Economics Division
School of Social Sciences
University of Southampton
Southampton SO17 1BJ UK

Discussion Papers in Economics and Econometrics

The Existence and Uniqueness of Monotone Pure Strategy Equilibrium in Bayesian Games

Robin Mason and Akos Valentinyi

No. 0710

ISSN 0966-4246

The Existence and Uniqueness of Monotone Pure Strategy Equilibrium in Bayesian Games*

Robin Mason
University of Southampton and CEPR
Ákos Valentinyi
University of Southampton, Institute of Economics of the Hungarian Academy of Sciences, and CEPR

12 October 2007

Abstract

This paper provides a sufficient condition for existence and uniqueness of equilibrium, which is in monotone pure strategies, in a broad class of Bayesian games. The argument requires that the incremental interim payoff—the expected payoff difference between any two actions, conditional on a player's realised type—satisfies two conditions. The first is *uniform strict single-crossing* with respect to own type. The second condition is *Lipschitz continuity* with respect to opponents' strategies. Our main result shows that, if these two conditions are satisfied, and the bounding parameters satisfy a particular inequality, then the best response correspondence is a contraction, and hence there is a unique equilibrium of the Bayesian game. Furthermore, this equilibrium is in monotone pure strategies. We characterize the uniform monotonicity and Lipschitz continuity conditions in terms of the model primitives. We also consider a number of examples to illustrate how the approach can be used in applications.

Keywords: Bayesian games, Existence, Uniqueness, Monotone pure strategy equilibrium, Contraction Mapping.

JEL classification: C72; D82.

*We are grateful to Grant Hillier, Godfrey Keller, Stephen Morris, John Quah, Hyun Shin and Juuso Välimäki for very helpful comments and tips. We thank seminar participants at Birmingham, Cardiff Business School, Essex, the EUI, Helsinki, Keele, Oxford and Southampton. We also thank the editor and three referees. Obviously, they are not responsible for any errors. Robin Mason acknowledges financial support from the ESRC Research Fellowship Award R000271265.

1 Introduction

This paper provides a sufficient condition for existence and uniqueness of equilibrium, which is in monotone pure strategies, in a broad class of games of incomplete information. A sufficient condition for existence and uniqueness has been established for global games (see among others Frankel, Morris, and Pauzner (2003)). More generally, existence, but not uniqueness, of monotone pure strategy equilibrium has been established for Bayesian games that satisfy a Spence-Mirrlees single-crossing property: see e.g., the seminal paper of Athey (2001). Our contribution is to establish a simple condition that ensures *both* existence and uniqueness of equilibrium in monotone pure strategies in a broad class of games.

The basic intuition for our result is relatively straightforward. Consider the incremental interim payoff—the expected payoff difference between any two actions, conditional on a player's realised type. Two factors affect this: a player's own type (a non-strategic effect), and the strategy profile of its opponents (a strategic interaction). We require a player's incremental interim payoff to be strictly increasing in its type. This means that a player's best response must be in monotone pure strategies, whatever strategy profile is played by its opponents. Uniqueness of equilibrium would clearly follow if opponents' strategies have no effect on a player's best response. More generally, we require in addition that a player's type has a greater effect than its opponents' strategy profile on its incremental interim payoff. A large number of papers have observed that multiple equilibria can arise when strategic interactions are important. (We discuss some of these papers below.) This second sufficient condition ensures that strategic interaction is dominated by non-strategic effects. Consequently, when our sufficient conditions are satisfied, there is a unique equilibrium, which is monotone pure strategies.

We formulate this intuition in a rigorous manner and show that if two bounds are satisfied, then the best response correspondence is a contraction, which ensures both existence and uniqueness of equilibrium. Our first bound is *uniform strict single-crossing* with respect to own type. This condition requires the incremental interim payoff to be strictly increasing in a player's type, with the rate of increase uniformly bounded from below by a strictly positive constant φ_1. An immediate consequence of this condition is that the strict single crossing property holds for any strategy profile played by opponents; hence each player's best response to any strategy profile is a monotone pure strategy. The second condition is *Lipschitz continuity* with respect to opponents' strategies. This condition requires a change in the strategy profile of a player's opponents to have a bounded effect on the incremental interim payoff, where the bound is a positive uniform (Lipschitz) constant φ_2. Our main result shows that, if the incremental interim payoff satisfies uniform strict single-crossing and Lipschitz continuity, and if the bounding constants satisfy $\varphi_2 < \varphi_1$, then the best response correspondence is a contraction, and hence there is a unique equilibrium of the game of incomplete information. Furthermore, this equilibrium is in monotone pure strategies.

Having established a sufficient condition for existence and uniqueness in terms of bounds on the incremental interim payoff, we relate the sufficient condition to bounds on *ex post* payoffs and conditional densities, for two classes of applications: continuous games (in which *ex post* payoffs are Lipschitz continuous in actions), and discontinuous games (such as auctions). The details of the bounds vary across the two classes. But they share common features. First, a player's payoff must be sufficiently sensitive to its own type. Secondly, the effect that the realised actions have on the *ex post* payoff of each player is bounded above; hence strategic interactions cannot be too important. Finally, players cannot have 'too much' information about the types of their opponents. (What 'too much' means varies according to the application.) These three features ensure that higher types prefer higher actions, and hence best responses are monotone pure strategies. They also ensure that best responses are sufficiently insensitive to opponents' strategies.

Another well-known approach to existence and uniqueness of equilibrium is developed in the literature on global games. Global games are games of incomplete information where type spaces are determined by the players each observing a noisy signal of an underlying state; see Carlsson and van Damme (1993), Morris and Shin (1998), Morris and Shin (2003) and Frankel, Morris, and Pauzner (2003). If players' actions are strict strategic complements, if there are "dominance regions" (i.e., types for which there is a strictly dominant action), and if players' signals are sufficiently informative about the true underlying state, then global games have a unique, dominance solvable equilibrium. Existence of equilibrium is assured by the results of Milgrom and Roberts (1990) on supermodular games. In the unique surviving strategy profile, each player's action is a nondecreasing function of its signal i.e., the unique equilibrium is in monotone pure strategies.

A major advantage of our approach, relative to global games, is that we require neither strategic complementarities nor dominance regions. Dispensing with these two assumptions means that iterated elimination of dominated strategies cannot be used to solve for equilibrium. Our approach therefore differs in terms of technical detail: instead of iterated deletion, we use a contraction mapping. It also differs in terms of the detailed intuition for the result. At one level, both approaches generate uniqueness by introducing *heterogeneity* of some type. In a global game, uniqueness requires that a player's assessment of the probability that an opponent's type is lower than his should be sufficiently insensitive to the player's type. This occurs when heterogeneity is very small and highly correlated. In contrast, our approach requires large heterogeneity, in two ways: a player's type is sufficiently uninformative about the types of its opponents; and conditional densities are bounded above. (See Morris and Shin (2005) for further discussion of this distinction.) In summary: our approach shares with global games the general feature of establishing a unique equilibrium, which is in monotone pure strategies; but in all other respects, the two approaches are distinct.

A number of papers have analysed conditions under which monotone pure strategy equilibria exist in class of incomplete information games that are broader than global games. In particular, Athey (2001) establishes existence of monotone pure strategy equilibria, using a single crossing condition (SCC) on incremental interim payoffs. This condition requires that, when higher types play weakly higher actions, the difference in a player's interim payoff from a high action versus a low one crosses zero at most once and from below, as a function of its type. She shows further that games in which *ex post* payoffs are supermodular or log-supermodular in all players' actions and types, and in which types are affiliated, satisfy the SCC.[1]

While there is some relation between our paper and this literature—both establish existence of monotone pure strategy equilibrium—there are several differences. Our objective of establishing uniqueness, rather than just existence, means that our assumptions and methods are quite different. We, like Athey and McAdams, require a single-crossing condition, but one which is stricter than theirs. Furthermore, we require that each player's incremental interim payoff is Lipschitz continuous in opponents' strategies. These different conditions on incremental interim payoffs translate to different assumptions on the model primitives. The technical details of our argument are quite different from those of Athey and McAdams, who both establish convexity of the best-response correspondence in order to apply a fixed point theorem. In contrast, we use a contraction mapping argument. We therefore see our approach and e.g., Athey's as complementary.

Finally, our analysis helps to clarify the mechanism at work in a number of previous papers that have found, in a variety of situations, that heterogeneity can ensure uniqueness of equilibrium. For example, in a canonical two-by-two public good model in Fudenberg and Tirole (1991, pp. 211–213), there are two pure strategy equilibria in the common knowledge game. If the distribution of types satisfies certain conditions, there is only one equilibrium in the incomplete information game. One such condition is that the maximum value of the density is sufficiently small; following Grandmont (1992), this can be interpreted as requiring a sufficient degree of heterogeneity between the players. Burdzy, Frankel, and Pauzner (2001) demonstrate that there can be a unique equilibrium in a

[1]Earlier work, e.g., Milgrom and Weber (1985), established existence of pure strategy equilibria in games with a finite number of actions and (conditionally) independent types, but without requiring strategic complementarity. Milgrom and Roberts (1990) and Vives (1990) use lattice-theoretic methods to establish the existence pure strategy equilibria in supermodular games; these equilibria need not be monotone. McAdams (2003) generalizes Athey (2001) to multidimensional action and type spaces. Van Zandt and Vives (2005) take a different approach to establish existence using lattice-theoretic methods. In recent work, Reny (2006) has shown that the SCC can be weakened by using a particular fixed point theorem, when *ex post* payoffs are continuous in actions.

model in which players face exogenous shocks, can change their action only occasionally, and are heterogeneous in the frequency with which they can change their action. Herrendorf, Valentinyi, and Waldmann (2000) show how heterogeneity in the manufacturing productivity (rather than the information) of agents in a two-sector, increasing returns-to-scale model can remove indeterminacy and multiplicity of equilibrium. Glaeser and Scheinkman (2003) show that if there is not too much heterogeneity among players, then there can be multiple equilibria in social interaction games. In all of these papers, heterogeneity plays some part in ensuring the uniqueness of equilibrium. Our analysis shows exactly what form of heterogeneity is needed, in terms of the informational structure of the game; and exactly what mechanism is at work when heterogeneity yields uniqueness.

The rest of the paper is structured as follows. Section 2 presents the general analysis, identifying the sufficient condition to ensure uniqueness of equilibrium. In sections 3 and 4, we characterize our sufficient condition for equilibrium existence and uniqueness for two classes of applications: continuous and discontinuous games. Section 5 concludes. Longer proofs are in the appendix.

2 The General Model

Consider a game of incomplete information between I players, $i \in I \equiv \{1, \ldots, I\}$, where each player first observes its own type, $t_i \in T_i \equiv [\underline{t}_i, \bar{t}_i] \subset \mathbb{R}$ and then takes an action a_i from an action set A_i that is a compact subset of the real line $A_i \subset \mathbb{R}$. Let $\boldsymbol{a}$ denote an action profile: $\boldsymbol{a} = (a_1, \ldots, a_I)$; and let $A \equiv \times A_i$ the space of action profiles. A type profile and the space of type profiles are similarly defined as $\boldsymbol{t} \equiv (t_1, \ldots, t_I)$ and $T \equiv \times T_i$. Finally, let $\boldsymbol{a}_{-i}$ denote the profile of actions of all other players, and A_{-i} the space of all such action profiles. A similar notation is adopted for type profiles, strategy profiles, marginals etc.. The joint distribution of players' types is given by the probability measure η on the (Borel) subsets of T. The marginal distribution on each T_i is denoted η_i.

