostow [18].

The result of Besson, Courtois, and Gallot [1] has itself been generalized by Connell and Farb [4] to n-manifolds that admit a complete, finite-volume metric which is locally isometric to a product of negatively curved (rank 1) symmetric spaces of dimension at least 3.

A positive solution to the minimal entropy problem appears to single out manifolds that have either a high degree of symmetry or a low topological complexity. What this means in the context of 3-manifolds will become apparent below. A similar phenomena is observed for closed simply connected manifolds of dimensions 4 and 5: there are essentially only nine manifolds for which the minimal entropy problem can be solved and they can be explicitly listed. These nine manifolds share the property that their loop space homology grows polynomially for any coefficient field, see Paternain and Petean [21].

The goal of this note is to classify those closed orientable geometrizable 3-manifolds with zero simplicial volume for which the minimal entropy problem can be solved. Specifically, in Section 4, we prove:

Theorem A. *Let M be a closed orientable irreducible 3-manifold whose fundamental group contains a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup. The following are equivalent:*

1. *the simplicial volume $\|M\|$ of M is zero and the minimal entropy problem for M can be solved;*
2. *M admits a geometric structure modelled on $\mathbb{E}^3$ or* Nil*;*
3. *M admits a smooth metric g with $\mathrm{h_{top}}(g) = 0$.*

In Section 5 we prove:

Theorem B. *Let M be a closed orientable geometrizable 3-manifold. The following are equivalent:*

1. *the simplicial volume $\|M\|$ of M is zero and the minimal entropy problem for M can be solved;*
2. *M admits a geometric structure modelled on $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{E}^3$, or* Nil*;*
3. *M admits a smooth metric g with $\mathrm{h_{top}}(g) = 0$.*

Recall that the *simplicial volume* of a closed orientable manifold M is defined as the infimum of $\sum_i |r_i|$ where the r_i are the coefficients of a real cycle that represents the fundamental class of M. For 3-manifolds, the positivity of the simplicial volume (which is a homotopy invariant) is closely related to the existence of compact hyperbolizable submanifolds in M. This is described in more detail in Section 2.5.

We close the introduction by describing some of the elements of the proofs of Theorems A and B, and by describing a conjectural picture. We will see in Section 2 that a closed orientable geometrizable 3-manifold M has zero simplicial volume if and only if M has zero minimal entropy. Therefore, the minimal entropy problem can be solved if and only if M admits a smooth metric with zero topological entropy. This in turn forces the fundamental group of M to have subexponential growth. We end up showing that this can occur only if M admits one of the four geometric structures listed in the statement of Theorem B. On the other hand, it is a calculation that the manifolds in the statement of Theorem B carry a metric of zero entropy. The proof of Theorem A follows a similar line, and makes use of the remarkable theorem, due essentially to Thurston, that a manifold satisfying the hypothesis of the theorem is geometrizable. The precise definition of geometrizable manifold is given in Subsection 2.4. Thurston's geometrization conjecture states that every closed orientable 3-manifold is geometrizable.

From this discussion and the above mentioned result of Besson, Courtois and Gallot it seems quite reasonable to speculate that the following statement holds:

Let M be a closed orientable geometrizable 3-manifold. Then, the minimal entropy problem for M can be solved if and only if M admits a geometric structure modelled on $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{E}^3$, Nil, *or $\mathbb{H}^3$.*

Indeed, suppose that the simplicial volume of M were *not* zero. This would imply that M contains a disjoint collection $H_1, \ldots, H_p$ of compact submanifolds whose interiors each admit a complete hyperbolic structure of finite volume. In particular, it should be that the minimal entropy of M is the maximum of the minimal entropies of the H_k. It would then seem reasonable that an entropy minimizing metric on M would try to be as hyperbolic as possible on the interiors of the H_k and would try as much as possible to be one of the other seven standard 3-dimensional geometries on the components of $M - (H_1 \cup \cdots \cup H_p)$. However, it would seem that the minimizer would have to be singular along the ∂H_k, and so there should be no metric of minimal entropy. Unfortunately, we do not yet know how to make this argument rigorous.

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2. Preliminaries

The purpose of this Section is to present some of the basic material from 3-manifold theory that we will need. We refer the interested reader to Hempel [8] for a more detailed introduction to 3-manifold topology. For a more detailed description of Seifert fibered spaces, and of the torus decomposition and the geometrization of 3-manifolds, we also refer the interested reader to the survey articles of Scott [26] and Bonahon [2], and the references contained therein.

2.1. **3-manifold basics.** We begin with some basic definitions. A 3-manifold is *closed* if it is compact with empty boundary.

An embedded 2-sphere $\mathbb{S}^2$ in a 3-manifold M is *essential* if M does not bound a closed 3-ball in M. A 3-manifold is *irreducible* if it contains no essential 2-sphere.

