Can Greater Uncertainty Hasten Investment?*

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Abstract

This paper examines irreversible investment in a project with uncertain returns, when there is an advantage to being the first to invest and externalities to investing when others also do so. We show that in a duopoly, greater uncertainty can actually hasten rather than delay investment, contrary to the usual outcome, due its effect on the equilibrium of the timing game between the players. In the presence of positive externalities, greater uncertainty can raise the leader's value more than the follower's; pre-emption then entails that the leader must act sooner. A switch in the pattern of equilibrium investment as uncertainty increases is also possible, which may hasten investment. These findings reinforce the importance of extending real options analysis to include strategic interactions and externalities between players.

Keywords: Real Options, Network Effects, Pre-emption.

JEL classification: C73, D81, L13, O31.

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1 Introduction

The literature on irreversible investment under uncertainty teaches three major lessons. First, the net present value (NPV) rule for investment is generally incorrect, since it considers only a now-or-never decision and fails to appreciate that investment can be delayed. Secondly, an option value is created by the fact that the return is bounded below by the payoff from not investing; the effect of this option value is to delay investment, relative to the NPV rule. Finally, the greater the degree of uncertainty, the larger this delay: an increase in uncertainty increases the upside potential from investment, and so increases the value of the investment option.

The early literature on the 'real options' approach analyses investment decisions for a single agent in isolation. In many real world cases, however, investment takes place in a more competitive environment in which there are externalities and strategic interactions between investing agents. The purpose of this paper is to demonstrate that such interactions can have important consequences for irreversible, uncertain investments, with effects that run counter to the standard results given above.

We analyse irreversible investment in a project with uncertain returns in a dynamic two-player model, with a general specification. Two types of strategic interactions are considered. The first is pre-emption: when there is some advantage to being the first to undertake an investment, there will be competition to be the first. In this situation, any benefit from delaying investment due to real option effects has to be balanced against the loss from being pre-empted. The second interaction arises when the value of an investment depends on the number of agents who have also invested. The interaction may affect value negatively: e.g., if it arises through a competitive effect; or it may have a positive effect, if there are complementarities between the agents' actions such as network externalities or demand expansion. In both cases, the timing of an agent's investment is influenced by the investment decisions of others.

The contribution of this paper is to show that, contrary to the standard result, the effect of uncertainty on investment is ambiguous in a duopoly. When strategic interactions determine the exercise of the option, greater uncertainty can hasten, rather than delay, investment. We show that if the payoff externality when the follower invests is positive and there is not too much uncertainty, then a small increase in uncertainty causes the leader's investment point to fall. If investment takes the form of entry into a product market, for example, then this would require that e.g., the demand expansion effect of an additional firm outweighs increased competition. Investment in the context of strong network externalities would be similarly affected.

To understand this surprising comparative static, note that when two agents invest sequentially, the first investment point is determined by rent equalization (see Fudenberg and Tirole (1985)): at the point at which the first investor acts it is indifferent between being the leader and the follower. At the leader's investment point, the leader's value function crosses the follower's from below (otherwise investment would have occurred sooner). Hence an increase in uncertainty lowers the leader's investment point if and only if the value function from investing first increases by more than the value to investing second.

With positive externalities between the agents, investment by the follower is valuable to the leader—particularly since it does not require any additional cost for the leader. But once the leader has acted, the leader is unable to affect the follower's investment decision, and must take the investment point of the follower as given. For the follower, investment is costly; but it chooses its investment point optimally. The first factor tends to make the follower's option less valuable than the leader's; the second factor makes it more valuable. When uncertainty is relatively small, the first factor dominates and the leader's option[1] is more valuable. Given the form of the value functions, this also means that the leader's value function increases by more than the follower's when uncertainty rises. (Both functions increase, due to the convexity of payoffs.)

There is an additional way in which greater uncertainty may hasten investment: by causing a switch in the pattern of equilibrium investment. Two equilibrium patterns of investment are possible. Either the agents invest sequentially (i.e., the 'leader' invests

[1]Strictly speaking, the leader does not hold an option: it simply waits for the follower to act. It is more accurate to refer to the option-like term of the leader. We use the more accurate terminology in the rest of the paper.

early while the 'follower' invests late), or they invest simultaneously. We also show that an increase in uncertainty can cause equilibrium to switch from sequential to simultaneous investment, or vice versa, in such a way that the first investment occurs sooner. We argue that these effects are present at plausible parameter values, and so can be empirically important.

Overall, therefore, strategic interactions in the presence of positive externalities give rise to significant qualitative effects that are omitted from the standard real options analysis of investment, even those models incorporating game-theoretic interactions. The relationship between uncertainty and investment, even for a single irreversible project available to each firm, is more complex than given by the standard result. This finding is relevant to cross-border investment in the face of exchange rate uncertainty, for example. Its implications are important to firms planning their investments, and for policy-makers wishing to understand and anticipate private sector investment behaviour.

Two general strands of literature are related to this paper. Real options models have been used to explain delay and hysteresis arising in a wide range of contexts. McDonald and Siegel (1986) and Pindyck (1988) consider irreversible investment opportunities available to a single agent. Dixit (1989) and Dixit (1991) analyse product market entry and exit in monopolistic and perfectly competitive settings respectively. The second strand of literature concerns timing games of entry or exit in a deterministic setting. There are several types of paper within this strand. Papers analysing pre-emption games include Fudenberg, Gilbert, Stiglitz, and Tirole (1983), Fudenberg and Tirole (1985), Katz and Shapiro (1987) and Lippman and Mamer (1993). Wars of attrition have been modelled by e.g., Fudenberg and Tirole (1986).

A number of real options models incorporating strategic interactions now exist. Smets (1991) examines irreversible market entry in a duopoly facing stochastic demand. Simultaneous investment may arise only when the leadership role is exogenously pre-assigned. Consequently, he does not consider fully the pre-emption externality. Weeds (2002) presents a model in which two firms may invest in competing research projects with uncertain returns. She does not impose an asymmetry between the firms, but allows the leader to emerge endogenously. She does not, however, include more general externalities. Other papers combining real options with game theory include Boyer, Lasserre, Mariotti, and Moreaux (2004) Huisman and Kort (1999) and Lambrecht and Perraudin (2003); these, however, do not generate the comparative static result we find. A survey of this literature is provided by Boyer, Gravel, and Lasserre (2004). The general specification of our model encompasses several of these contributions. Hoppe (2000) analyses a timing game of new technology investment in an uncertain environment. She considers second, rather than first, mover advantages and models uncertainty in a different way from our paper.

In a two-period model, Kulatilaka and Perotti (1998) find that greater uncertainty over market demand may increase cost-reducing investment undertaken in the first period. Their model is quite different from ours: there is an exogenous asymmetry between the firms—only firm 1 holds a strategic investment opportunity in the first period—and this firm exercises a subsequent option (over production) in the second period. Although their result has a superficial similarity to ours, it is driven by the strategic effect of first period investment in reducing the competitor's output in the second period (à la Cournot), or deterring entry altogether (as in Dixit (1980)), combined with optionality at the second stage. Since the first period investment is available to a single firm, there is no competition in exercising the option. In this paper, by contrast, our result is due to the effect of uncertainty on the equilibrium outcome of the timing game between the players. Dixit and Pindyck (1994) describe situations in which uncertainty can speed up investment, because investment itself reveals information about costs. We show that even in the absence of this 'shadow value', investment may be speeded up by uncertainty.