Players use behavioural strategies. A behavioural strategy for player i is a measurable function $\mu_i : \mathcal{A}_i \times T_i \to [0,1]$ where $\mathcal{A}_i$ is the collection of Borel subsets of A_i, with the following properties: (i) for every $B \in \mathcal{A}_i$, the function $\mu_i(B,\cdot) : T_i \to [0,1]$ is measurable; (ii) for every $t_i \in T_i$, the function $\mu_i(\cdot, t_i) : \mathcal{A}_i \to [0,1]$ is a probability measure. Hence when player i observes its type t_i, it selects an action in A_i according to the measure $\mu_i(\cdot, t_i)$. A pure strategy in behavioural form is simply a function that returns a probability measure that is concentrated on the graph of a classical pure strategy.[2] A *monotone pure strategy* is a pure strategy such that a player of higher type chooses a weakly higher action than a player of lower type. Denote the set of behavioural strategies for player i by M_i.

Let $\boldsymbol{\mu}_{-i} \in M_{-i}$ denote the vector of behavioural strategies played by the opponents of player i. The interim payoff of player i (i.e., when it knows its type t_i) is written as:

$$U_i(a_i, t_i, \boldsymbol{\mu}_{-i}) = \int_{T_{-i}} \int_{A_{-i}} u_i(\boldsymbol{a}, \boldsymbol{t}) \prod_{j \neq i} d\mu_j(\cdot, t_j) f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i}$$

where $f(\boldsymbol{t}_{-i}|t_i)$ is the conditional density of types. Let the incremental interim payoff be defined as

$$\Delta U_i(a_i, a_i', t_i, \boldsymbol{\mu}_{-i}) \equiv U_i(a_i, t_i, \boldsymbol{\mu}_{-i}) - U_i(a_i', t_i, \boldsymbol{\mu}_{-i}).$$

The following basic assumption is maintained throughout the paper:

A1 The payoff function $u_i : A \times T \to \mathbb{R}$ is bounded and measurable, and upper semi-continuous in a_i. The types have conditional densities with respect to the Lebesgue measure. The conditional density of $\boldsymbol{t}_{-i}$ given t_i, is denoted $f(\boldsymbol{t}_{-i}|t_i)$ for $i \in I$; it is strictly positive.

Assumption A1 is standard and ensures that the interim payoff $U_i(\cdot)$ exists and that players possess best responses.

[2] An alternative approach would use distributional strategies. A distributional strategy for player i is a probability measure μ_i on $A_i \times T_i$ such that the marginal distribution on T_i is η_i i.e., $\mu_i(A_i \times S) = \eta_i(S)$ for any Borel subset S of T_i; see Milgrom and Weber (1985). As Milgrom and Weber show, there is a many-to-one mapping from behavioural strategies to distributional strategies. In fact, there is little difference between the two approaches here, since we establish quickly (see lemma 1) a sufficient condition so that in equilibrium, only monotone pure strategies are used. It is slightly more convenient, however, to use behavioural strategies.

Two conditions are central to our argument.

Definition 1 (Uniform Strict Single-Crossing) *There is a constant $\varphi_1 > 0$ such that for all $a_i \geq a_i'$, $t_i \geq t_i'$ and $\boldsymbol{\mu}_{-i} \in M_{-i}$,*

$$\Delta U_i(a_i, a_i', t_i, \boldsymbol{\mu}_{-i}) - \Delta U_i(a_i, a_i', t_i', \boldsymbol{\mu}_{-i}) \geq \varphi_1 (t_i - t_i')(a_i - a_i'). \tag{1}$$

Note that definition 1 involves a stronger condition than the single-crossing property that is commonly used (see e.g., Athey (2001)). Uniform strict single-crossing implies single crossing: and in fact, it ensures that single-crossing holds for all $\boldsymbol{\mu}_{-i} \in M$, and not just for opponents' strategy profiles that are monotonic. Uniform strict single-crossing implies, in addition, that there is *strict* single crossing. Moreover, it requires that the same lower bound φ_1 can be used for all $a_i \geq a_i'$, $t_i \geq t_i'$ and $\boldsymbol{\mu}_{-i} \in M_{-i}$.

We use next the results of Milgrom and Shannon (1994) to establish that uniform strict single-crossing implies that a player's best response to any strategy profile of its opponents is a monotone pure strategy.

Lemma 1 *Suppose that assumption A1 holds. If uniform strict single-crossing holds, then any best response of player $i \in I$ to any profile of opponents' strategies is a monotone pure strategy.*

Proof The action set A_i is totally ordered (because $\{0,1\} \subseteq A_i \subset [0,1]$), implying that $U_i(a_i, t_i, \boldsymbol{\mu}_{-i})$ is quasi-supermodular in a_i.[3] Moreover, A_i is independent of t_i, and $T_i \in \mathbb{R}$ is also totally ordered. Given uniform strict single-crossing, $U_i(a_i, t_i, \boldsymbol{\mu}_{-i})$ satisfies the strict single crossing property. Therefore by the Monotone Selection Theorem 4' of Milgrom and Shannon (1994), every selection from the set $\arg\max_{a_i \in A_i} U_i(a_i, t_i, \boldsymbol{\mu}_{-i})$ is monotone non-decreasing in t_i. The strict single crossing property implies that there is indifference only on sets of measure zero. $\square$

[3] A function $h : X \to \mathbb{R}$ on a lattice X is quasi-supermodular if (i) $h(x) \geq h(x \wedge y)$ implies $h(x \vee y) \geq h(y)$ and (ii) $h(x) > h(x \vee y) > h(y)$. Here, $\wedge$ is the greatest lower bound, or meet operator; $\vee$ is the least lower bound, or join operator.

For the rest of this section, we maintain the assumption of uniform strict single-crossing. We can, therefore, restrict attention to monotone pure strategies for each player $i \in I$. Denote a monotone pure strategy by $\alpha_i : T_i \to A_i$, where $\alpha_i(t_i) \geq \alpha_i(t_i')$ for $t_i \geq t_i'$. Let $\boldsymbol{\alpha}(\boldsymbol{t})$ be the monotone pure strategy profile, and $\boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})$ be the strategy profile of the opponents of player i. Let S be the set of joint monotone pure strategies, and $\boldsymbol{\phi} : S \to S$ be the vector of best reply correspondences. A Bayesian Nash equilibrium is a fixed point of $\boldsymbol{\phi}$.

Next we introduce a metric that is used in stating our second important condition. Let d on S be defined as[4]

$$d(\boldsymbol{\alpha}, \boldsymbol{\alpha}') \equiv \sup_{i \in I} \sup_{\rho \in \mathbb{R}} \{t_i - t_i' | \alpha_i(\tau_i) < \rho < \alpha_i'(\tau_i) \ \vee \ \alpha_i'(\tau_i) < \rho < \alpha_i(\tau_i), \forall \tau_i \in [t_i', t_i] \subset T_i\}. \quad (2)$$

Thus, $d(\boldsymbol{\alpha}, \boldsymbol{\alpha}')$ is the supremum of the length of all intervals over which for some player i, and some $\rho \in \mathbb{R}$, one of $\alpha_i(t_i)$ and $\alpha_i'(t_i)$ is strictly above ρ and the other is strictly below ρ.[5] It is easy to see that d satisfies the properties of a metric, and that it renders the space of joint pure strategies a complete metric space. It is also noteworthy that in the case of discrete action space, it is related to Athey (2001) representation of monotone pure strategies. Let $\boldsymbol{x}_i = (x_{ij})_{j=1}^K$ be a vector of jump points in player i's monotone pure strategy, where the jump points indicate the type at which player i switches from action j to action j'; K is the cardinality of A_i. The joint vector of jump points $\boldsymbol{x}$ therefore represents $\boldsymbol{\alpha}$. Then $d(\boldsymbol{\alpha}, \boldsymbol{\alpha}') = \max_i \max_j |x_{ij} - x_{ij}'|$.

The second condition which is central to our argument is the following.

Definition 2 (Lipschitz Continuity) *There is a finite constant $\varphi_2 \geq 0$ such that for all $a_i \geq a_i'$ and any two monotone pure strategy profiles $\boldsymbol{\alpha}_{-i}, \boldsymbol{\alpha}_{-i}'$,*

$$|\Delta U_i(a_i, a_i', t_i, \boldsymbol{\alpha}_{-i}) - \Delta U_i(a_i, a_i', t_i, \boldsymbol{\alpha}_{-i}')| \leq \varphi_2 (a_i - a_i') d(\boldsymbol{\alpha}_{-i}, \boldsymbol{\alpha}_{-i}'), \quad (3)$$

[4] In this definition, $\vee$ is the logical operator 'or'.

[5] We are indebted to an anonymous referee who suggested this metric.

where $d(\cdot,\cdot)$ is the metric defined in equation (2)*.*

In sections 3 and 4, we derive conditions on the primitives of the model (*ex post* payoffs and conditional densities) that ensure that uniform strict single-crossing and Lipschitz continuity of the incremental interim payoff are satisfied.

We now prove that assumption A1, uniform strict single-crossing and Lipschitz continuity ensure existence and uniqueness of equilibrium. We do this in two steps. Lemma 1 means that any equilibrium must be in monotone pure strategies. Our main result in theorem 1 gives a sufficient condition (consistent with lemma 1) that ensures that the correspondence $\boldsymbol{\phi}(\boldsymbol{\alpha})$ is a contraction mapping, and hence that there is a unique equilibrium, which is in monotone pure strategies.

Theorem 1 *If assumption A1, uniform strict single-crossing and Lipschitz continuity hold, and if $\varphi_2 < \varphi_1$, then the best response correspondence is a contraction, and hence there is a unique equilibrium of the Bayesian game. Furthermore, this equilibrium is in monotone pure strategies.*

Proof See the appendix. □

The intuition for theorem 1 can be seen most clearly when there are two players, $i \in \{1,2\}$ and two actions, $\{0,1\}$. Uniform monotonicity means that, in equilibrium, both players use monotone pure strategies. For simplicity, suppose that there is no dominant action i.e., it is never the case that one of the actions is strictly preferred by all types. Hence high (low) types prefer to play action 1 (0); and there is a threshold type of player i who is indifferent between the two actions i.e., whose incremental interim payoff is zero. Now consider two strategies chosen by player $-i$, both of which can be summarised by the threshold types t'_{-i} and t''_{-i}, say. By Lipschitz continuity, the difference in player i's incremental interim payoffs, for player $-i$'s two strategies, is no greater than φ_2 times the distance between player $-i$'s strategies. The proof of the theorem uses the particular metric in equation (2); in this simple case with binary actions, this metric is just the difference between player $-i$'s threshold types in the two strategies: $|t'_{-i} - t''_{-i}|$.

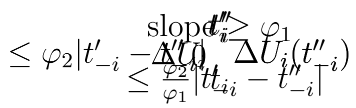

Figure 1: Illustration of Theorem 1

By uniform strict single-crossing, player i's incremental interim payoff increases in its type at a rate greater than φ_1. Hence the change in player i's threshold type can be no greater than φ_2/φ_1 times the difference in player $-i$'s threshold types. The sufficient condition $\varphi_2 < \varphi_1$ then ensures that the change in player i's threshold types is strictly less than the change in player $-i$'s thresholds. Consequently, the best reply of player i is a contraction. This argument is illustrated in figure 1, where, for clarity, player i's incremental interim payoff is drawn as being continuously differentiable and linear in type.[6]

The intuition for theorem 1 will be developed further in the next two sections, where we derive conditions on the primitives of the model. We conclude this section with three remarks. First, weak single-crossing, where the bound $\varphi_1 = 0$, is insufficient for our result, since the strict inequality $\varphi_2 < \varphi_1$ cannot then hold. Secondly, continuity, where the bound φ_2 can be arbitrarily large, is also insufficient for our result, for exactly the same reason. Thirdly, the uniform bounds involved in the uniform strict single-crossing and Lipschitz continuity conditions are stronger than is, strictly speaking, necessary.

[6]In the figure, $\Delta U_i(t'_{-i})$ denotes player i's incremental interim payoff when player $-i$ uses the monotone pure strategy with threshold t'_{-i}.