A 3-manifold is *prime* if it cannot be decomposed as a non-trivial connected sum. That is, M is prime if for every decomposition $M = M_1 \# M_2$ of M as a connected sum, one of M_1 or M_2 is homeomorphic to the standard 3-sphere $\mathbb{S}^3$. Every irreducible 3-manifold is prime, but the converse does not hold. However, the only closed orientable 3-manifold that is prime but not irreducible is $\mathbb{S}^2 \times \mathbb{S}^1$.

We note here that if the closed orientable 3-manifold M contains a non-separating essential 2-sphere, then M can be expressed as the connected sum $M = P\#(\mathbb{S}^2 \times \mathbb{S}^1)$ for some 3-manifold P. Hence, in what follows, we need only consider separating essential 2-spheres in 3-manifolds.

There is an inverse to the operation of connected sum for 3-manifolds, called the *prime decomposition*. The following statement is adapted from Bonahon [2], and follows from work of Kneser [12] and Milnor [16].

Let M be a closed orientable 3-manifold. Then, there exists a compact 2-submanifold Σ of M, unique up to isotopy, so that two conditions hold. First, each component of Σ is an embedded essential separating 2-sphere, and the 2-spheres in Σ are pairwise non-parallel, in that no two 2-spheres in Σ bound an embedded $\mathbb{S}^2 \times [0,1]$ in M. Second, if $M_0, M_1, \ldots, M_q$ are the closures of the components of $M - \Sigma$, then M_0 is homeomorphic to the 3-sphere $\mathbb{S}^3$ minus finitely many disjoint open 3-balls; while for $k \geq 1$, each M_k contains a unique component of Σ, and every separating essential 2-sphere in M_k is parallel to ∂M_k.

The *prime decomposition* of M is the collection of 3-manifolds that results by taking the complements of the 2-submanifold Σ in M as just described, and filling in each 2-sphere boundary component of $M_0, M_1, \ldots, M_p$ with a 3-ball. Each of the resulting 3-manifolds is then prime. (Note that both $\mathbb{S}^3$ and $\mathbb{S}^2 \times \mathbb{S}^1$ have trivial prime decompositions, as they do not contain a separating essential 2-sphere.) The prime decomposition is one of two standard decompositions of a closed orientable 3-manifold, the other being the torus decomposition, which is discussed in detail in Section 2.3.

In general, a closed orientable embedded surface S in a 3-manifold M is *2-sided* if there exists an embedding f of $S \times [-1,1]$ into M so that $f(S \times \{0\}) = S$. A closed orientable embedded surface S in a 3-manifold M is *incompressible* if the fundamental group of S is infinite and if the inclusion $S \hookrightarrow M$ induces an injection on fundamental groups. An incompressible surface S is *essential* if S is not homotopic into ∂M.

A compact orientable irreducible 3-manifold M is *sufficiently large* if it contains a 2-sided incompressible surface. Sufficiently large 3-manifolds are also known as *Haken* 3-manifolds.

2.2. **Seifert fibered spaces.** A *Seifert fibration* of a 3-manifold M is a decomposition of M into disjoint simple closed curves, called the *fibers* of the fibration, so that each fiber c has a neighborhood U in M of the following form: U is diffeomorphic

to the quotient of $\mathbb{S}^1 \times \mathbb{B}^2$ by the free action of a finite group respecting the product structure, where the fibers of the fibration correspond to the curves $\{x\} \times \mathbb{B}^2$ for $x \in \mathbb{S}^1$. (In this note, we only consider Seifert fibrations of closed 3-manifolds and of 3-manifolds without boundary that are homeomorphic to the interior of a compact 3-manifold with 2-torus boundary components.)

Since we are considering only orientable 3-manifolds in this note, the group of covering transformations of $\mathbb{S}^1 \times \mathbb{B}^2$ in the above definition is generated by $\tau_{p,q}$ for some pair (p, q) of relatively prime integers, where

$$\tau_{p,q}\left(e^{i\varphi}, re^{i\theta}\right) = \left(e^{i\left(\varphi+\frac{2\pi}{p}\right)}, re^{i\left(\theta+\frac{2\pi q}{p}\right)}\right).$$

A fiber is a *regular fiber* if it has a neighborhood diffeomorphic to $\mathbb{S}^1 \times \mathbb{B}^2$, and is a *singular fiber* otherwise. Note that the singular fibers of a Seifert fibration are necessarily isolated.

Let S be the space of fibers of a Seifert fibration of a 3-manifold M, equipped with the quotient topology coming from the projection map $p : M \to S$. We often refer to S as the *base orbifold* of the Seifert fibered space M. Using the neighborhoods of the fibers in M, we see that S is an orientable surface with one cone point for each singular fiber.

