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THE EXACT POWER ENVELOPE OF TESTS FOR A UNIT ROOT

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Abstract

We show how to obtain the exact power envelope of tests for a unit root against trend-stationary alternatives, under normality. This is in contrast to the asymptotic power envelope derived by Elliott, Rothenberg and Stock (1996), and is used to indicate the lack of power of unit root tests in fixed sample sizes.

1. INTRODUCTION

The issue of testing for a unit root in economic time series has generated a considerable literature, both theoretical and empirical. Stock (1994) provides a good up-to-date reference to this large and detailed literature. Here we address one

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particular feature of testing for a unit root, namely the exact power properties of a common class of tests of a unit root against trend-stationary alternatives. While this issue of exact power has received some attention in the recent literature (e.g. Bhargava (1986, 1996), Hwang and Schmidt (1993)), much of the remaining analysis has been asymptotic in nature.

We show how to obtain the exact power envelope of tests for a unit root against trend-stationary alternatives, under normality. This exact power envelope traces out the maximum possible power of any such test, in a sense to be made precise later. Evaluating the exact power envelope for this testing problem is both feasible (making use of Dufour and King's (1991) related analysis) and informative.

The next section outlines the precise testing problem under consideration. Section 3 discusses the derivation and interpretation of the exact power envelope for tests of a unit root against trend-stationary alternatives. The exact power envelope is based upon the notion of point optimal tests introduced to econometrics by King (1987a), following a related statistical literature on hypothesis testing (e.g. Lehmann (1959), Kadilaya (1970); see also Pere (1997)). This exact power envelope is in contrast to the asymptotic power envelope derived by Elliott, Rothenberg and Stock (1996), and considered in Section 4. Section 5 provides a numerical illustration of the exact power envelope, and compares it with the exact power function for a representative unit root test. The final section offers some conclusions about the usefulness of the exact power envelope.

2. TESTING FOR A UNIT ROOT

The simplest possible framework for testing for a unit root takes the form of testing the null hypothesis $H_0 : \beta = 1$ in (e.g.) the AR(1) model with an intercept

$$y_t = \alpha + \beta y_{t-1} + \varepsilon_t \ , \qquad t = 1, ..., T$$

where it is assumed (e.g.) that the errors $\varepsilon_t \sim IN(0, \sigma^2)$, and y_0 is an unknown constant. However, under the alternative that $\beta \neq 1$ this model does not allow for a trend, and so is not well-suited to testing the null hypothesis that y_t is difference-stationary against the alternative hypothesis that y_t is trend-stationary. Indeed, West (1988) shows that tests in such a model are inconsistent against trend-stationary alternatives.

Because of this, we consider an alternative framework proposed by Bhargava (1986), and subsequently used by DeJong et al. (1992) and Schmidt and Phillips (1989), that is better suited to testing difference-stationarity against trend-stationarity. For an observed time series y_t we postulate

$$\begin{aligned} y_t &= \mu + \alpha t + u_t 1 \\ u_t &= \beta u_{t-1} + \varepsilon_t \ , \qquad \varepsilon_t \sim IN(0, \sigma^2) \end{aligned} \tag{2.1}$$

together with some assumptions about an initial error u_0. As is well known, this can be rewritten to obtain

$$y_t = [\mu(1-\beta) + \alpha\beta] + \alpha(1-\beta)t + \beta y_{t-1} + \varepsilon_t \ .$$

Thus under the null hypothesis $H_0 : \beta = 1$, this simplifies to the difference-stationary model

$$y_t = \alpha + y_{t-1} + \varepsilon_t \ ,$$

i.e. a random walk with drift parameter α. However, under the (one-sided) alternative hypothesis $H_a^- : \beta < 1$, we obtain

$$y_t = \gamma + \delta t + \beta y_{t-1} + \varepsilon_t \ ,$$

where $\gamma = \mu(1-\beta)+\alpha\beta$ and $\delta = \alpha(1-\beta)$. With $|\beta| < 1$, this is a trend-stationary model. Note that the framework from which we can derive these simple difference-stationary and trend-stationary models is simply that of a static linear regression (of y_t upon an intercept and a linear trend) with AR(1) errors. Also, in all that follows, we assume that the errors ε_t are normally distributed.