Our paper is also related to the literature on technology investment with network externalities, such as Farrell and Saloner (1986) and Katz and Shapiro (1986). In Farrell and Saloner (1986), a model of technology investment with uncertainty about the timing of (rather than return from) investment, positive network effects, and irreversibility is analysed (see section II). Unlike Farrell and Saloner, we allow agents to invest at any time, not just at random opportunities. If this assumption were used in the Farrell and Saloner model, then many of the features would disappear (although the basic co-ordination problem due to network effects would remain). Here, delay is endogenously determined through the optimization decisions of the agents, rather than imposed exogenously. Choi (1994) examines a model in which there are positive network effects, uncertainty and the possibility of delay. In Choi's model, users are exogenously asymmetric: user 1 is able to choose which of two technologies (with random returns) to invest in either of two periods, while user 2 is able to invest only in the second period. This paper departs from Choi's in several respects. Most importantly, it does not impose exogenously an asymmetry between players, but instead allows the first mover to be determined endogenously. In our model, the leader invests at the point at which it is indifferent between leading and following; see section 3. The fact that investment by the leader is determined by indifference, rather than optimally (for the leader), makes an important difference to investment behaviour. We also allow for a more general payoff structure, including allowing for negative as well as positive externalities.

The rest of the paper is structured as follows. Section 2 describes the model. Section 3 analyses the non-co-operative equilibria of the model. Section 4 looks at the effect of uncertainty on investment delay when pre-emption can occur. Section 5 concludes. The appendix contains lengthier proofs.

2 The Model

This section develops a general model to capture the three effects that are the focus of this paper: (i) uncertainty, irreversibility and the possibility of delay in investment; (ii) investment externalities, where the return to investment depends on the number of investors; and (iii) pre-emption, where early investors have an advantage.

Two risk neutral agents, labelled $i \in \{1, 2\}$ each can invest in a project. There is a cost $K > 0$ to doing so, which is the same for both agents. Investment is irreversible (the cost K is entirely sunk) and can be delayed indefinitely. Time is continuous and labelled by $t \in [0, \infty)$. The timing of investment is the main concern of the analysis. Investment by the two agents may occur sequentially—that is, the two agents invest at distinctly different times—or simultaneously.

Consider first the outcome when the agents invest sequentially. Call the first investor the 'leader' and the second investor the 'follower'. The leader's instantaneous payoff at time t from investment, before the follower has invested, is

$$\pi_L^I = \theta_t,$$

where θ_t is the stand-alone benefit from investment—the instantaneous payoff received by an agent that is the sole investor. After the follower has invested, the leader's instantaneous payoff becomes

$$\pi_L^{II} = (1 + \delta_L)\theta_t.$$

The follower's instantaneous payoff at time t from investment is

$$\pi_2^{II} = (1 + \delta_F)\theta_t.$$

Now suppose that the agents invest simultaneously. The instantaneous payoff at time t from investment is the same for both agents:

$$\pi^{III} = (1 + \delta_S)\theta_t.$$

The model specification is a general one. We do not investigate all possible configurations of the payoff parameters δ_L, δ_F and δ_S. Instead, we restrict attention to cases described in the following assumption:

Assumption 1 $-1 \leq \delta_F \leq 0$; $\delta_F \leq \delta_S$; *and* $\delta_F \leq \delta_L$.

This assumption ensures two things. First, there may be a first-mover advantage, since $\delta_L \geq \delta_F$. Secondly, there may be a second-mover disadvantage, in the sense that δ_F is less than both δ_S and δ_L. The role of particular aspects of assumption 1 will be pointed out as the analysis progresses.

Even with assumption 1, our model encompasses many related papers. For example, in Fudenberg and Tirole (1985), when n firms have adopted the new technology, the payoff of a firm that has not adopted is $\pi_0(n)$, and of a firm that has adopted is $\pi_1(n)$. They assume that if $n' \geq n$, then $\pi_1(n') < \pi_1(n)$. A specific version of their payoffs can be represented in our model by supposing that $\pi_0(n) = 0 \ \forall n$, $\pi_1(1) = \theta$ and $\delta_L = \delta_F = \delta_S < 0$. Real options duopoly models such as Smets (1991), Weeds (2002) and Huisman and Kort (1999) employ functional forms equivalent to negative δ_L, δ_F and δ_S parameters. Similarly, some of the payoff structures used in Katz and Shapiro (1987) can be replicated within our model. What they term the 'stand-alone incentive' is measured by δ_L in this model; their 'pre-emption incentive' is measured by $\delta_L - \delta_F$; the degree of imitation that is possible can be captured by δ_F. Lippman and Mamer (1993) analyse a model in which the first firm to innovate spoils the market for its rival; in this case, $\delta_F = -1$. Notice also that by setting $\delta_S = (\delta_L + \delta_F)/2$, we can allow for the possibility that, in the event of simultaneous adoption, the roles of leader and follower are assigned randomly between the two agents.

θ_t is assumed to be exogenous and stochastic, evolving according to a geometric Brownian motion (GBM) with drift:

$$d\theta_t = \mu\theta_t dt + \sigma\theta_t dW_t \tag{1}$$

where $\mu \in [0, r)$ is the drift parameter, measuring the expected growth rate of θ, r is the continuous-time discount rate,[2] $\sigma > 0$ is the instantaneous standard deviation or volatility parameter, and dW is the increment of a standard Wiener process, $dW_t \sim N(0, dt)$. The parameters μ, σ and r are common knowledge and constant over time. The choice of continuous time and this representation of uncertainty is motivated by the analytical tractability of the value functions that result.

The strategies of the agents in the investment game are now defined. If agent i has not invested at any time $\tau < t$, its action set is $A_t^i = \{\text{invest, don't invest}\}$. If, on the other hand, agent i has invested at some $\tau < t$, then A_t^i is the null action 'don't move'.

[2]The restriction that $\mu < r$ ensures that there is a positive opportunity cost to holding the 'option' to invest, and so that the option is not held indefinitely.

The agent therefore faces a control problem in which its only choice is when to choose the action 'invest'. After taking this action, the agent can make no further moves.

A strategy for agent i is a mapping from the history of the game H_t (the sample path of the stochastic variable θ and the actions of both agents up to time t) to the action set A_t^i. Agents are assumed to use stationary Markovian strategies: actions depend on only the current state and the strategy formulation itself does not vary with time. Since θ follows a Markov process, Markovian strategies incorporate all payoff-relevant factors in this game. Furthermore, if one player uses a Markovian strategy, then its rival has a best response that is Markovian as well. Hence, a Markovian equilibrium remains an equilibrium when history-dependent strategies are also permitted, although other non-Markovian equilibria may then also exist. (For further explanation see Maskin and Tirole (1988) and Fudenberg and Tirole (1991).)

The formulation of the agents' strategies is complicated by the use of a continuous-time model. Fudenberg and Tirole (1985) point out that there is a loss of information inherent in representing continuous-time equilibria as the limits of discrete time mixed strategy equilibria. To correct for this, they extend the strategy space to specify not only the cumulative probability that player i has invested, but also the 'intensity' with which each player invests at times 'just after' the probability has jumped to one.[3] Although this formulation uses mixed strategies, the equilibrium outcomes are equivalent to those in which agents employ pure strategies. (See section 3 of Fudenberg and Tirole (1985).) Consequently, the analysis will proceed as if each agent uses a pure Markovian strategy, i.e., a stopping rule specifying a critical value or 'trigger point' for the exogenous variable θ at which the agent invests. Note, however, that this is for convenience only: underlying the analysis is an extended space with mixed strategies.