Let $p_1, \ldots, p_s$ be the cone points on S, and let n_j be the *order* at the cone point p_j, so that a neighbhorhood of p_j is diffeomorphic to the quotient $\mathbb{B}^2/\mathbb{Z}_{n_j}$, where $\mathbb{Z}_{n_j}$ acts by rotation. The *orbifold Euler characteristic* $\chi(S)$ of S is the quantity

$$\chi(S) = 2 - 2\,\mathrm{genus}(S) - \sum_{k=1}^{s}\left(1 - \frac{1}{n_j}\right).$$

(This discussion is also valid in the case that M is a 3-manifold without boundary that is homeomorphic to the interior of a compact 3-manifold with 2-torus boundary components. In this case, the base orbifold has punctures as well as cone points, and we view each puncture as a cone point of infinite order.)

There are two cases of particular interest. In the case that $\chi(S) < 0$, S has a hyperbolic structure, so that we can express S as the quotient $S = \mathbb{H}^2/\Gamma$, where $\mathbb{H}^2$ is the hyperbolic plane and Γ is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^2)$, where the fixed points of the action of non-trivial elements of Γ descend to the cone points on S. We refer to Γ as the *orbifold fundamental group* of S. In this case, we have that Γ contains a free subgroup of rank 2, and in particular Γ contains an element of infinite order.

In the case that $\chi(S) = 0$, S has a Euclidean structure, so that we can express S as the quotient $S = \mathbb{E}^2/\Gamma$, where $\mathbb{E}^2$ is the Euclidean plane and Γ is a discrete subgroup of $\mathrm{Isom}(\mathbb{E}^2)$, where the fixed points of the action of non-trivial elements of Γ descend to the cone points on S. As above, we refer to Γ as the *orbifold fundamental group* of S. In this case, we have that Γ contains an element of infinite order, but not a free subgroup of rank two.

In both of these cases, the orbifold fundamental group of the base orbifold S of the Seifert fibered space M is a subgroup of $\pi_1(M)$. In fact, there is a short exact

sequence

$$1 \to \mathbb{Z} \to \pi_1(M) \to \pi_1(S) \to 1,$$

where $\pi_1(S)$ is the orbifold fundamental group of S and where $\mathbb{Z}$ is generated by any regular fiber of the Seifert fibration.

The following follows immediately from this discussion.

Lemma 2.1. *Let M be a Seifert fibered space as above with base orbifold S. If $\chi(S) \leq 0$, then $\pi_1(M)$ contains a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup.*

Proof. The proof of Lemma 2.1 is standard, but we sketch it here for the sake of completeness. Let $p : M \to S$ be the quotient map. Since $\chi(S) \leq 0$, there is a closed curve c, not necessarily simple, on S that represents an infinite order element of the orbifold fundamental group of S. Let $T = p^{-1}(c)$ in M be the subset of M that consists of all the fibers in M corresponding to points of c. Then, T is an incompressible 2-torus in M, though not necessarily embedded. However, this is sufficient to guarantee that there exists a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup of $\pi_1(M)$, namely the fundamental group of T. □

2.3. **The torus decomposition.** Let M be a closed orientable irreducible 3-manifold with infinite fundamental group. There is then a canonical decomposition of M along embedded essential 2-tori, due to Jaco and Shalen [9] and Johannson [10]. (Note that the restriction to irreducible 3-manifolds causes no loss of generality, as we may first apply the prime decomposition to M, as described in Section 2.1. Also, we tend to not take the torus decomposition of $\mathbb{S}^2 \times \mathbb{S}^1$.) The statement given below is adapted from Theorem 3.4 of Bonahon [2].

Theorem 2.2. [2] *Let M be a closed orientable irreducible 3-manifold. Then, up to isotopy, there is a unique compact 2-submanifold T of M such that:*

1. *every component of T is a 2-sided essential 2-torus;*
2. *every component of $M - T$ either contains no essential embedded 2-torus or Klein bottle, or else admits a Seifert fibration (or possibly both);*
3. *property (2) fails when any component of T is removed.*

We refer to this 2-submanifold T as the *torus decomposition* of M. Note that condition (3) implies that no two of the 2-tori in the torus decomposition are isotopic.

Let M be a compact orientable 3-manifold, and let $M_0, M_1, \ldots, M_p$ be the components of its prime decomposition. Let T_k be the torus decomposition of M_k. Say that M is a *graph manifold* if, for each $1 \leq k \leq p$, every component of $M_k - T_k$ admits a Seifert fibration. Clearly, every Seifert fibered space is trivially a graph manifold. Also, every 2-torus bundle over $\mathbb{S}^1$ is a graph manifold.

Theorem 2.2 is a small part of the machinary of the *characteristic submanifold* of a 3-manifold developed by Jaco and Shalen and by Johannson. Note that this discussion includes the possibility that the torus decomposition T is empty, even though $\pi_1(M)$ may contain a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup.

A closely related result is the following torus theorem. For a discussion and proof of this result, see Scott [27].

