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Family Symmetries With Extra Dimensions

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ABSTRACT

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"Family Symmetries With Extra Dimensions"

by Toby John Burrows

Possibly one of the most interesting unanswered questions posed by the Standard Model is an explanation for the existence of three light generations of matter. Perhaps the most conservative extension to the Standard Model to offer an explanation is to include a symmetry between the families. One of the most promising candidates for this symmetry is the discrete group A_4, the symmetry group of the tetrahedron.

Extra dimensions have long been considered to be included in a final "Theory Of Everything". More recently research into String Theory has led to more interest in extra dimensional theories. The geometry of these extra dimensions has also been used to generate discrete symmetries which may be exploited as a family symmetry.

Grand Unified Theories seek to unify electromagnetism, the weak and the strong forces into a single unified force at high energy. If we wish for such a unification then restrictions are placed upon any family symmetry we may use.

We study models which seek to explain the large leptonic mixing angles together with the small quark mixing angles and large quark hierarchy by considering models which incorporate the use of extra dimensions together with Grand Unified and family symmetries.

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For Todd,

my brother.

Declaration of Authorship

I, Toby Burrows, declare that this thesis, entitled "Family Symmetries With Extra Dimensions" and the work presented in it are my own. I confirm that

- This work was done wholly or mainly while in candidature for a research degree at this university.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this university or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.
- Work contained in this thesis has previously been published in references [1] and [2].

Signed:

Date:

Acknowledgements

Firstly I would like to thank Naomi for putting up with me for so long.

I would also like to thank my parents for supporting me even when there is no end in sight.

Finally I would like to thank Steve whose support and guidance has been invaluable.

— T.J.B.

Chapter 1

Introduction

1.1 Motivation and outline

The Standard Model with the inclusion of right-handed neutrinos explains experimental data to date. However at a theoretical level there are many good reasons to suppose that there is more physics to be discovered beyond the Standard Model. Extensions to the Standard Model include Supersymmetry (SUSY) and Grand Unified Theories (GUTs) which are theoretically appealing, short reviews are contained in subsections 1.4 and 1.5.

The puzzle of why there are three generations of matter is still very much an open question. In the Standard Model the fermion masses and mixings are simply parameters to be determined by experiment. To go beyond the Standard Model we must propose some underlying mechanism which generates these masses and mixings. The picture is further complicated by neutrino data which shows that in contrast to the small mixing angles in the quark sector the leptons have quite large mixings. Perhaps the most conservative and minimal extension of the Standard Model to explain the existence of the three families is to propose a so-called family symmetry. As gauge symmetries relate different particles within a family a family symmetry relates particles between families.

The remainder of chapter 1 provides a brief introduction to the Standard Model along with brief introductions to Supersymmetry (section 1.4) and Grand Unified Theories (section 1.5). Section 1.6 serves as an introduction and review of family symmetries, a review of an important A_4 model is also given. Also included is a brief overview of the Froggatt-Nielsen mechanism along with a toy model to illustrate the concept of a family symmetry. Chapter 1 is concluded with a short introduction to the Seesaw mechanism in section 1.6.7.

Chapter 2 introduces the concept of extra dimensions in particle physics and the use of orbifolds is discussed. A brief review of a model presented in [3] is given where a family symmetry is generated from the geometry of the extra dimension. Recent models using extra dimensions are also reviewed.

Chapter 3 presents original work [1] on explaining the origin of the fermion masses and mixings using a discrete family symmetry. The model uses a family symmetry derived from the geometry of an extra dimension as in [3] but also extends it from a purely leptonic theory to an SU(5) Grand Unified Theory.

Chapter 4 presents original work [2] again using a discrete family symmetry namely A_4 and orbifolded extra dimensions. The family symmetry is not derived from the geometry of the extra dimensions but a mass hierarchy is generated in part by bulk suppression factors originating from the size of the extra dimensions. Another interesting feature of the model is the use of orbifolding to achieve the vacuum alignment of the flavons.

Chapter 5 serves as a brief conclusion and summary of the thesis.

1.1.1 Weyl spinors

As left-handed and right-handed particles are treated differently under the gauge group it is often more convenient to use a chiral basis using 2 component Weyl Spinors.

The familiar 4-component Dirac spinor is reducible into 2 2-component Weyl spinors

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \zeta_\alpha \\ \eta^{\dagger\dot{\alpha}} \end{pmatrix}. \tag{1.1.1}$$

If the Dirac spinor has the same undotted and dotted Weyl spinors ($\eta = \zeta, \psi_L \equiv \psi_R$) then it is called a Majorana* spinor. The hermitian conjugate of a left-handed spinor is a right-handed spinor and vice-versa:

$$(\eta^{\dagger\dot{\alpha}})^\dagger = \eta^\alpha. \tag{1.1.2}$$

Whether the indices α are raised or lowered is important, they are raised and lowered by the anti-symmetric Levi-Civita tensors $\epsilon^{\alpha\beta}$ or $\epsilon_{\alpha\beta}$ in the obvious way $\zeta_\alpha = \epsilon_{\alpha\beta}\zeta^\beta$, similarly for the dotted versions. In this thesis we will in general omit the indices for simplicity, with the understanding that two left-handed spinors contract as $\zeta\eta = \zeta^\alpha\eta_\alpha$ and for right-handed spinors as $\zeta^\dagger\eta^\dagger = \zeta^\dagger_{\dot{\alpha}}\eta^{\dagger\dot{\alpha}}$.

Dirac and Majorana masses

In the above notation a 4-component Dirac spinor Ψ_{D} is given by:

$$\Psi_{\mathrm{D}} = \begin{pmatrix} \xi_\alpha \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix}, \quad \text{and} \quad \bar{\Psi}_{\mathrm{D}} = \begin{pmatrix} \chi^\alpha & \xi^\dagger_{\dot{\alpha}} \end{pmatrix}. \tag{1.1.3}$$

We shall now rewrite the Dirac Lagrangian using this notation,

$$\mathcal{L}_{\mathrm{D}} = i\bar{\Psi}_{\mathrm{D}}\gamma^\mu\partial_\mu\Psi_{\mathrm{D}} - m_{\mathrm{D}}\bar{\Psi}_{\mathrm{D}}\Psi_{\mathrm{D}} \tag{1.1.4}$$

$$= i\xi^\dagger\bar{\sigma}^\mu\partial_\mu\xi + i\chi\bar{\sigma}^\mu\partial_\mu\chi^\dagger - m_{\mathrm{D}}(\xi\chi + \xi^\dagger\chi^\dagger). \tag{1.1.5}$$

If we contrast this with a Majorana spinor given by:

$$\Psi_{\mathrm{M}} = \begin{pmatrix} \xi_\alpha \\ \xi^{\dagger\dot{\alpha}} \end{pmatrix}, \quad \text{and} \quad \bar{\Psi}_{\mathrm{M}} = \begin{pmatrix} \xi^\alpha & \xi^\dagger_{\dot{\alpha}} \end{pmatrix}. \tag{1.1.6}$$

*After Ettore Majorana, born 1906, Catania, Sicily and presumed disappeared at sea 1938 [4].

then the Lagrangian may be rewritten in the 2-component Weyl form as:

$$\mathcal{L}_{\rm M} = \frac{i}{2}\bar{\Psi}_{\rm M}\gamma^{\mu}\partial_{\mu}\Psi_{\rm M} - \frac{1}{2}m_{\rm M}\bar{\Psi}_{\rm M}\Psi_{\rm M} \tag{1.1.7}$$

$$= i\xi^{\dagger}\bar{\sigma}^{\mu}\partial_{\mu}\xi - \frac{1}{2}m_{\rm M}(\xi\xi + \xi^{\dagger}\xi^{\dagger}). \tag{1.1.8}$$

We can now see that the Dirac mass couples left and right-handed fields ($\xi\chi$) together whereas the Majorana masses couple left-handed and right-handed fields to themselves, ($\xi\xi$) and ($\xi^{\dagger}\xi^{\dagger}$).

Right and Left-handed notation

Often in GUTs when we need to unify right-handed and left-handed fields within the same representation it is useful to remember that the charge conjugate of a right-handed field transforms as a left-handed field. In this thesis we shall either use the notation ψ_L and ψ_R to denote left and right-handed fields or we shall use ψ and ψ^c. The advantage of the latter notation is that we can place the charge conjugate of a right-handed field in a GUT representation with left-handed fields.

