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Exact Properties of the Maximum Likelihood Estimator in Exponential Regression Models: A Differential Geometric Approach

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Abstract

Using recently developed methods for obtaining exact distribution results for implicitly de ned estimators, we study the exact properties of the maximum likelihood estimator in exponential regression modes. The main technical problem is the evaluation of a surface integral over an $(n-k)$-dimensional hyperplane embedded in the n-dimensional sample space.

Details of the calculation are given in the cases $k=1$ and $k=2$, and some general properties of the densities for arbitrary k are indicated.

1 INTRODUCTION

In a recent paper Hillier and Armstrong (1996) have given an integral formula for the (exact) density of the maximum likelihood estimator (MLE). The formula, which expresses the density at a point t, say, as a surface integral over the manifold in the sample space upon which the MLE has the value t, does not require that the estimator be a known function of the data, but does require that the manifold on which the MLE is xed (i.e., the level set of the MLE) be known. One, but by no means the only, situation in which this is so occurs when the MLE is uniquely de ned by the vanishing of the score vector. The importance of this result lies in the fact that the formula can be used to obtain the exact density even when the estimator is only implicitly de ned in terms of the data. The exponential regression model is well-known to be of this type, and in this paper we apply the Hillier and Armstrong result to the MLE for this model.

The observations $x_1, \dots, x_n$ are assumed to be independent realizations of exponential random variables with means

$$\lambda_i = \exp\{\theta' w_i\}, i = 1, \dots, n, \tag{1}$$

where θ is a $k \times 1$ vector of parameters, and w_i is a $k \times 1$ vector of covariates, assumed non-random. The joint density of the data is thus:

$$pdf(x_1, \dots, x_n; \theta) = \exp\{-n\theta' \bar{w}\} \exp\{-\Sigma_{i=1}^n x_i \exp[-\theta' w_i]\}, \tag{2}$$

for $x_i > 0, i = 1, \dots, n$, where $\bar{w}$ is the vector of sample means of the w_i. Let $x = (x_1, \dots, x_n)'$, and abbreviate the condition $x_i > 0$ for $i = 1, \dots, n$ to simply $x > 0$. The log-likelihood, score vector, and observed information matrix, are:

$$\ell(x; \theta) = -n\theta' \bar{w} - \Sigma_{i=1}^n x_i \exp\{-\theta' w_i\}, \tag{3}$$

$$u(x; \theta) = u(x; \theta, W) = \partial \ell(x; \theta)/\partial \theta = -n\bar{w} + \Sigma_{i=1}^n x_i w_i \exp\{-\theta' w_i\}, \tag{4}$$

$$j(x; \theta) = j(x; \theta, W) = -\partial^2 \ell(x; \theta)/\partial \theta \partial \theta' = \Sigma_{i=1}^n x_i w_i w_i' \exp\{-\theta' w_i\}, \tag{5}$$

respectively. Provided the matrix $W(n \times k)$ with rows w_i', $i = 1, \dots, n$, has rank k, it is well-known that the MLE for θ is the unique solution to the equations $u(x; \theta) = 0$, but the MLE cannot be expressed directly in terms of $x_1, \dots, x_n$. This has hitherto prevented an analysis of the small-sample properties of the MLE in this model, but the Hillier/Armstrong formula makes such results accessible, at least for small values of k, as we shall see.

To the best of our knowledge the only other analytic study of the exact properties of the MLE in this model is Knight and Satchell (1996). This used an approach suggested by Huber (1964) (see also Shephard (1993)), and characteristic function inversion techniques, to deduce some properties of the density for the cases $k = 1$ and $k = 2$, but this approach does not generalise easily. In fact, our differential geometric formula can be regarded as a generalisation of the Huber approach to the multi-parameter case, avoiding the need for characteristic function inversion

We denote the MLE for θ by $T = T(x) = T(x; W)$, and a particular value of T by t. The density of T(with respect to Lebesgue measure, dt) at $T = t$ will be denoted by $pdf_T(t;\theta)$, or, if the dependence on W is important, by $pdf_T(t;\theta,W)$. From Hillier and Armstrong (1996), equation (26), we have the following expression for the density of T:

$$pdf_T(t;\theta) = \exp\{-n\theta'\bar{w}\} \mid \Sigma_{i=1}^n w_i w_i' \exp\{-2t'w_i\} \mid^{-1/2}$$

$$\int_{\mathcal{S}(t)} \mid \Sigma_{i=1}^n x_i w_i w_i' \exp\{-t'w_i\} \mid \exp[-\Sigma_{i=1}^n x_i \exp\{-\theta' w_i\}](d\mathcal{S}(t)), \tag{6}$$

where $\mathcal{S}(t) = \{x; x > 0, \Sigma_{i=1}^n x_i w_i \exp\{-t'w_i\} = n\bar{w}\}$, and $(d\mathcal{S}(t))$ denotes the (canonical) volume element on the manifold $\mathcal{S}(t)$(see Hillier and Armstrong (1996), Appendix A, for de nitions and technical details). That is, $\mathcal{S}(t)$ is the intersection of an $(n-k)$-dimensional hyperplane with the non-negative orthant.

The key problem, therefore, is the evaluation of the surface integral in (6). Because the surface $\mathcal{S}(t)$ is, in this case, at, $\mathcal{S}(t)$ admits a global coordinate chart, so that this surface integral can be reduced to an ordinary integral over a region in R^{n-k}. Nevertheless, the evaluation of this integral presents considerable difficulties: the region of interest consists of a polyhedral region in R^{n-k} bounded by the coordinate axes and k intersecting hyperplanes. In the present paper we give details of the evaluation of this integral for the cases $k = 1$ and $k = 2$. The completely general case can no doubt be dealt with similarly, see Schechter (1998) for one possible algorithm.

2 SOME PROPERTIES OF THE DENSITY IN THE GENERAL CASE

Before considering the evaluation of (6) in detail we make some general observations on the density of T that follow almost trivially from equations (4) - (6). Consider rst a transformation of the $w_i, w_i \to A'w_i, i = 1, \ldots, n$, where A is a $k \times k$ non-singular matrix, so that $W \to WA$. From (4) we see that $u(x;\theta,WA) = A'u(x;A\theta,W)$. Hence, $T(x;W) = AT(x;WA)$, so that the transformation $W \to WA$ induces the transformation $T \to A^{-1}T$ on the MLE. It follows from this observation that $pdf_T(t;\theta,W) = \|A\|^{-1} pdf_{T^*}\ t^*;\theta,WA)$, where $T^* = T(x;WA) = A^{-1}T(x;W)$, and $t = At^*$. That is, the density of $T = T(x;W)$ is trivially obtainable from the density of $T^* = T(x;WA)$, the MLE when W is replaced by WA, so that there is no loss of generality in standardising the w_i's so that, for instance, $W'W = I_k$. Note that $W'W$ is the Fisher information matrix for θ.

Next, for a xed value of t one can transform variables in equation (6) from x_i to $\tilde{x}_i = \exp\{-t'w_i\}x_i, i = 1, \ldots, n$. This maps the manifold $\mathcal{S}(t)$ into a new manifold $\tilde{\mathcal{S}}(t)$, say, and using Hillier and Armstrong (1996), equation (A4), the volume elements on $\mathcal{S}(t)$ and $\tilde{\mathcal{S}}(t)$ are related by:

$$(d\tilde{\mathcal{S}}(t)) = \exp\{-nt'\bar{w}\} \mid \Sigma_{i=1}^n x_i w_i w_i' \exp\{-2t'w_i\} \mid^{-1/2} \mid W'W \mid^{1/2} (d\mathcal{S}(t))$$

Rewriting (6) in terms of the transformed x_i we obtain:

$$pdf_T(t;\theta) = \exp\{n(t-\theta)'\bar{w}\} \mid W'W \mid^{-1/2}$$

$$\int_{\mathcal{S}} \mid \Sigma_{i=1}^n x_i w_i w_i' \mid \exp[-\Sigma_{i=1}^n x_i \exp\{(t-\theta)' w_i\}](d\mathcal{S}), \tag{7}$$

where $\mathcal{S} = \{x; x > 0, \Sigma_{i=1}^n x_i w_i = n\bar{w}\}$ does not depend on t. It follows at once from (7) that the density of T depends on (t, θ) only through their difference $(t - \theta)$.When regarded as a function of $d = (t-\theta)$, it is easy to see that, *at each point on the surface* $\mathcal{S}$, the integrand (with the term $\exp\{nd'\bar{w}\}$ attached) is maximized at $d = 0$. Hence, the mode of the (joint) density is at the point $t = \theta$.

Write the density of $T = T(x, W)$ in (7) as $f(d, W)$. It is clear from (7) that $f(d, W)$ is invariant under permutations of the rows of W, and also that the density of $T^* = T(x, WA) = A^{-1}T(x, W)$ is $f(d, WA) = \|A\|\, f(Ad, W)$, for any non-singular $k \times k$ matrix A. In particular, $f(d, -W) = f(-d, W)$(on choosing $A = -I_k$), and thus, if $-W = PW$ for some permutation matrix P, the density of $(t-\theta)$ is symmetric about the origin. If the model contains an intercept, and the remaining variables are symmetric about their means, the density of the estimates of the coefficients of those variables will be symmetric about the corresponding true values, and hence will be unbiased if their means exist.

3 THE ONE-PARAMETER CASE

We consider rst the case $k = 1$, and work from expression (7) for the density, which, in the case $k = 1$ becomes (on replacing w_i by z_i):

$$pdf_T(t;\theta) = \exp\{n(t-\theta)\bar{z}\}[\Sigma_{i=1}^n z_i^2]^{-1/2}$$

$$\int_{\mathcal{S}} [\Sigma_{i=1}^n x_i z_i^2] \exp\left[-\sum_{i=1}^n x_i \exp\{(t-\theta)z_i\}\right](d\mathcal{S}), \tag{8}$$

where $\mathcal{S}$ is the intersection of the hyperplane $\Sigma_{i=1}^n x_i z_i = n\bar{z}$ with the non-negative orthant. To simplify matters we assume that the z_i are all of the same sign, and there is no loss of generality in taking this to be positive. Results for the case where the $z_i's$ are of mixed signs require only minor modi cations of what follows. We also assume for convenience that the $z_i's$ are distinct.