Our analysis focuses on trigger points of the stochastic variable θ. These could also

[3] In Fudenberg and Tirole (1985), an agent's strategy is a *collection* of *simple strategies* satisfying an *intertemporal consistency condition*. A simple strategy for agent i in a game starting at a positive level θ of the state variable is a pair of real-valued functions $(G_i(\theta), \epsilon_i(\theta)) : (0, \infty) \times (0, \infty) \to [0, 1] \times [0, 1]$ satisfying certain conditions (see definition 1 in their paper) ensuring that G_i is a cumulative distribution function, and that when $\epsilon_i > 0$, $G_i = 1$ (so that if the intensity of atoms in the interval $[\theta, \theta + d\theta]$ is positive, the agent is sure to invest by θ). A collection of simple strategies for agent i, $(G_i^\theta(.), \epsilon_i^\theta(.))$, is the set of simple strategies that satisfy intertemporal consistency conditions.

be expressed in terms of expected stopping times; we do not, however, include this step. For our comparative static results it is sufficient to recall that, for a given time path of the stochastic variable, a lower trigger point corresponds to earlier investment.

The possible states of each agent are denoted $n_i \in \{0,1\}$ when the agent has not invested and has invested, respectively. The following assumptions are made:

Assumption 2 *If* $n_i(\tau) = 1$*, then* $n_i(t) = 1$ *for all* $t \geq \tau$*,* $i \in \{1,2\}$*.*

Assumption 3 $\max\{1, 1+\delta_L\}\mathbb{E}_0 \left[\int_0^\infty \exp(-rt)\theta_t dt\right] - K < 0.$

Assumption 2 formalizes the irreversibility of investment: if agent i has invested by date τ, it then remains active at all dates subsequent to τ. Assumption 3 states that the initial value of the project is sufficiently low that the expected return from investment is negative, thus ensuring that immediate investment is not worthwhile. (The operator $\mathbb{E}_0$ denotes expectations conditional on information available at time $t = 0$.)

3 Equilibrium

3.1 Sequential Investment

Start by assuming that the agents invest at different points. The possibility of simultaneous investment is considered below. As usual in dynamic games, the stopping time game is solved backwards; see e.g., Dixit (1989). Thus the first step is to consider the optimization problem of the follower who invests strictly later than the leader. Given that the leader has invested irreversibly, the follower's payoff on investing has two components: the flow payoff from the project, $(1+\delta_F)\theta_t$; and the cost of investment, $-K$. The follower's value function $F(\theta_t)$ at time t given a level θ_t of the state variable is therefore

$$F(\theta_t) = \max_{T_F} \mathbb{E}_t \left[\int_{T_F}^\infty \exp(-r(\tau - t))(1+\delta_F)\theta_\tau d\tau - K \exp(-r(T_F - t))\right]$$

where T_F is the random investment time for the follower, and the operator $\mathbb{E}_t$ denotes expectations conditional on information available at time t.

The value function F has two components, holding over different ranges of θ: one relating to the value of investment before the follower has invested, the other to the follower's value after investment. We derive these value functions in section A in the appendix. We show there that the follower's value function is

$$F(\theta) = \begin{cases} b_F \theta^\beta & \theta < \theta_F, \\ \frac{(1+\delta_F)\theta}{r-\mu} - K & \theta \geq \theta_F. \end{cases} \tag{2}$$

θ_F is the follower's optimally-chosen investment point. (The value function in equation (2) assumes that the leader invests at some level of θ less than θ_F. We verify below that this is the case in equilibrium.) By arbitrage, the critical value θ_F must satisfy a value-matching condition; optimality requires a second condition, known as 'smooth-pasting', to be satisfied. (See Dixit and Pindyck (1994) for an explanation.) This condition requires the two components of the follower's value function to meet smoothly at θ_F with equal first derivatives, which together with the value matching condition implies that

$$\begin{aligned} \theta_F &= \left(\frac{\beta}{\beta-1}\right)\left(\frac{K}{1+\delta_F}\right)(r-\mu), \\ b_F &= \frac{(1+\delta_F)\theta_F^{-(\beta-1)}}{\beta(r-\mu)}. \end{aligned} \tag{3}$$

Equation (3) for the follower's trigger point can be interpreted as the effective flow cost of investment with an adjustment for uncertainty. The sunk investment cost is K, but this yields a flow payoff of $(1+\delta_F)\theta$; hence the effective sunk cost is $\frac{K}{1+\delta_F}$. With an effective interest rate of $r-\mu$ (i.e., the actual interest rate r minus the expected proportional growth in the flow payoff μ), this gives an instantaneous cost of $\left(\frac{K}{1+\delta_F}\right)(r-\mu)$. If a Marshallian rule were used for the investment decision, the trigger point would be simply this cost. But with uncertainty, irreversibility and the option to delay investment, the Marshallian trigger point must be adjusted upwards by the factor $\frac{\beta}{\beta-1} > 1$. The follower's trigger can also be compared to the standard single-agent trigger,

$$\theta_L \equiv \left(\frac{\beta}{\beta-1}\right)K(r-\mu);$$

see e.g., Dixit and Pindyck (1994).

In section A in the appendix, we show that the leader's value function has the following form:

$$L(\theta) = \begin{cases} b_{L0}\theta^{\beta} & \theta < \theta_P, \\ \frac{\theta}{r-\mu} + b_{L1}\theta^{\beta} - K & \theta \in [\theta_P, \theta_F), \\ \frac{(1+\delta_L)\theta}{r-\mu} - K & \theta \geq \theta_F, \end{cases} \tag{4}$$

given the leader's trigger point θ_P and investment by the follower at the higher θ_F. Notice that the first and third components of the leader's value function mirror the follower's value function. The second component is new, and shows the leader's value after it has invested, but before the follower has invested.

The value of the unknown constant b_{L1} is found by considering the impact of the follower's investment on the payoff to the leader. When θ_F is first reached, the follower invests and the leader's expected flow payoff is altered. Since value functions are forward-looking, L_1 anticipates the effect of the follower's action and must therefore meet L_2 at θ_F. Hence, a value-matching condition holds at this point (for further explanation see Harrison (1985)); however, there is no optimality on the part of the leader, and so no corresponding smooth-pasting condition. This implies that

$$b_{L1} = \frac{\delta_L \theta_F^{-(\beta-1)}}{r-\mu}. \tag{5}$$

The remaining coefficient, b_{L0} is determined by value matching at θ_P:

$$b_{L0} = \frac{K}{\beta - 1}\theta_F^{-\beta}. \tag{6}$$

The next proposition describes the equilibrium in this case.

Proposition 1 *Given assumptions 1–3, when equilibrium investment is sequential, the leader invests at θ_P and the follower at $\theta_F > \theta_P$. $\theta_P \in (\theta_0, \theta_L)$ is the smallest solution to the equation*

$$\frac{\theta_P}{r-\mu} - K = \frac{K}{\beta-1}\left(\frac{1+\delta_F - \beta\delta_L}{1+\delta_F}\right)\left(\frac{\theta_P}{\theta_F}\right)^{\beta}. \tag{7}$$

The proof of the proposition is in section B in the appendix. The explanation of the equilibrium is as follows. The leader cannot choose its investment point optimally, as the follower can. Instead, the first agent to invest does so at the point at which it prefers to lead rather than follow, not the point at which the benefits from leading are largest. Clearly, it cannot be that the first agent invests when the value from following is greater than the value from leading—if this were the case, the agent would do better by waiting. Likewise, it cannot be that the first agent invests when the value from leading is strictly greater than the value from following, since in this case without pre-assigned roles, the other agent could pre-empt it and still gain. Hence the investment point is determined by indifference between leading and following. The trigger point θ_P in the pre-emption model is given by indifference: $L(\theta_P) = F(\theta_P)$. This is in contrast to the trigger point of the follower, which is determined by value matching and smooth pasting, i.e., is chosen optimally.