1.2 The Standard Model

Particles in nature exhibit similar properties, this is suggestive of symmetries which these particles obey. The weak interactions suggest that fermions be grouped into doublets, and the quarks must come in three colours. The need for three colours originally arose from the requirement that three quarks with the same quantum numbers live within hadrons, therefore to be compatible with the Pauli exclusion principle they each needed to be a different colour. Additional support for the three colours comes from decay widths and annihilation cross sections. This suggests that strong interactions could be described by SU(3) and weak interactions by SU(2). Electromagnetic interactions don't change the quantum numbers of the interacting particles so a U(1) group can also be used. The Standard Model [5] is based upon the gauge group:

$$\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1). \tag{1.2.1}$$

Though colour was initially an ad-hoc introduction it is now viewed on a much more fundamental level. In an analogous way to the electric charge being the source of the electric field then colour charge is the source of the colour field. Weak interactions have "charge" given by the third component of weak isospin T_3. Only left-handed particles are charged under weak isospin, right-handed particles are placed into singlet representations a summary of charges under the Standard Model is given in table 1.1. Excluding right-handed neutrinos which will be addressed in subsection 1.6.7, the Standard Model contains 15 matter fields within each generation: there are 2 left-handed lepton fields $(\nu_e, e^-)_L$ and 1 right-handed lepton e_R^+, there are 6 left-handed quark fields $3 \times (u, d)_L$ and 6 right-handed quark fields $3 \times u_R$ and $3 \times d_R$ in both cases the factor of 3 comes from the fact that there are 3 colours. The $\mathrm{SU}(3)_C$ is the gauge group of Quantum Chromodynamics (QCD) which describes the coloured particles i.e. quarks and gluons. The rest of the Standard Model gauge group is the Electroweak group $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ which is broken at low energies. The electroweak gauge bosons are the weak gauge bosons W^+, W^-, Z^0 and the photon γ of the electromagnetic interactions. Electromagnetic interactions originate from the interchange of the neutral gauge boson from $\mathrm{SU}(2)_L$ as well as the gauge boson from U(1), as such the charge of the U(1) group is not the same as the electric charge the electric charge is given by the Gell-Mann-Nishijima relation $Q_{\mathrm{em}} = T_3 + \frac{Y}{2}$ where T_3 is isospin, the third generator associated with $\mathrm{SU}(2)_L$, and Y is the hypercharge from the $\mathrm{U}(1)_Y$ gauge group.

The Higgs mechanism Fermion mass terms cannot simply arise in the Lagrangian as they are excluded by the Standard Model gauge group. Taking a Dirac electron mass term as an example, mee^c is not invariant under $\mathrm{SU}(2)_L$. Since electrons obviously do have mass this is solved by using the Higgs doublet ϕ. Since ϕ is an $\mathrm{SU}(2)_L$ doublet we can form invariant terms by using ϕ together with the lepton doublet l, and similarly for the quark Dirac mass terms. These terms are contained within the yukawa sector of the Lagrangian:

$$\mathcal{L}_{\mathrm{yuk}} = y_u^{ij} \bar{q}_{Li} \tilde{\phi} u_{Rj} + y_d^{ij} \bar{q}_{Li} \phi d_{Rj} + y_e^{ij} \bar{l}_{Li} \phi e_{Rj} + y_\nu^{ij} \bar{l}_{Li} \tilde{\phi} \nu_{Rj} + \mathrm{h.c.} \tag{1.2.2}$$

	Generations			Quantum Numbers		
helicity	1	2	3	Q	T_3	Y_W
L	$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L$	$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$	0 -1	$1/2$ $-1/2$	-1 -1
	$\begin{pmatrix} u \\ d' \end{pmatrix}_L$	$\begin{pmatrix} c \\ s' \end{pmatrix}_L$	$\begin{pmatrix} t \\ b' \end{pmatrix}_L$	$2/3$ $-1/3$	$1/2$ $-1/2$	$1/3$ $1/3$
R	e_R	μ_R	τ_R	-1	0	-2
	u_R	c_R	t_R	2/3	0	4/3
	d_R	s_R	b_R	-1/3	0	-2/3

Table 1.1: The particle content of the Standard Model

The quantum numbers under the electroweak gauge group. The electric charge is labelled Q, the third component of isospin is given by T_3 and weak hypercharge is given by Y_W. The up(u), down(d), strange(s), charm(c) top(t) and bottom(b) quark fields have three colours which have been omitted in the table. The weak isospin partners of the electron(e), muon(μ) and tau-on(τ) are the neutrinos v_e, v_μ, ν_τ. The primes on the down strange and bottom quarks are to label the interaction eigenstates which are superpositions of the mass eigenstates i.e. the observed particles. This superposition is described by the Cabibbo-Kobayashi-Maskawa (CKM) matrix. Not listed in the table is the Higgs boson which transforms as an SU$(2)_L$ doublet with hypercharge of 1.

The i and j are family indices with the SU$(2)_L$ and SU$(3)_C$ indices having been suppressed for simplicity. The Yukawa couplings y^{ij} of the Higgs boson to the fermions govern the masses of the Standard Model fermions, after spontaneous symmetry breaking where the Higgs boson obtains a vacuum expectation value (VEV).

The above Yukawa interactions can only give rise to Dirac masses, where a left-handed and right-handed fermion, or equivalently the charge conjugate of a right-handed fermion as displayed above, are coupled together to form a term $m_{LR} f_L f_R$. Majorana mass terms couple left-handed fields to left-handed fields and right-handed fields to right-handed fields. Of the fields introduced so far it is possible to introduce such masses for the right-handed neutrinos, ν_R, only. We are allowed to do this because the right-handed neutrinos are neutral under the Standard Model gauge group and so a mass term $M\nu_R\nu_R$ is not forbidden.

The down, strange and bottom quarks in table 1.1 are shown in their interaction eigenstates which are superpositions of the observed mass eigenstates, the mixing between these states is given by the CKM matrix. The CKM matrix is given by the product of two unitary matrices which diagonalise y_u and y_d. If $V_u^{L,R}$ diagonalises y_u by $V_u^L y_u V_u^{R\dagger}$ and similarly $V_d^{L,R}$ for y_d, then the CKM matrix $V_{\rm CKM}$ is given by $V_{\rm CKM} \equiv V_u^L V_d^{L\dagger}$. The norm of the elements of the CKM matrix are given by [6]:

$$|V_{\rm CKM}(M_{\rm weak})| \sim \begin{pmatrix} 0.97 & 0.23 & 0.004 \\ 0.23 & 0.97 & 0.04 \\ 0.008 & 0.04 & 0.99 \end{pmatrix}. \tag{1.2.3}$$

The CKM matrix can be parametrised a number of ways the most famous of which is probably the Wolfenstein parametrisation [7]:

$$V_{\rm CKM} = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{1.2.4}$$

where $\lambda \sim 0.22, A \sim 0.82, \rho \sim -0.22, \eta \sim 0.22$ at the weak scale. At one loop order only the parameter A changes significantly and even at two loop order A remains the same order of magnitude up to GUT scales [8]. The Yukawa couplings are the majority of the unknown parameters in the Standard Model. In the quark sector they correspond to 6 quark masses, 3 mixing angles and a complex phase. For the lepton sector we also have 6 lepton masses, 3 mixing angles and a complex phase assuming that the light neutrinos have only Dirac masses. If we include the right-handed neutrino Majornana masses then the number of free parameters obviously increases.

In addition to these parameters the Standard Model has the following parameters: in the Higgs sector the vacuum expectation value and quartic coupling coefficient. In the gauge sector: the $\mathrm{SU}(3)_C$ gauge coupling g_3, the $\mathrm{SU}(2)_L$ gauge coupling g and the $\mathrm{U}(1)_Y$ gauge coupling g'. There is also $\theta_{\rm QCD}$ which parametrises the CP violation of the strong interactions.

1.3 Neutrinos

Neutrinos are unique among the fermions of the Standard Model in that they are uncharged. This special status allows the neutrinos more freedom in the way a mass term may be written down. For charged fermions the only allowed mass term in the so-called Dirac mass. The Dirac mass term connects fields of opposite handedness, a Dirac mass term therefore looks like $m_D \nu \nu^c$. However for the neutral neutrinos don't have such a constraint and we may write down a so-called Majorana mass term connecting fields of the same chirality. The Majorana mass term has the form $m_M \nu \nu$ or $m_M \nu^c \nu^c$. We can immediately see that the Dirac mass is the only type of mass term that we may write down for a charged fermion without violating charge conservation. This stems from a Majorana particle being its own anti-particle, a Majorana mass vertex creates two identical fields, if the field carried a non-zero charge the mass vertex would clearly not conserve the charge. Charged fermions therefore must be Dirac particles where their anti-particle has opposite chirality and the mass vertex creates two oppositely charged particles.

Though we may write down neutrinos with Majorana mass, that doesn't necessarily mean that they are Majorana particles in the real world. One important prediction of Majorana neutrinos is the existence of neutrino-less double beta decay ($\beta\beta_{0\nu}$) which can only arise if the neutrinos are Majorana particles. Numerous experiments have been devised to look for such a decay, see [9] for a review of $\beta\beta_{0\nu}$. Even though we may write a Majorana mass for the neutrino there is nothing to stop us from also writing down a Dirac mass at the same time, such a possibility makes the various Seesaw mechanisms possible providing a natural explanation for the very small observed neutrino masses.