Theorem 2.3. [27] *Let M be a closed orientable irreducible 3-manifold whose fundamental group contains a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup. Then, either M contains an incompressible embedded 2-torus or M is a Seifert fibered space.*

2.4. **Geometric structures and geometrization.** A *3-dimensional geometry* is a pair (X, G), where X is a simply connected Riemannian 3-manifold with a complete homogeneous metric and G is a maximal transitive group of orientation-preserving isometries of X, with the proviso that there exists a subgroup H of G with compact quotient X/H. Note that since G is a maximal group of isometries, it suffices to specify X and set $G = \mathrm{Isom}(X)$.

It is a result of Thurston that there exist exactly eight 3-dimensional geometries, namely $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}_2}$, Nil, and Sol, with their respective groups of (orientation preserving) isometries. (A proof of this result, and a detailed description of the eight geometries, is given in Scott [26].)

Let M be an orientable 3-manifold that is homeomorphic to the interior of a compact 3-manifold with 2-torus boundary components. (This includes the possibility that M is closed.) Say that *M admits a geometric structure modelled on X* if M is diffeomorphic to the quotient X/Γ, where X is one of the eight 3-dimensional geometries and Γ is a fixed point free subgroup of $\mathrm{Isom}(X)$. It is known that if a 3-manifold admits a geometric structure, then it admits a unique geometric structure.

More generally, let M be a closed orientable irreducible 3-manifold with torus decomposition T. Say that M is *geometrizable* if each component of $M - T$ admits a geometric structure. (Note that we do not require that different components of $M - T$ admit the same geometric structure.)

Finally, say that a closed orientable 3-manifold is *geometrizable* if every component of its prime decomposition is geometrizable. (This causes no difficulties, as $\mathbb{S}^2 \times \mathbb{S}^1$, which may arise as a component of the prime decomposition but is not irreducible, admits a geometric structure modelled on $\mathbb{S}^2 \times \mathbb{R}$.)

Thurston's *geometrization conjecture* states that every closed orientable 3-manifold is geometrizable. For a more complete discussion of the geometrization conjecture, see Scott [26], Bonahon [2], or Thurston [30].

There are a number of manifolds for which the geometrization conjecture is known to be true. If M is a closed orientable irreducible sufficiently large 3-manifold, then M is geometrizable; this is Thurston's geometrization theorem; see Morgan [17] or Otal [19] for a discussion of this theorem.

In particular, if M has a non-empty torus decomposition, then it is geometrizable. In this case, each component of the complement of the torus decomposition of M either is a Seifert fibered space or admits a hyperbolic structure, that is the geometric structure modelled on $\mathbb{H}^3$. We encode in the following theorem the parts of this discussion we make the most use of.

Theorem 2.4. *Let M be a closed orientable irreducible sufficiently large 3-manifold. Then, M admits a torus decomposition T. Moreover, each component of $M - T$ either is a Seifert fibered space or admits a hyperbolic structure.*

Additionally, the geometrization of Seifert fibered spaces, and in fact of irreducible graph manifolds, is completely understood.

Theorem 2.5. [26, Theorem 5.3] *Let M be a closed orientable 3-manifold. Then,*

(1) *M possesses a geometric structure modelled on* Sol *if and only if M is finitely covered by a 2-torus bundle over $\mathbb{S}^1$ with hyperbolic glueing map;*
(2) *M possesses a geometric structure modelled on one of $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}}_2$, or* Nil *if and only if M is a Seifert fibered space.*

We note here that the two unresolved cases of the geometrization conjecture are that the fundamental group of M is finite, in which case M should admit a geometric structure modelled on $\mathbb{S}^3$ [the Poincaré conjecture and the spherical space form problem], and that the fundamental group of M is infinite, does not contain $\mathbb{Z} \oplus \mathbb{Z}$, and does not contain a normal cyclic subgroup, in which case M should admit a geometric structure modelled on $\mathbb{H}^3$ [the hyperbolization conjecture].

2.5. **Simplicial volume.** Let M be a closed manifold. Denote by C_* the real chain complex of M: a chain $c \in C_*$ is a finite linear combination $\sum_i r_i \sigma_i$ of singular simplices σ_i in M with real coefficients r_i. Define the *simplicial l^1-norm* in C_* by setting $|c| = \sum_i |r_i|$. This norm gives rise to a pseudo-norm on the homology $H_*(M, \mathbb{R})$ by setting

$$|[\alpha]| = \inf\{|z| \,:\, z \in C_* \text{ and } [z] = [\alpha]\}.$$

When M is orientable, define the *simplicial volume* of M, denoted $\|M\|$, to be the simplicial norm of the fundamental class. The simplicial volume is also called *Gromov's invariant*, since it was first introduced by Gromov [7].

The following lower bound on $\|M\|$ is due to Thurston [29].