The manifold $\mathcal{S}$ admits a global coordinate chart, and we can use $x_2, \ldots, x_n$ as coordinates, setting

$$x_1 = z_1^{-1}(n\bar{z} - \Sigma_{i=2}^n x_i z_i). \tag{9}$$

We then have (from Hillier and Armstrong (1996), equation (A3)):

$$(d\mathcal{S}) = z_1^{-1}[\Sigma_{i=1}^n z_i^2]^{1/2} dx_2 dx_3 \ldots dx_n,$$

and, in view of (9), the region of integration becomes:

$$\mathcal{R} = \{x_i > 0, i = 2, \ldots, n; \Sigma_{i=2}^n x_i z_i < n\bar{z}\}.$$

To further simplify the integration it will be helpful to lift the term $[\Sigma_{i=1}^n x_i z_i^2]$ in the integrand in (8) into the exponential. This can be done by writing

$$[\Sigma_{i=1}^n x_i z_i^2] \exp\{-\Sigma_{i=1}^n x_i r_i\} = (\partial/\partial w)[\exp\{-\Sigma_{i=1}^n x_i (r_i - w z_i^2)\}]_{w=0},$$

where we have put $r_i = \exp\{(t-\theta)z_i\}, i = 1, \ldots, n$. Substituting for x_1 from (9), and noting that the differentiation (with respect to w) commutes with the integration, the density becomes:

$$pdf_T(t;\theta) = z_1^{-1} \exp\{n(t-\theta)\overline{z}\}$$

$$(\partial/\partial w) \left[\exp\{-(n\overline{z}a_1/z_1)\} \int_R \exp\{-\Sigma_{i=2}^n x_i [a_i z_1 - a_1 z_i]/z_1 dx_2 ... dx_n\right]_{w=0} \tag{10}$$

where we have now set $a_i = r_i - w z_i^2, i = 1, ..., n$ .The essential problem, therefore, is the evaluation of the integral in (10).

De ne

$$d_{si} = (z_s a_i - z_i a_s)/z_s, i = 2, \ldots, n; s < i. \tag{11}$$

The x_2-integral in (10) is (with the term $\exp\{-n\overline{z}a_1/z_1\}$ attached):

$$\exp\{-n\overline{z}a_1/z_1 - \Sigma_{i=3}^n x_i d_{1i}\} \left\{\int_0^{u_2} \exp\{-x_2 d_{12}\} dx_2\right\}, \tag{12}$$

where $u_2 = z_2^{-1}[n\bar{z} - \Sigma_{i=3}^n x_i z_i]$, giving, on integrating out x_2,

$$\exp\{-n\overline{z}a_1/z_1 - \Sigma_{i=3}^n x_i d_{1i}\} d_{12}^{-1} \{1 - \exp(-d_{12}u_2)\}$$

$$= d_{12}^{-1}\{\exp\{-n\overline{z}a_1/z_1 - \Sigma_{i=3}^n x_i d_{1i}\} - \exp\{-n\overline{z}a_2/z_2 - \Sigma_{i=3}^n x_i d_{2i}\}\}, \tag{13}$$

since $a_1/z_1 + d_{12}/z_2 = a_2/z_2$ and $d_{1i} - z_i d_{12}/z_2 = d_{2i}$. The integral is interpreted as zero if $u_2 \leq 0$.

Integrating now with respect to x_3 we have:

$$d_{12}^{-1}\{\exp\{-n\overline{z}a_1/z_1 - \Sigma_{i=4}^n x_i d_{1i}\} d_{13}^{-1} [1 - \exp\{-d_{13}u_3\}]$$

$$- \exp\{-n\overline{z}a_2/z_2 - \Sigma_{i=4}^n x_i d_{2i}\} d_{23}^{-1} [1 - \exp\{-d_{23}u_3\}]\}, \tag{14}$$

where $u_3 = z_3^{-1}[n\bar{z} - \Sigma_{i=4}^n x_i z_i]$. There appear to be four distinct terms here, but identities similar to those below (13) yield:

$$(d_{12}d_{13})^{-1} \exp\{-n\overline{z}a_1/z_1 - \Sigma_{i=4}^n x_i d_{1i}\} - (d_{12}d_{23})^{-1} \exp\{-n\overline{z}a_2/z_2 - \Sigma_{i=4}^n x_i d_{2i}\}$$

$$-[(d_{12}d_{13})^{-1} - (d_{12}d_{23})^{-1}] \exp\{-n\overline{z}a_3/z_3 - \Sigma_{i=4}^n x_i d_{3i}\}. \tag{15}$$

The iterative relation is now clear: the p-th step in the integration yields a linear combination of terms

$$\exp\{-n\overline{z}a_s/z_s - \Sigma_{i=p+1}^n x_i d_{si}\}, s = 1, \ldots, p,$$

in which the coefficients are simply the coefficients at the previous step multiplied by $d_{1p}^{-1}, d_{2p}^{-1}, \ldots, d_{p-1,p}^{-1}$, respectively, except for the last term ($s = p$), whose coefficient is minus the sum of the coefficients of all lower terms. Thus, if we denote by c_p the $p \times 1$ vector of coefficients of the terms $\exp\{-n\bar{z}a_s/z_s - \Sigma_{i=p+1}^{n} x_i d_{si}\}$ after integrating out x_p, we may write:

$$c_p = L_p c_{p-1},$$

where L_p is the $p \times (p-1)$ matrix:

$$L_p = \begin{bmatrix} d_{1p}^{-1} & 0 & .. & 0 \\ 0 & d_{2p}^{-1} & .. & . \\ . & . & .. & . \\ 0 & 0 & .. & d_{p-1,p}^{-1} \\ -d_{1p}^{-1} & -d_{2p}^{-1} & .. & -d_{p-1,p}^{-1} \end{bmatrix}. \tag{16}$$

After integrating out $x_2 \ldots, x_n$, therefore, we are left with a linear combination of the terms

$$g_n(j) = \exp\{-n\bar{z}a_j/z_j\}, j = 1, \ldots, n, \tag{17}$$

with vector of coefficients, c_n, given by the recursive relation:

$$c_n = L_n L_{n-1} \ldots .. L_2 = \prod_{i=1}^{n-1} L_{n-i+1}, \tag{18}$$

starting with

$$L_2 = \begin{bmatrix} d_{12}^{-1} \\ -d_{12}^{-1} \end{bmatrix}$$

We therefore have a very simple expression for the density:

$$\begin{aligned} pdf_T(t;\theta) &= z_1^{-1} \exp\{n\bar{z}(t-\theta)\}(\partial/\partial w)\,[c_n' g_n]_{w=0} \\ &= z_1^{-1} \exp\{n\bar{z}(t-\theta)\}\,[(\partial c_n/\partial w)' g_n + c_n'(\partial g_n/\partial w)]_{w=0}\,, \end{aligned} \tag{19}$$

where g_n is the $n \times 1$ vector with elements $g_n(j)$ given in (17).

It remains to evaluate the derivatives in (19), and then set $w = 0$. It is easy to see that

$$\partial g_n(j)/\partial w \mid_{w=0} = n\bar{z}z_j \exp\{-n\bar{z}r_j/z_j\} = n\bar{z}z_j \tilde{g}_n(j), \text{say.} \tag{20}$$

De ning $\tilde{L}_p$ as L_p has been de ned above, but with the d_{sp} replaced by $\tilde{d}_{sp} = [z_s r_p - z_p r_s]/z_s$, (since $a_i = r_i$ when $w = 0$), we de ne

$$\tilde{c}_n \equiv c_n \mid_{w=0} = \prod_{i=1}^{n-1} \tilde{L}_{n-i+1}. \tag{21}$$

This deals with the second term in the $[\cdot]$ in (19).

Now, from the de nition of c_n in terms of the L_p in (18) we have that:

$$\partial c_n/\partial w = \sum_{i=1}^{n-1} \left[L_n L_{n-1} \ldots L_{n-i+2} (\partial L_{n-i+1}/\partial w) L_{n-i} L_{n-i-1} \ldots L_2 \right].$$

The matrices

$$\partial L_p/\partial w \mid_{w=0}$$

that occur here have the same structure as the $\tilde{L}_p$ except that the elements $\tilde{d}_{sp}^{-1}$ are replaced by $z_p(z_p - z_s)\tilde{d}_{sp}^{-2}$. Denote these matrices by $\tilde{L}_p^*$, $p = 2, \ldots, n$. Then clearly

$$\partial c_n/\partial w \mid_{w=0} = \sum_{i=1}^{n-1} [\tilde{L}_n \ldots \tilde{L}_{n-i+2} \tilde{L}_{n-i+1}^* \tilde{L}_{n-i} \ldots \tilde{L}_2] = \tilde{c}_n^*, \text{ say.}$$

Hence we nally have an expression for the density in the form:

$$pdf_T(t;\theta) = z_1^{-1} \exp\{n\bar{z}(t-\theta)\} \sum_{j=1}^{n} \tilde{g}_n(j) \{\tilde{c}_n^*(j) + n\bar{z} z_j \tilde{c}_n(j)\}. \tag{22}$$

Unfortunately, because the vectors $\tilde{c}_n^*$ and $\tilde{c}_n$ are both de ned only by recursive formulae, it is difficult to study the properties of the density (22) analytically. In Section 5 below we brie y summarize some results obtained by direct numerical evaluation of (22).

4 INCLUSION OF A CONSTANT TERM.

Suppose now that $E(x_i) = \exp\{\alpha + \theta z_i\}$, so that $w_i = (1, z_i)'$ in the notation used in the Introduction. We denote the (xed values of) the MLE s for (α, θ) by (a, t). From equation (7) we have:

$$pdf_{A,T}(a,t;\alpha,\theta) = \lambda_0^n \exp\{n(t-\theta)\bar{z}\} \mid \Sigma_{i=1}^n \begin{bmatrix} 1 \\ z_i \end{bmatrix} \begin{bmatrix} 1 \\ z_i \end{bmatrix}' \mid^{-1/2}$$

$$\int_{\mathcal{S}} \mid \Sigma_{i=1}^n x_i \begin{bmatrix} 1 \\ z_i \end{bmatrix} \begin{bmatrix} 1 \\ z_i \end{bmatrix}' \mid \exp\{-\lambda_0 \Sigma_{i=1}^n x_i r_i\}(d\mathcal{S}) \tag{23}$$

where $\lambda_0 = \exp(a - \alpha)$, and we have again put $r_i = \exp\{(t-\theta)z_i\}, i = 1, \ldots n$. The integral is now over the surface $\mathcal{S}$ de ned by:

$$\Sigma_{i=1}^n x_i = n, \text{ and } \Sigma_{i=1}^n x_i z_i = n\bar{z},$$

and $x_i > 0, i = 1, \ldots$,n. In what follows we assume that the $z_i's$ are distinct, and are ordered so that $z_1 < z_2 < \ldots < z_n$. It is clear from (23) that the density is invariant to the order of the $z_i's$, so the assumption that the $z_i's$ are ordered is not restrictive. The assumption that the $z_i's$ are distinct is restrictive, but unlikely to be important in practice.

We first choose $n-2$ coordinates for the surface $\mathcal{S}$, and for this purpose it will be convenient to use $x_2, \ldots, x_{n-1}$. Writing x_1 and x_n in terms of $x_2, \ldots, x_{n-1}$ we have:

$$x_1 = n(z_n - \bar{z})/(z_n - z_1) - \Sigma_{i=2}^{n-1} x_i[(z_n - z_i)/(z_n - z_1)] \tag{24}$$

$$x_n = n(\bar{z} - z_1)/(z_n - z_1) - \Sigma_{i=2}^{n-1} x_i[(z_i - z_1)/(z_n - z_1)] \tag{25}$$

Note that the constants, and the coefficients of the x_i, in both of these expressions are all positive, because of our ordering of the z_i's.