Neutrino oscillations imply the existence of neutrino mass, thus the Standard Model (without the right-handed neutrino) must be an incomplete description of nature. The existence of a neutrino mass is to date the only evidence of physics beyond the Standard Model in the realm of particle physics. The Standard Model doesn't contain a right-handed neutrino and as such there is no coupling of the form $y_\nu H l \nu^c$ which would give the neutrinos mass after symmetry breaking in the same manner as other particles of the Standard Model. A straightforward way to extend the Standard Model to include neutrino mass would therefore seem to be to include a right-handed neutrino, this would also make the model symmetric with respect to quarks and leptons. However there are problems with this seemingly easy extension of the Standard Model. The first problem we see immediately is that the Yukawa coupling $y_\nu H l \nu^c$ we would naturally expect to be of the same order as the quark and charged leptons. However experiment suggests that neutrino masses are at least a factor of 10^6 smaller than the smallest of the quark and charged lepton masses. Therefore neutrino mass not only implies the existence of the right-handed neutrinos but also the existence of some new physics which would enable us to understand why we have such small neutrino masses. A plausible explanation for the small neutrino masses lies in the Seesaw mechanism which makes use of the neutrinos being unique among known fermions in that they can have Majorana mass as described earlier. It has been commented [10] that we may have a better explanation of the 10^6 factor in neutrino masses than we do for the similar 10^6 factor between the top quark and electron masses for which at this present time there is no accepted explanation. The Seesaw mechanism and varieties of it are described below in section 1.6.7. Though the Seesaw mechanism provides an elegant and natural explanation for the lightness of neutrinos there are alternatives to the Seesaw which also seek to explain the small neutrino masses. In such theories the neutrinos can be either Dirac or Majorana fields, they often predict observable charged lepton lepton-flavour violating signals, detection of which could help eliminate some of several theoretical explanations for the origin of the neutrino masses.

1.3.1 Data

As neutrinos are massive, the masses and mixings are as important parameters to understanding nature as the masses and mixings of the charged lepton and quark sectors. Neutrino oscillations arise from a simple quantum mechanical phenomenon during their propagation causing flavour changes. Oscillations are possible due to the existence of lepton mixing in an entirely analogous manner to quark mixing. In quark mixing we have the CKM matrix describing the mixing whereas the leptonic version is called the Pontecorvo-Maki-Nakagawa-Sakata matrix (commonly MNS but also known as the PMNS or MNSP matrix). If we consider a basis where the charged lepton mass matrix is diagonal then we may write the neutrino mass and flavour states as:

$$\nu_i = U_{i\alpha}\nu_\alpha. \tag{1.3.1}$$

The neutrinos with Roman indices are the mass (observed) eigenstates and the neutrinos with Greek indices are the flavour (interaction) eigenstates. The matrix $U_{i\alpha}$ is the MNS matrix which relates the two sets of states, in this way we can easily see how neutrino oscillations occur. Every flavour eigenstate is a linear combination of mass eigenstates which will change during propagation as each mass eigenstate will have a phase factor e^{iE_it}. Since neutrinos are very light then we may take $m \ll p_i$ and therefore $E_i = \sqrt{p^2 + m_i^2} \sim p(1 + \frac{m_i^2}{2E^2} + \dots)$, then each mass state has an energy given by $E_i \sim E + \frac{m_i^2}{2E}$. We can now calculate approximately the probability of the oscillation between two flavour states when a neutrino propagates a given distance. If we call this distance L then the transition probability from state α to β is given by

$$P_{\alpha\beta} = \left| U_{i\alpha}U^*_{i\beta}e^{-i\frac{m_i^2 L}{2E}} \right|^2. \tag{1.3.2}$$

We can express $P_{\alpha\beta}$ in terms of deviations from the identity matrix as $P_{\alpha\beta} = \delta_{\alpha\beta} + D_{\alpha\beta}$ with the deviation $D_{\alpha\beta}$ given by:

$$D_{\alpha\beta} = -4\sum_{i>j} \Re(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}) \sin^2\left(\frac{\Delta m^2_{ij}}{4E}\right) + 2\Im(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}) \sin\left(\frac{\Delta m^2_{ij}}{2E}\right) \tag{1.3.3}$$

The key point is that although the overall mass scale isn't measurable the squared difference $\Delta m^2_{ij} = m_i^2 - m_j^2$ is, a summary of the oscillation data is given in table 1.2.

The angles in table 1.2 refer to the standard parametrisation of the neutrino

Parameter	best fit	2σ	3σ
Δm^2_{21} $[10^{-5}\text{eV}^2]$	$7.65^{+0.23}_{-0.20}$	7.25-8.11	7.05-8.34
Δm^2_{31} $[10^{-3}\text{eV}^2]$	$2.40^{+0.12}_{-0.11}$	2.18-2.64	2.07-2.75
$\sin^2\theta_{12}$	$0.304^{+0.022}_{-0.016}$	0.27-0.35	0.25-0.37
$\sin^2\theta_{23}$	$0.50^{+0.07}_{-0.06}$	0.39-0.63	0.36-0.67
$\sin^2\theta_{13}$	$0.01^{+0.016}_{-0.011}$	≤ 0.040	≤ 0.056

Table 1.2: Neutrino Data
Best-fit values, 1σ errors, 2σ and 3σ intervals (1 d.o.f.) for the three flavour neutrino oscillation parameters from global data (from [11]).

mixing matrix:

$$U = \begin{pmatrix} c_{13}c_{12} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - s_{23}s_{13}c_{12} & c_{23}c_{12} - s_{23}s_{12}s_{13} & s_{23}c_{13} \\ s_{23}s_{12} - s_{13}c_{23}c_{12} & -s_{23}c_{12} - s_{13}s_{12}c_{23} & c_{23}c_{13} \end{pmatrix}, \tag{1.3.4}$$

where c_{ij}, s_{ij} refer to $\cos\theta_{ij}$ and $\sin\theta_{ij}$ respectively. The angle θ_{12} is referred to as the solar angle $\theta_\odot$, θ_{23} is the atmospheric angle $\theta_@$ and finally the angle θ_{13} is the reactor angle θ_r. The names of the various angles refer to the types of experiment used to measure them.

1.3.2 Tri-Bimaximal mixing

The data in table 1.2 is consistent with the so-called Tri-Bimaximal mixing scheme first proposed by Harrison, Perkins and Scott [12]. This scheme has:

$$\sin^2\theta_\odot = 1/3, \tag{1.3.5}$$

$$\sin^2\theta_@ = 1/2, \tag{1.3.6}$$

$$\sin^2\theta_r = 0, \tag{1.3.7}$$

leading to the MNS mixing matrix of:

$$U_{\rm MNS} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{6}} & -\sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}, \tag{1.3.8}$$

contrast these large mixing angles with the quark sector and we see that the mixings in the lepton sector are very much larger than the quark sector (the Cabibbo angle is the largest quark mixing angle at $\sin\theta_C \sim 0.23$). Such a mismatch is a challenge for models which seek to unify leptons and quarks within some shared symmetry(s). The focus of this thesis will be the construction of models which predict Tri-Bimaximal mixing.

1.4 Supersymmetry

Supersymmetry is a symmetry between bosons and fermions. It can be realised in nature if we assume that each particle with spin j has a supersymmetric partner with spin $j \pm 1/2$. The particle spectrum is therefore doubled, we can assign the particles to supermultiplets. The vector supermultiplet contains the gauge bosons and the chiral multiplet contains the matter fields. However the supersymmetric particles “sparticles” have so far not been observed in nature which leaves us with two possibilities: 1) Supersymmetry is an nice idea but it has nothing to do with reality or 2) Supersymmetry is not exact and the sparticles are heavier than the particles and thus haven’t been observed yet. Support for 2 is widespread as there are many good reasons for believing in supersymmetry:

Supersymmetry solves the hierarchy problem: The hierarchy problem [13] of the Standard Model stems from the Higgs mass being quadratically dependent on the cutoff at which new physics appears. The Higgs mass as yet hasn’t been measured experimentally however we know that since it sets the scale of electroweak breaking it must be $\mathcal{O}(10^2)$ GeV. If the new physics appears at the Planck scale then the ratio

Multiplet			
Chiral		Vector	
$J = 1/2$	$J = 0$	$J = 1$	$J = 1/2$
q_L, u_R, d_R	$\tilde{q}_L, \tilde{u}_R, \tilde{d}_R$	g	$\tilde{g}$
l_L, e_R	$\tilde{l}_L, \tilde{e}_R$	$W^{\pm}, W^0$	$\tilde{W}^{\pm}, \tilde{W}^0$
$\tilde{H}_1, \tilde{H}_2$	H_1, H_2	B	$\tilde{B}$

Table 1.3: A list of the Standard Model particles alongside their supersymmetric partners in the MSSM

of the Higgs mass and the cutoff is $\mathcal{O}(10^{-17})$ which would require fine tuning between the tree level mass and the radiative corrections. Supersymmetry solves the hierarchy problem by introducing new diagrams to the contribution to the quadratic divergence of the Higgs mass, these diagrams exactly cancel the Standard Model contributions due to the -1 introduced due to the fermion loop.

Supersymmetry offers an explanation for dark matter: There is a commonly assumed symmetry called “R-parity” in supersymmetric models, for example see [13] for details. A consequence of this symmetry is that there should exist a stable supersymmetric particle, to so-called LSP (lightest supersymmetric particle). such a particle could be a viable candidate for a dark matter WIMP (weakly interacting massive particle).

Supersymmetry is required by String Theory: String theory is one of the most promising candidates for a theory of everything. In many string theories supersymmetry is a natural part of the theory. If we wish to reconcile some quantum field theory with general relativity at some high energy scale in some string theory then supersymmetry will have to be included at some point.

Supersymmetry unifies the coupling constants: Another extremely nice feature of supersymmetry at low energy is that there is apparent unification of the coupling constants. The Standard Model doesn’t quite unify the coupling constants however if superpartners are introduced at around the TeV scale then the coupling constants evolve differently and the three couplings run together.