The rent equalization condition $L(\theta_P) = F(\theta_P)$ gives the non-linear equation (7) for θ_P. A number of possibilities arise: there may be no, one or multiple solutions to this equation. In the proof of the proposition, we show that there is at least one solution which lies between θ_0 and θ_L. We have also assumed that the initial value of the project is sufficiently low that immediate investment is not worthwhile (see assumption 3). Hence in equilibrium, there is no investment before θ hits θ_P. In other words, the leader's value function hits the follower's from below.

One possibility for a solution to equation (7) is illustrated in figure 1 (in which it is assumed that $1 + \delta_F - \beta\delta_L > 0$). The left-hand side of equation (7) is the increasing, linear function; the right-hand side is the increasing, convex function. There are two intersection points of the two functions; the lower point is the relevant solution for the leader's equilibrium trigger point θ_P. ($\theta_M \equiv K(r - \mu)$ in the figure is the myopic Marshallian trigger, i.e., the investment point of an agent who ignores both uncertainty and any subsequent investment by other agents.)

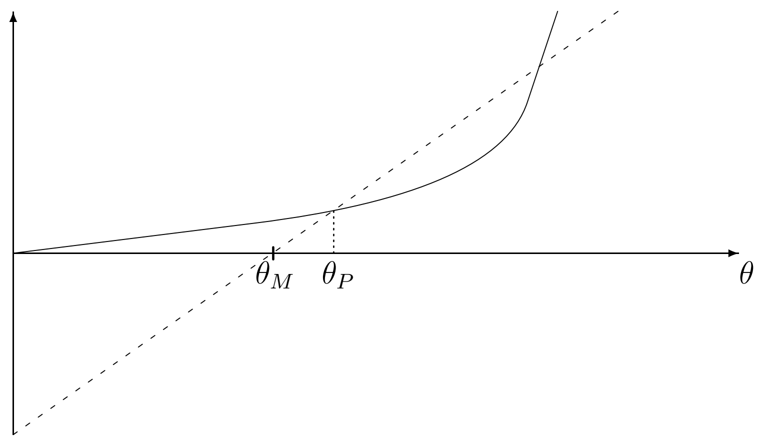

Figure 1: The solution for θ_P

3.2 Simultaneous Investment

Now consider the alternative case, in which investment is simultaneous at the trigger point θ_S. The previous analysis indicates that the value function of each agent is then

$$S(\theta) = \begin{cases} b_S \theta^\beta & \theta < \theta_S, \\ \frac{(1+\delta_S)\theta}{r-\mu} - K & \theta \geq \theta_S. \end{cases}$$

(This value function can be derived from the appropriate Bellman equation, following the steps shown in the appendix.) There is a continuum of simultaneous solutions; it is straightforward to show that they can be Pareto ranked, with higher trigger points yielding higher value functions. In this case, it seems reasonable that the agents invest at the Pareto optimal point, given by both value matching and smooth pasting. So

Proposition 2 *The Pareto optimal trigger point for the simultaneous equilibrium is*

$$\theta_S = \left(\frac{\beta}{\beta - 1}\right)\left(\frac{K}{1 + \delta_S}\right)(r - \mu).$$

The coefficient in the value function is

$$b_S \quad = \quad \frac{(1+\delta_S)\theta_S^{-(\beta-1)}}{\beta(r-\mu)}. \tag{8}$$

The next proposition describes when simultaneous investment is an equilibrium.

Proposition 3 *Simultaneous investment occurs in equilibrium iff*

$$\lambda_E \equiv (1+\delta_S)^\beta - \left(1+\beta\delta_L(1+\delta_F)^{\beta-1}\right) \geq 0. \tag{9}$$

A sufficient condition is $\delta_S \geq 0 \geq \delta_L$.

Proof. For equilibrium simultaneous investment, it must be that $S(\theta) \geq L(\theta)$ for $\theta \in [\theta_P, \theta_S]$. Due to the convexity of the value functions, this requires that $S(\theta) \geq L(\theta)$ for $\theta \in [0, \theta_P]$, and so that $b_S \geq b_{L0}$. Substituting the expressions for these two coefficients gives the necessary and sufficient condition of equation (9). The sufficient condition follows directly from equation (9). □

Whether simultaneous investment occurs in equilibrium is determined by whether the leader wishes to invest before the follower, or at the same time (i.e., by the comparison of $L(\theta)$ and $S(\theta)$). The proposition shows the reasonable condition that, in order for simultaneous investment to occur in equilibrium, it must be the case that δ_S is sufficiently large and/or δ_L and δ_F sufficiently small. (This is clearest in the sufficient condition.) Note that the simultaneous investment equilibrium, when it exists, Pareto dominates the sequential outcome; this is an immediate consequence of the condition for existence of the simultaneous investment equilibrium: $S(\theta) \geq L(\theta)$ for $\theta \in [0, \theta_S]$.

4 Uncertainty and Delay with Pre-emption

Real options analysis for monopoly or perfectly competitive industries concludes that:

1. The net present value (NPV) rule for investment is incorrect since it ignores the option value created by irreversibility and uncertainty.

2. The effect of this option value is to delay investment, relative to the NPV rule.

3. The greater the degree of uncertainty, the larger the option value and the greater the extent of delay.

In this section, we show that the third conclusion need not hold when pre-emption is possible; in particular, more uncertainty can hasten investment. Our results demonstrate that the combination of uncertainty and pre-emption can result in complex interactions.

First note that the triggers θ_F and θ_S are increasing in σ, for the familiar real options reason. The intuition is that delay allows for the possibility that the random process (1) will go up; if it goes down, then the agent need not invest. The greater the variance of the process, the more valuable is the option created by this asymmetric situation, and so the more delay occurs for both agents. Notice that this result relies on the fact that all of these triggers are chosen optimally by the relevant agent(s).

There are two ways in which greater uncertainty can hasten investment. First, when equilibrium investment is sequential, the trigger point θ_P of the leader may decrease as σ increases. This possibility is examined in proposition 4. Secondly, a rise in σ can cause the pattern of equilibrium investment to switch, with investment in the new equilibrium pattern occurring earlier. This possibility is considered in proposition 5.

Proposition 4 *Joint sufficient conditions for the leader's investment trigger θ_P to be decreasing in the volatility parameter σ are*

$$1 + \beta \ln(1 + \delta_F) < 0 \quad \text{and} \quad 0 \leq \frac{(1 + \delta_F) \ln(1 + \delta_F)}{1 + \beta \ln(1 + \delta_F)} \leq \delta_L.$$

The proof is in section C in the appendix.

The result therefore raises the striking possibility that greater uncertainty lowers the leader's trigger point. The possibility arises from the lack of optimality in the choice of the pre-emption trigger point. An optimal trigger point is such that the marginal benefit from delaying investment for a period equals the marginal cost. The marginal benefit is the interest saved on the investment cost plus the expected gain from the possibility that the flow payoff increases. The marginal cost is the flow payoff foregone by not investing. In this marginal calculation, the agent does not consider the effect of its delay on the investment decision of the other agent, since in the models considered in this paper, each agent's trigger point (with the exception of θ_P) does not depend on the other's. Increased uncertainty raises the expected gain from delay, causing the (optimally chosen) trigger point to increase. This reasoning does not apply in the case of θ_P, however: it is not chosen according to a marginal equality, but an absolute equality between the value from leading and the value from following. The proposition shows that this difference in the determination of the trigger point can lead to θ_P decreasing as uncertainty increases.