3. DERIVATION OF THE EXACT POWER ENVELOPE

Dufour and King (1991) introduced the *most powerful invariant* (MPI) test of $H_0 : \beta = 1$, against the *particular* (point) alternative $H_a : \beta = \beta_1$ (where $|\beta_1| < 1$). In the regression error specification $u_t = \beta u_{t-1} + \varepsilon_t$, we permit the AR coefficient β to take any finite value. Following Dufour and King (1991), we specify the initial regression error as $u_0 = d_0 \varepsilon_0$, where $d_0 \neq 0$ is unknown, and $\varepsilon_0 \sim N(0, \sigma^2)$ independent of ε_t, $t = 1, ..., T$. We consider tests which satisfy two conditions. Firstly, they are *invariant* to transformations of $\mathbf{y} = (y_0,\ y_1,\ y_2\ ,...,.\ y_T)'$ of the form

$$\mathbf{y}^* = \gamma_0 \mathbf{y} + \mathbf{X}\gamma \qquad \mathbf{X} = [\iota : \mathbf{t}]$$

where ι and $\mathbf{t}$ denote vectors of ones, and a linear trend respectively. Durbin and Watson (1971) show the optimality of the Durbin-Watson statistic under such a transformation group. Secondly, we require that the *null* distribution of the test does not depend on d_0, defined above. Dufour and King (1991) show that this can be achieved by extending the transformation group by including in the transformed model a dummy variable for the initial observation.

Dufour and King's (1991) Theorem 5 provides a MPI test of the null hypothesis $H_0 : \beta = \beta_0$ against the point alternative $H_a : \beta = \beta_1$, by defining a *point optimal invariant* (POI) test (see King (1987b) for further details) that is invariant under the group of transformations described above, and thus has the correct size for any d_0 value. The MPI test is *most powerful* against the point alternative, given the above assumptions, and is *invariant* by design, as described above. For our case where $\beta_0 = 1$ and $\beta_1 < 1$, the MPI test of $H_0 : \beta = 1$ against $H_a : \beta = \beta_1$

(where $|\beta_1| < 1$) takes the form of rejecting H_0 for small values of

$$MPI(\beta_1, d_0^*) = \frac{v'Q'MQv}{v'v} \tag{2}$$

where v is a $T \times 1$ normally distributed random vector depending upon the observed $y = (y_1,\ y_2\ ,...,\ y_T)'$, and Q and M are particular $T \times T$ fixed matrices, given β_1 and d_0^*. See Dufour and King (1991) for the definitions of v, Q and M.

Note that to apply Dufour and King's (1991) Theorem 5, we have to specify both a value of the point alternative β_1 and some value (not necessarily the true value) d_0^* of d_0. Although it would be possible to avoid the arbitrary choice of d_0^* by considering a larger invariance group such that the MPI test (and not just its null distribution) does not depend on d_0, Dufour and King (1991) indicate that such tests have very inferior power properties compared with the MPI tests considered here. In practice (as we shall see) the arbitrary choice of d_0^* appears to make little difference to test properties.

Computationally, the principal advantage of the MPI test is that it can be expressed as a ratio of quadratic forms in normal variables. Hence using numerical methods (see, e.g., King (1987a) and Ansley et al. (1992)) we can calculate to any desired accuracy both (i) the *exact critical value* for any desired test size; and (ii) the *exact power* against any fixed alternative.

Of course, operationally the MPI test requires knowledge of, or priors upon, the values of the fixed point alternative β_1 (< 1). In most practical circumstances this is an unrealistic assumption, and this limits that practical appeal of such MPI tests. However, this MPI test can be used more generally to construct (empirically or graphically) the **power envelope** of tests of $H_0 : \beta = 1$, against $H_a^- : \beta < 1$, by considering a range of β_1 values. For each point alternative within this range, the MPI test is (by design) the most powerful test (within the invariance group) against that point alternative. This power envelope gives the *maximum attainable* *power* of any test (satisfying the given invariance property) over the region $\beta < 1$. It is important to note that this power envelope delineates *exact* powers for any given sample size T. The power envelope indicates how deficient (relative to this power envelope) a particular unit root test's power is for specific alternatives β_1. King (1990) summarises computational issues involved in evaluating the power envelope.