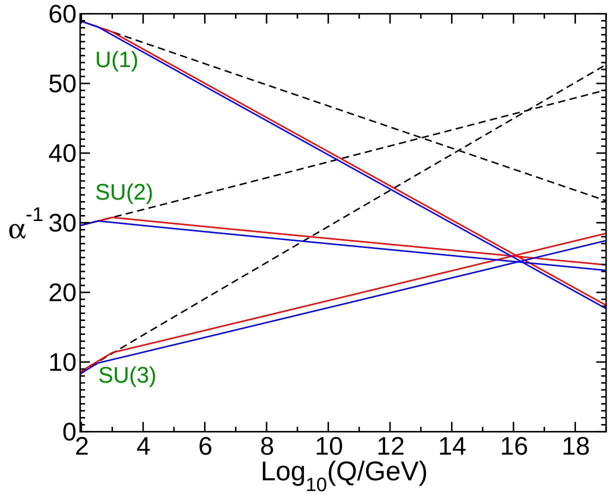

Figure 1.1: Running coupling constants, from [13]
The dotted lines represent the evolution of the coupling constants in the absence of Supersymmetry. The solid lines show that when Supersymmetry is included the coupling constants unify.

1.5 Review of Grand Unified Theories

The Standard Model relates charged leptons and neutrinos under the $SU(2)_L$ symmetry which is broken via the Higgs mechanism. In a similar way it may be possible to relate quarks and leptons under some larger symmetry at a higher energy scale which is broken to the Standard Model at low energies. Apart from gravity each of the known forces, electromagnetism, the weak and strong forces, is associated with a Lie algebra, this suggests that it may be possible to unify the forces within a single simple Lie algebra, such theories are called Grand Unified Theories (GUTs).

If such a GUT were to exist then because the Standard Model gauge group is rank 4 therefore any GUT group must be rank 4 or larger. Of rank 4 Lie groups there are nine which have one coupling strength. Georgi and Glashow argued [14] that seven of these nine groups may be excluded since they don't have complex representations, this leaves is with SU(5) and $SU(3) \otimes SU(3)$. As $SU(3) \otimes SU(3)$ cannot accommodate integer and fractional charges SU(5) is the only viable rank 4 GUT group. Though we will mainly use SU(5) other gauge groups are available of particular note is SO(10) which has a 16 dimensional representation which naturally contains a right-handed neutrino (15 Standard Model fields + right-handed neutrino). Many string theories make use of the group E_8 whose dynkin diagram is shown in figure 1.2, by removing the right-most root we eventually get to SU(5) and the Standard Model. Though proving nothing this does suggest the route to take to build from the Standard Model through GUTs to a String Theory.

1.5.1 The SU(5) Grand Unified Theory

The SU(5) grand unified model was one of the first attempts to unify the Standard Model within an larger gauge group, the model is often referred to as the Georgi-Glashow model after Howard Georgi and S. L. Glashow [14]. The 15 left-handed fields of the Standard Model may be placed into a $\bar{\mathbf{5}} \oplus \mathbf{10}$ of SU(5):

$$\bar{\mathbf{5}} = \begin{pmatrix} d^c_{\mathrm{r}} \\ d^c_{\mathrm{g}} \\ d^c_{\mathrm{b}} \\ e^- \\ \nu \end{pmatrix}, \mathbf{10} = \begin{pmatrix} 0 & u^c_{\mathrm{b}} & -u^c_{\mathrm{g}} & u_{\mathrm{r}} & d_{\mathrm{r}} \\ -u^c_{\mathrm{b}} & 0 & u^c_{\mathrm{r}} & u_{\mathrm{g}} & d_{\mathrm{g}} \\ u^c_{\mathrm{g}} & -u^c_{\mathrm{r}} & 0 & u_{\mathrm{b}} & d_{\mathrm{b}} \\ -u_{\mathrm{r}} & -u_{\mathrm{g}} & -u_{\mathrm{b}} & 0 & e^c \\ -d_{\mathrm{r}} & -d_{\mathrm{g}} & -d_{\mathrm{b}} & -e^c & 0 \end{pmatrix}. \tag{1.5.1}$$

The SU(5) group has 24 generators which can be represented by generalised Gell-Mann matrices. The 24 gauge bosons transform as the adjoint representation as:

$$\mathbf{24} = \left(\begin{array}{ccc|cc} G_{\mathrm{rr}} - \frac{2B}{\sqrt{30}} & G_{\mathrm{rg}} & G_{\mathrm{rb}} & X^c_{\mathrm{r}} & Y^c_{\mathrm{r}} \\ G_{\mathrm{gr}} & G_{\mathrm{gb}} - \frac{2B}{\sqrt{30}} & G_{\mathrm{gb}} & X^c_{\mathrm{g}} & Y^c_{\mathrm{g}} \\ G_{\mathrm{br}} & G_{\mathrm{bg}} & G_{\mathrm{bb}} - \frac{2B}{\sqrt{30}} & X^c_{\mathrm{b}} & Y^c_{\mathrm{b}} \\ \hline X_{\mathrm{r}} & X_{\mathrm{g}} & X_{\mathrm{b}} & \frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} & W^+ \\ Y_{\mathrm{r}} & Y_{\mathrm{g}} & Y_{\mathrm{b}} & W^- & -\frac{W^3}{\sqrt{2}} + \frac{3B}{\sqrt{30}} \end{array}\right). \tag{1.5.2}$$

In addition to the 12 gauge bosons of the Standard Model the Georgi-Glashow theory also includes 12 new Baryon-Lepton number violating X and Y bosons.

Charge prediction: A useful feature of the SU(5) GUT is the prediction of charges of the particles. Since the quarks and leptons are assigned to the same multiplet then

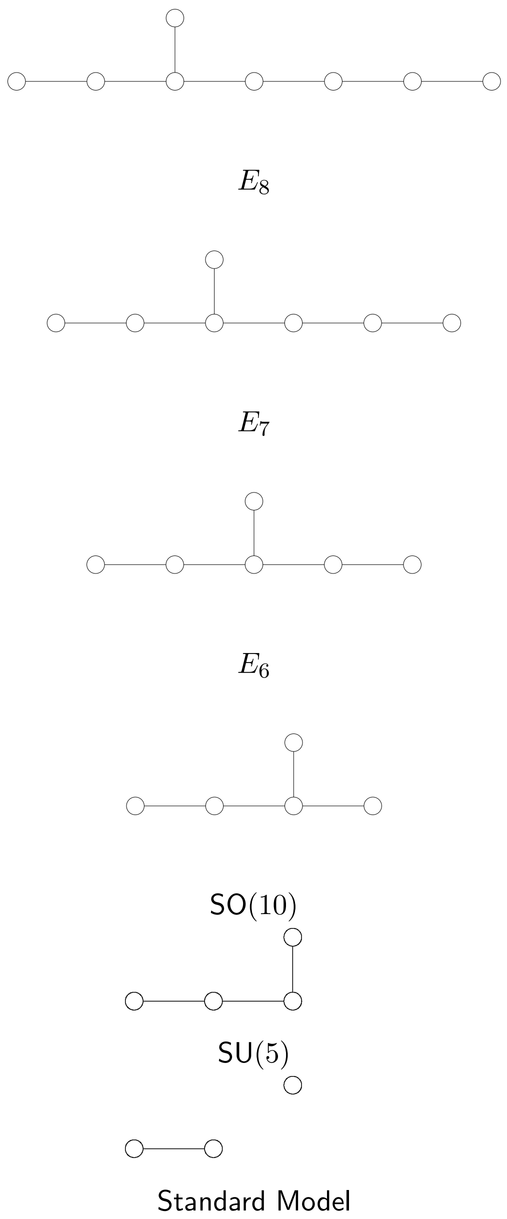

Figure 1.2: Dynkin diagrams of GUT groups
Dynkin diagrams of GUT groups: by removing the right-most root from the diagram of E_8 we find that we go through the gauge groups $E_7 \rightarrow E_6 \rightarrow \mathrm{SO}(10) \rightarrow \mathrm{SU}(5) \rightarrow$ SM reaching the Standard Model gauge group

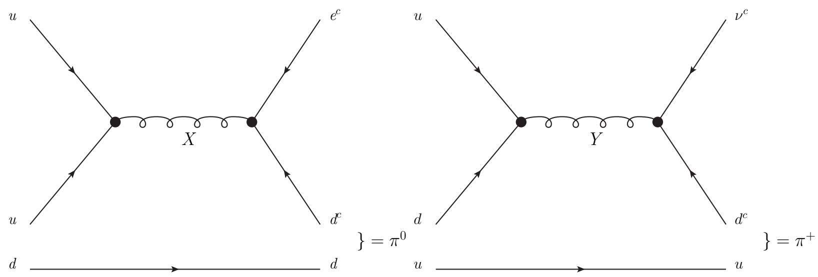

Figure 1.3: Proton Decay via SU(5) Gauge Bosons
The new SU(5) gauge bosons, often called leptoquark bosons, introduce new transitions between quarks and leptons. These new transitions result in proton decay.

their charges must be related as the trace of any generator of SU(5) must be zero. Acting the charge operator on the fundamental representation gives us:

$$\operatorname{Tr} Q = \operatorname{Tr}(q_{d^c}, q_{d^c}, q_{d^c}, e, 0) = 0. \tag{1.5.3}$$

This gives us the prediction that the charge on the d-quark must be $\frac{1}{3}$ the charge of the electron i.e. $-\frac{1}{3}$. The theory also predicts the charge of the u-quark being $+\frac{2}{3}$.