In order for this unusual comparative static to hold, it must be that in the region of the intersection point, the leader's value function increases by more than the follower's when uncertainty rises, holding constant the leader's trigger point θ_P. (This statement follows directly from using the implicit function theorem on the non-linear equation (7) defining θ_P.) There are, therefore, two necessary and sufficient conditions for θ_P to be decreasing in σ:

1. The leader's value function L_1 is increasing in σ.

2. The increase in the leader's value function is larger than the increase in the follower's value function F_0.

The leader's value function depends on uncertainty due to the option-like term that anticipates investment by the follower: $b_{L1}\theta^{\beta}$, where $b_{L1} \equiv \delta_L \theta_F^{-(\beta-1)}/(r-\mu)$ and $\theta \in (\theta_P, \theta_F)$. Hence this option-like term is positive if and only if $\delta_L > 0$; this implies that the follower's investment benefits the leader, e.g. when demand expansion outweighs the effect of competition, or in a setting with network externalities. When this is the case, the option-like term increases in value with the degree of uncertainty (for the usual reasons), and so condition 1 holds. The follower's value function also depends on uncertainty, due to the option value of its investment: $b_F\theta^{\beta}$, where $b_F = (1+\delta_F)\theta_F^{-(\beta-1)}/\beta(r-\mu)$ and $\theta < \theta_F$. This option value increases with the degree of uncertainty.

The sufficient conditions in the proposition ensure that $\beta\delta_L - (1 + \delta_F) > 0$. So, when the conditions are satisfied, the value of the leader's option-like term is greater than the option value of the follower. Both values are convex functions of θ; the leader's value is more convex than the follower's, since it lies above it. Therefore the same condition ensures that the value of the leader's option-like term, $b_{L1}\theta^{\beta}$, increases by more than the option value of the follower, $b_F\theta^{\beta}$, for any increase in σ and any value of $\theta \in (\theta_P, \theta_F)$.

The sufficient conditions require that δ_L is sufficiently large (certainly positive), δ_F is sufficiently small (i.e., negative), and σ is sufficiently small (so that β is large). The result is illustrated in figure 2, which plots the triggers θ_F and θ_P against the volatility parameter σ. As the figure shows, θ_F is increasing in σ—the standard comparative static. But θ_P is decreasing in σ for low values of the parameter, but eventually increases in σ for values above around 1.75%.[4]

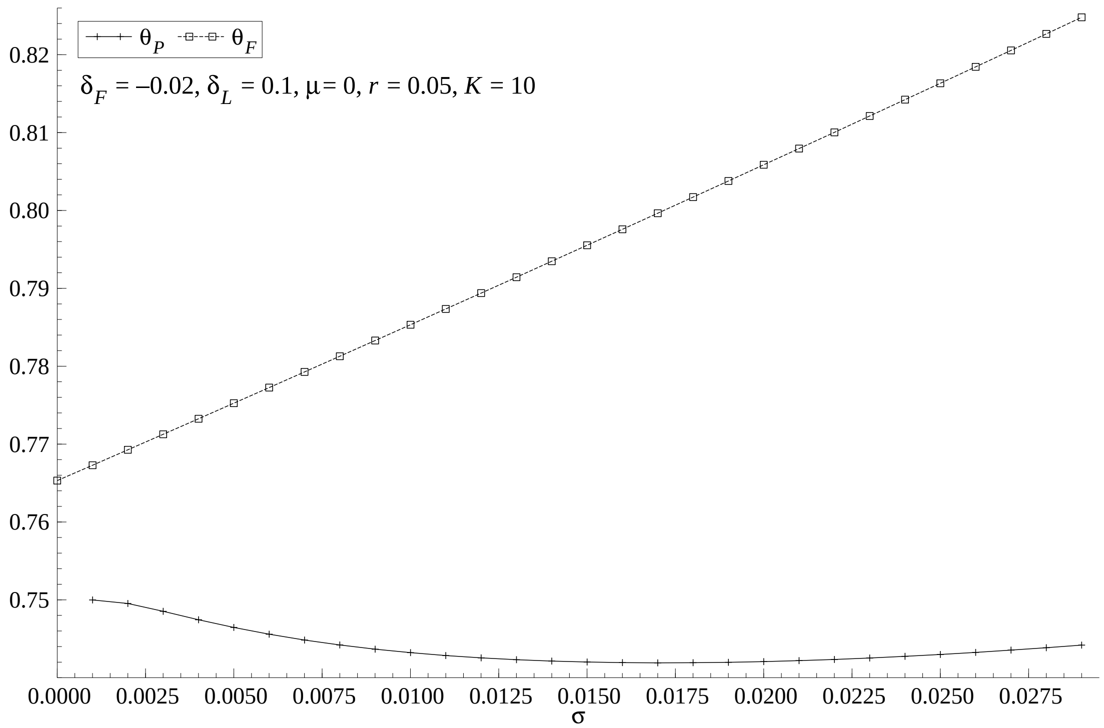

Figure 2: Triggers against the volatility parameter σ

[4]Numerical results and figures are generated using Ox version 3.30 (see Doornik (1999)).

Hence this result relies on the existence of a positive externality from the follower's investment to the leader's payoff. This factor is absent from earlier duopoly real options models, which consider only competitive interactions between the players. By expanding the framework to include positive, as well as negative, externalities, we have demonstrated that contrary comparative statics can arise.

We now consider the second possibility for greater uncertainty to hasten investment: as a result of a switch in the equilibrium pattern of investment as uncertainty increases. There are two cases to consider. First, equilibrium investment switches from simultaneous to sequential, and $\theta_S > \theta_P$. In this case, the investment point of the first investor decreases; but the follower adopts at a higher value of θ, since $\theta_F > \theta_S$. Secondly, equilibrium investment switches from sequential to simultaneous, and $\theta_S < \theta_P$. In this second case, the investment points of both agents unambiguously decrease. Two steps are needed to obtain sufficient conditions for these results, set out in proposition 5. The first step analyses whether the necessary and sufficient condition in proposition 3 for equilibrium to be simultaneous is easier or more difficult to satisfy as σ increases (i.e., whether λ_E is increasing or decreasing in σ). The second analyses whether θ_S is greater or less than θ_P.

Lemma 1 1. *Joint sufficient conditions for λ_E to be a decreasing function of σ are: $\delta_S \geq 0$ and either (i) $\delta_L \geq 0$ and $\delta_F \leq e^{-1} - 1$ or (ii) $\delta_L \leq 0$ and $\delta_F \geq e^{-1} - 1$.*

2. *Joint sufficient conditions for λ_E to be an increasing function of σ are: $\delta_S < 0$ and either (i) $\delta_L \geq 0$ and $\delta_F \geq e^{-1} - 1$ or (ii) $\delta_L \leq 0$ and $\delta_F \leq e^{-1} - 1$.*

The proof is in section D in the appendix.

Recall that two terms in θ appear in the two parts of the leader's value function before the follower's investment: L_0 contains a direct option value associated with the leader's own investment, while L_1 has an option-like term relating to the follower's investment.[5] Consider the effect of an increase in σ when $\delta_L < 0$. The leader's value increases due to

[5] Refer to equation (4). Notice that both terms are important for $\theta \leq \theta_F$. This is explicit over the range $\theta \in [\theta_P, \theta_F)$, and implicit for $\theta < \theta_P$: for the latter, the two factors show up in the expression for b_{L0}—see equation (6).

the first, direct option term—this is the standard comparative static of an option value. But the leader's value decreases due to the second term: the magnitude of the option-like value increases, but it is a negative value, since $\delta_L < 0$. Hence there are two conflicting effects when σ increases, and consequently the comparative static with respect to σ may be (and in fact is) non-monotonic.

In the cases identified in the lemma, however, the comparative statics are unambiguous. Consider part 1(i) of the lemma, in which $\delta_S \geq 0$ and $\delta_L \geq 0$. The value from simultaneous investment increases with σ, in line with the standard option value comparative static. The marginal effect on the simultaneous investment value function of an increase in σ is therefore positive; but it is decreasing in δ_S. This is because as δ_S increases, for any given level of σ, simultaneous investment occurs sooner (θ_S decreases). Hence an increase in δ_S acts in the opposite direction to an increase in σ, which increases θ_S.