4. COMPARISON WITH THE ASYMPTOTIC POWER ENVELOPE

Elliott et al. (1996) adopt a related, but more particular, analysis. They employ a local-to-unity asymptotic approximation to derive the *asymptotic* power envelope of tests for a unit root. As with Dufour and King's (1991) approach, and that detailed here, they assume a normally distributed model. They argue (p. 814) that their asymptotic power envelope analysis provides "simpler and more interpretable results" than Dufour and King's exact POI tests.

Their analysis covers a wider range of models than that considered here (where attention is restricted to testing $H_0 : \beta = 1$ in (1)). For example, they consider cases which allow for linear models (as in (1)) where the deterministic components are slowly evolving. Nonetheless, a brief comparison with their analysis in the case where the deterministic components are an intercept and a linear trend (i.e. equation (1) above) is useful.

Their local-to-unity asymptotic approximation allows the parameter space to be a shrinking neighbourhood of unity (i.e. $\beta = 1$) as the sample size $T \to \infty$. In particular they reparameterise the model so that $c \equiv T(\beta - 1)$, and then let c be constant as $T \to \infty$. Then a likelihood ratio test statistic yields an asymptotically most powerful test of $H_0 : \beta = 1$ against the alternative that

$c = c_1 \equiv T(\beta_1 - 1)$. This test is asymptotically *point optimal* against the point alternative that $c = c_1$. The family of asymptotically point optimal tests (for different c_1 values) defines the asymptotic power envelope function $\Pi(c)$. Each of these tests has an asymptotic power curve tangent to the power envelope at one point ($c = c_1$). For the case of interest here, the asymptotic power functions of the standard Dickey-Fuller tests are "well below" the power envelope (p. 822).

Aside from this use of the asymptotic power envelope to obtain refined tests, a direct comparison with the exact power envelope is not straightforward. Because of the local-to-unity asymptotic approximation employed, i.e. $c \equiv T(\beta - 1)$, where c is held constant as $T \to \infty$, Elliott et al.'s tabulations and graphed asymptotic power envelopes are not readily interpretable without further direct evaluation. Nonetheless, we can make some rough comparisons. For example, by inspection from their Figure 3, which deals with the case of interest here (i.e. equation (1)), we can see that the asymptotic power functions for the Dickey-Fuller statistics are very close to the asymptotic power envelope for values of c between 0 and about -5 (or about -7 for the $\widehat{\rho}_\tau$ statistic). If we take $T \approx 50$, then this implies a range of values for $\beta \approx 1 + \frac{c}{T}$ between approximately $(0.9 - 1)$, or about $(0.86 - 1)$. Thus for this local (to unity) range of β values, the asymptotic power functions lie "close" to the asymptotic power envelope. Of course, this is not directly interpretable in terms of the *exact* power functions.

Elliott et al. (1996) indicate an advantage of their asymptotic power envelope analysis over Dufour and King's (1991) exact approach: namely that it avoids having to make strong assumptions about the initial error u_0 (and thus d_0^*). The cost of this simplification, however is that their analysis is purely *asymptotic.* We now consider an illustration of the alternative *exact* power envelope approach argued here.

5. NUMERICAL ILLUSTRATION

We consider two tests for the null hypothesis $H_0 : \beta = 1$. One of these, based upon the Sargan-Bhargava statistic (see Sargan and Bhargava (1983) and the extension by Bhargava (1986)) is used as a representative test for a unit root. For example, Edmonds et al. (1992) conclude that the Sargan-Bhargava test has comparatively good power properties. The other test statistic is the MPI test described above, and is used to construct the power envelope of tests for a unit root.

The Sargan-Bhargava statistic is

$$SB = \frac{\sum_{t=2}^{T}(y_t - y_{t-1})^2 - \frac{1}{T-1}(y_T - y_1)^2}{\frac{1}{(T-1)^2}\sum_{t=1}^{T}\left\{(T-1)y_t - (t-1)y_T - (T-t)y_1 - (T-1)[\overline{y} - \frac{1}{2}(y_1 + y_T)]\right\}^2}$$

This is a *locally most powerful invariant* test in the neighbourhood of $\beta = 1$ of the difference stationarity null hypothesis $H_0 : \beta = 1$, against the trend-stationary alternative $H_a^- : \beta < 1$ in (1), and is invariant to the values of μ and σ^2. Table I of Bhargava (1986) gives 5% (exact) critical values for sample sizes $T = \{20\ (5)\ 50\ (10)\ 100\}$. Note that as SB can be written as a quadratic form in normal variates, its size and exact power function can be evaluated by numerical methods.