Proton Decay and Doublet-Triplet splitting: The X and Y gauge bosons can induce proton decay because they introduce transitions between quarks and leptons, such transitions violate lepton and baryon number but the difference $B - L$ is conserved in these transitions. Some example decays are shown in figure 1.3.

In SU(5) the breaking of the electroweak symmetry is achieved by a **5**-plet of Higgs fields, the minimum of the potential is chosen to be:

$$\langle H_{\mathbf{5}} \rangle = v \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{1.5.4}$$

Where the fourth and fifth entries correspond to the SU(2) doublet of the Standard

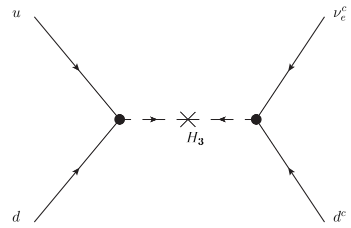

Figure 1.4: Triplet Higgs mediate nucleon decay
A rather troublesome feature of the SU(5) model is the appearance of coloured triplet Higgs. These transitions must be suppressed to get a realistic rate of proton decay.

Model. Since the colour triplet Higgs fields couple to all fermions with mass they can induce proton decay via the diagram in figure 1.4.

GUT relations: In the SU(5) GUT theory both the charged lepton and down quark yukawa couplings are given by terms of the form $y_{ij} H \bar{\mathbf{5}}_i \mathbf{10}_j$ if the Higgs multiplet is taken to be in the fundamental representation (i.e. $\bar{\mathbf{5}}$) then the yukawa matrices $y_{\text{charged lepton}}, y_{\text{down}}$ will be transposes of each other this leads to the relation, at the GUT scale, that the masses will be related:

$$m_e = m_d \tag{1.5.5}$$

$$m_\mu = m_s \tag{1.5.6}$$

$$m_\tau = m_b. \tag{1.5.7}$$

Such a relation is in conflict with data which is a problem for the SU(5) theory. However mechanisms have been proposed [15] that allow an SU(5) GUT to evade these relations.

1.5.2 Georgi-Jarlskog mechanism

The GUT mass relations (equations (1.5.5),(1.5.6),(1.5.7)) predicted by the SU(5) theory are in contradiction with experiment. Though we won't make use of it in this thesis, a mechanism exists [15] which allows an SU(5) gauge theory to correctly predict the GUT mass relations. The key idea is that in addition to the $\mathbf{5}$ of Higgs a $\mathbf{45}$ of Higgs fields is introduced. This is the other choice of the representation we can choose for the Higgs since $\bar{\mathbf{5}} \otimes \mathbf{10} = \mathbf{5} \oplus \mathbf{45}$. The particular form of the VEV of the **45** introduces a factor of 3 in the mass matrices for the charged leptons relative to the down quark mass matrix which gives correct GUT relations. Perhaps the best way to understand the mechanism is to review the model presented in [15], the field content is as follows (note:group indices have been omitted for simplicity): We have right-handed **5**-plets of SU(5)

$$F_{jR}, j = 1 - 3 \tag{1.5.8}$$

and also left-handed **10**-plets

$$T_{jL}, j = 1 - 3. \tag{1.5.9}$$

There are three **5**s and a **45** of Higgs

$$H_{5j} \ , \qquad \langle H_{5j} \rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v_j \end{pmatrix} \quad , \quad j = 1 - 3 \tag{1.5.10}$$

$$H_{45} \ , \langle H_{45}{}^{a5}_{c} \rangle = K \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.5.11}$$

where the matrix representing the VEV of the **45**-plet is the projection in the 5^{th} direction in the third tensor index where there is a non-zero VEV. If we consider the yukawa sector of the Lagrangian is given by:

$$\begin{aligned} \mathcal{L}_y &= \{A\bar{T}_{1L}F_{2R} + A'\bar{T}_{2L}F_{1R} + B\bar{T}_{3L}F_{3R}\}H_{51}/\mu_1 \\ &+ C\bar{T}_{2L}F_{2R}H_{45}/K \\ &+ \{DT_{2L}^T\gamma^0 T_{2L} + ET_{3L}^T\gamma^0 T_{3L}\}H_{52}/\mu_2 \\ &+ FT_{2L}^T\gamma^0 T_{3L}H_{53}/\mu_3 + \text{h.c.} \end{aligned} \tag{1.5.12}$$

The Higgs multiplets obtain VEVs (equations (1.5.10,1.5.11)) giving mass matrices of the form:

$$m_{\mathrm{up}} = \begin{pmatrix} 0 & D & 0 \\ D & 0 & F \\ 0 & F & E \end{pmatrix} , \; m_{\mathrm{down}} = \begin{pmatrix} 0 & A' & 0 \\ A & C & 0 \\ 0 & 0 & B \end{pmatrix} , \tag{1.5.13}$$

$$m_{\text{charged lepton}} = \begin{pmatrix} 0 & A & 0 \\ A' & -3C & 0 \\ 0 & 0 & B \end{pmatrix} . \tag{1.5.14}$$

The required factor of -3 arises because of the particular form of the VEV we have chosen for the **45**. The choice of VEV for $H_{45}{}^{a5}_{c}$ isn't entirely random, The first three entries in the "matrix" $\delta^a_c - 4\delta^{a4}\delta_{b4}$ must be equal to preserve colour symmetry and because the **45** is an irreducible representation it must be traceless resulting in the factor of -3:

$$\delta^a_c - 4\delta^{a4}\delta_{b4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} . \tag{1.5.15}$$

Note when we diagonalise m_{down} and $m_{\text{charged lepton}}$ we find the down quark mass is given by $\frac{AA'}{C}$ and the electron mass by $\frac{AA'}{3C}$ resulting in the correct GUT mass relations:

$$m_e = \frac{1}{3} m_{\mathrm{down}} \tag{1.5.16}$$

$$m_\mu = 3 m_{\mathrm{strange}} \tag{1.5.17}$$

$$m_\tau = m_{\mathrm{bottom}} . \tag{1.5.18}$$

To complete the story we would also have to predict the size of the coefficients in the Lagrangian and forbid a term in the 11 entry of m_{down} and $m_{\text{charged lepton}}$, but the mechanism shows that SU(5) GUTs can give correct GUT mass relations. The mechanism isn't restricted to SU(5) GUTs models have been proposed using GUT groups other than SU(5) for example Pati-Salam and SO(10) [16, 17].

1.6 Review of family symmetries

In contrast to GUTs a family symmetry is a symmetry between the different generations of matter, that is to say it is some symmetry between electrons, muons and tau-ons or between down, strange and bottom quarks. If we consider what this family symmetry may be then we must look at what is consistent with data so far, the largest family symmetry group that is consistent with the Standard Model is $U(3)_f^5$†. This corresponds to and independent $U(3)_f$ for the left-handed quark doublet q, the quark singlets u^c and d^c, the left-handed lepton doublet l and the lepton singlet e^c. If we include the right-handed neutrino ν^c then the maximal family symmetry grows to $U(3)_f^6$. On the other hand if the family symmetry is to be made compatible with a GUT group then the maximal family symmetry is reduced. For example an SO(10) GUT has a maximal family symmetry group of $U(3)_f$ as all the Standard Model families belong to the same SO(10) representation, for SU(5) including a right-handed neutrino ν^c the maximal family symmetry is $U(3)_f^3$ (a $\bar{\mathbf{5}}, \mathbf{10}$ and $\mathbf{1}$).

In order to explain the observed masses and mixing angles the family symmetry must be broken. We break the family symmetry by using fields which acquire a VEV giving mass terms in the Lagrangian. Such fields are called "flavons" due to the connection with flavour‡.In the following subsection we shall give a review of several recent family symmetry models and give a very simple model illustrating a family symmetry leading directly to fermion masses and mixing angles.

1.6.1 Non-Abelian family symmetries

We need not restrict ourselves to continuous groups, we may also use discrete groups. Discrete groups have more lower dimensional representations than continuous groups, non-Abelian groups also have irreducible representations with dimension greater than

†The subscript f denotes a family symmetry rather than a gauge symmetry

‡Such fields are alternatively termed "familons" for the same reason, or sometimes "spurions".

one allowing them to relate different generations. For these reasons non-Abelian discrete groups would seem to be a promising candidate for a theory of flavour.

The basic idea is to assign the gauge representations to the flavour representation of the non-Abelian group and then write down the yukawa couplings. Because of the choice of representations the structure of the Yukawa sector is restricted and, after the Higgs boson(s) obtain a VEV, the mass matrices are restricted.

Among the non-Abelian discrete groups A_4 is very useful. A_4 allows the two quark mass matrices ($m_{\text{up}}, m_{\text{down}}$) to be diagonalised by the same unitary transformation giving no mixing at leading order. However large mixing can be achieved in the lepton sector because of the Majorana nature of the neutrinos. This gives the possibility of achieving TBM (section 1.3.2) which is as noted earlier is a good approximation to the available data.