The direct option term in the leader's value function increases with σ; and the marginal effect of an increase in uncertainty is independent of δ_L and δ_F. The second term increases with uncertainty, since $\delta_L \geq 0$. In this case, the marginal effect of an increase in uncertainty is decreasing in δ_F: as δ_F increases, for any given level of σ, the follower invests sooner (θ_F decreases). Hence an increase in δ_F acts in the opposite direction to an increase in σ, which increases θ_F. This argument establishes that the value of the leader increases with uncertainty by more than the value of a simultaneous investor if (i) δ_S is sufficiently large; (ii) δ_L is sufficiently large; and (iii) δ_F is sufficiently small. Similar considerations underlie the sufficient conditions in the other parts of the lemma.

The second step is to compare θ_S and θ_P (the proof of the following lemma comes directly from substitution of θ_S into equation (7)).

Lemma 2 *θ_S is greater (less) than θ_P iff*

$$\frac{\delta_S}{1+\delta_S} < (>) \frac{\delta_L}{1+\delta_F}.$$

The lemma gives the intuitive condition that θ_S is greater than θ_P if and only if δ_S is sufficiently small (since $\delta_S/(1+\delta_S)$ is increasing in δ_S) and/or δ_L sufficiently large and δ_F sufficiently small.

Lemmas 1 and 2 can be combined to give sufficient conditions for the trigger point of the first investor to decrease as σ rises, as a result of a change in the equilibrium pattern of investment.

Proposition 5 1. *Suppose that the conditions in part 1 of lemma 1 hold, and that $\delta_S/(1+\delta_S) < \delta_L/(1+\delta_F)$. Then there exists a $\sigma'' > \sigma' > 0$ such that $\lambda_E(\sigma') > 0 > \lambda_E(\sigma'')$; and $\theta_S > \theta_P$.*

2. *Suppose that the conditions in part 2 of lemma 1 hold, and that $\delta_S/(1+\delta_S) > \delta_L/(1+\delta_F)$. Then there exists a $\sigma'' > \sigma' > 0$ such that $\lambda_E(\sigma') < 0 < \lambda_E(\sigma'')$; and $\theta_S < \theta_P$.*

Both cases give sufficient conditions for an increase in uncertainty from σ' to σ'' to cause the trigger point of the first investor to decrease. In the second case the trigger points of both investors decrease.

(Proposition 5 follows directly from the two preceding lemmas, and so is stated without proof.)

The proposition gives, then, a second reason why a model of investment under uncertainty with strategic interaction can be very different from the single-agent case. The reason now is that there are two types of equilibrium in the multi-agent case. An increase in uncertainty can cause a switch from one type to another in such a way as to decrease the trigger point of the first investor. Of course, this factor cannot arise in the single-agent case.

The final issue to consider is: how empirically relevant is this unusual result? To focus the discussion, we concentrate on proposition 4 (which gives sufficient conditions for uncertainty to reduce the leader's investment trigger, when investment is sequential). Recall that the proposition requires that the first-mover advantage δ_L must be large (certainly positive), and δ_F and σ small. The first part of this condition may seem unusual—it requires that investment by a second agent increase the flow payoff to the first investor. If investment takes the form of entry into a product market, then this would require, for example, that the demand expansion effect of an additional firm outweighs increased competition.

Note, however, that δ_L does not need to be very large at all. Figure 3 plots the critical value of δ_L as a function of δ_F, for three different values of σ: 1%, 2% and 2.5%. δ_L must be greater than this value for the sufficient conditions of proposition 4 to be satisfied. The figure illustrates that when σ is low, the sufficient conditions can be satisfied for values of δ_L and δ_F close to 0. For example, when $\sigma = 1\%$ and $\delta_F = -0.04$, δ_L must be greater than about 0.01 for the sufficient conditions to be satisfied. In words: investment by the second agent must increase the flow payoff of the first investor by 1% or more. The ultimate test of the relevance of the proposition is how it matches data: the pattern of investment and the level of profits observed in a particular market. Nevertheless, these parametric conditions do not seem implausible.

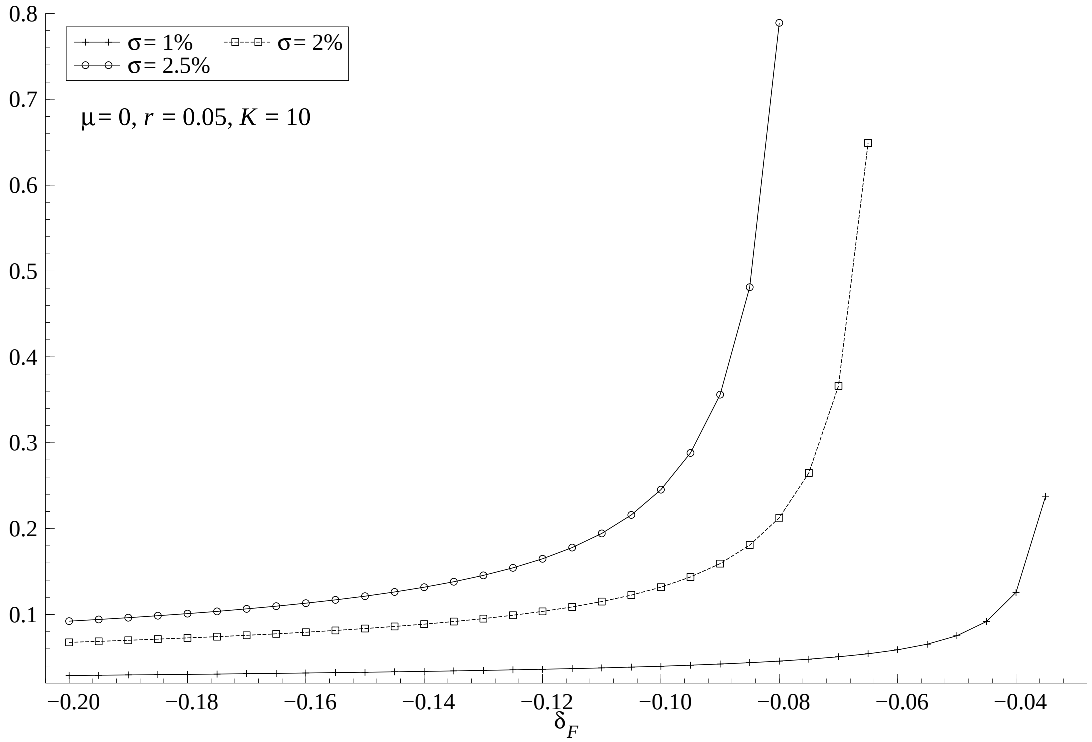

Figure 3: The critical value of δ_L

Furthermore, the result and its empirical relevance is not specific to our model. The ratio of the leader's and follower's values anticipating the follower's investment is key for the result. In our model, the ratio is $\beta\delta_L/(1+\delta_F)$; when $\delta_L > 0$, this ratio is positive and tends to infinity as σ tends to zero (so that β tends to infinity). More generally, the result requires that, when the first-mover advantage is sufficiently large, the ratio increases above 1 as uncertainty decreases. The follower's option value at any level of the state variable below its trigger point decreases as uncertainty is reduced. This fact is not specific to the particular form of process (see equation (1)) that we use, or the particular payoffs assumed.