The bounding parameters φ_1 and φ_2 could depend on the action pairs a_i, a_i', the type pairs t_i, t_i' and the strategy profile pairs $\boldsymbol{\mu}_{-i}, \boldsymbol{\mu}_{-i}'$. The sufficient condition in theorem 1 would then be $\varphi_2(a_i, a_i', t_i, \boldsymbol{\mu}_{-i}, \boldsymbol{\mu}_{-i}') < \varphi_1(a_i, a_i', t_i, t_i', \boldsymbol{\mu}_{-i})$ for all $a_i \geq a_i'$, $t_i \geq t_i'$, and monotone pure strategy profiles $\boldsymbol{\mu}_{-i}, \boldsymbol{\mu}_{-i}'$. This sufficient condition would be very difficult to check in applications. Hence we consider only uniform strict single-crossing and Lipschitz continuity, where the bounding parameters are uniform.

3 Characterizing the existence and uniqueness condition: continuous games

The aim of this section is to find conditions on the primitives of the model—the *ex post* payoff $u_i(\boldsymbol{a}, \boldsymbol{t})$ and the conditional density $f(\boldsymbol{t}_{-i}|t_i)$ for each player $i \in I$—that ensure that the incremental interim payoff satisfies monotonicity and Lipschitz continuity. There are two reasons to do this. The first is that it provides further intuition for how we can ensure existence and uniqueness of equilibrium, in monotone pure strategies. The second is that the conditions on the *ex post* payoff and conditional density are easier to check in applications.

We first note that, if there are types that have a strictly dominant action, then clearly the best response correspondence is uniquely defined for these types. Any assumptions on payoffs and conditional densities that are imposed to ensure existence and uniqueness of equilibrium need apply, therefore, only for types that do not have a strictly dominant action. Hence, define

$$D_i(a_i) \equiv \big\{t_i \in T_i \mid a_i = \arg\max_{a \in A_i} u_i(a, \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i}) \ \forall \boldsymbol{a}_{-i} \in A_{-i} \text{ and } \boldsymbol{t}_{-i} \in T_{-i}\big\}.$$

That is, $D_i(a_i)$ is the set of types for player i over which a_i is a dominant action. Notice that $D_i(a_i)$ could be empty i.e., $\emptyset \subseteq D_i(a_i) \subset T_i$. Let

$$D_i \equiv \cup_{a_i \in A_i} D_i(a_i).$$

D_i is therefore the set of dominance regions for player i. Finally, let $\hat{T}_i \equiv T_i \setminus D_i$, so that $\hat{T}_i$ is the set of types for player i over which there is no dominant action.

Our first step is to bound payoff effects in the non-dominance regions. In the following, actions $a_i, a_i' \in A_i$ and types $t_i, t_i' \in \hat{T}_i$, for all $i \in I$. Let

$$\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, \boldsymbol{t}) \equiv u_i(a_i, \boldsymbol{a}_{-i}, \boldsymbol{t}) - u_i(a_i', \boldsymbol{a}_{-i}, \boldsymbol{t})$$

denote the incremental *ex post* payoff.

U1 Uniformly Positive Sensitivity to Own Action and Type. There is a $\delta \in (0, \infty)$ such that for all $a_i \geq a_i'$, $t_i \geq t_i'$, $\boldsymbol{a}_{-i}, \boldsymbol{t}_{-i}$ and $i \in I$,

$$\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i}) - \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i', \boldsymbol{t}_{-i}) \geq \delta(a_i - a_i')(t_i - t_i').$$

U2 Lipschitz Continuity to Own Action. There is an $\omega \in (0, \infty)$ such that for all $a_i \geq a_i'$, $\boldsymbol{a}_{-i}, \boldsymbol{t}$, and $i \in I$,

$$|\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, \boldsymbol{t})| \leq \omega(a_i - a_i').$$

U3 Uniformly Bounded Sensitivity to Opponents' Action. There is a $\kappa \in (0, \infty)$ such that for all $a_i \geq a_i', \boldsymbol{a}_{-i}, \boldsymbol{a}'_{-i}, \boldsymbol{t}$ and $i \in I$,

$$|\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, \boldsymbol{t}) - \Delta u_i(a_i, a_i', \boldsymbol{a}'_{-i}, \boldsymbol{t})| \leq \kappa(a_i - a_i').$$

Assumption U1 essentially requires that a higher type makes a higher action more appealing to a player. It is similar to, but stronger than, an assumption that a player's payoff function $u_i(a_i, \boldsymbol{a}_{-i}, \boldsymbol{t})$ is supermodular in (a_i, t_i).[7] In our case, supermodularity of u_i in (a_i, t_i) implies that $\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i}) \geq \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i', \boldsymbol{t}_{-i})$; clearly, therefore, the uniform boundedness assumption is stronger. Nevertheless, the assumption is satisfied in a large number of games, including many supermodular games.

Assumptions U1 and U2 place restrictions on the incremental *ex post* payoff, illustrated in figure 2. The incremental *ex post* payoff $\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i})$ must lie in the shaded area drawn in the figure, bounded from below by $-\omega(a_i - a_i')$ and above by $-\omega(a_i - a_i')$ (by assumption U2), with the boundaries having slope δ (by assumption U1). Moreover, $\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i})$ must have a slope of at least δ (again by assumption U1). The curve in the figure illustrates a possibility for the function $\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i})$.

In addition to the assumptions on *ex post* payoffs, we make the following assumptions about the conditional density:

D1 There is a $\iota \in (0, \infty)$ such that for any $t_i > t_i'$ and $i \in I$, $\sqrt{I(t_i, t_i')} \leq \iota(t_i - t_i')$, where

$$I(t_i, t_i') \equiv \mathrm{Var}_{T_{-i}} \left(\frac{f(\boldsymbol{t}_{-i}|t_i) - f(\boldsymbol{t}_{-i}|t_i')}{f(\boldsymbol{t}_{-i}|t_i)} \right).$$

D2 There is a $\nu \in [0, \infty)$ such that $f_j(t_j|t_i) \leq \nu$ for all $i, j \in I$ and $j \neq i$ where

$$f_j(t_j|t_i) = \int_{\times_{k \neq i,j} T_k} f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i}.$$

[7] Let X be a lattice i.e., a partially ordered set that includes both the meet $\wedge$ (the greatest lower bound) and join $\vee$ (the least upper bound) of any two elements in the set. A function $h : X \to \mathbb{R}$ is supermodular if, for all $\boldsymbol{x}, \boldsymbol{y} \in X$, $h(\boldsymbol{x} \vee \boldsymbol{y}) + h(\boldsymbol{x} \wedge \boldsymbol{y}) \geq h(\boldsymbol{x}) + h(\boldsymbol{y})$. In the case that h is twice differentiable, h is supermodular if and only if

$$\frac{\partial^2}{\partial x_i \partial x_j} h(\boldsymbol{x}) \geq 0$$

for all i, j; see Topkis (1998).

$-\omega(a_i - \Delta t_i)$ Slope $= \delta$

t_i

Figure 2: Assumptions U1 and U2

The function defined in assumption D1 is the expectation of the square of a likelihood ratio:

$$\mathbb{E}_{T_{-i}}\left[\left(\frac{f(\boldsymbol{t}_{-i}|t_i')}{f(\boldsymbol{t}_{-i}|t_i)}\right)^2\right],$$

and so is a measure of *differential information.* In the case that the conditional density $f(\boldsymbol{t}_{-i}|t_i)$ is differentiable in t_i, the function is related to the *Fisher information* of a player's type about the types of the opponents. To see this, consider the limit as $t_i' \to t_i$:

$$\lim_{t_i \to t_i'} \frac{I(t_i, t_i')}{t_i - t_i'} \to \mathcal{I}(t_i) \equiv \mathrm{Var}_{T_{-i}}\left(\frac{\partial \ln f(\boldsymbol{t}_{-i}|t_i)}{\partial t_i}\right).$$

$\mathcal{I}(t_i)$ is the variance of a score function and so is the Fisher information, measuring how sensitive the likelihood of other players' types is to the type of player i. Hence assumption D1 bounds the Fisher information in the model.

Assumption D2 introduces a particular type of heterogeneity, in terms of the upper bound ν on the conditional density. This condition is similar to the one used by Grandmont (1992): we, like him, require the density function to be sufficiently flat.

These assumptions on *ex post* payoffs and conditional densities allow us to relate conditions on the primitives of the model to monotonicity and Lipschitz continuity, which are properties of the incremental interim payoff.

Theorem 2 *Suppose that assumptions U1–U2 and D1 hold. If*

$$\delta > \iota\omega, \tag{4}$$

then uniform strict single-crossing is satisfied, with $\varphi_1 \equiv \delta - \iota\omega > 0$.

Proof See the appendix. □

Theorem 2 shows that uniform strict single-crossing can be related to assumptions U1–U2 and D1 on the primitives of the model. Assumption U1 implies that, all other things equal, a higher type prefers a higher action. This is the basic force towards players' incremental interim payoffs satisfying a strict single crossing property, and hence towards players using monotone pure strategies. This basic force can, however, be overturned by strategic interaction. A player with a higher type has a different posterior over the types of its opponents; and therefore different beliefs about the actions that will be played by its opponents. The higher-type player may therefore evaluate the incremental interim payoff between a higher and lower action differently from a lower-type player. This strategic effect may reinforce the non-strategic force; but it may counteract it.

Assumption D1 ensures that a higher type's posterior cannot be too different from a lower type's. Assumption U2 ensures that, even when posteriors are different, a higher type's evaluation of the incremental interim payoff between a higher and lower action is not too different from a lower type's. Hence, if $\delta > \iota\omega$, then the strategic effect is strictly smaller than the non-strategic effect.

Assumptions U1 and D1 can be contrasted to the conditions used by Athey (2001). In our paper and Athey's, the interim payoff must satisfy a single crossing property in incremental returns (SCP-IR).[8] Athey shows that this condition is satisfied in games where

[8] A function $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies single crossing of incremental returns in (x, θ) if, for all $x_H > x_L$ and $\theta_H > \theta_L$, $h(x_H, \theta_L) - h(x_L, \theta_L) \geq (>) 0$ implies $h(x_H, \theta_H) - h(x_L, \theta_H) \geq (>) 0$. See Milgrom and Shannon (1994).

agents' *ex post* utility is supermodular in $\boldsymbol{a}$ and (a_i, t_j), $j \in I$ and types are affiliated (see Athey (2001, theorem 3)). In contrast, we require that the *ex post* utility function u_i is uniformly increasing in own action and type, (a_i, t_i), a condition slightly stronger than supermodularity in (a_i, t_i); and that types are not too associated. We can then show that the interim payoff satisfies a SCP-IR for *any* strategy profile of opponents.

Note that our assumptions are neither weaker nor stronger than Athey's. Our assumption on payoffs is stronger in one sense, since it requires more than supermodularity; but is weaker in another sense, in that it involves only own action and type. Similarly, our distributional assumptions are stronger, since they limit the degree of association between types; but they are weaker, since they allow for negative as well as positive correlation between types. (Affiliation allows only for the latter.)

Theorem 3 *Suppose that assumptions U1–U3 and D1–D2 hold; and that $\delta > \iota\omega$. Then Lipschitz continuity is satisfied, with $\varphi_2 \equiv \nu\kappa$.*

Proof See the appendix. □

The next theorem is an immediate corollary of theorems 2 and 3 and is therefore stated without proof.

Theorem 4 *If assumptions U1–U3 and D1– D2 hold, and if*

$$\delta > \iota\omega + \nu\kappa, \tag{5}$$

then the best response correspondence is a contraction; and hence there is a unique equilibrium of the Bayesian game. Furthermore, this equilibrium is in monotone pure strategies.