Theorem 2.6. [29, Theorem 6.5.5] *Suppose that M is a closed orientable 3-manifold and that $H \subset M$ is a 3-dimensional submanifold whose interior admits a complete hyperbolic structure of finite volume. Suppose further that $\overline{H}$ is embedded in M and that $\partial \overline{H}$ is incompressible in M. Then,*

$$\|M\| \geq \frac{\mathrm{Vol}(H)}{v_3} > 0,$$

where v_3 is the volume of the regular ideal tetrahedron in $\mathbb{H}^3$.

The next theorem follows immediately from Theorems 2.6, 2.4, and 2.5.

Theorem 2.7. *Let M be a closed orientable geometrizable 3-manifold. Suppose that $\|M\| = 0$. Then M is a graph manifold.*

Proof. The proof of Theorem 2.7 is essentially contained in Soma [28]; we include it here solely for the sake of completeness.

We begin by considering the prime decomposition of M. That is, write M as the connected sum $M = M_0 \# \cdots \# M_p$, where each M_i is a prime 3-manifold. (Note that we are including in this discussion the case that M is itself prime, and so has trivial prime decomposition.)

Since simplicial volume behaves additively with respect to connected sums (cf. Gromov [7]), the hypothesis that M has zero simplicial volume implies that each M_i has zero simplicial volume as well. Since the connected sum of graph manifolds is again a graph manifold (cf. Soma [28]), it suffices to show that each M_i is a graph manifold. Since each M_i is prime, it is either irreducible or diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$, which is a Seifert fibered space. So, we may assume without loss of generality that M is irreducible.

Let T be the torus decomposition of M. Recall that M is assumed to be geometrizable. If T is empty, then M admits a geometric structure other than the one modelled on $\mathbb{H}^3$ (which is excluded by the assumption on the simplicial volume of M), and so M is a graph manifold, by Theorem 2.5.

If T is non-empty, then M is sufficiently large, and so Thurston's geometrization conjecture holds for M. Since $\|M\| = 0$, each component of $M - T$ is a Seifert fibered space, as no piece can be hyperbolic, by Theorem 2.6. It follows that M must be a graph manifold. □

2.6. **Topological entropy.** We recall in this subsection the definition of the topological entropy of the geodesic flow of a smooth Riemannian metric g on a closed manifold M. For a more detailed discussion, we refer the interested reader to Paternain [20].

The geodesic flow of g is a flow ϕ_t that acts on SM, the unit sphere bundle of M, which is a closed hypersurface of the tangent bundle of M. Let d be any distance function compatible with the topology of SM. For each $T > 0$ we define a new distance function

$$d_T(x, y) := \max_{0 \le t \le T} d(\phi_t(x), \phi_t(y)).$$

Since SM is compact, we can consider the minimal number of balls of radius $\varepsilon > 0$ in the metric d_T that are necessary to cover SM. Let us denote this number by $N(\varepsilon, T)$. We define

$$\mathrm{h}(\phi, \varepsilon) := \limsup_{T \to \infty} \frac{1}{T} \log N(\varepsilon, T).$$

Observe now that the function $\varepsilon \mapsto \mathrm{h}(\phi, \varepsilon)$ is monotone decreasing and therefore the following limit exists:

$$\mathrm{h}_{\mathrm{top}}(g) := \lim_{\varepsilon \to 0} \mathrm{h}(\phi, \varepsilon).$$

The number $\mathrm{h}_{\mathrm{top}}(g)$ thus defined is the *topological entropy* of the geodesic flow of g. Intuitively, this number is a measure of the orbit complexity of the flow. The positivity of $h_{top}(\phi)$ indicates complexity or 'chaos' of some kind in the dynamics of ϕ_t.

There is a formula, known as Mañé's formula, that gives a nice alternative description of $\mathrm{h}_{\mathrm{top}}(g)$. Given points p and q in M and $T > 0$, define $n_T(p, q)$ to be the number of geodesic arcs joining p and q with length $\le T$. Mañé [14] showed that

$$\mathrm{h}_{\mathrm{top}}(g) = \lim_{T \to \infty} \frac{1}{T} \log \int_{M \times M} n_T(p, q)\, dp\, dq.$$

Finally we note that entropy behaves well under scaling of the metric. Namely, if c is any positive constant, then $\mathrm{h_{top}}(cg) = \frac{\mathrm{h_{top}}(g)}{\sqrt{c}}$.

2.7. **Minimal volume and collapsing.** The *minimal volume* $\mathrm{MinVol}(M)$ of a Riemannian manifold M is defined to be the infimum of $\mathrm{Vol}(M, g)$ over all smooth metrics g such that the sectional curvature K_g of g satisfies $|K_g| \leq 1$. This differential invariant was introduced by M. Gromov in [7].

We shall need the following result, see Cheeger and Gromov [3, Example 0.2 and Theorem 3.1] and Rong [23].

Proposition 2.8. *Let M be a closed orientable 3-manifold. If M is a graph manifold, then M admits a polarized F-structure, and hence* $\mathrm{MinVol}(M) = 0$.