With these coordinates the integral (23) becomes an ordinary integral over $x_2, \ldots, x_{n-1}$, and the volume element becomes:

$$(d\mathcal{S}) = [(z_n - z_1)]^{-1} \mid \Sigma_{i=1}^{n} \begin{bmatrix} 1 \\ z_i \end{bmatrix} \begin{bmatrix} 1 \\ z_i \end{bmatrix}' \mid^{1/2} (dx_2 dx_3 \ldots dx_{n-1}) \tag{26}$$

Because x_1 and x_n in (24) and (25) must be positive, the region of integration becomes that part of the non-negative orthant (for $x_2, \ldots, x_{n-1}$) within which:

$$\Sigma_{i=2}^{n-1} x_i(z_n - z_i) < n(z_n - \bar{z}) \text{ and } \Sigma_{i=2}^{n-1} x_i(z_i - z_1) < n(\bar{z} - z_1). \tag{27}$$

That is, the region of integration becomes the subset, $\mathcal{R}$ say, of the $(n-2)$-dimensional non-negative orthant below both of the hyperplanes defined by replacing the inequalities in (27) by equalities. It is straightforward to check that the two hyperplanes involved must intersect, so neither lies entirely below the other. This obviously complicates the integration problem to be dealt with.

We first set out a notation that will be helpful in the evaluation of the integral in (23). First, we define $b_i = z_i - \bar{z}, i = 1, \ldots, n$, and $b_{jk} = z_j - z_k, j, k = 1, \ldots, n, j \neq k$. Note that b_{jk} will be positive for $j > k$ because of the ordering of the z_i, and that the b_i will necessarily be negative for i less than some integer p, say, $(1 \leq p < n)$, and positive thereafter. This property of the $b_i's$ will be important in what follows. The following identities, easily derived from the definitions of the b_i and b_{jk}, will be used repeatedly in what follows to combine products of terms:

$$b_r b_{is} + b_s b_{ri} - b_i b_{rs} = 0 \tag{28}$$

$$b_{ir} b_{js} - b_{ij} b_{rs} - b_{is} b_{jr} = 0 \tag{29}$$

Next we define

$$\ell_{rs} = nb_s + \Sigma_{i=r}^{n-1} x_i b_{is} \; (r = 2, \ldots, n-2, s < r), \tag{30}$$

and

$$\bar{\ell}_r = nb_n - \Sigma_{i=r}^{n-1} x_i b_{ni}. \tag{31}$$

Substituting for x_1 and x_n from (24) and (25) into the determinantal factor in the integrand of (23) we find that:

$$\left| \Sigma_{i=1}^{n} x_i \begin{bmatrix} 1 \\ z_i \end{bmatrix} \begin{bmatrix} 1 \\ z_i \end{bmatrix}' \right| = n^2 \{(b_n z_1^2 - b_1 z_n^2)/b_{n1} - \bar{z}^2\}$$

$$+n\Sigma_{i=2}^{n-1}x_i\{b_{n1}z_i^2 - b_{ni}z_1^2 - b_{i1}z_n^2\}/b_{n1} \tag{32}$$

Note that this is linear in $x_2, \ldots, x_{n-1}$, *not* quadratic.

As before, it will be helpful to lift the determinantal factor (32) into the exponential term in the integrand. We thus write the integrand in the form:

$$n\lambda_0^{-1}[\partial/\partial w][\exp\{-n\lambda_0 w\bar{z}^2\} \times \exp\{-n\lambda_0([b_n a_1 - b_1 a_n]/b_{n1}) - \lambda_0\Sigma_{i=2}^{n-1}x_i d_{n1i}\}]_{w=0}$$

where we have de ned

$$d_{ijk} = [a_k b_{ij} - a_i b_{kj} - a_j b_{ik}]/b_{ij}, \tag{33}$$

and

$$g_{ij} = [b_i a_j - b_j a_i]/b_{ij}, \tag{34}$$

with

$$a_i = (r_i - wz_i^2), i = 1, \ldots, n. \tag{35}$$

Assuming that differentiation with respect to w commutes with the integration, we therefore have:

$$pdf_{A,T}(a, t; \alpha, \theta) = n\lambda_0^{n-1}\exp\{n(t-\theta)\overline{z}\}[b_{n1}]^{-1}(\partial/\partial w)[\exp\{-n\lambda_0 w\bar{z}^2\}$$

$$\exp\{-n\lambda_0 g_{n1}\}\int_{\mathcal{R}}\exp\{-n\lambda_0\Sigma_{i=2}^{n-1}x_i d_{n1i}\}dx_2 \ldots .dx_{n-1}]_{w=0}. \tag{36}$$

Our rst task is therefore to evaluate the integral in the last line of (36), though in what follows we shall include the term $\exp\{-n\lambda_0 g_{n1}\}$ in the derivations of the results, because this will facilitate their simpli cation as we proceed.

Consider now the integral with respect to x_2. From (27), the range of x_2 is restricted by

$$x_2 < [nb_n - \Sigma_{i=3}^{n-1}x_i b_{ni}]/b_{n2} = \bar{\ell}_3/b_{n2},$$

and

$$x_2 < [-nb_1 - \Sigma_{i=3}^{n-1}x_i b_{i1}]/b_{21} = -\ell_{31}/b_{21},$$

Now, the difference between these two upper bounds is

$$\bar{\ell}_3/b_{n2} + \ell_{31}/b_{21} = (b_{n1}/(b_{n2}b_{21}))[nb_2 + \Sigma_{i=3}^{n-1}x_i b_{i2}]$$

$$= (b_{n1}/(b_{n2}b_{21}))\ell_{32}. \tag{37}$$

For $x_3, \ldots, x_{n-1}$ such that $\ell_{32} > 0$ and $\ell_{31} < 0$, the x_2-integral is over the interval $(0, -\ell_{31}/b_{21})$, while for $x_3, \ldots, x_{n-1}$ such that $\ell_{32} < 0$ and $\bar{\ell}_3 > 0$, it is over the interval $(0, \bar{\ell}_3/b_{n2})$. Notice that, because all coefficients other than b_2 in ℓ_{32} are positive, if $b_2 = z_2 - \bar{z} > 0$ the only possibility is $\ell_{32} > 0$. However, assume for the moment that $b_2 < 0$, so that ℓ_{32} can be either positive or negative. Since the regions of subsequent integration with respect to $x_3, \ldots, x_{n-1}$ are disjoint, the integral with respect to x_2 may be expressed as a sum of two terms, each to be subsequently integrated over *different* regions for $x_3, \ldots, x_{n-1}$. The result of integrating out x_2 is thus the sum of two terms:

$$[\lambda_0 d_{n12}]^{-1} \exp\{-\lambda_0[ng_{n1} + \Sigma_{i=3}^{n-1} x_i d_{n1i}]\}[1 - \exp(\lambda_0 d_{n12}\ell_{31}/b_{31})],$$

to be integrated over the region $\{\ell_{32} > 0, \bar{\ell}_3 < 0\}$, and

$$[\lambda_0 d_{n12}]^{-1} \exp\{-\lambda_0[ng_{n1} + \Sigma_{i=3}^{n-1} x_i d_{n1i}]\}[1 - \exp(-\lambda_0 d_{n12}\bar{\ell}_3/b_{n2})],$$

to be integrated over the region $\{\ell_{32} < 0, \bar{\ell}_3 > 0\}$.

After some tedious algebra the two results above for the x_2-integral become (apart from the factor $[\lambda_0 d_{n12}]^{-1}$):

$$\left[\exp\{-\lambda_0[ng_{n1} + \Sigma_{i=3}^{n-1} x_i d_{n1i}]\} - \exp\{-\lambda_0[ng_{21} + \Sigma_{i=3}^{n-1} x_i d_{21i}]\}\right], \tag{38}$$

to be integrated over $\{\ell_{32} > 0, \ell_{31} < 0\}$, plus

$$\left[\exp\{-\lambda_0[ng_{n1} + \Sigma_{i=3}^{n-1} x_i d_{n1i}]\} - \exp\{-\lambda_0[ng_{n2} + \Sigma_{i=3}^{n-1} x_i d_{n2i}]\}\right], \tag{39}$$

to be integrated over $\{\ell_{32} < 0, \bar{\ell}_3 > 0\}$.

Now, the rst terms in (38) and (39) are the same, but are to be integrated over two disjoint regions. When added, therefore, the integral of this term will be over the union of the regions $\{\ell_{32} > 0, \ell_{31} > 0\}$ and $\{\ell_{32} < 0, \bar{\ell}_3 > 0\}$, i.e., over the region $\{\ell_{31} > 0, \bar{\ell}_3 > 0\}$. The result of the x_2-integration is therefore a sum of three terms, each to be integrated over a different region of $(x_3, \ldots, x_{n-1})$-space. These are (again apart from the term $[\lambda_0 d_{n12}]^{-1}$), together with their respective regions of subsequent integration:

$$+\exp\{-\lambda_0[ng_{n1} + \Sigma_{i=3}^{n-1} x_i d_{n1i}]\}, \qquad (\ell_{31} < 0, \bar{\ell}_3 > 0); \tag{40}$$

$$-\exp\{-\lambda_0[ng_{n2} + \Sigma_{i=3}^{n-1} x_i d_{n2i}]\}, \qquad (\ell_{32} < 0, \bar{\ell}_3 > 0); \tag{41}$$

$$-\exp\{-\lambda_0[ng_{21} + \Sigma_{i=3}^{n-1} x_i d_{21i}]\}, \qquad (\ell_{32} > 0, \ell_{31} < 0). \tag{42}$$

Note that if ℓ_{32} cannot be negative (i.e., if $z_2 > \bar{z}$), the second term here is missing.