Condition (5) is similar to condition (4). Both conditions ensure that a player's own type dominates strategic interaction effects in payoff terms enough to make any best response a monotone pure strategy. Roughly speaking, if condition (5) is satisfied, then each player places more weight on its own type than on the possible actions of its opponents when choosing its best action. It does so by ensuring that the direct effect of a player's type (measured by δ, according to assumption U1) is sufficiently large. It also ensures that the interaction effect is sufficiently weak, by limiting the size of the effects of both a player's own action (measured by ω, according to assumption U2) and its opponents' actions (measured by κ, from assumption U3). Finally, it ensures that a player's type is sufficiently uninformative about the types (and hence likely action) of others (measured by ι and ν, according to assumptions D1 and D2).

Condition (5) is, however, stricter than condition (4), since it must both ensure that players choose monotone pure strategies; and that the best response correspondence is a contraction. The latter introduces two additional assumptions: U3 (bounding the effect of opponents' actions) and D2 (bounding the conditional density). The proof makes clear why these additional assumptions are required. Intuitively, to establish a contraction, a player's expected payoff difference between two actions must be sufficiently insensitive to a change in the strategies of its opponents. This requires first that the realised actions of opponents should not affect the *ex post* payoff of a player too much. Assumption U3 ensures this. It also requires that the change in opponents' strategies should not result in a change in realised actions that is too large. Assumption D2 achieves this by ensuring that there is not too much mass placed on any profile of opponents' types.

3.1 Applications

Given theorem 4, we must verify two types of condition in order to apply our results. The first is that the *ex post* payoffs and conditional densities in the application have uniform bounds, as required by assumptions U1–U3 and D1–D2. The second is that the sufficient condition in theorem 4 is satisfied. In this section, we consider a small number of applications to see how this can be done.

Consider a variant of a Diamond-type search model. There are a finite number of players N who exert effort searching for trading partners. Any trader's probability of finding another particular trader is proportional to his own effort and the total effort of others. Let $a_i \in [0,1]$ be the effort of player i. The *ex post* payoff to player i is

$$u_i = a_i \left(1 + \sum_{j \neq i} a_j\right) v(t_i) - C(a_i).$$

t_i is the type of player i, drawn from the compact interval $[0, \bar{t}]$. $v(t_i) : [0, \bar{t}] \to [0, \bar{v}]$ is a continuous and hence bounded function. It is also differentiable and uniformly increasing, so that there exists a $\delta > 0$ such that $v'(t) \geq \delta$ for all $t \in [0, \bar{t}]$. $C(\cdot)$ is a strictly increasing, convex, differentiable function. Note that it is critical that in this example, a player can increase the probability of a match through its own effort, even if all other players exert no effort. If this were not true, then our approach could not be applied.

With these assumptions, this is a supermodular game, since $\partial^2 u_i / \partial a_i \partial a_j = v(t_i) > 0$. Moreover,

$$\begin{aligned} &\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i}) - \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i', \boldsymbol{t}_{-i}) \\ &\quad = (a_i - a_i')(1 + \sum_{j \neq i} a_j)(v(t_i) - v(t_i')) \geq (a_i - a_i')(v(t_i) - v(t_i')) \geq \delta(a_i - a_i')(t_i - t_i') \end{aligned}$$

and so the game satisfies assumption U1. Assumption U2 is satisfied with $\omega \equiv N\bar{v}$. Assumption U3 is also satisfied, with $\kappa \equiv N\bar{v}$.

To complete the application, suppose that there are two players whose types may take one of two values: $t_i \in \{\underline{t}, \bar{t}\}$ for $i \in \{1, 2\}$, where $0 < \underline{t} < \bar{t} < +\infty$. Let the conditional densities be as follows: conditional on player i being type $\underline{t}$ ($\bar{t}$), the probability of player $j \neq i$ being type $\underline{t}$ ($\bar{t}$) is $q \in [0.5, 1]$.[9] A straightforward calculation shows that, in this case, the measure of differential information used in assumption D1 is

$$I = \frac{(1-2q)^2}{q(1-q)}.$$

Assumption D1 requires that there exists a finite constant, ι, such that $\sqrt{I} \leq \iota$; clearly,

[9] This example can be extended easily to allow for different conditional probabilities, so that the probability of player j being type $\underline{t}$ when player i is type $\underline{t}$ differs from the probability of player j being type $\bar{t}$ when player i is type $\bar{t}$.

this requires that $q < 1$. Alternatively, for any $\iota > 0$, there exist $0 < \underline{q}_\iota < \bar{q}_\iota < 1$ such that $\sqrt{I} \leq \iota$ for all $q \in [\underline{q}_\iota, \bar{q}_\iota]$. Assumption D2 is satisfied with $\nu = 1$; alternatively, for any given ν, the conditional density is less than ν for all $q \in [\underline{q}_\nu, \bar{q}_\nu]$. So, for a given $\iota > 0$ and $\nu > 0$, assumptions D1 and D2 are satisfied if $q \in [\underline{q}_\iota, \bar{q}_\iota] \cap [\underline{q}_\nu, \bar{q}_\nu]$.

The condition in theorem 4 is satisfied if

$$\frac{\delta}{\bar{v}} > 2\left(\frac{(1-2q)}{\sqrt{q(1-q)}} + 2\right). \tag{6}$$

For example, when $q = 0.5$, condition (6) requires that $\delta > 4\bar{v}$. More generally, the condition in theorem 4 is easier to satisfy when q is closer to 0.5.

Consider next a Cournot quantity game in which actions are output or investment decisions, and types are (the negative of) marginal cost. The *ex post* payoff of agent i in this game is

$$u_i(\boldsymbol{a}, \boldsymbol{t}) = a_i(P(a_i, \boldsymbol{a}_{-i}) + t_i)$$

where $P(\cdot, \cdot)$ is the inverse demand function. (This formulation allows for differentiated goods and a general inverse demand function.) Then it is straightforward to show that

$$\Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i, \boldsymbol{t}_{-i}) - \Delta u_i(a_i, a_i', \boldsymbol{a}_{-i}, t_i', \boldsymbol{t}_{-i}) = (a_i - a_i')(t_i - t_i'),$$

which satisfies assumption U1, with $\delta = 1$. Note that, since the inverse demand function drops out of the expression in payoff differences, it is not even necessary that demand be downward-sloping. To complete the example, suppose that (inverse) demand is linear: let $A \equiv \sum_{i=1}^N a_i$ be aggregate output, where $N \geq 2$ is the number of firms. Let inverse demand be

$$P(\boldsymbol{a}) = \begin{cases} \alpha - \beta \sum_{i=1}^N a_i & \sum_{i=1}^N a_i < \frac{\alpha}{\beta} \\ 0 & \sum_{i=1}^N a_i \geq \frac{\alpha}{\beta} \end{cases}$$

where α and β are strictly positive constants, and $N \geq 2$ is the number of firms. Suppose that firms' marginal costs $-t_i$ are drawn independently from a lognormal distribution, with a shaping parameter $\sigma > 0$. Note that is a dominant strategy for any firm with a marginal cost greater than α to produce zero output. In this application, the bounding parameters in our assumptions take the values: $\delta = 1$; $\omega = \max_{\{i|-t_i \leq \alpha\}}(\alpha + t_i) = \alpha$; $\kappa = \beta$; $\iota = 0$; and

$$\nu = \frac{\exp(\frac{\sigma^2}{2})}{\sigma\sqrt{2\pi}}$$

By theorem 4, there is a unique equilibrium, which is in monotone pure strategies, if

$$1 > 2\beta\frac{\exp(\frac{\sigma^2}{2})}{\sigma\sqrt{2\pi}}. \tag{7}$$

The right-hand side of this inequality is a non-monotonic function of σ. Hence, for any given $\beta > 0$, there exist $0 \leq \underline{\sigma}_\beta < \bar{\sigma}_\beta$ such that for all $\sigma \in (\underline{\sigma}_\beta, \bar{\sigma}_\beta)$, there is a unique equilibrium in the general Cournot oligopoly game, which is in monotone pure strategies (i.e., firms with higher marginal costs produce less).

Our approach therefore establishes conditions for uniqueness of equilibrium in Cournot (and other rent-seeking) games. There are few existing results in this area. Uniqueness can be established with a standard contraction argument with a small number of firms; the (sufficient) condition becomes harder to satisfy as the number grows. For example, with a linear inverse demand curve, the sufficient condition is violated if there are more than two firms. See Vives (1999). With two firms, Athey (2001)'s results can be used, since in this case, the Cournot game is supermodular. Van Long and Soubeyran (2000) is a recent contribution to the subject, also using a contraction mapping approach, but based on a function involving costs.

In summary: our approach requires that two types of condition hold in a continuous game. The first there are uniform bounds, as required by assumptions U1–U3 and D1–D2. This first condition is relatively mild for Lipschitz continuous games, but does rule out e.g., auctions. The second condition is that the sufficient condition in theorem 4 is satisfied. This second condition restricts the range of (Lipschitz continuous) applications covered by our result. Our sufficient condition is likely to be violated in applications in which players' types are highly correlated, and in which the effect of a player's own type on its *ex post* payoff is dominated by the effect of players' actions.

4 Characterizing the existence and uniqueness condition: discontinuous games

In the previous section, we derived conditions on *ex post* payoffs and the conditional density that ensured uniform strict single-crossing and Lipschitz continuity of the interim payoff function. Assumptions U2 and U3 require that players' payoffs are Lipschitz continuous in their own and opponents' actions. These assumptions are violated in discontinuous games, such as auctions, with a continuum of actions, in which a small change in players' actions can lead to a large change in payoffs.

In this section, we consider how our approach can be applied to these types of games. We restrict attention to "standard auctions":

Definition 3 *A standard auction model has the following features:*

- *Player i's action set is $A_i \equiv [\underline{a}_i, \bar{a}_i]$.*
- *Player i's* ex post *payoff function when losing is $\underline{v}_i(a_i, \boldsymbol{t}) : A_i \times T \to \mathbb{R}$. The function $\Delta\underline{v}_i(a_i, a_i', t_i, \boldsymbol{t}_{-i}) \equiv \underline{v}_i(a_i, t_i, \boldsymbol{t}_{-i}) - \underline{v}_i(a_i', t_i, \boldsymbol{t}_{-i})$ is a function only of a_i, a_i' and t_i and is non-decreasing in $(-a_i, t_i)$.*[10]
- *Player i's* ex post *payoff function when winning is $\bar{v}_i(a_i, \boldsymbol{t}) : A_i \times T \to \mathbb{R}$.*
- *The payoff functions $\underline{v}_i(\cdot, \cdot)$ and $\bar{v}_i(\cdot, \cdot)$ are bounded, measurable, and continuous in $a_i, \boldsymbol{t}$.*
- *The function $v_i(a_i, \boldsymbol{t}) \equiv \bar{v}_i(a_i, \boldsymbol{t}) - \underline{v}_i(a_i, \boldsymbol{t})$ is strictly increasing in $(-a_i, t_i)$. Payoffs are normalised so that for all $i \in I$, $|v_i(\cdot, \cdot)| \leq 1$.*
- *Let the allocation rule be denoted $\psi_i(\boldsymbol{a})$, which specifies the probability that player i*

[10] For example, this feature is clearly satisfied in first-price auctions, where $\underline{v}_i = 0$; and all-pay auctions, where $\underline{v}_i = -a_i$.

wins given the vectors of actions $\boldsymbol{a}$. $\psi_i(\boldsymbol{a}) : A \to [0,1]$ is such that, with k units to be allocated, player i receives the object with probability zero if k or more opponents choose actions such that $a_j > a_i$, and with probability 1 if $I-k$ opponents choose actions such that $a_j < a_i$. The remaining events are ties, resolved randomly.