The (exact) MPI test given in (2) above is used to construct the (exact) power envelope in the following way. For a given value of d_0^*, set $\beta_1 = 1$, and use (2) to obtain the critical value appropriate to a test of desired size. Then use this critical value together with (2) to evaluate the rejection probability (i.e. test power) of the MPI test against the alternative that $H_a : \beta = \beta_1$ (for some β_1 satisfying $|\beta_1| < 1$). This is, by design, a POI test against that specific alternative $H_a : \beta = \beta_1$. Of course, given β_1 and the appropriate critical value, the power function can be evaluated over a range of alternative values for β, but only at $\beta = \beta_1$ will the exact power function coincide with the power envelope. This is then repeated for a range of β_1 values, at each stage providing another tangency point to the power envelope.

In the calculations reported in Table 1, all critical values are evaluated at exact size 0.05. The range of values for β is $\{1.0\ 0.95\ 0.9\ 0.8\ 0.85\ 0.75\ 0.5\}$, and the powers are evaluated for two sample sizes, $T = \{25, 50\}$. Note that the values reported here for the power envelope all refer to calculations assuming $d_0^* = 1$. Alternative values for d_0^* of 0.1 or 10 typically made no practical difference to the calculations, and so are not reported.

The tabulations of the exact power envelope and exact power function for the Sargan-Bhargava statistic SB indicate two key properties. First, for the sample sizes considered here the Sargan-Bhargava test has an exact power function that lies remarkably close to the exact power envelope, especially for β close to unity. To the extent that this test is a representative test of a unit root, with reasonably favourable power properties compared to many competitor tests (see, e.g., Stock (1994)), this suggests that there is little to gain (by way of increased power) by seeking alternative tests. Note that this is not clear from Elliott et al.'s (1996) *asymptotic* power envelope analysis; there the power function/power envelope discrepancies seemed larger.

Second, this exact power envelope emphasises the following point which has been made repeatedly in the literature but is often still ignored by practitioners. This is that these unit root tests are **not** at all powerful for these sample sizes, even against "plausible" trend-stationary alternatives not too close to the null β value of unity. For example, with $T = 50$, the exact power envelope indicates that no well-behaved unit root test will have power against the point alternative $\beta = 0.85$ better than 0.193 ! Thus this reinforces earlier warnings (see, e.g., Campbell and Perron (1991) and Stock (1994)) about the possible inappropriate use of unit root tests in these circumstances, with conventional significance levels.

TABLE 1

Exact Powers for Exact Size 0.05

	T=25	
β	Sargan-Bhargava	Power Envelope
0.95	0.054	0.055
0.9	0.066	0.067
0.85	0.084	0.087
0.8	0.109	0.115
0.75	0.141	0.149
0.5	0.432	0.460

	T=50	
β	Sargan-Bhargava	Power Envelope
0.95	0.066	0.068
0.9	0.107	0.114
0.85	0.177	0.193
0.8	0.277	0.302
0.75	0.404	0.442
0.5	0.918	0.962

6. CONCLUSIONS

This paper attempts to make two simple points. Firstly, obtaining information about some exact features of unit root tests is feasible. For a standard framework for testing for a unit root against trend-stationary alternatives, Dufour and King's (1991) results can be adapted to permit numerical evaluation of the *exact* power envelope, without recourse to simulation methods. Secondly, this exact power envelope can be very informative about unit root tests. This was illustrated by a simple numerical example, with the following conclusions that reinforce earlier simulation-based findings. For alternative values of β near to 1, but "representative" of trend-stationary models, best possible power is awful. Also, existing tests are "close" to attaining "best possible" power, so there is little to gain in searching for more sophisticated (powerful) tests. If higher power is desired, what is needed is more information, in particular more observations. Alternatively, Hansen (1995) suggests that adding correlated covariates to the regression can yield large power gains[1].

The techniques illustrated here can be generalised, while still making use of Dufour and King's (1991) results. For example, their linear regression model with AR(1) errors could be extended to "bigger" design matrices (to allow for, e.g., seasonal dummies, or structural breaks).

[1] We are grateful to Niels Haldrup for this observation.

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