1.6.2 The A_4 group and its representations

In this section we will provide a brief overview of the A_4 group. In particular deriving the explicit calculation of terms containing products of A_4 representations. The A_4 group is the group of even permutations of 4 objects. There are $\frac{4!}{2} = 12$ elements. This group is also the symmetry group of the tetrahedron, the odd permutations can be seen as the exchange of two vertices which can't be obtained with a rigid solid. If we let a generic permutation be denoted by $(1, 2, 3, 4) \to (n_1, n_2, n_3, n_4) = (n_1 n_2 n_3 n_4)$. A_4 can be generated by the two basic permutations S and T where $S = (4321)$ and $T = (2314)$. We can check that the following relation holds:

$$S^2 = T^3 = (ST)^3 = 1. \tag{1.6.1}$$

This relation is characteristic of A_4 and is called the presentation of the group.

Equivalence classes of A_4

There are 4 equivalence classes (h and k belong to the same equivalence class if there is a member of the group g such that $ghg^{-1} = k$):

$$C1 : I = (1234) \tag{1.6.2}$$

$$C2 : T = (2314), ST(4132), TS = (3241), STS = (1423) \tag{1.6.3}$$

$$C3 : T^2 = (3124), ST^2 = (4213), T^2S = (2431), TST = (1342) \tag{1.6.4}$$

$$C4 : S = (4321), T^2ST = (3412), TST^2 = (2143). \tag{1.6.5}$$

For a finite group the squared dimensions for each inequivalent representation sum to N, the number of transformations in the group (N=12 for A_4). There are 4 inequivalent representations of A_4 three singlets 1,$1'$,$1''$ and a triplet 3. The three singlets representations are:

$$1 : \; S = 1 \;\; T = 1 \tag{1.6.6}$$

$$1' : \; S = 1 \;\; T = e^{2\pi i/3} = \omega \tag{1.6.7}$$

$$1'' : \; S = 1 \;\; T = e^{4\pi i/3} = \omega^2. \tag{1.6.8}$$

The triplet representation in the basis where S is diagonal is constructed from:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{1.6.9}$$

Characters of A_4

The characters of a group χ_g^R of each element g are defined as the trace of the matrix that maps the element in a representation R. Equivalent representations have the same characters and the characters have the same value for all the elements in an equivalence class. Characters satisfy $\sum_g \chi_g^R \chi_g^{S*} = N\delta^{RS}$. Also the character for an element h in a direct product of representations is a product of characters

Class	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_3
C_1	1	1	1	3
C_2	1	ω	ω^2	0
C_3	1	ω^2	ω	0
C_4	1	1	1	-1

Table 1.4: The A_4 Character table
From the character table we can see, by using $\sum_g \chi_g^R \chi_g^{S^*} = N\delta^{RS}$, that there are no more irreducible representations other than $\mathbf{1}, \mathbf{1}', \mathbf{1}''$ and $\mathbf{3}$.

$\chi_h^{R\otimes S} = \chi_h^R \chi_h^S$and is also equal to the sum of characters in each representation that appears in the decomposition of $R \otimes S$.

From the character table 1.4 we can see that there are no more inequivalent irreducible representations of A_4 than $1, 1', 1''$ and 3. We can also see the multiplication rules:

$$3 \times 3 = 1 + 1' + 1'' + 3 + 3 \tag{1.6.10}$$

$$1' \times 1' = 1'' \tag{1.6.11}$$

$$1' \times 1'' = 1 \tag{1.6.12}$$

$$1'' \times 1'' = 1'. \tag{1.6.13}$$

If we have two triplets $3_a \sim (a_1, a_2, a_3)$ and $3_b \sim (b_1, b_2, b_3)$ we can obtain the irreducible representations from their product:

$$1 = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1.6.14}$$

$$1' = a_1 b_1 + \omega^2 a_2 b_2 + \omega a_3 b_3 \tag{1.6.15}$$

$$1'' = a_1 b_1 + \omega a_2 b_2 + \omega^2 a_3 b_3 \tag{1.6.16}$$

$$3_\mathrm{s} \sim (a_2 b_3, a_3 b_1, a_1 b_2) \tag{1.6.17}$$

$$3_\mathrm{a} \sim (a_3 b_2, a_1 b_3, a_2 b_1). \tag{1.6.18}$$

Another representation

So far we have used the representation where the matrix S is diagonal. In this thesis we will construct models in a different basis where we arrange T to be diagonal through a unitary transformation:

$$T' = VTV^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad S' = VSV^\dagger = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \tag{1.6.19}$$

where

$$V = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}. \tag{1.6.20}$$

In this basis the product composition rules are different:

$$1 = a_1b_1 + a_2b_3 + a_3b_2 \tag{1.6.21}$$

$$1' = a_3b_3 + a_1b_2 + a_2b_1 \tag{1.6.22}$$

$$1'' = a_2b_2 + a_1b_3 + a_3b_1 \tag{1.6.23}$$

$$3_{\mathrm{s}} \sim \frac{1}{3}(2a_1b_1 - a_2b_3 - a_3b_2, 2a_3b_3 - a_1b_2 - a_2b_1, 2a_2b_2 - a_1b_3 - a_3b_1) \tag{1.6.24}$$

$$3_{\mathrm{a}} \sim \frac{1}{2}(a_2b_3 - a_3b_2, a_1b_2 - a_2b_1, a_1b_3 - a_3b_1). \tag{1.6.25}$$

1.6.3 Recent models

By way of introducing the concept of family symmetries we will give a short (and by no means complete) list of recent papers which contain "family symmetry" in the title.

Both Abelian and non-Abelian groups have been considered as possible candidates for a family symmetry. A brief search of the literature indicates that non-Abelian groups seem to be favoured at the present time. We may split the non-Abelian groups into models using continuous and discrete groups. Of the models using continuous groups [18] uses an extended GUT model based on the Pati-Salam GUT group, the model uses a SO(3) family symmetry. A slightly larger family symmetry group, SU(3) is used in [19], an additional feature of the model is the prediction of Bi-maximal mixing. The model given in [20] uses an SU(2) family symmetry along with a supersymmetric extension of the Standard Model, in this model the first two families transform as doublets with the third family transforming as a singlet. The model given in [21] uses an O(2) family symmetry in the leptonic sector, the model predicts a vanishing θ_{13} mixing angle.

Discrete family symmetries have also been the subject of much interest in the literature. We will consider the group A_4 later on and indeed it is a good candidate for a discrete family symmetry as it is the smallest discrete group with a triplet representation. The model given in [22] uses A_4 as a family symmetry, [22] uses a SUSY GUT model based on $SU(5)$ and predicts Tri-Bimaximal mixing. The larger group A_5 is the symmetry group of the icosahedron and is considered as a family symmetry in [23], the model predicts golden ratio neutrino mixing. Smaller discrete groups are also candidates for a family symmetry S_3 the group of all permutations of 3 objects is considered in [24] in combination with an E_6 GUT group. Finally [25] considers D_6 as a candidate for a family symmetry and identifies a cold dark matter candidate.

1.6.4 Review of an A_4 model given in [26]

An important model regarding A_4 family symmetry is given in [26] which we shall, by way of an introduction to A_4 models, briefly review here. The model predicts Tri-Bimaximal neutrino mixing and is of the direct kind. The right-handed leptons e^c, μ^c, τ^c are assigned to the A_4 singlet representations $\mathbf{1}, \mathbf{1}'', \mathbf{1}'$ respectively. The Higgs doublets $h_{u,d}$ are invariant under the A_4 symmetry. The Yukawa interactions in the leptonic sector are as follows:

$$\mathcal{L}_l = y_e e^c(\varphi_T l) + y_\mu \mu''^c(\varphi_T l)' + y_\tau \tau'^c(\varphi_T l)'' + x_a \xi(ll) + x_b(\varphi_S ll) + \text{h.c.} + \dots \quad (1.6.26)$$

where the dots indicate higher order terms. As in [26] we shall omit the Higgs fields $h_{u,d}$ and the cut-off scale Λ, for example the term $y_e e^c(\varphi_T l)$ means $y_e \frac{h_d}{\Lambda} e^c(\varphi_T l)$ and similarly $\xi(ll)$ means $\frac{\xi}{\Lambda^2}(h_u l h_u l)$. The reader will note that terms allowed by the flavour symmetry such as interchanging $\varphi_T \leftrightarrow \varphi_S$ and (ll) are absent, this is crucial to the model and their absence is motivated by extra discrete symmetries. We then assume that the flavon fields φ_T,φ_S,ξ develop VEVs of:

$$\langle\varphi_T\rangle = (v_T, 0, 0) \tag{1.6.27}$$

$$\langle\varphi_S\rangle = (v_S, v_S, v_S) \tag{1.6.28}$$

$$\langle\xi\rangle = u. \tag{1.6.29}$$

After the Higgs and flavon fields obtain their VEVs from equation (1.6.26) we are left with the mass terms:

$$\begin{aligned}\mathcal{L}_l &= v_d\frac{v_T}{\Lambda}(y_e e e^c + y_\mu \mu\mu^c + y_\tau \tau\tau^c)\\ &+ x_a v_u^2 \frac{u}{\Lambda^2}(\nu_e\nu_e + 2\nu_\mu\nu_\tau)\\ &+ x_b v_u^2 \frac{2v_S}{3\Lambda^2}(\nu_e\nu_e + \nu_\mu\nu_\mu + \nu_\tau\nu_\tau - \nu_e\nu_\mu - \nu_\mu\nu_\tau - \nu_\tau\nu_e) + \text{h.c.} + \ldots\end{aligned} \tag{1.6.30}$$