With positive externalities between the agents, investment by the follower is valuable to the leader—particularly since it does not require any additional cost for the leader. But once the leader has acted, the leader is unable to affect the follower's investment decision, and must take the investment point of the follower as given. For the follower, investment is costly; but it chooses its investment point optimally. The first factor tends to make the follower's option less valuable than the leader's option-like term; the second factor makes it more valuable. When uncertainty is relatively small, the first factor dominates, and the leader's option-like term is more valuable. Given the form of the value functions, this also means that the leader's value function increases by more than the follower's when uncertainty rises. This result is robust and extends beyond the assumptions used here.

5 Conclusions

This paper has analysed irreversible investment in a project with uncertain returns, when there may be an advantage to being the first investor, and externalities to investing when others also invest. It therefore extends standard 'real options' analysis to a setting where there are general strategic interactions and externalities between investing agents. This framework captures a variety of strategic situations and industry settings, and encompasses a number of earlier contributions.

We believe that this is an important area of research. The real options literature has taught us that an option value is created by irreversibility and uncertainty; this option value typically leads to delayed investment, where the degree of delay increases with uncertainty. Strategic interactions and externalities, omitted from the standard real options analysis, can have important qualitative effects on investment behaviour. In particular, we have shown that due to the interaction of pre-emption with positive externalities, greater uncertainty can actually hasten, rather than delay, investment, contrary to the usual presumption.

Appendix

A Value Functions

Let the follower's value functions be denoted F_0 and F_1, before and after its investment respectively.

Prior to investment, the follower holds an option to invest but receives no flow payoff. In this 'continuation' region, in any short time interval dt starting at time t the follower experiences a capital gain or loss dF_0 . The Bellman equation for the value of the investment opportunity is therefore

$$F_0 = \exp(-rdt)\mathbb{E}_t\left[F_0 + dF_0\right]. \tag{A.10}$$

Itô's lemma and the GBM equation (1) gives the ordinary differential equation (ODE)

$$\frac{1}{2}\sigma^2\theta^2 F_0''(\theta) + \mu\theta F_0'(\theta) - rF_0(\theta) = 0. \tag{A.11}$$

From equation (1), it can be seen that if θ ever goes to zero, then it stays there forever. Therefore the option to invest has no value when $\theta = 0$, and must satisfy the boundary condition $F_0 = 0$. Solution of the differential equation subject to this boundary condition gives $F_0 = b_F\theta^\beta$, where b_F is a positive constant and $\beta > 1$ is the positive root of the quadratic equation $\mathcal{Q}(z) = \frac{1}{2}\sigma^2 z(z-1) + \mu z - r$; i.e., $\beta = \frac{1}{2}\left(1 - \frac{2\mu}{\sigma^2} + \sqrt{\left(1 - \frac{2\mu}{\sigma^2}\right)^2 + \frac{8r}{\sigma^2}}\right)$.

Now consider the value of the agent in the 'stopping' region, in which the value of θ is such that it is optimal to invest at once. Since investment is irreversible, the value of the agent in the stopping region is given by the expected value alone with no option value terms. When the level at time t of the state variable is θ_t, this is

$$F_1(\theta_t) = \mathbb{E}_t\left[\int_t^{\infty} \exp\left(-r(\tau - t)\right)(1+\delta_F)\theta_\tau d\tau - K\right].$$

θ is expected to grow at rate μ, so that

$$F_1(\theta) = \frac{(1+\delta_F)\theta}{r-\mu} - K. \tag{A.12}$$

The boundary between the continuation region and the stopping region is given by a trigger point θ_F of the stochastic process such that continued delay is optimal for $\theta < \theta_F$ and immediate investment is optimal for $\theta \geq \theta_F$. The optimal stopping time T_F is then defined as the first time that the stochastic process θ hits the interval $[\theta_F, \infty)$ from below.

Putting together the two regions gives the follower's value function:

$$F(\theta) = \begin{cases} b_F\theta^{\beta} & \theta < \theta_F, \\ \frac{(1+\delta_F)\theta}{r-\mu} - K & \theta \geq \theta_F, \end{cases} \tag{A.13}$$

given that the leader invests at $\theta_P < \theta_F$.

There are three components to the leader's value function holding over different ranges of θ. The first L_0 describes the value of investment before the leader (and so the follower) has invested; the second L_1 after the leader has invested, but before the follower has done so; and the third L_2, after the follower has invested. The first and third components are equivalent to those of the follower, determined previously. The second component is new, and so is derived in detail.

After the leader has invested, it has no further decision to take and its payoff is given by the expected value of its investment. This payoff is affected, however, by the action of the follower investing later at θ_F. Taking account of subsequent investment by the follower, the leader's post-investment payoff is given by

$$L_1(\theta_t) = \mathbb{E}_t\left[\int_t^{T_F} \exp\left(-r(\tau - t)\right)\theta_\tau d\tau + \int_{T_F}^{\infty} \exp\left(-r(\tau - t)\right)(1+\delta_L)\theta_\tau d\tau - K\right]. \quad \text{(A.14)}$$

The Bellman equation for the leader is

$$L_1 = \theta dt + \exp\left(-rdt\right)\mathbb{E}_t\left[L_1 + dL_1\right]. \quad \text{(A.15)}$$

Using Itô's lemma and equation (1) gives

$$\frac{1}{2}\sigma^2\theta^2 L_1''(\theta) + \mu\theta L_1'(\theta) - rL_1(\theta) + \theta = 0. \quad \text{(A.16)}$$

As before, investment has no value when $\theta = 0$, and so $L_1 = \frac{\theta}{r-\mu} + b_{L1}\theta^\beta$, where b_{L1} is a constant. The first part of the value function L_1 gives the expected value of investment before the follower invests, while the second is an option-like term reflecting the value (due to externalities) to the leader of future investment by the follower.

B Proof of Proposition 1

The follower's equilibrium investment point, θ_F, is given by equation (3). In this proof, we derive the leader's investment point and establish that it is given by rent equalization.

Define

$$\Delta(\theta) \equiv \frac{\theta}{r-\mu} - K - \left(\frac{\theta}{\theta_F}\right)^\beta \left(\frac{1-\beta\delta_L+\delta_F}{1+\delta_F}\right)\frac{K}{\beta-1} \quad \text{(B.17)}$$

i.e., $L(\theta) - F(\theta)$, where $L(\theta)$ is conditional on the leader having invested, and $F(\theta)$ is conditional on the leader having invested but not the follower. There are three possibilities: that there are (i) no, (ii) one or (iii) multiple solutions to equation (B.17). We use the following facts: (i) $\Delta(\theta)$ is a continuously differentiable function of θ; (ii) $\Delta(0) = -K < 0$; (iii) $\Delta(\theta_L) = \frac{K}{(\beta-1)(1+\delta_F)}\left(\left(\frac{\theta_L}{\theta_F}\right)^\beta \beta\delta_L + (1 - \left(\frac{\theta_L}{\theta_F}\right)^\beta)(1+\delta_F)\right)$; (iv) since, from assumption 1, $\delta_L \geq \delta_F$, $\Delta(\theta_L) \geq \frac{K}{(\beta-1)(1+\delta_F)}\left(\left(\frac{\theta_L}{\theta_F}\right)^\beta \beta\delta_F + (1 - \left(\frac{\theta_L}{\theta_F}\right)^\beta)(1+\delta_F)\right)$; (v) for all $\delta_F \in [-1, 0]$ (see assumption 1) and $\beta \geq 1$, $\left(\frac{\theta_L}{\theta_F}\right)^\beta \beta\delta_F + (1 - \left(\frac{\theta_L}{\theta_F}\right)^\beta)(1+\delta_F) \geq 0$. Hence, by the intermediate value theorem, there exists a value $\theta_P < \theta_L$ such that $\Delta(\theta_P) = 0$, and $\Delta(\theta)$ is less (greater) than 0 for θ immediately less (greater) than θ_P.