The expected utility of player i of type t_i taking action a_i, when its opponents play the (behavioural) strategy profile $\boldsymbol{\mu}_{-i}$, is therefore

$$U_i(a_i, t_i; \boldsymbol{\mu}_{-i}) = \int_{T_{-i}} \underline{v}_i(a_i, \boldsymbol{t}) f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} + \int_{T_{-i}} \Psi_i(a_i, \boldsymbol{t}_{-i}; \boldsymbol{\mu}_{-i}) v_i(a_i, \boldsymbol{t}) f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \quad (8)$$

where

$$\Psi_i(a_i, \boldsymbol{\mu}_{-i}(\boldsymbol{t}_{-i})) \equiv \int_{A_{-i}} \psi_i(a_i, \boldsymbol{a}_{-i}) \prod_{j \neq i} d\mu_j(\cdot, t_j).$$

Our objective is to find assumptions that ensure that the expected utility in equation (8) satisfies the uniform strict single-crossing and Lipschitz continuity conditions. We need to make some assumptions on the payoff function $v_i(a_i, \boldsymbol{t})$, the conditional density $f(\boldsymbol{t}_{-i}|t_i)$, and most crucially, the strategies that players use.

U1' Uniformly Positive Sensitivity to Own Type. There is a $\eta \in (0, \infty)$ such that for all a_i, $t_i \geq t_i'$, $\boldsymbol{a}_{-i}, \boldsymbol{t}_{-i}$ and $i \in I$,

$$v_i(a_i, t_i, \boldsymbol{t}_{-i}) - v_i(a_i, t_i', \boldsymbol{t}_{-i}) \geq \eta(t_i - t_i').$$

U2' Lipschitz Continuity to Own Action. Let $\Delta v_i(a_i, a_i', \boldsymbol{t}) \equiv v_i(a_i, \boldsymbol{t}) - v_i(a_i', \boldsymbol{t})$. There is an $\omega \in (0, \infty)$ such that for all $a_i \geq a_i'$, $\boldsymbol{a}_{-i}, \boldsymbol{t}$, and $i \in I$,

$$|\Delta v_i(a_i, a_i', \boldsymbol{t})| \leq \omega(a_i - a_i').$$

D1' There is a $\bar{\nu} \in (0, \infty)$ such that $f(\boldsymbol{t}_{-i}|t_i) \leq \bar{\nu}$ for all $t_i, \boldsymbol{t}_{-i}$ and $i \in I$.

D2' There is a $\underline{\nu} \in (0, \infty)$ such that $f(\boldsymbol{t}_{-i}|t_i) \geq \underline{\nu}$ for all $t_i, \boldsymbol{t}_{-i}$ and $i \in I$.

D3' There is a $\tau \in (0, \infty)$ such that for any $t_i \geq t_i'$ and $i \in I$,

$$|f(\boldsymbol{t}_{-i}|t_i) - f(\boldsymbol{t}_{-i}|t_i')| \leq \tau(t_i - t_i').$$

S1 Fix $K \geq 1$. $\forall\ i$, let Σ_i^K be the set of admissible strategies for player i. If $\mu_i \in \Sigma_i^K$, then μ_i is a uniformly increasing pure strategy with uniform constant K: for any $t_i \in T_i$, $\mu_i(\cdot, t_i)$ assigns probability 1 to some action $\alpha(t_i) \in A_i$, and probability 0 to all other actions; and

$$\frac{1}{K} \leq \frac{\alpha(t_i) - \alpha(t_i')}{t_i - t_i'} \leq K \qquad \forall\ t_i \neq t_i'.$$

Assumptions U1' and U2' are fairly mild. Consider the case of risk neutral bidders with private values in a first-price auction for a single unit. Suppose that the payoff from losing is zero and from winning is $t_i - a_i$. With these payoffs, assumptions U1' and U2' are satisfied, with $\eta = 1$ and $\omega = 1$.

Assumptions D1' and D2' require the conditional density to be sufficiently flat, but uniformly bounded away from zero. Both assumptions effectively require that a player's type does not contain too much information about its rivals' types, in terms of both what types they might have (since the conditional density is bounded above), and what types that do not have (since the density is bounded below). Assumption D1' is stronger than its counterpart in section 3, assumption D2, since the former places a condition on $f(\boldsymbol{t}_{-i}|t_i')$ instead of $f_j(t_j|t_i)$. Assumption D2' has no counterpart in section 3. Assumption D3' replaces assumption D1 as a measure of differential information, and again is related to (but stronger than) Grandmont (1992)'s notion of heterogeneity.

To illustrate these assumptions, suppose that there are two players whose types may take one of two values: $t_i \in \{\underline{t}, \bar{t}\}$ for $i \in \{1, 2\}$, where $0 < \underline{t} < \bar{t} < +\infty$. Let the conditional densities be as follows: conditional on player i being type $\underline{t}$ ($\bar{t}$), the probability of player $j \neq i$ being type $\underline{t}$ ($\bar{t}$) is $q \in [0.5, 1]$. Assumption U1' is then satisfied, with $\bar{\nu} = 1$. Assumption U2' requires that q be bounded away from 0 and 1: for a given $\underline{\nu}$, $q \in [\underline{\nu}, 1-\underline{v}]$. Assumption D3' is satisfied with $\tau = 1$.

Assumption S1 has previously been used by Cho (2005). It is the most awkward of the assumptions. It requires that all players use strategies that are strictly increasing, Lipschitz continuous pure strategies. As $K \to \infty$, any strictly increasing strategy can be approximated; the smaller is K, the more restrictive is the assumption. But note that, whatever value of K is specified, the assumption still requires that strategies be monotonic and pure. Hence, in this section, we shall be able to establish conditions for the existence and uniqueness of equilibrium, conditional on players using monotone pure strategies that satisfy assumption S1. The result for discontinuous games is therefore weaker than for continuous games, where uniqueness can be established without any restriction on strategies. But, as the proofs of theorems 5 and 6 make clear, this restriction on strategies is unavoidable if we are to accommodate discontinuous games.

First, we need to establish the existence of best responses within the set of strategies $\Sigma^K \equiv \prod_{i=1}^{I} \Sigma_i^K$. Cho (2005) shows that for any given K, $\forall i, \Sigma_i^K$ is compact. Since strategies are required to be strictly increasing, the probability of a tie (multiple winning bids) is zero. Hence the interim expected utility function is continuous; and so the best response correspondence is nonempty and compact. That is, for any vector of strategies $\boldsymbol{\alpha} \in \Sigma^K$, the best response correspondence $\boldsymbol{\phi}(\boldsymbol{\alpha})$ is non-empty and is in Σ^K.

We now turn to the conditions that are required to establish uniform strict single-crossing and Lipschitz continuity—the counterparts to theorems 2 and 3 for continuous games.

Theorem 5 *Consider a standard auction (see definition 3); and suppose that assumptions U1', D2', D3' and S1 hold. If*

$$\frac{\eta\underline{\nu}}{K} > \tau K,$$

then uniform strict single-crossing is satisfied, with $\varphi_1 \equiv \eta\underline{\nu}/K - \tau K > 0$.

Theorem 6 *Consider a standard auction (see definition 3); and suppose that assumptions U2', D1' and S1. Then Lipschitz continuity is satisfied, with* $\varphi_2 \equiv (K + \omega)\bar{\nu}$.

Theorems 5 and 6 can be combined to give the main result for discontinuous games.

Theorem 7 *Consider a standard auction (see definition 3). If assumptions U1'–U2', D1'–D3' and S1 hold; and if*

$$\eta\underline{\nu}/K > \tau K + (K + \omega)\bar{\nu}, \tag{9}$$

then the best response correspondence is a contraction; and hence there is a unique equilibrium in monotone pure strategies of the Bayesian game.

The sufficient condition (9) makes clear the problems that arise without restricting players' strategy sets. As $K \to \infty$, so that any monotone pure strategy is allowed, the left-hand side of the condition tends to zero. Condition (9) would therefore be violated unless $\tau = \bar{\nu} = 0$. But the latter is ruled out by the requirement in assumptions D1' and D2' that $0 < \underline{\nu} \leq \bar{\nu}$. For any finite K, condition (9) is satisfied if (i) the conditional density $f(\boldsymbol{t}_{i-}|t_i)$ is sufficiently 'flat', in the sense that the ratio $\underline{\nu}/\bar{\nu} \leq 1$ is sufficiently large and/or τ is sufficiently small; (ii) a player's valuation is sufficiently sensitive to its type (η is sufficiently large); and (iii) ω is sufficiently small: a player's action does not affect its payoff too much.

Theorem 7 offers the possibility of establishing the existence of a monotone pure strategy equilibrium in a broad range of auctions. Reny and Zamir (2004) establish the existence of this type of equilibrium in first-price auctions for a single unit under very general conditions: asymmetric bidders, interdependent values, and affiliated one-dimensional signals. Outside of single-unit first-price auctions, results are more limited. For example, for all-pay auctions (in which players pay their bids regardless of whether they win or lose), existence has been established only with independent private values and (weakly) risk averse bidders; or positive value interdependence but independent information. See Athey (2001). Theorem 7 can be used to establish existence of monotone pure strategy equilibrium for all-pay auctions with interdependent values *and* information. For multi-unit first-price auctions, in which each bidder demands a single unit, Athey (2001) establishes existence of monotone pure strategy equilibrium when there are independent private values and bidders are not too risk-loving. Theorem 7 can be used to establish existence for cases with interdependent values and information.

The theorem has some bite in terms of uniqueness, although less than the result for continuous games. The logic of theorem 7 is as follows: for a fixed K which, given the other model parameters, satisfies condition 9, there is a unique equilibrium in monotone pure strategies which belong to the set Σ^K. This conclusion can be repeated for different values of K, all of which satisfy condition 9. The result is a set of equilibria, parameterised by K. This may be of use, particularly for numerical analysis of auctions.

5 Conclusions

In this paper, we have provided a sufficient condition for there to be a unique equilibrium, which is in monotone pure strategies, in games of incomplete information. The condition involves uniform strict single-crossing and Lipschitz continuity of the incremental interim payoff, and ensures that the equilibrium mapping is a contraction. We provide a characterization of uniform strict single-crossing and Lipschitz continuity in terms of the model primitives, for continuous and discontinuous games. The characterization is easy to check in applications, as well as having a clear economic interpretation.