We will not give here the precise definition of a polarized F-structure, because it is too technical. Instead we give an informal description, and we refer the interested reader to Cheeger and Gromov [3] for a more detailed discussion.

An *F-structure* on a manifold M is a natural generalization of a torus action on M. Different tori, possibly of different dimensions, act on subsets of M in such a way that M is partioned into disjoint orbits. The F-structure is said to be *polarized* if the local actions are locally free.

Consider the following example of a polarized F-structure on a graph manifold. Take a compact surface S with non-empty connected boundary, and consider two copies of $S \times \mathbb{S}^1$, each of which has a 2-torus boundary. Fixing an identification of ∂S with $\mathbb{S}^1$, glue the boundaries of two copies of $S \times \mathbb{S}^1$ by a map that interchanges the $\mathbb{S}^1$ factors, so that $(x, z) \in \partial S \times \mathbb{S}^1$ on one copy is glued to $(z, x) \in \partial S \times \mathbb{S}^1$ on the other copy.

The resulting manifold admits a free circle action on each copy of $\mathrm{int}(S) \times \mathbb{S}^1$, but at their common boundary the actions do not agree. However, they do generate a 2-torus action which acts locally near their common boundary, thus defining a polarized F-structure on the whole manifold.

2.8. **An important chain of inequalities.** Let M be a closed Riemannian manifold with smooth metric g, and let $\widetilde{M}$ be its universal covering endowed with the induced metric. For each $x \in \widetilde{M}$, let $V(x, r)$ be the volume of the ball with center x and radius r. Set

$$\lambda(g) := \lim_{r \to +\infty} \frac{1}{r} \log V(x, r).$$

Manning [13] showed that this limit exists and is independent of x.

Set

$$\lambda(M) := \inf\{\lambda(g) \mid g \text{ is a smooth metric on } M \text{ with } \mathrm{Vol}(M, g) = 1\}.$$

It is well known, see Milnor [15], that $\lambda(g)$ is positive if and only if $\pi_1(M)$ has exponential growth. Manning's inequality [13] asserts that for any metric g,

$$\lambda(g) \leq \mathrm{h_{top}}(g). \tag{1}$$

In particular, it follows that if $\pi_1(M)$ has exponential growth, then $\mathrm{h_{top}}(g)$ is positive for any metric g. (This fact was first observed by Dinaburg [5]). Gromov [7] showed that if $\mathrm{Vol}(M, g) = 1$, then

$$\frac{1}{C_n\, n!}\|M\| \leq [\lambda(g)]^n, \tag{2}$$

where

$$C_n = \Gamma\left(\frac{n}{2}\right) / \sqrt{\pi}\, \Gamma\left(\frac{n+1}{2}\right).$$

Finally it was observed by Paternain [20] that

$$[\mathrm{h}(M)]^n \leq (n-1)^n \mathrm{MinVol}(M). \tag{3}$$

Combining equations (1), (2), and (3), we obtain the following chain of inequalities:

$$\frac{1}{C_n n!}\|M\| \leq [\lambda(M)]^n \leq [\mathrm{h}(M)]^n \leq (n-1)^n \mathrm{MinVol}(M). \tag{4}$$

We note here that the only known 3-manifolds with $\mathrm{h}(M) > 0$ are those with $\|M\| \neq 0$. In fact it follows from Theorem 2.7, Proposition 2.8, and the chain of inequalities (4) that if M is a closed orientable geometrizable 3-manifold, then the vanishing of the simplicial volume implies that $\mathrm{h}(M) = 0$.

We encode this information in the following theorem.

Theorem 2.9. *Let M a closed orientable geometrizable 3-manifold. Then the following are equivalent:*

1. *the minimal volume* $\mathrm{MinVol}(M)$ *of M vanishes;*
2. *the minimal entropy* $\mathrm{h}(M)$ *of M vanishes;*
3. *the simplicial volume* $\|M\|$ *of M vanishes;*
4. *M is a graph manifold.*

3. Geometric structures and the minimal entropy problem

In this section, we consider the minimal entropy problem for those 3-manifolds that admit a single geometric structure. Namely, we prove the following.

Proposition 3.1. *Let M be a closed orientable 3-manifold. Suppose that M admits a geometric structure. Then, the minimal entropy problem for M can be solved if and only if M admits a geometric structure modelled on $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$,* Nil*, or $\mathbb{H}^3$. Moreover, if M admits a geometric structure modelled on $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$, or* Nil*, then M admits a smooth metric g with $\mathrm{h_{top}}(g) = 0$.*

Proof. We start by showing that if M admits a geometric structure modelled on one of these 5 geometries, then the minimal entropy problem for M can be solved. Observe first that if M admits a geometric structure modelled on $\mathbb{H}^3$, then the minimal entropy problem can be solved by the results of Besson, Courtois and Gallot [1].