From assumption 3, immediate investment is not profitable for any agent. This implies that $L(\theta) < 0 \; \forall \theta \leq \theta_0$. Hence $L(\theta_0) < F(\theta_0)$, since $F(\theta) \geq 0 \; \forall \theta$. Therefore $\Delta(\theta_0) < 0$ and $\theta_P > \theta_0$. Hence in the sequential equilibrium, no agent invests when $\theta \in [\theta_0, \theta_p)$, where θ_P is the smallest solution to $\Delta(\theta_P) = 0$. At $\theta = \theta_P$, the leader invests; at $\theta_F > \theta_P$, the follower invests.

C Proof of Proposition 4

The difference between the values of the leader's option-like term and the follower's option associated with the follower's investment is

$$\Delta(\theta, \beta) \equiv (b_{L1} - b_F)\theta^\beta = \left(\frac{\beta\delta_L - (1 + \delta_F)}{1 + \delta_F} \right) F(\theta)$$

where $F(\theta) \equiv b_F \theta^\beta > 0$ for $\theta \in (\theta_P, \theta_F)$. The objective of the proof is to establish that $\partial\Delta(\theta_P, \beta)/\partial\beta \leq 0$, so that $\partial\Delta(\theta_P, \beta)/\partial\sigma \geq 0$, which means that the leader's value function increases by more than the follower's (evaluated at $\theta = \theta_P$) for a small increase in σ. If this is the case, then θ_P must decrease in σ.

We start by evaluating the derivative of $\Delta(\theta, \beta)$ with respect to β:

$$\frac{\partial\Delta(\theta, \beta)}{\partial\beta} = \frac{\delta_L F(\theta) + (\beta\delta_L - (1 + \delta_F))\frac{\partial F(\theta)}{\partial\beta}}{1 + \delta_F}.$$

But

$$\frac{\partial F(\theta)}{\partial\beta} = F(\theta) \ln\left(\frac{\theta}{\theta_F}\right).$$

Hence

$$\frac{\partial\Delta(\theta, \beta)}{\partial\beta} = \frac{F(\theta)}{1 + \delta_F} \left(\delta_L + (\beta\delta_L - (1 + \delta_F)) \ln\left(\frac{\theta}{\theta_F}\right) \right) \tag{C.18}$$

for $\theta \in [\theta_P, \theta_F]$.

Now note that $\theta_P \leq \theta_L$ (see the proof of proposition 1 in section B). Hence

$$\ln\left(\frac{\theta_P}{\theta_F}\right) \leq \ln(1+\delta_F).$$

There are two cases to consider: (i) $\beta\delta_L - (1+\delta_F) \leq 0$ and (ii) $\beta\delta_L - (1+\delta_F) > 0$. We now show that the first case would violate assumption 1 if it lead to $\partial\Delta(\theta_P, \beta)/\partial\beta \leq 0$. In order for the latter inequality to hold in case (i), it must be that

$$\delta_L \leq (\beta\delta_L - (1+\delta_F)) \ln\left(\frac{\theta_P}{\theta_F}\right); \tag{C.19}$$

but the right-hand side of this inequality is less than or equal to $-(\beta\delta_L - (1+\delta_F))\ln(1+\delta_F) \leq -(\beta\delta_F - (1+\delta_F))\ln(1+\delta_F)$, where the second part of the statement follows from assumption 1, that $\delta_F \leq \delta_L$. Hence $\partial\Delta(\theta_P, \beta)/\partial\beta \leq 0$ only if $\delta_L \leq -(\beta\delta_F - (1+\delta_F))\ln(1+\delta_F)$. But the right-hand side of this inequality, $-(\beta\delta_F - (1+\delta_F))\ln(1+\delta_F)$, is less than or equal δ_F when $\beta = 1$: $\ln(1+\delta_F) \leq \delta_F$ $\forall\delta_F \in [-1, 0]$, with equality only if $\delta_F = 0$. And the right-hand side is decreasing in β; hence the right-hand side is less than or equal to δ_F for all $\delta_F \in [-1, 0]$ and $\beta \geq 1$, with equality only if $\delta_F = 0$. Therefore $\partial\Delta(\theta_P, \beta)/\partial\beta \leq 0$ in case (i) only if $\delta_L \leq \delta_F$, with equality only if $\delta_F = 0$. This is consistent with assumption 1 if and only if $\delta_L = \delta_F = 0$; but then, equation (C.18) shows that $\partial\Delta(\theta_P, \beta)/\partial\beta > 0$, which is a contradiction.

Hence $\partial\Delta(\theta_P, \beta)/\partial\beta \leq 0$ can hold, if at all, only in case (ii), when $\beta\delta_L - (1+\delta_F) > 0$. The necessary and sufficient condition for $\partial\Delta(\theta_P, \beta)/\partial\beta \leq 0$ is again given by equation (C.19). In this case, this means that $\delta_L \leq -(\beta\delta_F - (1+\delta_F))\ln(1+\delta_F)$ is a sufficient condition for $\partial\Delta(\theta_P, \beta)/\partial\beta \leq 0$. Re-arranging this inequality yields

$$\delta_L(1 + \beta\ln(1+\delta_F)) \leq (1+\delta_F)\ln(1+\delta_F). \tag{C.20}$$

This inequality cannot be satisfied if $1 + \beta\ln(1+\delta_F) > 0$ and assumption 1 holds (in particular, $\delta_L \geq \delta_F$). To see why, notice that equation (C.20) would require in this case that $\delta_L \leq \underline{\delta_L}$, where, as in the proposition,

$$\underline{\delta_L} \equiv \frac{(1+\delta_F)\ln(1+\delta_F)}{1+\beta\ln(1+\delta_F)}$$

and $\underline{\delta_L} \leq 0$. Assumption 1 then requires that $\underline{\delta_L} \geq \delta_F$. But this in turn requires that $(\beta - 1)(1+\delta_F)\ln(1+\delta_F) - \beta\ln(1+\delta_F) + \delta_F \leq 0$. When $\beta = 1$, this inequality requires that $-\ln(1+\delta_F) + \delta_F \leq 0$, which is violated for all $\delta_F \in [-1, 0)$ and holds with equality only when $\delta_F = 0$. Since $(\beta-1)(1+\delta_F)\ln(1+\delta_F) - \beta\ln(1+\delta_F) + \delta_F$ is increasing in β, this means that $(\beta-1)(1+\delta_F)\ln(1+\delta_F) - \beta\ln(1+\delta_F) + \delta_F \geq 0$, with equality only when $\delta_F = 0$.

Hence inequality (C.20) requires that $1 + \beta\ln(1+\delta_F) < 0$; and hence that $\delta_L \geq \underline{\delta_L}$, where $\underline{\delta_L} \geq 0$.

D Proof of Lemma 1

Differentiate λ_E with respect to β:

$$\frac{\partial \lambda_E}{\partial \beta} = (1+\delta_S)^{\beta}\ln(1+\delta_S) - \delta_L(1+\delta_F)^{\beta-1}(1+\ln(1+\delta_F)). \tag{D.21}$$

It is sufficient for λ_E to be an increasing function of β that all terms in equation (D.21) be positive. Hence joint sufficient conditions are: (i) $\delta_S \geq 0$, so that $\ln(1+\delta_S) \geq 0$; (ii) $-\delta_L(1+\ln(1+\delta_F)) \geq 0$, which in turn requires that either (a) $\delta_L \geq 0$ and $1 + \ln(1+\delta_F) \leq 0$, i.e., $\delta_F \leq e^{-1} - 1$, or (b) the converse. To complete the proof of the first part, note that β is decreasing in σ. The proof of the second part is very similar, and so is omitted.

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