Appendix

A Proof of Theorem 1

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ be two distinct joint monotone pure strategies. Moreover, suppose that $\boldsymbol{\phi}(\boldsymbol{\alpha})$ and $\boldsymbol{\phi}(\boldsymbol{\alpha}')$ are distinct. The definition of the metric in (2) implies that, for $i \in I$

$$d(\boldsymbol{\phi}(\boldsymbol{\alpha}), \boldsymbol{\phi}(\boldsymbol{\alpha}')) = (t_i - t_i') \tag{A.10}$$

for some t_i, t_i'. The definition of the metric implies that $\alpha_i(t_i' + \varepsilon) > \rho > \alpha_i'(t_i - \varepsilon)$ or $\alpha_i(t_i' + \varepsilon) < \rho < \alpha_i'(t_i - \varepsilon)$ for all $\varepsilon > 0$ such that $t_i - \varepsilon > t_i' + \varepsilon$ and for some $\rho \in \mathbb{R}$. Without loss of generality suppose that the first inequality holds. Then the best reply $a_{i\varepsilon}$ of player i at $t_i' + \varepsilon$ against $\boldsymbol{\alpha}_{-i}$ is strictly greater than the best reply $a_{i\varepsilon}'$ of player i at $t_i - \varepsilon$ against $\boldsymbol{\alpha}_{-i}'$. This also implies that

$$\Delta U_i(a_{i\varepsilon}, a_{i\varepsilon}', t_i' + \varepsilon, \boldsymbol{\alpha}_{-i}) \geq 0, \tag{A.11a}$$

$$\Delta U_i(a_{i\varepsilon}, a_{i\varepsilon}', t_i - \varepsilon, \boldsymbol{\alpha}_{-i}') \leq 0. \tag{A.11b}$$

It follows from uniform strict single-crossing that

$$\Delta U_i(a_{i\varepsilon}, a_{i\varepsilon}', t_i - \varepsilon, \boldsymbol{\alpha}_{-i}) - \Delta U_i(a_{i\varepsilon}, a_{i\varepsilon}', t_i' + \varepsilon, \boldsymbol{\alpha}_{-i}) \geq \varphi_1(a_{i\varepsilon} - a_{i\varepsilon}')(t_i - t_i' - 2\varepsilon), \tag{A.12a}$$

$$\Delta U_i(a_{i\varepsilon}, a_{i\varepsilon}', t_i - \varepsilon, \boldsymbol{\alpha}_{-i}') - \Delta U_i(a_{i\varepsilon}, a_{i\varepsilon}', t_i' + \varepsilon, \boldsymbol{\alpha}_{-i}') \geq \varphi_1(a_{i\varepsilon} - a_{i\varepsilon}')(t_i - t_i' - 2\varepsilon). \tag{A.12b}$$

Now equations (A.11a) and (A.12a) and imply that

$$\Delta U_i(a_{i\varepsilon}, a_{i\varepsilon}', t_i - \varepsilon, \boldsymbol{\alpha}_{-i}) \geq 0. \tag{A.13a}$$

Similarly, equations (A.11b) and (A.12b) and imply that

$$\Delta U_i(a_{i\varepsilon}, a'_{i\varepsilon}, t'_i + \varepsilon, \boldsymbol{\alpha}'_{-i}) \le 0. \tag{A.13b}$$

Next it follows from Lipschitz-continuity that

$$|\Delta U_i(a_{i\varepsilon}, a'_{i\varepsilon}, t_i - \varepsilon, \boldsymbol{\alpha}_{-i}) - \Delta U_i(a_{i\varepsilon}, a'_{i\varepsilon}, t_i - \varepsilon, \boldsymbol{\alpha}'_{-i})| \le \varphi_2(a_{i\varepsilon} - a'_{i\varepsilon}) d(\boldsymbol{\alpha}_{-i}, \boldsymbol{\alpha}'_{-i}), \tag{A.14a}$$

$$|\Delta U_i(a_{i\varepsilon}, a'_{i\varepsilon}, t'_i + \varepsilon, \boldsymbol{\alpha}_{-i}) - \Delta U_i(a_{i\varepsilon}, a'_{i\varepsilon}, t'_i + \varepsilon, \boldsymbol{\alpha}'_{-i})| \le \varphi_2(a_{i\varepsilon} - a'_{i\varepsilon}) d(\boldsymbol{\alpha}_{-i}, \boldsymbol{\alpha}'_{-i}). \tag{A.14b}$$

Now equations (A.11b), (A.13a) and (A.14a) and imply that

$$0 \le \Delta U_i(a_{i\varepsilon}, a'_{i\varepsilon}, t_i - \varepsilon, \boldsymbol{\alpha}_{-i}) \le \varphi_2(a_{i\varepsilon} - a'_{i\varepsilon}) d(\boldsymbol{\alpha}_{-i}, \boldsymbol{\alpha}'_{-i}). \tag{A.15a}$$

Similarly, equations (A.11a) and (A.13b) and (A.14b) imply that

$$0 \le \Delta U_i(a_{i\varepsilon}, a'_{i\varepsilon}, t'_i + \varepsilon, \boldsymbol{\alpha}_{-i}) \le \varphi_2(a_{i\varepsilon} - a'_{i\varepsilon}) d(\boldsymbol{\alpha}_{-i}, \boldsymbol{\alpha}'_{-i}). \tag{A.15b}$$

Combining equations (A.15a) and (A.15b) with (A.12a) leads to

$$\varphi_1(a_{i\varepsilon} - a'_{i\varepsilon})(t_i - t'_i - 2\varepsilon) \le \varphi_2(a_{i\varepsilon} - a'_{i\varepsilon}) d(\boldsymbol{\alpha}_{-i}, \boldsymbol{\alpha}'_{-i}).$$

Dividing both sides by $(a_{i\varepsilon} - a'_{i\varepsilon}) > 0$ and taking the limit $\varepsilon \to 0$ leads to

$$\varphi_1(t_i - t'_i) \le \varphi_2 d(\boldsymbol{\alpha}_{-i}, \boldsymbol{\alpha}'_{-i}).$$

Using the fact (A.10) and the definition of the metric, we obtain

$$\varphi_1 d(\boldsymbol{\phi}(\boldsymbol{\alpha}), \boldsymbol{\phi}(\boldsymbol{\alpha}')) \le \varphi_2 d(\boldsymbol{\alpha}_{-i}, \boldsymbol{\alpha}'_{-i}) \le \varphi_2 d(\boldsymbol{\alpha}, \boldsymbol{\alpha}') \tag{A.16}$$

which proves our theorem. $\square$

B Proof of Theorem 2

By definition,

$$
\begin{aligned}
&\Delta U_i(a_i, a_i', t_i, \boldsymbol{\alpha}_{-i}) - \Delta U_i(a_i, a_i', t_i', \boldsymbol{\alpha}_{-i}) \\
&= \int_{T_{-i}} \Delta u_i(a_i, a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}), t_i, \boldsymbol{t}_{-i}) f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \\
&\quad - \int_{T_{-i}} \Delta u_i(a_i, a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}), t_i', \boldsymbol{t}_{-i}) f(\boldsymbol{t}_{-i}|t_i') d\boldsymbol{t}_{-i} \\
&= \int_{T_{-i}} \left[\Delta u_i(a_i, a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}), t_i, \boldsymbol{t}_{-i}) - \Delta u_i(a_i, a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}), t_i', \boldsymbol{t}_{-i})\right] f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \\
&\quad - \int_{T_{-i}} \Delta u_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}), t_i', \boldsymbol{t}_{-i}) \left[f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)\right] d\boldsymbol{t}_{-i}. \qquad \text{(B.17)}
\end{aligned}
$$

From assumption U1, we obtain for the first term that

$$
\begin{aligned}
&\int_{T_{-i}} [\Delta u_i(a_i, a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}), t_i, \boldsymbol{t}_{-i}) - \Delta u_i(a_i, a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}), t_i', \boldsymbol{t}_{-i})] f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \\
&\qquad\qquad \geq \delta(a_i - a_i')(t_i - t_i'). \qquad \text{(B.18)}
\end{aligned}
$$

Now consider the second term in equation (B.17). The integral can be separated, so that

$$
\begin{aligned}
&\int_{T_{-i}} \Delta u_i(a_i, a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}), t_i', \boldsymbol{t}_{-i}) \left[f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)\right] d\boldsymbol{t}_{-i} \\
&= \int_{T_{-i}} [\Delta u_i(a_i, a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}), t_i', \boldsymbol{t}_{-i})] \frac{f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)}{f(\boldsymbol{t}_{-i}|t_i)} f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \\
&\leq \left(\int_{T_{-i}} [\Delta u_i(a_i, a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}), t_i', \boldsymbol{t}_{-i})]^2 f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i}\right)^{1/2} \\
&\qquad \times \left(\int_{T_{-i}} \left(\frac{f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)}{f(\boldsymbol{t}_{-i}|t_i)}\right)^2 f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i}\right)^{1/2} \qquad \text{(B.19)}
\end{aligned}
$$

where in the last line, we use the Cauchy-Schwarz inequality.

Using assumption U2 and the fact $a_i \geq a_i'$ yields an upper bound on the first term of the product in equation (B.19),

$$\left(\int_{T_{-i}} [\Delta u_i(a_i, a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}), t_i', \boldsymbol{t}_{-i})]^2 f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i}\right)^{1/2} \leq \omega(a_i - a_i'). \tag{B.20}$$

For the second term of the product in equation (B.19),

$$\left(\int_{T_{-i}} \left(\frac{f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)}{f(\boldsymbol{t}_{-i}|t_i)}\right)^2 f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i}\right)^{1/2} = \sqrt{\mathrm{Var}_{T_{-i}} \left(\frac{f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)}{f(\boldsymbol{t}_{-i}|t_i)}\right)}$$

because

$$\begin{aligned}\mathbb{E}_{T_{-i}} \left[\frac{f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)}{f(\boldsymbol{t}_{-i}|t_i)}\right] &= \int_{T_{-i}} \frac{f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)}{f(\boldsymbol{t}_{-i}|t_i)} f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \\ &= \int_{T_{-i}} (f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)) d\boldsymbol{t}_{-i} = 0\end{aligned}$$

since $\int_{T_{-i}} f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} = \int_{T_{-i}} f(\boldsymbol{t}_{-i}|t_i') d\boldsymbol{t}_{-i} = 1$. Therefore from assumption D1,

$$\left(\int_{T_{-i}} \left(\frac{f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)}{f(\boldsymbol{t}_{-i}|t_i)}\right)^2 f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i}\right)^{1/2} \leq \iota(t_i - t_i') \tag{B.21}$$

Combining equation (B.17) with equations (B.18)–(B.21) yields

$$\Delta U_i(a_i, a_i', t_i, \boldsymbol{\alpha}_{-i}) - \Delta U_i(a_i, a_i', t_i', \boldsymbol{\alpha}_{-i}) \geq (\delta - \iota\omega)(a_i - a_i')(t_i - t_i'). \tag{B.22}$$

This proves the theorem. □

C Proof of Theorem 3

By definition,

$$\begin{aligned}&\left|\Delta U_i(a_i, a_i', t_i, \boldsymbol{\alpha}_{-i}) - \Delta U_i(a_i, a_i', t_i, \boldsymbol{\alpha}_{-i}\right| \\ &\quad \leq \int_{T_{-i}} \left|\Delta u_i(a_i, a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}), \boldsymbol{t}) - \Delta u_i(a_i, a_i', \boldsymbol{\alpha}'_{-i}(\boldsymbol{t}_{-i}), \boldsymbol{t})\right| f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i}.\end{aligned} \tag{C.23}$$

Next let

$$\tilde{T}_j(\rho,\alpha_j,\alpha'_j)=\{t_j\in T_j:\alpha_j(t_j)<\rho<\alpha'_j(t_j)\ \vee\ \alpha'_j(t_j)<\rho<\alpha_j(t_j),\rho\in\mathbb{R}\} \tag{C.24}$$

and let the indicator function $\chi_j(t_j,\rho,\alpha_j,\alpha'_j)$ be defined as

$$\chi_j(t_j,\rho,\alpha_j,\alpha'_j)=\begin{cases}1 & \text{if } t_j\in\tilde{T}_j(\rho,\alpha_j,\alpha'_j)\\ 0 & \text{otherwise.}\end{cases} \tag{C.25}$$

First note that if $\boldsymbol{\alpha}_{-i}=\boldsymbol{\alpha}'_{-i}$, then $\sup_{j\neq i}\sup_{\rho\in\mathbb{R}}\chi_j(t_j,\rho,\alpha_j,\alpha'_j)=0$ and the right hand side of equation (C.23) is zero too. Otherwise, consider a $\tilde{\boldsymbol{t}}_{-i}$ such that $\boldsymbol{\alpha}_{-i}(\tilde{\boldsymbol{t}}_{-i})\neq\boldsymbol{\alpha}'_{-i}(\tilde{\boldsymbol{t}}_{-i})$. Then $\sup_{j\neq i}\sup_{\rho\in\mathbb{R}}\chi_j(\tilde{t}_j,\rho,\alpha_j,\alpha'_j)=1$ and the right hand side of equation (C.23) is positive. Hence we can write (C.23) as

$$\begin{aligned}\Big|\Delta U_i(a_i,a'_i,t_i,\boldsymbol{\alpha}_{-i})-&\Delta U_i(a_i,a'_i,t_i,\boldsymbol{\alpha}'_{-i})\Big|\\ &\leq\int_{T_{-i}}\big|\Delta u_i(a_i,a'_i,\boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}),\boldsymbol{t})-\Delta u_i(a_i,a'_i,\boldsymbol{\alpha}'_{-i}(\boldsymbol{t}_{-i}),\boldsymbol{t})\big|\\ &\qquad\qquad\times\sup_{j\neq i}\sup_{\rho\in\mathbb{R}}\chi_j(t_j,\rho,\alpha_j,\alpha'_j)f(\boldsymbol{t}_{-i}|t_i)d\boldsymbol{t}_{-i}\\ &\leq\int_{T_{-i}}\kappa(a_i-a'_i)\sup_{j\neq i}\sup_{\rho\in\mathbb{R}}\chi_j(t_j,\rho,\alpha_j,\alpha'_j)f(\boldsymbol{t}_{-i}|t_i)d\boldsymbol{t}_{-i}\end{aligned}$$

where in the last step we used assumption U3.