The nal form of the result we seek is certainly not yet apparent, so we need to proceed to integrate out x_3 in the same way. To do so we need to deal with the three terms in (40) to (42) separately, since each has a different region of integration for $(x_3, \ldots, x_{n-1})$. Proceeding as above for the x_2-integration, after isolating x_3 each region gives rise to a sum of two x_3-integrals, and each of these yields two distinct terms. At this stage we have:
from (40)

$+[\lambda_0 d_{n13}]^{-1}$ multiplied by

$$\exp\{-\lambda_0[ng_{n1} + \Sigma_{i=4}^{n-1} x_i d_{n1i}]\}\,[1 - \exp(\lambda_0 d_{n13}\ell_{41}/b_{31})] \qquad (\ell_{43} > 0, \ell_{41} < 0)$$

plus

$$\exp\{-\lambda_0[ng_{n1} + \Sigma_{i=4}^{n-1} x_i d_{n1i}]\} \left[1 - \exp(-\lambda_0 d_{n13}\overline{\ell}_4/b_{n3})\right] \qquad (\ell_{43} < 0, \bar{\ell}_4 > 0)$$

from (41)

$-[\lambda_0 d_{n23}]^{-1}$ multiplied by

$$\exp\{-\lambda_0[ng_{n2} + \Sigma_{i=4}^{n-1} x_i d_{n2i}]\} \left[1 - \exp(\lambda_0 d_{n23}\ell_{42}/b_{32})\right] \qquad (\ell_{43} > 0, \ell_{42} < 0)$$

plus

$$\exp\{-\lambda_0[ng_{n2} + \Sigma_{i=4}^{n-1} x_i d_{n2i}]\} \left[1 - \exp(-\lambda_0 d_{n23}\overline{\ell}_4/b_{n3})\right] \qquad (\ell_{43} < 0, \bar{\ell}_4 > 0)$$

from (42)

$-[\lambda_0 d_{213}]^{-1}$ multiplied by

$$\exp\{-\lambda_0[ng_{21} + \Sigma_{i=4}^{n-1} x_i d_{21i}]\} \left[\exp(\lambda_0 d_{213}\ell_{42}/b_{32}) - \exp(\lambda_0 d_{213}\ell_{41}/b_{31})\right]$$

$$(\ell_{43} > 0, \ell_{42} < 0)$$

plus

$$\exp\{-\lambda_0[ng_{21} + \Sigma_{i=4}^{n-1} x_i d_{21i}]\} \left[1 - \exp(\lambda_0 d_{213}\ell_{41}/b_{31})\right] \qquad (\ell_{41} < 0, \ell_{42} > 0)$$

From the rst of these sets of results we get three terms (ignoring the factor $[\lambda_0]^{-1}$ which occurs in all terms):

d_{n13}^{-1} multiplied by:

$$+\exp\{-\lambda_0[ng_{n1} + \Sigma_{i=4}^{n-1} x_i d_{n1i}]\} \qquad (\ell_{41} < 0, \bar{\ell}_4 > 0) \qquad (43)$$

$$-\exp\{-\lambda_0[ng_{31} + \Sigma_{i=4}^{n-1} x_i d_{31i}]\} \qquad (\ell_{43} > 0, \ell_{41} < 0) \qquad (44)$$

$$-\exp\{-\lambda_0[ng_{n3} + \Sigma_{i=4}^{n-1} x_i d_{n3i}]\} \qquad (\ell_{43} < 0, \bar{\ell}_4 > 0) \qquad (45)$$

From the second group we get:

$-d_{n23}^{-1}$ multiplied by:

$$+\exp\{-\lambda_0[ng_{n2} + \Sigma_{i=4}^{n-1} x_i d_{n2i}]\} \qquad (\ell_{42} < 0, \bar{\ell}_4 > 0) \qquad (46)$$

$$-\exp\{-\lambda_0[ng_{32} + \Sigma_{i=4}^{n-1} x_i d_{32i}]\} \qquad (\ell_{43} > 0, \ell_{42} < 0) \qquad (47)$$

$$-\exp\{-\lambda_0[ng_{n3} + \Sigma_{i=4}^{n-1} x_i d_{n3i}]\} \qquad (\ell_{43} < 0, \bar{\ell}_4 > 0) \qquad (48)$$

Finally, from the third group we get:

$-d_{213}^{-1}$ multiplied by:

$$+\exp\{-\lambda_0[ng_{21} + \Sigma_{i=4}^{n-1} x_i d_{21i}]\} \qquad (\ell_{41} < 0, \ell_{42} > 0) \qquad (49)$$

$$+\exp\{-\lambda_0[ng_{32} + \Sigma_{i=4}^{n-1} x_i d_{32i}]\} \qquad (\ell_{43} > 0, \ell_{42} < 0) \qquad (50)$$

$$-\exp\{-\lambda_0[ng_{31} + \Sigma_{i=4}^{n-1} x_i d_{31i}]\} \qquad (\ell_{43} > 0, \ell_{41} < 0) \qquad (51)$$

To simplify the summary of these results, write

$$f_{ns}^{(k)} = \exp\{-\lambda_0[ng_{ns} + \Sigma_{i=k}^{n-1} x_i d_{nsi}]\}, \quad k = 3, \ldots, n-1, s < k; \tag{52}$$

and, for $j > s$,

$$f_{js}^{(k)} = \exp\{-\lambda_0[ng_{js} + \Sigma_{i=k}^{n-1} x_i d_{jsi}]\}, \quad k = 3, \ldots, n-1, s < k. \tag{53}$$

From (40) - (42), the result after integrating out x_2 may, with this notation, be expressed as:

$$[\lambda_0 d_{n12}]^{-1}\{-f_{21}^{(3)} + f_{n1}^{(3)} - f_{n2}^{(3)}\}, \tag{54}$$

with respective regions of integration: for $f_{n1}^{(3)}$, $\{\ell_{31} < 0, \bar{\ell}_3 > 0\}$, for $f_{n2}^{(3)}, \{\ell_{32} < 0, \bar{\ell}_3 > 0\}$, and for $f_{21}^{(3)}, \{\ell_{32} > 0, \ell_{31} < 0\}$. Likewise, from (43)-(51), the result after integrating out x_3 may be expressed in the form:

$$[\lambda_0^2 d_{n12}]^{-1}\{d_{n13}^{-1} f_{n1}^{(4)} - d_{n23}^{-1} f_{n2}^{(4)} - [d_{n13}^{-1} - d_{n23}^{-1}] f_{n3}^{(4)} - d_{213}^{-1} f_{21}^{(4)}$$
$$-[d_{n13}^{-1} - d_{213}^{-1}] f_{31}^{(4)} - [d_{213}^{-1} - d_{n23}^{-1}] f_{32}^{(4)}\}, \tag{55}$$

with subsequent regions of integration:

for $f_{ns}^{(4)}$, $\{\ell_{4s} < 0, \bar{\ell}_4 > 0\}$, $s = 1, 2, 3$;
for $f_{js}^{(4)}$, $\{\ell_{4j} > 0, \ell_{4s} < 0\}$, $j = 2, 3$, $s < j$.

Evidently, so long as $(k-1) < p$ (recall that p is the rst value of i for which $b_i = z_i - \bar{z} > 0$), the result of integrating out $x_2, \ldots, x_{k-1}$ will be a linear combination of $k(k-1)/2$ terms, the $(k-1)$ terms $f_{ns}^{(k)}$, $s = 1, \ldots, k-1$, together with the $(k-1)(k-2)/2$ terms $f_{js}^{(k)}$, $j = 2, \ldots, (k-1)$, $s = 1, \ldots, (j-1)$, each term to be subsequently integrated over a different region for $(x_k, \ldots, x_{n-1})$. As in the one-parameter case, the coefficients in this linear combination can be generated recursively. To deduce the transition rules for the recursion, write the result of integrating out $x_2, \ldots, x_{k-1}$ (assuming $k-1 < p$) in the form:

$$\sum_{s=1}^{k-1} a_{ns}^{(k)} f_{ns}^{(k)} + \sum_{r=2}^{k-1}\sum_{s=1}^{r-1} a_{rs}^{(k)} f_{rs}^{(k)}, \tag{56}$$

with regions of subsequent integration:

$$\text{for } f_{ns}^{(k)} : (\ell_{ks} < 0, \bar{\ell}_k > 0)$$

$$\text{for } f_{rs}^{(k)} : (\ell_{kr} > 0, \ell_{ks} < 0).$$

We proceed now to integrate out x_k. There are two cases to consider: (i) the case $k < p$, and (ii) the case $k = p$.

Case (i): $k < p$

We begin with the rst sum in (56). Isolating x_k in the inequalities $(\ell_{ks} < 0, \bar{\ell}_k > 0)$, we nd that x_k must satisfy both of the inequalities:

$$x_k < -\ell_{k+1,s}/b_{ks} \text{ and } x_k < \bar{\ell}_{k+1}/b_{nk}.$$

Using the identities (28) and (29), the difference between these upper bounds is

$$\bar{\ell}_{k+1}/b_{nk} + \ell_{k+1,s}/b_{ks} = (b_{ns}/(b_{nk}b_{ks}))\ell_{k+1,k}.$$

The x_k-integral of an $f_{ns}^{(k)}$ term in (56) therefore splits into a sum of two terms (each multiplied by $f_{ns}^{(k+1)}$):

$$\int_0^{-\ell_{k+1,s}/b_{ks}} \exp(-\lambda_0 x_k d_{nsk}) dx_k$$
$$= [\lambda_0 d_{nsk}]^{-1} [1 - \exp(\lambda_0 d_{nsk}\ell_{k+1,s}/b_{ks})] \quad (\text{if } \ell_{k+1,k} > 0, \quad \ell_{k+1,s} < 0), \tag{57}$$

plus

$$\int_0^{\bar{\ell}_{k+1}/b_{nk}} \exp(-\lambda_0 x_k d_{nsk}) dx_k$$
$$= [\lambda_0 d_{nsk}]^{-1} \left[1 - \exp(\lambda_0 d_{nsk}\bar{\ell}_{k+1}/b_{nk})\right] \quad (\text{if } \ell_{k+1,k} < 0, \quad \bar{\ell}_{k+1} > 0). \tag{58}$$

As before, the sum of the two equal terms $f_{ns}^{(k+1)}$ to be integrated over the disjoint regions $(\ell_{k+1,k} > 0, \ell_{k+1,s} < 0)$ and $(\ell_{k+1,k} < 0, \bar{\ell}_{k+1} > 0)$ is simply the integral over the union of those regions, i.e., over the region $(\ell_{k+1,s} < 0, \bar{\ell}_{k+1} > 0)$. Using the identities (28) and (29) again, we see that

$$f_{ns}^{(k+1)} \exp(\lambda_0 \ell_{k+1,s} d_{nsk}/b_{ks}) = f_{ks}^{(k+1)},$$

and

$$f_{ns}^{(k+1)} \exp(-\lambda_0 \bar{\ell}_{k+1} d_{nsk}/b_{nk}) = f_{nk}^{(k+1)}.$$

Hence, integration of the rst sum in (56) yields the sum:

$$\lambda_0^{-1} \sum_{s=1}^{k-1} [a_{nsk}/d_{nsk}]\{f_{ns}^{(k+1)} - f_{ks}^{(k+1)} - f_{nk}^{(k+1)}\}, \tag{59}$$

with regions of subsequent integration:

$$\text{for } f_{ns}^{(k+1)} : \{\ell_{k+1,s} < 0, \bar{\ell}_{k+1} > 0\},$$
$$\text{for } f_{ks}^{(k+1)} : \{\ell_{k+1,k} > 0, \ell_{k+1,s} < 0\},$$
$$\text{for } f_{nk}^{(k+1)} : \{\ell_{k+1,k} < 0, \bar{\ell}_{k+1} > 0\}.$$