It follows immediately from Theorem 2.5 that if M admits a geometric stucture modelled on one of the seven geometries $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}}_2$, Nil, or Sol, then

M is a graph manifold. Hence by Proposition 2.8 and the chain of inequalities (4), we have that for such an M, the minimal entropy satisfies $\mathrm{h}(M) = 0$.

We now show that if M admits a geometric structure modelled on one of $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$, or Nil, then the minimal entropy problem for M can be solved. To do this, we need to show that M admits a smooth metric g with $\mathrm{h}_{\mathrm{top}}(g) = 0$.

(1) $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$: All the Jacobi fields in these geometries grow at most linearly (in the case of $\mathbb{S}^3$ they are actually bounded), and hence all the Liapunov exponents of *every* geodesic in M are zero. It follows from Ruelle's inequality [24] that all the measure entropies are zero. Hence, by the variational principle, the topological entropy of the geodesic flow of M must be zero.
(2) Nil: This geometry can be described as $\mathbb{R}^3$ with the metric

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2.$$

Here, not all the Jacobi fields grow linearly, but they certainly grow polynomially. Again this implies that all the Liapunov exponents of *every* geodesic in M are zero and hence the topological entropy of the geodesic flow of M must be zero.

Since we have assumed that M admits a geometric structure, we complete the proof by showing that if M admits a geometric structure modelled on one of the remaining geometries, namely $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}}_2$, and Sol, then M cannot admit a metric of zero topological entropy. To do this, we use the next lemma, together with the fact described in Subsection 2.8, that if $\pi_1(M)$ grows exponentially, then $\mathrm{h}_{\mathrm{top}}(g) > 0$ for any smooth metric g on M.

Lemma 3.2. *Let M be a closed orientable 3-manifold, and suppose that M admits a geometric structure modelled on one of $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}}_2$, or* Sol. *Then $\pi_1(M)$ grows exponentially.*

Proof. In the case that M admits a geometric structure modelled on $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}}_2$, we start by recalling from Theorem 2.5 that M is then a Seifert fibered space. The base orbifold of the Seifert fiber space admits a hyperbolic structure, and so the orbifold fundamental group of the base orbifold contains a free subgroup of rank 2, and hence so does $\pi_1(M)$. Hence, $\pi_1(M)$ grows exponentially.

In the case that M admits a geometric structure modelled on Sol, we have that M is finitely covered by the mapping torus N of a hyperbolic automorphism of a 2-torus. Note that a hyperbolic automorphism of a 2-torus is an Anosov diffeomorphism, and so the suspension flow on N is an Anosov flow. It is known that the fundamental group of a 3-manifold with an Anosov flow has exponential growth (see for example Plante and Thurston [22]). □

This completes the proof of Proposition 3.1. □

4. Proof of Theorem A

Up to this point, we have been considering the minimal entropy problem for closed 3-manifolds that admit a single geometric structure. In this section, we consider a more general geometrizable 3-manifold.

Theorem A. *Let M be a closed orientable irreducible 3-manifold whose fundamental group contains a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup. The following are equivalent:*

1. *the simplicial volume $\|M\|$ of M is zero and the minimal entropy problem for M can be solved;*
2. *M admits a geometric structure modelled on $\mathbb{E}^3$ or* Nil*;*
3. *M admits a smooth metric g with $\mathrm{h_{top}}(g) = 0$.*

Proof. Let us show that item 1 implies item 2. Suppose then that M has zero simplicial volume and that the minimal entropy problem for M can be solved. We show that M must then admit a geometric structure modelled on either $\mathbb{E}^3$ or Nil. Since the fundamental group of M contains a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup, Theorem 2.3 ensures that either M contains an incompressible embedded 2-torus or M is a Seifert fibered space. We now split the proof into two cases:

- Suppose first that M contains an incompressible embedded 2-torus, and so is sufficiently large. Since we have assumed that $\|M\| = 0$, Theorem 2.7 yields that M is a graph manifold. Hence, by Theorem 2.9, we have that $\mathrm{h}(M) = 0$.

  However, using work of Evans and Moser [6], specifically Theorem 4.2 and Corollary 4.10 in [6], we see that either $\pi_1(M)$ contains a free subgroup of rank 2 or M is finitely covered by a 2-torus bundle over $\mathbb{S}^1$. In the former case, $\pi_1(M)$ grows exponentially and therefore the minimal entropy problem cannot be solved for M.

  In the latter case, M admits a geometric structure modelled on one of $\mathbb{E}^3$, Nil, or Sol (cf. Theorem 5.5 of Scott [26]). However, in the case that M admits a geometric structure modelled on Sol, we know from Proposition 3.1 that the minimal entropy problem cannot be solved for M.