It follows from this that

$$\begin{aligned}\Big|\Delta U_i(a_i,&a'_i,t_i,\boldsymbol{\alpha}_{-i})-\Delta U_i(a_i,a'_i,t_i,\boldsymbol{\alpha}'_{-i})\Big|\\ &\leq\kappa(a_i-a'_i)\int_{T_{-i}}\sup_{j\neq i}\sup_{\rho\in\mathbb{R}}\chi_j(t_j,\rho,\alpha_j,\alpha'_j)f(\boldsymbol{t}_{-i}|t_i)d\boldsymbol{t}_{-i}\\ &\leq\kappa(a_i-a'_i)\sup_{j\neq i}\sup_{\rho\in\mathbb{R}}\int_{T_{-i}}\chi_j(t_j,\rho,\alpha_j,\alpha'_j)f(\boldsymbol{t}_{-i}|t_i)d\boldsymbol{t}_{-i}\\ &\leq\kappa(a_i-a'_i)\sup_{j\neq i}\sup_{\rho\in\mathbb{R}}\int_{T_j}\chi_j(t_j,\rho,\alpha_j,\alpha'_j)f(t_j|t_i)dt_j\end{aligned}$$

Finally, assumption D2 requires that $f_j(t_j|t_i) \leq \nu$; this leads to

$$\begin{aligned}
\Big|\Delta U_i(a_i,&a_i', t_i, \boldsymbol{\alpha}_{-i}) - \Delta U_i(a_i, a_i', t_i, \boldsymbol{\alpha}'_{-i})\Big| \\
&\leq \kappa(a_i - a_i') \sup_{j\neq i} \sup_{\rho\in\mathbb{R}} \int_{T_j} \chi_j(t_j, \rho, \alpha_j, \alpha_j')\nu dt_j \\
&\leq \nu\kappa(a_i - a_i') \sup_{j\neq i} \sup_{\rho\in\mathbb{R}} \int_{T_j} \chi_j(t_j, \rho, \alpha_j, \alpha_j')dt_j \\
&= \nu\kappa(a_i - a_i')d(\boldsymbol{\alpha}_{-i}, \boldsymbol{\alpha}'_{-i}).
\end{aligned}$$

The last step follows from the observation that $\int_{T_j} \chi_j(t_j, \rho, \alpha_j, \alpha_j')dt_j$ is an interval satisfying the inequality conditions in the definition of the metric with respect to ρ.

Hence there exists a $\varphi_2 \equiv \nu\kappa > 0$ such that Lipschitz continuity is satisfied. □

D Proof of Theorem 5

For $a_i > a_i'$ and $t_i > t_i'$,

$$\begin{aligned}
&\Delta U_i(a_i, a_i', t_i, \boldsymbol{\mu}_{-i}) - \Delta U_i(a_i, a_i', t_i', \boldsymbol{\mu}_{-i}) = \\
&\qquad \int_{T_{-i}} \Delta \underline{v}_i(a_i, a_i', t_i, \boldsymbol{t}_{-i}) f(\boldsymbol{t}_{-i}|t_i)d\boldsymbol{t}_{-i} - \int_{T_{-i}} \Delta \underline{v}_i(a_i, a_i', t_i', \boldsymbol{t}_{-i}) f(\boldsymbol{t}_{-i}|t_i')d\boldsymbol{t}_{-i} \\
&\quad + \int_{T_{-i}} [\Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i};))v_i(a_i, t_i, \boldsymbol{t}_{-i}) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))v_i(a_i', t_i, \boldsymbol{t}_{-i})] f(\boldsymbol{t}_{-i}|t_i)d\boldsymbol{t}_{-i} \\
&\quad - \int_{T_{-i}} [\Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))v_i(a_i, t_i', \boldsymbol{t}_{-i}) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i};))v_i(a_i', t_i', \boldsymbol{t}_{-i})] f(\boldsymbol{t}_{-i}|t_i')d\boldsymbol{t}_{-i}.
\end{aligned}$$

From the properties of a standard auction in definition 3, the first line in this expression is non-negative. Let

$$\Delta \hat{v}_i(a_i, a_i', t_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})) \equiv \Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))v_i(a_i, t_i, \boldsymbol{t}_{-i}) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))v_i(a_i', t_i, \boldsymbol{t}_{-i}).$$

Then

$$\begin{aligned}\int_{T_{-i}} \Delta\hat{v}_i(a_i, a_i', t_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))f(\boldsymbol{t}_{-i}|t_i)d\boldsymbol{t}_{-i} - \int_{T_{-i}} \Delta\hat{v}_i(a_i, a_i', t_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))f(\boldsymbol{t}_{-i}|t_i')d\boldsymbol{t}_{-i} \\ = \int_{T_{-i}} [\Delta\hat{v}_i(a_i, a_i', t_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})) - \Delta\hat{v}_i(a_i, a_i', t_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))]f(\boldsymbol{t}_{-i}|t_i)d\boldsymbol{t}_{-i} \\ - \int_{T_{-i}} \Delta\hat{v}_i(a_i, a_i', t_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))[f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)]d\boldsymbol{t}_{-i}. \quad \text{(D.26)}\end{aligned}$$

Consider the first term in the above expression:

$$\begin{aligned}\Delta\hat{v}_i(a_i,&a_i', t_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})) - \Delta\hat{v}_i(a_i, a_i', t_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})) \\ &= [\Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))v_i(a_i, t_i, \boldsymbol{t}_{-i}) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))v_i(a_i', t_i, \boldsymbol{t}_{-i})] \\ &\qquad - [\Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))v_i(a_i, t_i', \boldsymbol{t}_{-i}) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))v_i(a_i', t_i', \boldsymbol{t}_{-i})] \\ &\geq \big(\Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))\big)\big(v_i(a_i, t_i, \boldsymbol{t}_{-i}) - v_i(a_i, t_i', \boldsymbol{t}_{-i})\big),\end{aligned}$$

where we use the fact that $v_i(\cdot, \cdot)$ is strictly increasing in $(-a_i, t_i)$. Therefore for the first term we have

$$\begin{aligned}\int_{T_{-i}} &[\Delta\hat{v}_i(a_i, a_i', t_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})) - \Delta\hat{v}_i(a_i, a_i', t_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))]f(\boldsymbol{t}_{-i}|t_i)d\boldsymbol{t}_{-i} \\ &\geq \int_{T_{-i}} \big(\Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))\big)\big(v_i(a_i, t_i, \boldsymbol{t}_{-i}) - v_i(a_i, t_i', \boldsymbol{t}_{-i})\big)f(\boldsymbol{t}_{-i}|t_i)d\boldsymbol{t}_{-i} \\ &\geq \eta(t_i - t_i') \int_{T_{-i}} \big(\Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))\big)f(\boldsymbol{t}_{-i}|t_i)d\boldsymbol{t}_{-i} \\ &\geq \eta\underline{\nu}(t_i - t_i') \int_{T_{-i}} \big(\Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))\big)d\boldsymbol{t}_{-i} \\ &\geq \frac{\eta\underline{\nu}}{K}(a_i - a_i')(t_i - t_i').\end{aligned}$$

In these successive inequalities, we use assumptions U1', D2', and S1 respectively. To see the last step, note that

$$\int_{T_{-i}} \big(\Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))\big) f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i}$$
$$= \int_{T_{-i}} \int_{A_{-i}} \big(\psi_i(a_i, \boldsymbol{a}_{-i}) - \psi_i(a_i', \boldsymbol{a}_{-i})\big) \prod_{j \neq i} d\mu_j(\cdot, t_j) f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i}.$$

Given $\boldsymbol{t}_{-i} \in T_{-i}$, let $\bar{a}(\boldsymbol{\alpha}_{-i}, \boldsymbol{t}_{-i})$ be the largest element of the vector $\boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})$. Then

$$\psi_i(a_i, \mathbf{a}_{-i}(\boldsymbol{\alpha}_{-i}, \boldsymbol{t}_{-i})) = \begin{cases} 1 & a_i > \bar{a}(\boldsymbol{\alpha}_{-i}, \boldsymbol{t}_{-i}) \\ 0 & a_i < \bar{a}(\boldsymbol{\alpha}_{-i}, \boldsymbol{t}_{-i}). \end{cases}$$

(Ties can be ignored, since all strategies are strictly increasing and $f(\boldsymbol{t}_{-i}|t_i)$ is atomless.) Hence

$$\psi_i(a_i, \mathbf{a}_{-i}(\boldsymbol{\alpha}_{-i}, \boldsymbol{t}_{-i})) - \psi_i(a_i', \mathbf{a}_{-i}(\boldsymbol{\alpha}_{-i}, \boldsymbol{t}_{-i})) = \begin{cases} 1 & a_i > \bar{a}(\boldsymbol{\alpha}_{-i}, \boldsymbol{t}_{-i}) > a_i' \\ 0 & \text{otherwise.} \end{cases}$$

So, define $\boldsymbol{t}_{-i}(a_i, \boldsymbol{\alpha}_{-i})$ to be such that $\bar{a}(\boldsymbol{\alpha}_{-i}, \boldsymbol{t}_{-i}(a_i, \boldsymbol{\alpha}_{-i})) = a_i$, and $\boldsymbol{t}_{-i}(a_i', \boldsymbol{\alpha}_{-i})$ similarly. By assumption S1, for any $j \neq i$,

$$t_j(a_i, \boldsymbol{\alpha}_{-i}) - t_j(a_i', \boldsymbol{\alpha}_{-i}) \geq \frac{1}{K}(a_i - a_i').$$

Therefore

$$\int_{T_{-i}} \big(\Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))\big) d\boldsymbol{t}_{-i} \geq \frac{1}{K}(a_i - a_i').$$

Now consider the second term in equation (D.26):

$$
\begin{aligned}
&\left| \int_{T_{-i}} \Delta\hat{v}_i(a_i, a_i', t_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))[f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)]d\boldsymbol{t}_{-i} \right| \\
&\quad = \left| \int_{T_{-i}} \big(\Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))v_i(a_i, t_i, \boldsymbol{t}_{-i}) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))v_i(a_i', t_i, \boldsymbol{t}_{-i})\big)[f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)]d\boldsymbol{t}_{-i} \right| \\
&\quad \le \int_{T_{-i}} \Big|\big(\Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))v_i(a_i, t_i, \boldsymbol{t}_{-i}) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))v_i(a_i', t_i, \boldsymbol{t}_{-i})\big)\Big| \Big|f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)\Big| d\boldsymbol{t}_{-i} \\
&\quad \le \int_{T_{-i}} \Big|\big(\Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))\big)v_i(a_i', t_i, \boldsymbol{t}_{-i})\Big| \Big|f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)\Big| d\boldsymbol{t}_{-i} \\
&\quad \le \int_{T_{-i}} \Big|\big(\Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))\big)\Big| \Big|f(\boldsymbol{t}_{-i}|t_i') - f(\boldsymbol{t}_{-i}|t_i)\Big| d\boldsymbol{t}_{-i} \\
&\quad \le \tau(t_i - t_i') \int_{T_{-i}} \Big|\big(\Psi_i(a_i, \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i})) - \Psi_i(a_i', \boldsymbol{\alpha}_{-i}(\boldsymbol{t}_{-i}))\big)\Big| d\boldsymbol{t}_{-i} \\
&\quad \le \tau K(a_i - a_i')(t_i - t_i')
\end{aligned}
$$

where in the successive inequalities, we have used the property of norms; $v_i(\cdot,\cdot)$ is non-increasing in a_i; $|v_i(\cdot,\cdot)| \le 1$; assumption D3'; and finally assumption S1. This completes the proof. □