Consider now the second sum in (56). Isolating x_k in the inequalities $(\ell_{kr} > 0, \ell_{ks} < 0)$ gives:

$$x_k > -\ell_{k+1,r}/b_{kr} \text{ and } x_k < -\ell_{k+1,s}/b_{ks}.$$

The difference between the upper and lower limits is

$$\ell_{k+1,r}/b_{kr} - \ell_{k+1,s}/b_{ks} = (b_{rs}/(b_{kr}b_{ks}))\ell_{k+1,k}.$$

The integral vanishes, of course, if this is non-positive, i.e., if $\ell_{k+1,k} \le 0$. We again get a sum of two terms (each to be multiplied by $f_{rs}^{(k+1)}$):

$$\int_{-\ell_{k+1,r}/b_{kr}}^{-\ell_{k+1,s}/b_{ks}} \exp(-\lambda_0 x_k d_{rsk}) dx_k$$

$$= [\lambda_0 d_{rsk}]^{-1} \left[\exp(\lambda_0 \ell_{k+1,r} d_{rsk}/b_{kr}) - \exp(\lambda_0 \ell_{k+1,s} d_{rsk}/b_{ks})\right]$$

$$(\text{if } \ell_{k+1,r} < 0, \quad \ell_{k+1,k} > 0) \tag{60}$$

plus

$$\int_{0}^{-\ell_{k+1,s}/b_{ks}} \exp(-\lambda_0 x_k d_{rsk}) dx_k$$

$$= [\lambda_0 d_{rsk}]^{-1} \left[1 - \exp(\lambda_0 \ell_{k+1,s} d_{rsk}/b_{ks})\right]$$

$$(\text{if } \ell_{k+1,r} < 0, \quad \ell_{k+1,s} < 0) \tag{61}$$

Again using the identities (28) and (29) we nd that:

$$f_{rs}^{(k+1)} \exp(\lambda_0 \ell_{k+1,r} d_{rsk}/b_{kr}) = f_{kr}^{(k+1)},$$

and

$$f_{rs}^{(k+1)} \exp(\lambda_0 \ell_{k+1,s} d_{rsk}/b_{ks}) = f_{ks}^{(k+1)}.$$

The two equal terms in (60) and (61) combine as usual to give the term $f_{ks}^{(k+1)}$, to be integrated over $\{\ell_{k+1,k} > 0, \quad \ell_{k+1,s} < 0\}$. Hence the second sum in (56) becomes, after integrating out x_k,

$$\lambda_0^{-1} \sum_{r=2}^{k-1} \sum_{s=1}^{r-1} \left[a_{rs}^{(k)}/d_{rsk}\right] \{f_{rs}^{(k+1)} + f_{kr}^{(k+1)} - f_{ks}^{(k+1)}\} \tag{62}$$

with regions of subsequent integration:

$$\text{for } f_{rs}^{(k+1)} : \{\ell_{k+1,r} > 0, \quad \ell_{k+1,s} < 0\},$$

$$\text{for } f_{kr}^{(k+1)} : \{\ell_{k+1,r} < 0, \quad \ell_{k+1,k} > 0\},$$

$$\text{for } f_{ks}^{(k+1)} : \{\ell_{k+1,s} < 0, \quad \ell_{k+1,k} > 0\}.$$

Case (ii): $k = p$

In the case $k = p$, $\ell_{p+1,p}$ cannot be negative, so only (57) occurs when x_p is integrated out in the rst term of (56). We therefore get, in place of (59)

$$\lambda_0^{-1} \sum_{s=1}^{p-1} \left[a_{ns}^{(p)}/d_{nsp}\right] \{f_{ns}^{(p+1)} - f_{ps}^{(p+1)}\}, \tag{63}$$

with both terms to be integrated over the region (for $x_{p+1}, \ldots, x_{n-1}$) determined by the single condition $\ell_{p+1,s} < 0$. Notice that the term $f_{np}^{(p+1)}$ does not appear in (63).

Turning to the second term in (56), equations (60) and (61) apply with $k = p$, but the condition $\ell_{p+1,p} > 0$ in (60) is automatically satis ed. Hence the only change needed for this case is that the regions of subsequent integration of the terms $f_{pr}^{(p+1)}$ and $f_{ps}^{(p+1)}$ in (62) are determined by the single inequalities $\ell_{p+1,r} < 0$ and $\ell_{p+1,s} < 0$, respectively.

Combining these results, the result of integrating out x_k in (56) is, apart from the factor $[\lambda_0]^{-1}$,
For case (i): $k < p$

$$\sum_{s=1}^{k-1} \left[a_{ns}^{(k)}/d_{nsk}\right] \{f_{ns}^{(k+1)} - f_{ks}^{(k+1)} - f_{nk}^{(k+1)}\}$$

$$+\sum_{r=2}^{k-1}\sum_{s=1}^{r-1} \left[a_{rs}^{(k)}/d_{rsk}\right] \{f_{rs}^{(k+1)} + f_{kr}^{(k+1)} - f_{ks}^{(k+1)}\} \tag{64}$$

For case (ii): $k = p$

$$\sum_{s=1}^{p-1} \left[a_{ns}^{(p)}/d_{nsp}\right] \{f_{ns}^{(p+1)} - f_{ps}^{(p+1)}\}$$

$$+\sum_{r=2}^{p-1}\sum_{s=1}^{r-1} \left[a_{rs}^{(p)}/d_{rsp}\right] \{f_{rs}^{(p+1)} + f_{pr}^{(p+1)} - f_{ps}^{(p+1)}\}. \tag{65}$$

Identifying (64) with the analogue of (56):

$$\sum_{s=1}^{k} a_{ns}^{(k+1)} f_{ns}^{(k+1)} + \sum_{r=2}^{k}\sum_{s=1}^{r-1} a_{rs}^{(k+1)} f_{rs}^{(k+1)} \tag{66}$$

the new terms are $f_{nk}^{(k+1)}$(in the rst sum) and $f_{k1}^{(k+1)}, \ldots, f_{k,k-1}^{(k+1)}$, (in the second), a total of k new terms. Now, for $s = 1, \ldots, k-1$, $f_{ns}^{(k+1)}$ occurs only in the rst line of (64). Hence, if $k < p$,

$$a_{ns}^{(k+1)} = a_{ns}^{(k)}/d_{nsk}, \quad s = 1, \ldots, (k-1). \tag{67}$$

Also, the term $f_{nk}^{(k+1)}$ occurs only in the rst line of (64), so that

$$a_{nk}^{(k+1)} = -\sum_{s=1}^{k-1} \left[a_{ns}^{(k)}/d_{nsk}\right] \tag{68}$$

In the second line of (64), the terms $f_{rs}^{(k+1)}$ with $r < k$ occur with coefficients $a_{rs}^{(k)}/d_{rsk}$, so that

$$a_{rs}^{(k+1)} = a_{rs}^{(k)}/d_{rsk}, r = 2, \ldots, (k-1); \quad s = 1, \ldots, (r-1). \tag{69}$$

The new terms $f_{kj}^{(k+1)}, j = 1, \ldots, k-1$, occur in the rst line of (64) with coefficients $-a_{ns}^{(k)}/d_{nsk}$, and twice in the second line of (64), in the second term with coefficients

$$+\sum_{s=1}^{j-1}\left[a_{js}^{(k)}/d_{jsk}\right], \quad (j>1)$$

and in the third term with coefficients

$$-\sum_{r=j+1}^{k-1}\left[a_{rj}^{(k)}/d_{rjk}\right], \quad (j<k-1).$$

Hence the coefficients of the terms $f_{kj}^{(k+1)}$ in (64) are:

$$a_{k1}^{(k+1)} = -a_{n1}^{(k)}/d_{n1k} - \sum_{r=2}^{k-1}\left[a_{r1}^{(k)}/d_{r1k}\right];$$

$$a_{kj}^{(k+1)} = -a_{nj}^{(k)}/d_{njk} + \sum_{s=1}^{j-1}\left[a_{js}^{(k)}/d_{jsk}\right] - \sum_{r=j+1}^{k-1}\left[a_{rj}^{(k)}/d_{rjk}\right], \quad j=2,\ldots,k-2;$$

$$a_{k,k-1}^{(k+1)} = -a_{n,k-1}^{(k)}/d_{n,k-1,k} + \sum_{s=1}^{k-2}\left[a_{k-1,s}^{(k)}/d_{k-1,s,k}\right]. \tag{70}$$

Equations (67) - (70) specify the recursive relations between the coefficients in equation (56) up to the integration with respect to x_{p-1}. For the next step, integration with respect to x_p, we need to identify (65) with (66) (with k replaced by p). This gives:

$$a_{ns}^{(p+1)} = a_{ns}^{(p)}/d_{nsp}, \quad s=1,\ldots,p-1; \tag{71}$$

$$a_{np}^{(p+1)} = 0; \tag{72}$$

$$a_{rs}^{(p+1)} = a_{rs}^{(p)}/d_{rsp}, \quad r=2,\ldots,p-1; \quad s=1,\ldots,r-1; \tag{73}$$

$$a_{p1}^{(p+1)} = -a_{n1}^{(p)}/d_{n1p} - \sum_{r=2}^{p-1}\left[a_{r1}^{(p)}/d_{r1p}\right]$$

$$a_{pj}^{(p+1)} = -a_{nj}^{(p)}/d_{njp} + \sum_{s=1}^{j-1}\left[a_{js}^{(p)}/d_{jsp}\right] - \sum_{r=j+1}^{p-1}\left[a_{rj}^{(p)}/d_{rjp}\right], \quad j=2,\ldots,p-2$$

$$a_{p,p-1}^{(p+1)} = -a_{n,p-1}^{(p)}/d_{n,p-1,p} + \sum_{s=1}^{p-2}\left[a_{p-1,s}^{(p)}/d_{p-1,s,p}\right] \tag{74}$$

Thus, after integrating out x_p we shall have an expression:

$$\sum_{s=1}^{p-1} a_{ns}^{(p+1)} f_{ns}^{(p+1)} + \sum_{r=2}^{p-1}\sum_{s=1}^{r-1} a_{rs}^{(p+1)} f_{rs}^{(p+1)} + \sum_{s=1}^{p-1} a_{ps}^{(p+1)} f_{ps}^{(p+1)} \tag{75}$$

with regions of subsequent integration:

$$\text{for } f_{ns}^{(p+1)} : \ell_{p+1,s} < 0;$$

$$\text{for } f_{rs}^{(p+1)} : \{\ell_{p+1,r} > 0, \ell_{p+1,s} < 0\} \quad (r \le p-1)$$
$$\text{for } f_{ps}^{(p+1)} : \ell_{p+1,s} < 0, \quad s = 1, \ldots, p-1.$$