In the charged lepton sector the A_4 symmetry is broken to G_T a subgroup of A_4 generated by T and isomorphic to $\mathbb{Z}_3$. In the neutrino sector the A_4 is broken to G_S which is generated by S and isomorphic to $\mathbb{Z}_2$. The mass matrices are then given by:

$$m_e = v_d\frac{v_T}{\Lambda}\begin{pmatrix} y_e & 0 & 0\\ 0 & y_\mu & 0\\ 0 & 0 & y_\tau\end{pmatrix} \tag{1.6.31}$$

$$m_\nu = \frac{v_u^2}{\Lambda}\begin{pmatrix} a+2b/3 & -b/3 & -b/3\\ -b/3 & 2b/3 & a-b/3\\ -b/3 & a-b/3 & 2b/3\end{pmatrix} \tag{1.6.32}$$

where a and b are given by:

$$a \equiv 2x_a\frac{u}{\Lambda}, \qquad b \equiv 2x_b\frac{v_S}{\Lambda}. \tag{1.6.33}$$

The neutrino mass matrix is diagonalised by the familiar HPS matrix given in equation (1.3.8). The vacuum alignment proceeds via the introduction of driving fields

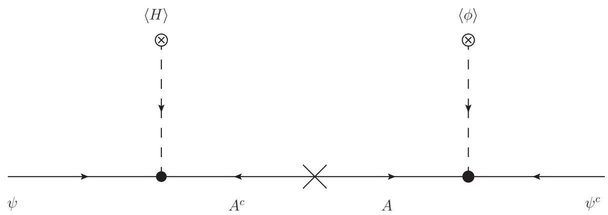

Figure 1.5: The Froggatt-Nielsen mechanism
The Froggatt-Nielsen mechanism gives rise to an effective mass term for ψ via a heavy messenger field.

and minimising the resulting scalar potential. The details are given in [26] and the same procedure is used in section 3.4.

1.6.5 The Froggatt-Nielsen mechanism

A mass generation mechanism we shall make use of in later chapters is the Froggatt-Nielsen mechanism [27]. The mechanism makes use of higher order diagrams via tree-level diagrams using heavy fields, the so-called messenger fields.

The diagram in figure 1.5 shows the simplest example of the Froggatt-Nielsen mechanism. The fields labelled A, A^c are the heavy Froggatt-Nielsen messenger fields. These fields have a mass M_A given by the mass term $M_A A^c A$ (represented by the $\times$ vertex). The messengers must also have appropriate Standard Model (or indeed GUT group) and family symmetry charge assignments, this is relevant to the placement of the Higgs and ϕ insertions. The heavy messenger fields are integrated out, in the case of figure 1.5 this gives rise to an effective superpotential term of:

$$w = \frac{\langle \phi \rangle}{M_A} \langle H \rangle \psi \psi^c = m_\psi \psi \psi^c \qquad (1.6.34)$$

the effective mass is therefore $m_\phi = \frac{\langle \phi \rangle}{M_A} \langle H \rangle$. We are not restricted by the number of messenger fields we choose to include in the theory, figure 1.6 gives a more general diagram of the mechanism. The diagram features two messenger fields A and B with associated mass terms $M_A A^c A$ and $M_B B^c B$ and the flavons ϕ_a and ϕ_b. We could of course go on and include messengers $C, D, \ldots$ and extra flavons however we must bear

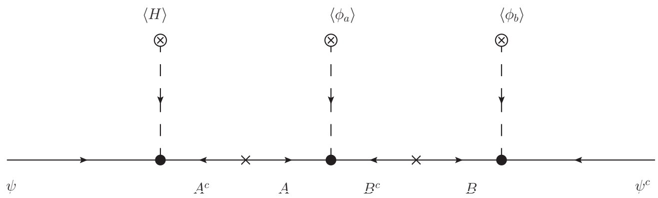

Figure 1.6: A more general diagram of the Froggatt-Nielsen mechanism

In this diagram we have a more general Froggatt-Nielsen mechanism with two flavons and two different messenger fields.

in mind that the charges of the messenger fields must be such that the diagrams are allowed. For example in figure 1.6 if ϕ_a and ϕ_b have $\mathrm{U}(1)_{\mathrm{Froggatt-Nielsen}}$ charges $-1, +1$ respectively, the Higgs and matter fields ψ, ψ^c are uncharged then the messenger fields A^c, A, B^c, B must be charged $0, 0, 1, -1$ respectively. The superpotential term giving the effective mass is given by:

$$w = \frac{\langle \phi_a \rangle \langle \phi_b \rangle}{M_A M_B} \langle H \rangle \psi \psi^c = m'_\psi \psi \psi^c \tag{1.6.35}$$

giving the effective mass of $m'_\psi = \frac{\langle \phi_a \rangle \langle \phi_b \rangle}{M_A M_B}$.

1.6.6 A U(1) toy model family symmetry

To illustrate the use of family symmetries we introduce a simple toy model using a U(1) family symmetry commuting with the Standard Model gauge group. The family symmetry is broken by introducing a flavon ϕ which acquires a vacuum expectation value $\langle \phi \rangle$. Since the model is only being used to illustrate the use of a family symmetry we will only concern ourselves with the down type quarks. The charges under the family symmetry are given in table 1.5.

According to the charge assignment the mass terms include powers of the flavon field ϕ in order to be invariant under the flavour symmetry in addition to the gauge

Field	U(1)
H_d	0
ϕ	-1
d_1	4
d_2	2
d_3	0
d_1^c	2
d_2^c	1
d_3^c	0

Table 1.5: U(1) Charge Assignments
Charge assignments for the toy model given in section 1.6.6 using a simple U(1) family symmetry assignment.

symmetry. The effective superpotential is given:

$$W \sim d_3 d_3^c H_d + \left(\frac{\phi}{M}\right) d_3 d_2^c H_d + \left(\frac{\phi}{M}\right)^2 d_2 d_3^c H_d$$
$$\left(\frac{\phi}{M}\right)^3 d_2 d_2^c H_d + \left(\frac{\phi}{M}\right)^2 d_3 d_1^c H_d + \left(\frac{\phi}{M}\right)^2 d_3 d_1^c H_d$$
$$\left(\frac{\phi}{M}\right)^4 d_2 d_1^c H_d + \left(\frac{\phi}{M}\right)^5 d_1 d_2^c H_d + \left(\frac{\phi}{M}\right)^6 d_1 d_1^c H_d. \tag{1.6.36}$$

The above superpotential generates the entries of the mass matrix, for example: the $(m_{\text{down}})_{31}$ term includes 4 powers of the ratio $\frac{\phi}{M}$. We take M to be some large mass scale relative to the VEV of the flavon ϕ, in actual fact the scale M will be the mass of some Froggatt-Nielsen messenger particle. By making the ratio $\frac{\langle\phi\rangle}{M} = \epsilon$ small enough then we can generate a hierarchy in the down quark mass matrix:

$$m_d \propto \begin{pmatrix} \epsilon^6 & \epsilon^5 & \epsilon^4 \\ \epsilon^4 & \epsilon^3 & \epsilon^2 \\ \epsilon^2 & \epsilon & 1 \end{pmatrix}. \tag{1.6.37}$$

If a theory of family symmetry is to be compatible with Grand Unified Theories then all members of a given GUT multiplet must have the same $U(1)_f$ charge. To give a flavour of how family symmetry may be extended into GUT theories we can simply extend the above toy model. In the above case since u_i and u_i^c both belong to the

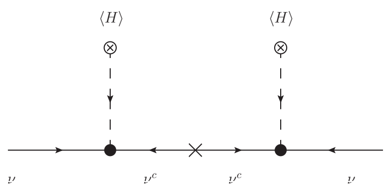

Figure 1.7: The Type I seesaw diagram.
The Seesaw mechanism provides a natural explanation of the smallness of the observed Neutrino masses.

10-plet representation then they must carry the same family charge as d_i this leads to an effective superpotential of:

$$\begin{aligned} W \sim\ & u_3 u_3^c H_u + \left(\frac{\phi}{M}\right)^2 u_3 u_2^c H_u + \left(\frac{\phi}{M}\right)^2 u_2 u_3^c H_u \\ & \left(\frac{\phi}{M}\right)^4 u_2 u_2^c H_u + \left(\frac{\phi}{M}\right)^4 u_3 u_1^c H_u + \left(\frac{\phi}{M}\right)^4 u_3 u_1^c H_u \\ & \left(\frac{\phi}{M}\right)^6 u_2 u_1^c H_u + \left(\frac{\phi}{M}\right)^6 u_1 u_2^c H_u + \left(\frac{\phi}{M}\right)^8 u_1 u_1^c H_u. \end{aligned} \tag{1.6.38}$$

This effective superpotential gives us a mass matrix proportional to:

$$m_u \propto \begin{pmatrix} \epsilon^8 & \epsilon^6 & \epsilon^4 \\ \epsilon^6 & \epsilon^4 & \epsilon^2 \\ \epsilon^4 & \epsilon^2 & 1 \end{pmatrix}. \tag{1.6.39}$$

The above toy model is simple but unfortunately not realistic, however it does illustrate the use of a family symmetry in generating fermion masses and mixings.