E Proof of Theorem 6

By definition,

$$
\begin{aligned}
&\Big|\Delta U_i(a_i, a_i', t_i, \boldsymbol{\mu}_{-i}) - \Delta U_i(a_i, a_i', t_i, \boldsymbol{\mu}'_{-i})\Big| \\
&\qquad = \Bigg| \int_{T_{-i}} \Big[\big(\Psi(a_i, \boldsymbol{\alpha}_{-i})v_i(a_i, \boldsymbol{t}) - \Psi(a_i', \boldsymbol{\alpha}_{-i})v_i(a_i', \boldsymbol{t})\big) \\
&\qquad\qquad - \big(\Psi(a_i, \boldsymbol{\alpha}'_{-i})v_i(a_i, \boldsymbol{t}) - \Psi(a_i', \boldsymbol{\alpha}' - i)v_i(a_i', \boldsymbol{t})\big)\Big] f(\boldsymbol{t}_{-i}|t_i)d\boldsymbol{t}_{-i}\Bigg|.
\end{aligned}
$$

Rearranging gives

$$
\begin{aligned}
&\Big|\Delta U_i(a_i, a_i', t_i, \boldsymbol{\mu}_{-i}) - \Delta U_i(a_i, a_i', t_i, \boldsymbol{\mu}_{-i}')\Big| \\
&\quad = \Bigg| \int_{T_{-i}} \Big[\Big(\big(\Psi(a_i, \boldsymbol{\alpha}_{-i}) - \Psi(a_i', \boldsymbol{\alpha}_{-i})\big) - \big(\Psi(a_i, \boldsymbol{\alpha}_{-i}') - \Psi(a_i', \boldsymbol{\alpha}_{-i}')\big) \Big) v_i(a_i, \boldsymbol{t}) \\
&\qquad\qquad + \big(\Psi(a_i', \boldsymbol{\alpha}_{-i}) - \Psi(a_i', \boldsymbol{\alpha}_{-i}')\big)\big(v_i(a_i, \boldsymbol{t}) - v_i(a_i', \boldsymbol{t})\big) \Big] f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \Bigg|. \qquad \text{(E.27)}
\end{aligned}
$$

Starting with the first term,

$$
\begin{aligned}
&\left| \int_{T_{-i}} \big(\Psi(a_i,\boldsymbol{\alpha}_{-i}) - \Psi(a_i', \boldsymbol{\alpha}_{-i})\big) - \big(\Psi(a_i, \boldsymbol{\alpha}_{-i}') - \Psi(a_i', \boldsymbol{\alpha}_{-i}')\big) v_i(a_i, \boldsymbol{t}) f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \right| \\
&\qquad \leq \int_{T_{-i}} \Big| \big(\Psi(a_i, \boldsymbol{\alpha}_{-i}) - \Psi(a_i', \boldsymbol{\alpha}_{-i})\big) - \big(\Psi(a_i, \boldsymbol{\alpha}_{-i}') - \Psi(a_i', \boldsymbol{\alpha}_{-i}')\big) \Big| |v_i(a_i, \boldsymbol{t})| f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \\
&\qquad \leq \int_{T_{-i}} \Big| \big(\Psi(a_i, \boldsymbol{\alpha}_{-i}) - \Psi(a_i', \boldsymbol{\alpha}_{-i})\big) - \big(\Psi(a_i, \boldsymbol{\alpha}_{-i}') - \Psi(a_i', \boldsymbol{\alpha}_{-i}')\big) \Big| f(\boldsymbol{t}_{-i}|t_i) d\boldsymbol{t}_{-i} \\
&\qquad \leq \bar{\nu} \int_{T_{-i}} \Big| \big(\Psi(a_i, \boldsymbol{\alpha}_{-i}) - \Psi(a_i', \boldsymbol{\alpha}_{-i})\big) - \big(\Psi(a_i, \boldsymbol{\alpha}_{-i}') - \Psi(a_i', \boldsymbol{\alpha}_{-i}')\big) \Big| d\boldsymbol{t}_{-i},
\end{aligned}
$$

The final integral can be bounded above:

$$
\int_{T_{-i}} \Big| \big(\Psi(a_i, \boldsymbol{\alpha}_{-i}) - \Psi(a_i', \boldsymbol{\alpha}_{-i})\big) - \big(\Psi(a_i, \boldsymbol{\alpha}_{-i}') - \Psi(a_i', \boldsymbol{\alpha}_{-i}')\big) \Big| d\boldsymbol{t}_{-i} \leq \int_{T_{-i}} \Big| \Psi(a_i, \boldsymbol{\alpha}_{-i}) - \Psi(a_i', \boldsymbol{\alpha}_{-i}) \Big| d\boldsymbol{t}_{-i}.
$$

Using the same step as in the proof of theorem 3, this can be written as

$$
\int_{T_{-i}} \Big| \Psi(a_i, \boldsymbol{\alpha}_{-i}) - \Psi(a_i', \boldsymbol{\alpha}_{-i}) \Big| \sup_{j \neq i} \sup_{\rho \in \mathbb{R}} \chi_j(t_j, \rho, \alpha_j, \alpha_j') d\boldsymbol{t}_{-i}.
$$

Hence the first term in equation (E.27) can be bounded above by

$$
\bar{\nu} K (a_i - a_i') d(\boldsymbol{\alpha}_{-i}, \boldsymbol{\alpha}_{-i}'),
$$

so that the Lipschitz constant for this term is $\bar{\nu} K$.

Turning now to the second term

$$\begin{aligned}
&\left| \int_{T_{-i}} \big(\Psi(a_i',\boldsymbol{\alpha}_{-i}) - \Psi(a_i',\boldsymbol{\alpha}'_{-i})\big)(v_i(a_i,\boldsymbol{t}) - v_i(a_i',\boldsymbol{t}))f(\boldsymbol{t}_{-i}|t_i)d\boldsymbol{t}_{-i} \right| \\
&\qquad \le \int_{T_{-i}} \Big|\big(\Psi(a_i',\boldsymbol{\alpha}_{-i}) - \Psi(a_i',\boldsymbol{\alpha}'_{-i})\big)\Big|\Big|(v_i(a_i,\boldsymbol{t}) - v_i(a_i',\boldsymbol{t}))\Big| f(\boldsymbol{t}_{-i}|t_i)d\boldsymbol{t}_{-i} \\
&\qquad \le \omega(a_i - a_i') \int_{T_{-i}} \Big|\big(\Psi(a_i',\boldsymbol{\alpha}_{-i}) - \Psi(a_i',\boldsymbol{\alpha}'_{-i})\big)\Big| f(\boldsymbol{t}_{-i}|t_i)d\boldsymbol{t}_{-i} \\
&\qquad \le \omega\bar{\nu}(a_i - a_i') \int_{T_{-i}} \Big|\big(\Psi(a_i',\boldsymbol{\alpha}_{-i}) - \Psi(a_i',\boldsymbol{\alpha}'_{-i})\big)\Big| d\boldsymbol{t}_{-i} \\
&\qquad \le \omega\bar{\nu}(a_i - a_i') d(\boldsymbol{\alpha}_{-i},\boldsymbol{\alpha}'_{-i})
\end{aligned}$$

where in the successive inequalities, we use the property of norms; assumption U2'; assumption D1'; and, in the final step, the definition of the metric $d(\cdot,\cdot)$. This completes the proof. □

References

Athey, S. (2001): "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica*, 69, 861–889.

Burdzy, K., D. Frankel, and A. Pauzner (2001): "Fast Equilibrium Selection by Rational Players Living in a Changing World," *Econometrica*, 69, 163–190.

Carlsson, H., and E. van Damme (1993): "Global Games and Equilibrium Selection," *Econometrica*, 61, 989–1018.

Cho, I.-K. (2005): "Monotonicity and Rationalizability in a Large First Price Auction," *Review of Economic Studies*, 72, 1031–1055.

Frankel, D., S. Morris, and A. Pauzner (2003): "Equilibrium Selection in Global Games with Strategic Complementarities," *Journal of Economic Theory*, 108, 1–44.

Fudenberg, D., and J. Tirole (1991): *Game Theory*. MIT Press, Cambridge.

Glaeser, E. L., and J. A. Scheinkman (2003): "Non-Market Interaction," in *Advances in Economics and Econometrics*, ed. by M. Dewatripont, L. P. Hansen, and S. Turnovsky, Proceedings of the Eighth World Congress of the Econometric Society. Cambridge University Press, Cambridge.

Grandmont, J.-M. (1992): "Transformation of Commodity Space, Behavioral Heterogeneity, and the Aggregation Problem," *Journal of Economic Theory*, 57, 1–35.

Herrendorf, B., Á. Valentinyi, and R. Waldmann (2000): "Ruling Out Multiplicity and Indeterminacy: The Role of Heterogeneity," *Review of Economic Studies*, 67, 295–307.

McAdams, D. (2003): "Isotone Equilibrium in Games of Incomplete Information," *Econometrica*, 71(4), 1191–1214.

Milgrom, P. R., and J. M. Roberts (1990): "Rationalizability, Learning and Equilibrium in Games with Strategic Complementarities," *Econometrica*, 58, 1255–1277.

Milgrom, P. R., and C. Shannon (1994): "Monotone Comparative Statics," *Econometrica*, 62(1), 157–180.

Milgrom, P. R., and R. J. Weber (1985): "Distributional Strategies For Games with Incomplete Information," *Mathematics of Operations Research*, 10, 619–632.

Morris, S., and H. S. Shin (1998): "Unique Equilibrium in a Model of Self-Fulfilling Currency Attacks," *American Economic Review*, 88, 587–597.

——— (2003): "Global Games: Theory and Applications," in *Advances in Economics and Econometrics*, ed. by M. Dewatripont, L. Hansen, and S. Turnovsky, Proceedings of the Eighth World Congress of the Econometric Society. Cambridge University Press, Cambridge.

——— (2005): "Heterogeneity and Uniqueness in Interaction Games," in *The Economy* *as an Evolving Complex System III*, ed. by L. Blume, and S. Durlauf, Santa Fe Institute Studies in the Sciences of Complexity, pp. 207–242. Oxford University Press.

Reny, P. J. (2006): "On the Existence of Monotone Pure Strategy Equilibria in Bayesian Games," Mimeo, Chicago University.

Reny, P. J., and S. Zamir (2004): "On The Existence of Pure Strategy Monotone Equilibria in Asymmetric First-Price Auctions," *Econometrica*, 72, 1105–1126.

Topkis, D. M. (1998): *Supermodularity and Complementarity.* Princeton University Press, Princeton.

Van Long, N., and A. Soubeyran (2000): "Existence and Uniqueness of Cournot Equilibrium: A Contraction Mapping Approach," *Economic Letters*, 67, 231–6.

Van Zandt, T., and X. Vives (2005): "Monotone Equilibria in Bayesian Games of Strategic Complementarities," *Journal of Economic Theory (forthcoming).*

Vives, X. (1990): "Nash Equilibrium with Strategic Complementarities," *Journal of Mathematical Economics*, 19(3), 305–321.

——— (1999): *Oligopoly Pricing: Old Ideas and New Tools.* MIT Press, Cambridge.