In general, after integrating out x_{k-1}, with $p < k-1 < n-1$, we shall have an expression of the form:

$$\sum_{s=1}^{p-1} a_{ns}^{(k)} f_{ns}^{(k)} + \sum_{r=2}^{p-1} \sum_{s=1}^{r-1} a_{rs}^{(k)} f_{rs}^{(k)} + \sum_{r=p}^{k-1} \sum_{s=1}^{p-1} a_{rs}^{(k)} f_{rs}^{(k)} \tag{76}$$

with regions of subsequent integration:

$$\text{for } f_{ns}^{(k)} : \ell_{ks} < 0; \quad s = 1, \ldots, p-1;$$
$$\text{for } f_{rs}^{(k)} : \ell_{kr} > 0, \ell_{ks} < 0; \quad r = 2, \ldots, p-1, \quad s = 1, \ldots, r-1;$$
$$\text{for } f_{rs}^{(k)} : \ell_{ks} < 0; \quad r = p, \ldots, n-2, \quad s = 1, \ldots, p-1.$$

Integration of (76) with respect to x_k then yields (apart from the factor λ_0^{-1}):

$$\begin{aligned} &\sum_{s=1}^{p-1} \left[a_{ns}^{(k)}/d_{nsk}\right] \{f_{ns}^{(k+1)} - f_{ks}^{(k+1)}\} \\ &+ \sum_{r=2}^{p-1} \sum_{s=1}^{r-1} \left[a_{rs}^{(k)}/d_{rsk}\right] \{f_{rs}^{(k+1)} + f_{kr}^{(k+1)} - f_{ks}^{(k+1)}\} \\ &+ \sum_{r=p}^{k-1} \sum_{s=1}^{p-1} \left[a_{rs}^{(k)}/d_{rsk}\right] \{f_{rs}^{(k+1)} - f_{ks}^{(k+1)}\} \end{aligned} \tag{77}$$

Identifying this with the analogue of (76) with $k-1$ replaced by k:

$$\sum_{s=1}^{p-1} a_{ns}^{(k+1)} f_{ns}^{(k+1)} + \sum_{r=2}^{p-1} \sum_{s=1}^{r-1} a_{rs}^{(k+1)} f_{rs}^{(k+1)} + \sum_{r=p}^{k} \sum_{s=1}^{p-1} a_{rs}^{(k+1)} f_{rs}^{(k+1)}, \tag{78}$$

the only additional terms are the $p-1$ terms $f_{ks}^{(k+1)}, \quad s = 1, \ldots, p-1$.

Comparison of (77) with (78) yields the recursive relations for the coefficients for terms beyond the p-th, but with $k < n-1$:

$$a_{ns}^{(k+1)} = a_{ns}^{(k)}/d_{nsk}, \quad s = 1, \ldots, p-1; \tag{79}$$
$$a_{rs}^{(k+1)} = a_{rs}^{(k)}/d_{rsk}, \quad r = 2, \ldots, p-1, \quad s = 1, \ldots, r-1; \tag{80}$$
$$a_{rs}^{(k+1)} = a_{rs}^{(k)}/d_{rsk}, \quad r = p, \ldots, k-1, \quad s = 1, \ldots, p-1; \tag{81}$$
$$a_{k1}^{(k+1)} = -a_{k1}^{(k)}/d_{n1k} - \sum_{r=2}^{p-1} \left[a_{r1}^{(k)}/d_{r1k}\right],$$
$$a_{kj}^{(k+1)} = -a_{kj}^{(k)}/d_{njk} - \sum_{r=j+1}^{p-1} \left[a_{rj}^{(k)}/d_{rjk}\right] + \sum_{s=1}^{j-1} \left[a_{js}^{(k)}/d_{jsk}\right], \quad j = 2, \ldots, p-2$$
$$a_{k,p-1}^{(k+1)} = -a_{k,p-1}^{(k)}/d_{n,p-1,k} + \sum_{s=1}^{p-2} \left[a_{p-1,s}^{(k)}/d_{p-1,s,k}\right],$$

$$a_{ks}^{(k+1)} = 0, \quad s > p-1. \tag{82}$$

Note that each integration with respect to an x_k, with $k > p-1$, adds only $p-1$ terms to the sum, not k.

Consider now the integral with respect to the last variable, x_{n-1}. We assume that $p < n-2$; slight modi cations of what follows are needed in the cases $p = n-1$ or $p = n$, but since these cases are unlikely in practice we omit those details. We need to integrate (78) (with k replaced by $n-2$) with respect to x_{n-1}. For the terms $f_{ns}^{(n-1)}$ the range of integration is $\ell_{n-1,s} < 0$, or $nb_s + x_{n-1}b_{n-1,s} < 0$, so that

$$x_{n-1} < -nb_s/b_{n-1,s}, \quad s = 1, \ldots, p-1.$$

Hence, integration of these terms with respect to x_{n-1} yields the sum:

$$\lambda_0^{-1} \sum_{s=1}^{p-1} \left[a_{ns}^{(n-1)}/d_{ns,n-1}\right] \{f_{ns} - f_{n-1,s}\}, \tag{83}$$

where here and below we set $f_{rs} = \exp\{-n\lambda_0 g_{rs}\}$.

For the second sum in (78), the inequalities $\ell_{n-1,r} > 0$ and $\ell_{n-1,s} > 0$ give $x_{n-1} < -nb_r/b_{n-1,r}$ and $x_{n-1} > -nb_s/b_{n-1,s}$. The difference between the upper and lower limits here is a positive multiple of b_{n-1}, and is thus positive unless $p = n$, which we rule out. Since the lower limit is certainly positive (because $b_s < 0$ when $s < p-1$), integration with respect to x_{n-1} yields:

$$\lambda_0^{-1} \sum_{r=2}^{p-1} \sum_{s=1}^{r-1} \left[a_{rs}^{(n-1)}/d_{rs,n-1}\right] \{f_{n-1,r} - f_{n-1,s}\} \tag{84}$$

Note particularly that (84) yields no terms f_{rs} with $r <$ p.

Finally, the third sum in (78) yields

$$\lambda_0^{-1} \sum_{r=p}^{n-2} \sum_{s=1}^{p-1} \left[a_{rs}^{(n-1)}/d_{rs,n-1}\right] \{f_{rs} - f_{n-1,s}\}. \tag{85}$$

Hence, after integrating out the nal variable x_{n-1} we shall have a linear combination of $(n-p+1)(p-1)$ terms:

$$f_{rs} = \exp\{-n\lambda_0 g_{rs}\}, \quad r = p, \ldots, n, \quad s = 1, \ldots, p-1. \tag{86}$$

If we write this linear combination in the form:

$$\sum_{r=p}^{n} \sum_{s=1}^{p-1} a_{rs} f_{rs}, \tag{87}$$

equations (79)-(85) yield the relations between the a_{rs} and the $a_{rs}^{(n-1)}$:

$$a_{rs} = a_{rs}^{(n-1)}/d_{rs,n-1}, \quad r = p, \ldots, n-2, \quad s = 1, \ldots, p-1; \tag{88}$$

$$a_{n-1,1} = -a_{n1}^{(n-1)}/d_{n1,n-1} - \sum_{r=2}^{n-2} \left[a_{r1}^{(n-1)}/d_{r1,n-1}\right]; \tag{89}$$

$$a_{n-1,s} = -a_{ns}^{(n-1)}/d_{ns,n-1} + \sum_{j=1}^{s-1} \left[a_{sj}^{(n-1)}/d_{sj,n-1}\right] - \sum_{r=s+1}^{n-2} \left[a_{rs}^{(n-1)}/d_{rs,n-1}\right]$$

$$s = 2, \ldots, p-1; \tag{90}$$

$$a_{ns} = a_{ns}^{(n-1)}/d_{ns,n-1}, \quad s = 1, \ldots, p-1; \tag{91}$$

As in the single-parameter case, these recursive relations can be expressed in terms of a product of matrices of increasing dimension. To do so, rst assume that $k < p$, and let c_k be the $[k(k+1)/2] \times 1$ vector of coefficients of the $f_{rs}^{(k+1)}$ after integrating out x_k, with an analogous de nition of c_{k-1}(which, of course, is $[k(k-1)/2] \times 1$). Assume that the elements of c_k are arranged in lexicographic order, i.e., in the order (for $k < p$):

$$((21), (31), (32), \ldots\ldots, (k1), (k2), \ldots, (k, k-1), (n1), \ldots, (nk)).$$

In the case $k \geq p$ only pairs (rs) with $s \leq p-1$ occur. In the transition from c_{k-1} to c_k $(k < p)$ the terms $((k1), \ldots, (k, k-1))$ and (nk) are added. Let L_k denote the $[k(k+1)/2] \times [k(k-1)/2]$ matrix that takes c_{k-1} to $c_k : c_k = L_k c_{k-1}$. From the results in (67) - (70), the structure of the matrix L_k, for $k < p$, is as follows:

$$L_k = \begin{bmatrix} L_{k11} & 0 \\ L_{k21} & L_{k22} \\ 0 & L_{k32} \\ 0 & L_{k42} \end{bmatrix}, \tag{92}$$

where

$$L_{k11} = diag\{d_{21k}^{-1}, d_{31k}^{-1}, d_{32k}^{-1}, \ldots\ldots\ldots, d_{k-1,1k}^{-1}, \ldots\ldots, d_{k-1,k-2,k}^{-1}\}, \tag{93}$$

is a $[(k-1)(k-2)/2] \times [(k-1)(k-2)/2]$ diagonal matrix

$$L_{k32} = \text{diag}\{d_{n1k}^{-1}, \ldots, d_{n,k-1,k}^{-1}\}, \tag{94}$$

is a $(k-1) \times (k-1)$ diagonal matrix, $L_{k22} = -L_{k32}$,

$$L_{k42} = (-d_{n1k}^{-1}, \ldots\ldots, -d_{n,k-1,k}^{-1}) \tag{95}$$

is a $1 \times (k-1)$ vector, and L_{k21} is a $(k-1) \times [(k-1)(k-2)/2]$ matrix with the following structure:

for $j = 1, \ldots, (k-1)$, the non-zero elements in row j are, in their lexicographic positions, the terms $-d_{rjk}^{-1}$ for $r > j$, and the terms d_{jsk}^{-1} for $s = 1, \ldots, j-1$.