  Hence, if the minimal entropy problem can be solved for M and if M contains an incompressible embedded 2-torus, then M admits a geometric structure modelled on either $\mathbb{E}^3$ or Nil.
- The other case is that M is a Seifert fibered space. Here, Theorem 2.5 ensures that M possesses a geometric structure modelled on one of $\mathbb{S}^3$, $\mathbb{E}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\widetilde{\mathrm{SL}}_2$ or Nil.

  Since the fundamental group of M admits a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup, the geometric structure on M cannot be modelled on $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$. Since we have assumed that the minimal entropy problem can be solved for M, Proposition 3.1 yields that M must admit a geometric structure modelled on either $\mathbb{E}^3$ or Nil, as desired.

To see that item 2 implies item 3, recall from Proposition 3.1 that if M admits a geometric structure modelled on $\mathbb{E}^3$ or Nil, then M admits a smooth metric g with $\mathrm{h_{top}}(g) = 0$.

Finally to prove that item 3 implies item 1, observe that if M admits a smooth metric g with $\mathrm{h_{top}}(g) = 0$ it then follows from inequalities (1) and (2) that M has zero simplicial volume.

This completes the proof of Theorem A. □

5. Proof of Theorem B

We are now ready to consider the minimal entropy problem for a general geometrizable 3-manifold with zero simplicial volume.

Theorem B. *Let M be a closed orientable geometrizable 3-manifold. The following are equivalent:*

1. *the simplicial volume $\|M\|$ of M is zero and the minimal entropy problem for M can be solved;*
2. *M admits a geometric structure modelled on $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{E}^3$, or* Nil*;*
3. *M admits a smooth metric g with $\mathrm{h_{top}}(g) = 0$.*

Proof. Let us prove that item 1 implies item 2. Suppose that M has zero simplicial volume and that the minimal entropy problem for M can be solved. Since M is geometrizable and its simplicial volume vanishes, Theorem 2.7 tells us that M is a graph manifold. Hence, by Theorem 2.9, M has zero minimal entropy.

Since we are assuming that the minimal entropy problem can be solved for M, the fact that M has zero minimal entropy in turn implies there exists a smooth metric on M with zero topological entropy. This in turn implies, by the discussion in Section 2.8, that $\pi_1(M)$ does not have exponential growth.

However, it is a fact from combinatorial group theory (which follows immediately from the existence of normal forms for free products, for instance) that if A and B are two finitely generated groups, then the free product $A * B$ contains a free subgroup of rank two unless A is trivial or B is trivial, or A and B are both of order two. Since the fundamental group of a connected sum is the free product of the fundamental groups of the summands, we conclude that either the prime decomposition is trivial or there are only two summands both of which have fundamental group $\mathbb{Z}_2$.

In the former case, it follows that M must be either irreducible or $\mathbb{S}^2 \times \mathbb{S}^1$, while in the latter case M must be $\mathbb{P}^3 \# \mathbb{P}^3$, where $\mathbb{P}^3$ is the 3-dimensional real projective space. Since $\mathbb{S}^2 \times \mathbb{S}^1$ and $\mathbb{P}^3 \# \mathbb{P}^3$ both admit a geometric structure modelled on $\mathbb{S}^2 \times \mathbb{R}$, we may assume from now on that M is irreducible.

There are now several cases, depending on $\pi_1(M)$. Suppose first that $\pi_1(M)$ is finite. Since M is geometrizable, we have that M admits a geometric structure modelled on $\mathbb{S}^3$.

In the case that $\pi_1(M)$ is infinite and contains a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup, the assumption that the simplicial volume of M is zero, together with the fact that the minimal entropy problem can be solved for M, allows us to apply Theorem A to see that M admits a geometric structure modelled on $\mathbb{E}^3$ or Nil.

The remaining case is that $\pi_1(M)$ is infinite and does not contain a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup. Since M is geometrizable, either M admits a hyperbolic structure or M is Seifert fibered. (Since $\pi_1(M)$ does not contain a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup, M cannot admit a geometric

structure modelled on Sol, as Sol manifolds are finitely covered by 2-torus bundles over the circle.) However, since $\|M\| = 0$, M cannot admit a hyperbolic structure.

Note though that M cannot admit a geometric structure modelled on $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{E}^3$, $\widetilde{\mathrm{SL}_2}$, or Nil, as such manifolds always have a $\mathbb{Z} \oplus \mathbb{Z}$ in their fundamental groups, by Lemma 2.1. Hence, the only possibilities remaining are that M admits a geometric structure modelled on either $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{S}^3$, as desired.

To see that item 2 implies item 3, recall from Proposition 3.1 that if M admits a geometric structure modelled on $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{E}^3$, or Nil, then M admits a smooth metric g with $\mathrm{h_{top}}(g) = 0$.

Finally to prove that item 3 implies item 1, observe that if M admits a smooth metric g with $\mathrm{h_{top}}(g) = 0$, it then follows from inequalities (1) and (2) that M has zero simplicial volume.

This completes the proof of Theorem B. □

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