Row 3, for instance, is:

$$\{0, d_{31k}^{-1}, d_{32k}^{-1}, 0, 0, -d_{43k}^{-1}, 0, 0, -d_{53k}^{-1}, \ldots\ldots, -d_{k-1,3,k}^{-1}, 0, \ldots, 0\}.$$

For $p \leq k < n-1$ the following modi cations to L_k must be made: (1) the last row is absent; (2) $L_{k32} = \text{diag}\{d_{n1k}^{-1}, \ldots, d_{n,p-1,k}^{-1}\}$ is $(p-1) \times (p-1)$, and hence so is $L_{k22} = -L_{k32}$; (3) in L_{k11}, the diagonal terms d_{rsk}^{-1} for $r > p$ appear only for $s = 1, \ldots, (p-1)$, so that L_{k11} is square of dimension $[p(p-1)/2 + (k-p)(p-1)]$, and, correspondingly; (4) L_{k21} is now $(p-1) \times [p(p-1)/2 + (k-p)(p-1)]$, with the same structure as above except that the d^{-1}_{rsk} that occur are for $r = 1, \ldots, (p-1)$ only. Hence, for $p \leq k < n-1$, L_k is

$$[p(p-1)/2 + (k-p+1)(p-1)] \times [p(p-1)/2 + (k-p+2)(p-1)]$$

Finally, the matrix L_{n-1} is $[(n-p+1)(p-1)] \times [(n-p)(p-1) + p(p-1)/2]$ with the following structure:

$$L_n = \begin{bmatrix} L_{n-1,11} & 0 \\ L_{n-1,21} & L_{n-1,22} \\ 0 & L_{n-1,32} \end{bmatrix}$$

where $L_{n-1,32} = \mathrm{diag}\{d^{-1}_{ns,n-1};\ s = 1, \ldots, p-1\}$ is $(p-1) \times (p-1)$, $L_{n-1,22} = -L_{n-1,32}$, $L_{n-1,21}$ is $(p-1) \times [(n-p-1)(p-1) + p(p-1)/2]$ with the same structure as in the case $p \leq k < n-1$ above, and $L_{n-1,11}$ is $(n-p-1)(p-1) \times [(n-p-1)(p-1) + p(p-1)/2]$ with the form:

$$L_{n-1,11} = \left[0, \mathrm{diag}\{d^{-1}_{rs,n-1}; r = p, .., n-2,\ s = 1, .., p-1\}\right],$$

where the initial block of zeros is $(n-p-1)(p-1) \times [(p-1)(p-2)/2]$.

The nal vector c_{n-1}, of dimension $[(n-p+1)(p-1)] \times 1$, is then given by the recursive formula:

$$c_{n-1} = L_{n-1} L_{n-2} \ldots . L_2, \tag{96}$$

starting with

$$L_2 = d^{-1}_{n12} \begin{bmatrix} -1 \\ +1 \\ -1 \end{bmatrix}, \tag{97}$$

(see (54) above). Letting f_{n-1} denote the vector of functions

$$f_{rs} = \exp\{-n\lambda_0[b_r a_s - b_s a_r]/b_{rs}\},$$

ordered lexicographically as above, we have:

$$pdf_{A,T}(a, t; \alpha, \theta) = (n\lambda_0 b^{-1}_{n1}) \exp\{n\bar{z}(t-\theta)\}(\partial/\partial w) \left[\exp\{-n\lambda_0 w \bar{z}^2\} c'_{n-1} f_{n-1}\right]_{w=0}$$

It remains now to evaluate the differential operator, and set $w = 0$. The results are exactly analogous to those for the single-parameter case given earlier. First we de ne

$$\tilde{c}_{n-1} = c_{n-1} \mid_{w=0} = \tilde{L}_{n-1} \tilde{L}_{n-2} \ldots . \tilde{L}_2, \tag{98}$$

where the $\tilde{L}_p$ are de ned exactly as the L_p are de ned above, but with d_{ijk} replaced by

$$\tilde{d}_{ijk} = [b_{ij} r_k - b_{kj} r_i - b_{ik} r_j]/b_{ij}. \tag{99}$$

Before proceeding we note that, at the point $t = \theta, r_i = 1$ for all i, and $\tilde{d}_{ijk} = 0$, so that the $\tilde{L}_r$ are not de ned at $t = \theta$. The results that follow therefore hold everywhere except at $t = \theta$. At the point $t = \theta$ we have, directly from (23),

$$pdf_{A,T}(a, t = \theta; \alpha, \theta) = \lambda_0^n \exp\{-n\lambda_0\} \times c_n(z), \tag{100}$$

where $c_n(z)$ is a constant. (100) follows from (23) when $t = \theta$ because, in the integrand of (23) the exponential term becomes

$$\exp\{-\lambda_0 \Sigma_{i=1}^n x_i\} = \exp\{-n\lambda_0\}$$

on $\mathcal{S}$ (since, on $\mathcal{S}$, $\Sigma_{i=1}^n x_i = n$). The integral is then a function only of $z = (z_1, \ldots, z_n)'$ and n. Since (100) is proportional to the conditional density of A given that $T = \theta$, which must integrate to one, we also obtain an expression for the density of T at $t = \theta$:

$$pdf_T(t = \theta; \theta) = n^{-n}\Gamma(n)c_n(z) \tag{101}$$

The constant $c_n(z)$ in (100) and (101) can be evaluated by methods like those above, but we omit these details.

Next, let $\tilde{f}_{n-1}$ be de ned as f_{n-1} which has been de ned above, but with the f_{rs} replaced by

$$\tilde{f}_{rs} = \exp\{-n\lambda_0[b_r r_s - b_s r_r]/b_{rs}\}, \tag{102}$$

and de ne $\tilde{f}^*_{n-1}$ to be the vector with elements

$$(\partial/\partial w)[\exp\{-n\lambda_0 w\bar{z}^2\}f_{rs}] \mid_{w=0} = -n\lambda_0 b_r b_s \tilde{f}_{rs}. \tag{103}$$

As before,

$$\partial c_{n-1}/\partial w \mid_{w=0} = \sum_{i=1}^{n-2}[\tilde{L}_{n-1} \ldots \tilde{L}^*_{n-i+1} \ldots \tilde{L}_2] = \tilde{c}^*_{n-1}, \text{ say,} \tag{104}$$

where $\tilde{L}^*_p$ is again de ned as L_p is de ned above but with d^{-1}_{ijk} replaced by

$$[b_{ij}z_k^2 - b_{kj}z_i^2 - b_{ik}z_j^2]/[b_{ij}\tilde{d}^2_{ijk}] = b_{ki}b_{kj}/\tilde{d}^2_{ijk} \tag{105}$$

Combining these results, we have a relatively simple expression for the density of (A, T):

$$pdf_{A,T}(a, t; \alpha, \theta) = \exp\{n\bar{z}(t - \theta)\}(n\lambda_0 b_{n1}^{-1})\left[c^{*\prime}_{n-1}\tilde{f}_{n-1} + \tilde{c}'_{n-1}\tilde{f}^*_{n-1}\right] \tag{106}$$

4.1 The Marginal Density of T

Remarkably, it is straightforward to integrate out a in (106) to obtain the marginal density of T, because the two terms in the $[\cdot]$ in (106) are linear combinations of terms $\exp\{-n\lambda_0 g_{rs}\}$, and $(-nb_r b_s\lambda_0)\exp\{-n\lambda_0 g_{rs}\}$, respectively, with coefficients that do not depend on a. Transforming from $(a-\alpha)$ to $\lambda_0 = \exp\{(a-\alpha)\} > 0$, and integrating out λ_0 we obtain:

$$pdf_T(t;\theta) = \exp\{n\bar{z}(t-\theta)\}b_{n1}^{-1}\left[c_{n-1}^{*\prime}h_{n-1} + \tilde{c}_{n-1}^{\prime}h_{n-1}^{*}\right], \qquad (107)$$

where h_{n-1} has elements g_{rs}^{-1}, and h_{n-1}^{*} has elements $-b_r b_s g_{rs}^{-2}$. Again, the density is de ned by (107) at all points other than $t = \theta$. At $t = \theta$ expression (101) must be used.

5 PROPERTIES OF THE EXACT DENSITIES

Because the coefficients in the exact expressions (22), (106), and (107) are generated recursively it is difficult to study the properties of the densities analytically. However, give a choice for the vector z, it is straightforward to analyse the densities numerically, although in the two-parameter case we did have some difficulty with the numerical stability of the calculations near $t = \theta$ (see below).

5.1 Properties in the case $k = 1$

From the remarks in Section 2, the density depends only on $d = (t-\theta)$, has its mode at $t = \theta$, and the density at d when $z > 0$ is the density at $-d$ with $z < 0$, so that the density with negative z s is simply the density with positive z s re ected about the origin.

It also follows from the remarks in section 2 that there is no loss of generality in scaling the z_i s so that $z'z$, the (expected) Fisher information for θ, is unity, corresponding to an asymptotic variance of one. Figure 1 shows the density, calculated from equation (22), for the case of equispaced positive z s, scaled so that $z'z = 1$, for the cases $n = 2, 4, 8$, and 16. Different patterns of z s produce little change in the graphs. In Table 1 we give the means, variances, and skewness for the cases that appear in Figure 1 (calculated by numerical integration).

TABLE 1
Means, Variances and Skewness for Figure 1

n	mean	variance	skewness
2	-.429	1.386	-.841
4	-.315	1.216	-.628
8	-.227	1.114	-.454
16	-.161	1.058	-.323

Both Figure 1 and Table 1 suggest that the approach to the asymptotic distribution of the MLE is quite rapid, that (with positive z s) the estimator is slightly negatively biased, and that the asymptotic variance ($= 1$ in this example) slightly understates the true variance.

5.2 Properties for the model with a constant term

In view of the remarks in Section 2, the joint density has its mode at the point $(a, t) = (\alpha, \theta)$. Further properties must be derived from the formulae above.

Consider rst the case $n = 3$. In this case no recursion is needed, and we have directly from (54):

$$pdf_{A,T}(a, t; \alpha, \theta) = \exp\{3(t - \theta)\bar{z} - 3\lambda_0 \tilde{g}_{31}\}[3\lambda_0 b_{31}^{-1}] \\ \times \left[-b_{21} b_{32} \phi_2(\tilde{d}_{312}) - \lambda_0 b_1 b_3 \phi_1(\tilde{d}_{312})\right] \quad (108)$$

where

$$\phi_1(y) = [1 - e^{-\gamma y}]/y, \quad (109)$$

and

$$\phi_2(y) = [1 - (1 + \gamma y)e^{-\gamma y}]/y^2. \quad (110)$$

with $\gamma = n\lambda_0 b_3/b_{32}$ if $p = 3$ and $\gamma = -n\lambda_0 b_1/b_{21}$ if $p = 2$. Note that (108) is well-de ned for all (a, t), including $t = \theta$, since both ϕ_1 and ϕ_2 in fact do not involve negative powers of $\tilde{d}_{312}$.

The marginal density of T is readily obtained from (108) by direct integration, giving:

$$pdf_T(t; \theta) = \exp\{3(t - \theta)\bar{z}\} b_{31}^{-1} \\ \times \left[-\beta^2 b_{21} b_{32} \tilde{g}_{31} - b_1 b_3 \beta(6\tilde{g}_{31} + \beta\tilde{d}_{312}\right] \Big/ \left[\tilde{g}_{31}^2 (3\tilde{g}_{31} + \beta\tilde{d}_{312})\right], \quad (111)$$

where now $\beta = \gamma/\lambda_0$. The marginal density of A does not seem to be obtainable analytically from (108), but is easily obtained from the joint density by numerical integration.

Figure 2 presents three cases of the marginal density of t in (111) corresponding to vectors $z' = (-1, -.9, 1), (-1, 0, 1)$, and $(-1, .9, 1)$ that were subsequently standardised to have $\bar{z} = 0$ and $\Sigma z_i^2 = 1$ (so that the asymptotic covariance matrix of $(\sqrt{n}(a - \alpha), (t - \theta))$ is an identity matrix). The rst and third cases, of course, are identical except for re ection about the origin. Even for such a small sample size, Fig. 2 reveals that the density is quite concentrated around zero, showing slight skewness (depending on the pattern of the z s. Table 2 presents some properties of the marginal densities for the case $n = 3$, and the two (unstandardised) z-vectors (a) $(-1, -.9, 1)$, (b) $(1, 0, 1)$.

FIGURE 2 ABOUT HERE

TABLE 2
Properties of the marginal densities: $n = 3$

	$pdf_T(t;\theta)$		$pdf_A(a;\alpha)$	
Case	(a)	(b)	(a)	(b)
Mean	-0.25	0.0	-0.64	-0.62
Variance	1.53	1.65	1.39	1.38
Skewness	-0.56	0.0	-0.61	-0.62
Kurtosis	4.22	4.15	3.64	3.68
Correlation	-0.03	0.03	-0.03	0.03

(a) $z' = (-1, -.9, 1)$; (b) $z' = (1, 0, 1)$.

Figure 3 shows the joint density $pdf_{A,T}(a, t; \alpha, \theta; z)$ for case (b). The marginal density of A, $pdf_A(a; \alpha; z)$, can be obtained by numerical integration, and is shown for all three cases in Figure 4 (although cases (a) and (c) are exactly superimposed). It does not exhibit the mirror symmetry of $pdf_T(t; \theta; z)$,being always negatively skewed. The density for (b) lies slightly to the right. The density of A has variance and kurtosis closer to the asymptotic values than those for T, but the mean and skewness are further from their asymptotic values. The correlation between A and T is negative for case (a), and positive for cases (b) and (c), which is the mirror image of (a). Given the small sample size, the results look well-behaved.

For $n = 4$, the marginal density of T is symmetric if the data is symmetric about its mean (*e.g.* $z' = [0.1, 0.2, 0.3, 0.4]$), and positively skewed z s (*e.g.* $z' = [0.1, 0.26, 0.33, 0.4]$) give the mirror image of the density with negatively skewed z s (*e.g.* $z' = [0.1, 0.17, 0.24, 0.4]$). Accordingly we can illustrate almost the full range of behaviour by using the symmetric z above, and the negatively skewed case $z' = [0.1, 0.11, 0.12, 0.4]$). Table 3 shows the resulting moments in its rst two columns. For $n = 5$, the cases $z' = [0.1, 0.2, 0.29, 0.4, 0.5]$, which gives the mirror image of the results with z_3 changed to 0.31, and $z' = [0.1, 0.11, 0.12, 0.13, 0.5]$, are illustrated in columns 3 and 4 of Table 3. The z-vectors given for Table 3 (and elsewhere) are in their unstandardised form. Progress towards limiting normality is masked by the possibility of greater skewness for $n = 5$ observations than for $n = 4$.

TABLE 3: Moments of the marginal density of T

	$n = 4$		$n = 5$	
	(a)	(b)	(a)	(b)
Mean	0.00	-0.329	-0.016	-0.393
Variance	1.43	1.45	1.40	1.55
Skew	0.00	-0.56	-0.05	-0.77
Kurtosis	3.48	3.58	3.71	4.22

For $n = 4$: (a) $z' = [0.1, 0.2, 0.3, 0.4]$), (b) $z' = [0.1, 0.11, 0.12, 0.4]$
For $n = 5$: (a) $z' = [0.1, 0.2, 0.29, 0.4, 0.5]$, (b) $z' = [0.1, 0.11, 0.12, 0.13, 0.5]$

FIGURES 3 AND 4 ABOUT HERE

Because of the numerical instability of the calculations near $t = \theta$, and because the cases above for small n suggest reasonably rapid convergence to the asymptotic distribution, we now concentrate on the tail area of the densities. Table 4 gives the exact tail area in the marginal density of T (obtained by numerical integration) for $n = 5$, and $n = 10$, and for three nominal (*i.e.* asymptotic) levels.

TABLE 4: marginal density of T
Tail Areas: $n = 5$ and $n = 10$

Nominal Tail area	lower%	upper%	sum%	$n = 10$, sum%
10	14.4	2.3	16.7	17.2
5	10.4	0.9	11.3	12.0
1	7.0	0.3	7.3	8.7

$n = 5$: $z' = [0.1, 0.11, 0.12, 0.13, 0.5]$
$n = 10$: $z' = [0.1, 0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 1.0]$

If one chooses z in this way, so that, for example, when $n = 10$, all the observations except the largest are crowded together at the lower end of the range, the skewness increases, and this acts against the effect of increasing n, to leave the tail areas more or less unchanged over the range $n = 5, .., 10$, as the last column of Table 4 illustrates. However, for uniformly distributed z s, for example

$$z' = [0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0]$$

the distribution is symmetric, the tail probabilities are equal, and are given in Table 5.

TABLE 5: Marginal Density of T
Tail area (uniformly distributed z s); combined tails

Nominal %	1	5	10
$n = 6$	4.7	8.7	14.3
$n = 8$	4.0	7.8	13.3
$n = 10$	3.6	7.3	12.7
$n = 12$	3.3	6.9	12.3

There is evidently steady progress towards the nominal values as n increases, slightly slower the further one is into the tails. For larger values of n the computational difficulties mentioned above have so far prevented us from carrying out a more detailed analysis of the densities.

6 CONCLUSION

We have shown that the surface integral formula for the exact density of the MLE given by Hillier and Armstrong (1996) provides a tractable expression for the exact density in the case of an exponential regression model with a k-covariate exponential mean function, at least for small values of k. It seems clear that an algorithm could, in principle, be written to provide similar results for arbitrary k.

The discussion in section 2 also show that, even for arbitrary k, the general formula can, by itself, provide considerable information about the properties of the exact density. It is worth noting, too, that the general approach used here extends easily to more general speci cations for the mean function (*i.e.* non-exponential functions of the w_i), provided only that the level set of the MLE is known. It will remain true under more general models that the surface integral to be evaluated is over an $(n-k)$-dimensional hyperplane.

Finally, as far as the results for the speci c model under consideration are concerned, our main conclusion is that the exact densities are well behaved, and well approximated by the asymptotic densities, even for quite small sample sizes. The sample behaviour of the covariates certainly has an impact on the properties of the estimator, as one would expect, but this effect is not dramatic.

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Figure 1.
Densities for n = 2, 4, 8, and 16 equispaced points, z'z = 1

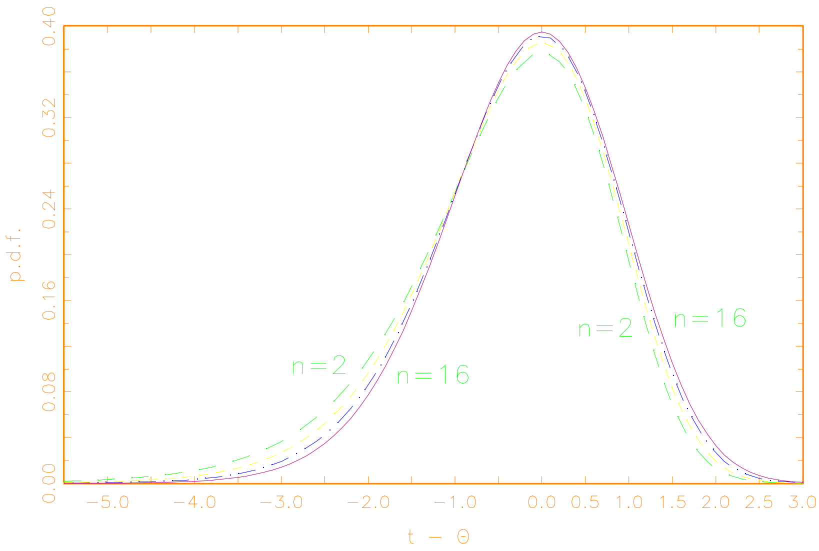

Fig 2. Marginal densities for t−Θ, n = 3

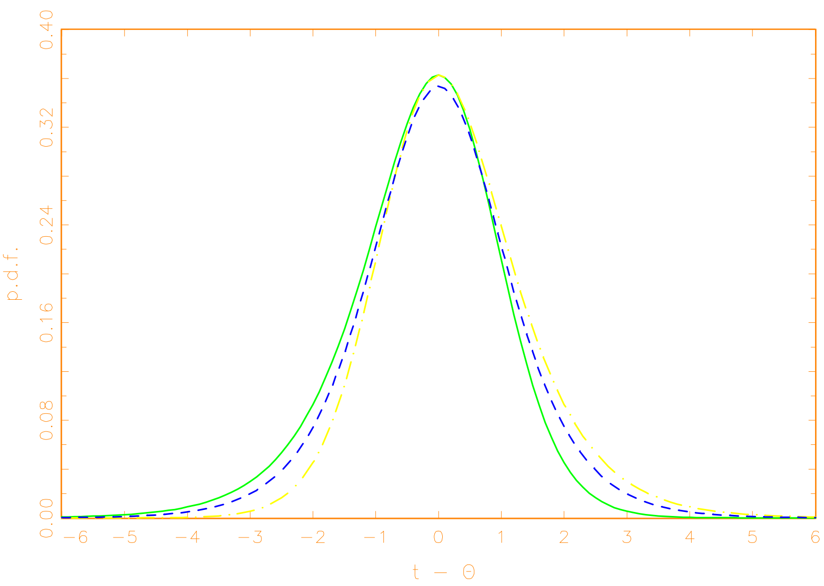

Fig 3. Joint density for t−Θ, a − α, n = 3

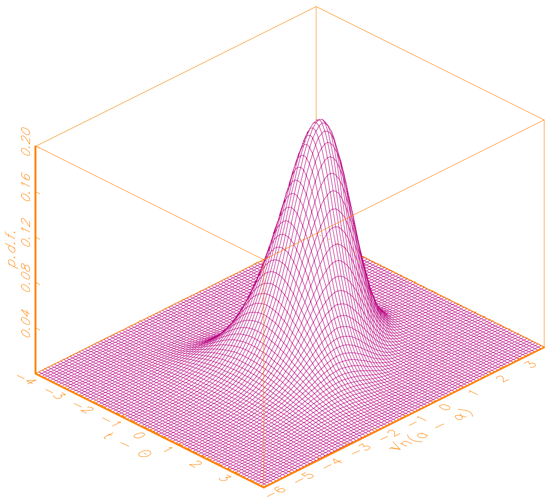

Fig 4. Marginal densities for a − α, n = 3