1.6.7 The Seesaw mechanism

Similar to the Froggatt-Nielsen mechanism there is the well known seesaw mechanism [28–30]. This mechanism generates effective light neutrino masses by integrating out a heavy right-handed neutrino. The diagram in Figure 1.7 shows a type I seesaw mechanism.

We expect the right-handed neutrino Majorana masses to be heavier than the Dirac neutrino masses. This is because the Standard Model gauge group doesn't protect the right-handed mass M_{RR}. As the right-handed neutrino transforms as a singlet under the Standard Model the mass term $M_{RR}\nu^c\nu^c$ is invariant unlike the Dirac mass terms which are generated by the spontaneous symmetry breaking of the Higgs mechanism and are only non-zero after $SU(2)_L$ has been broken. As we expect M_{RR} to be large we can then integrate out the right-handed neutrinos to obtain effective masses for the light neutrinos. The masses are given approximately by:

$$m_{LL} = -(m_{LR})(M_{RR})^{-1}(m_{LR})^T. \tag{1.6.40}$$

To see where equation 1.6.40 originates it is instructive to consider a simple case with only one family. In this case we have only one left-handed neutrino ν and only one singlet right-handed neutrino [§] ν^c. As discussed before the mass term $m_{LL}\nu\nu$ is forbidden by the Standard Model gauge group whereas the right-handed Majorana mass $M_{RR}\nu^c\nu^c$ is allowed. The Dirac mass term $m_{LR}\nu\nu^c$ is allowed as we can make use of the Higgs doublet to construct a yukawa term $y_\nu Hl\nu^c$ which results in a Dirac mass after H obtains a VEV. We therefore have a mixture of Dirac and Majorana mass terms which we can express in a 2×2 matrix:

$$\mathcal{L} \sim \underline{\nu}^{\mathrm{T}}\underline{\underline{M}}^{\nu}\underline{\nu} = \begin{pmatrix} \nu & \nu^c \end{pmatrix} \begin{pmatrix} 0 & m_{LR} \\ m_{LR} & M_{RR} \end{pmatrix} \begin{pmatrix} \nu \\ \nu^c \end{pmatrix}. \tag{1.6.41}$$

Since we expect $m_{LR} \ll M_{RR}$ as m_{LR} will be at the electroweak scale and M_{RR} will be at the GUT scale we can immediately find the approximate eigenvalues of the neutrino mass matrix. Using the trace of the matrix we can see that the largest eigenvalue will be approximately given by M_{RR} and the smallest eigenvalue will be given by $-\frac{(m_{LR})^2}{M_{RR}}$ as the determinant $-(m_{LR})^2$ must remain invariant and is given by the product of the eigenvalues. This is exactly the result quoted above in equation 1.6.40 when applied to the simpler case of only one generation. The exact result for

[§]Note that ν^c is not the charge conjugate of ν but rather the charge conjugate of the right-handed neutrino. To make a link to section 1.1.1: $\nu = \nu_L = \begin{pmatrix} \xi_\alpha \\ 0 \end{pmatrix}$ whereas $\nu^c = (\nu_R)^c = \begin{pmatrix} 0 \\ \chi^{\dagger\dot{\alpha}} \end{pmatrix}^c = \begin{pmatrix} \chi_\alpha \\ 0 \end{pmatrix}$.

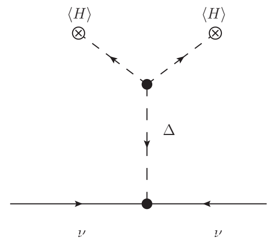

Figure 1.8: The type II seesaw mechanism

the one generation case are given below:

$$m_1 = \frac{1}{2}\left(M_{RR} + \sqrt{M_{RR}^2 + 4(m_{LR})^2}\right) \tag{1.6.42}$$

$$m_2 = \frac{1}{2}\left(M_{RR} - \sqrt{M_{RR}^2 + 4(m_{LR})^2}\right) \tag{1.6.43}$$

by expanding the square root we can extract the approximate result we derived above. Equation 1.6.40 is obtained by generalising to the three generations of neutrino where now m_{LR} and M_{RR} are 3×3 matrices.

The mechanism described above is not the only way to obtain the light neutrino masses. Though we will not make use of them there are other seesaw mechanisms which we shall include here for completeness. The type II seesaw mechanism requires the use of a $SU(2)_L$ triplet Higgs Δ as shown in Figure 1.8.

The seesaw formula equation 1.6.40 must now be modified to include a new term m_{LL} which was previously absent. The new formula reads:

$$M_{\text{effective}} = m_{LL} - (m_{LR})(M_{RR})^{-1}(m_{LR})^{\mathrm{T}}. \tag{1.6.44}$$

Again taking the simple example of only one generation of matter the new mass term m_{LL} appears in the top left of the neutrino mass matrix:

$$\underline{\underline{M}}^{\nu} = \begin{pmatrix} m_{LL} & m_{LR} \\ m_{LR} & M_{RR} \end{pmatrix}. \tag{1.6.45}$$

We can intuitively see from Figure 1.8 that m_{LL} will be $\mathcal{O}(\langle H\rangle^2/M_\Delta)$.

We refer the reader to [10] for a detailed review of neutrino physics. Neutrino mass mechanisms have also been proposed [31] using one-loop diagrams rather than the tree level diagrams we have reviewed here, further details are given in [32].

Chapter 2

Extra dimensions

In this section we will provide a brief introduction to extra dimensions and describe some recent models. We will also describe models which use both extra dimensions and the family symmetries introduced in section 1.6.

2.1 Introduction

Extra dimensional theories are not new, they have been around since the 1920's when they were introduced by Kaluza and Klein [33, 34]. The original motivation was to unify electromagnetism with gravity by identifying the photon field with the fifth component ($g_{\mu 5}$) of the (five dimensional) metric tensor. More recently it was realised that consistent string theories will require extra dimensions this led to a resurgence of work on theories with extra dimensions in the 1980's. Regardless of their motivation all extra dimensional theories must be able to hide the extra dimensions from observation. One possible mechanism for hiding the extra dimensions is to assume that unlike the four large dimensions we know about the extra dimensions are finite in size and compactified. In order to detect these compact extra dimensions one would then need to probe the length scales at which the compact dimensions live. Thus in order to hide the extra dimensions we simply make the length scales of the compact dimensions small enough that the energies required to probe them are sufficiently high. The consequences of the extra dimensions will then be hidden from observers living at lower energy scales.

2.2 A 5D toy model

One important consequence of extra dimensions is the existence of so-called Kaluza-Klein (KK) modes, we can use a simple model to illustrate how these KK modes arise. We shall consider a 5 dimensional theory with the extra dimension parametrised by y. A massless Klein-Gordon particle has an equation of motion of:

$$\partial^M \partial_M \phi(x^\mu, y) = (\partial^\mu \partial_\mu - \partial^y \partial_y)\phi(x^\mu, y) = 0 \tag{2.2.1}$$

where M runs over all the spacetime indices and μ running over the usual four dimensional spacetime indices (t, x_1, x_2, x_3). We then compactify the extra dimension on a circle of radius R, i.e. we make the identification:

$$y \to y + 2\pi R \tag{2.2.2}$$

We are then able to expand the field ϕ as a Fourier series on the extra dimensional space

$$\phi(x^\mu, y) = \sum_n \phi^{(n)}(x^\mu) e^{inky} \tag{2.2.3}$$

with k given by

$$\phi(x^\mu, y) = \phi(x^\mu, y + 2\pi R) \tag{2.2.4}$$

$$\Rightarrow \sum_n \phi^{(n)}(x^\mu) e^{inky} = \sum_n \phi^{(n)}(x^\mu) e^{ink(y+2\pi R)} \tag{2.2.5}$$

$$\Rightarrow e^{ink2\pi R} = 1 \tag{2.2.6}$$

$$\Rightarrow k = \frac{1}{R}. \tag{2.2.7}$$

Applying the equation of motion (equation 2.2.1) to this expansion gives us:

$$\partial^M \partial_M \phi = (\partial_t \partial^t - \partial_{x_i} \partial^{x_i} - \partial_y \partial^y)\phi = 0 \quad (2.2.8)$$

$$\sum_n (\partial_\mu \partial^\mu - \partial_y \partial^y)\phi^{(n)}(x^\mu)e^{iny/R} = 0 \quad (2.2.9)$$

$$\sum_n \left(\partial_\mu \partial^\mu \phi^{(n)}(x^\mu)e^{iny/R} + \frac{n^2}{R^2}\phi^{(n)}(x^\mu)e^{iny/R} \right) = 0 \quad (2.2.10)$$

$$\sum_n (\partial_\mu \partial^\mu + m^{(n)^2})\phi^{(n)}(x^\mu)e^{iny/R} = 0. \quad (2.2.11)$$

We are then left with an equation of motion for a set of particles $\phi^{(n)}(x^\mu)$ with a mass $m^{(n)} = \frac{n}{R}$. Thus one 5D particle has been split into an infinite set of 4D particles with ever increasing mass. If want a 4D theory where the extra dimension is hidden we need to require that the KK modes are too heavy to be observed, since the first mode has a mass of $\frac{1}{R}$ this allows us to set a limit on how large the compact dimension may be.

2.3 The $\mathbb{S}^1/\mathbb{Z}_2$ orbifold

In the previous section we compactified the extra dimensions on a circle with the identification $y \to y + 2\pi R$, where R is the radius of the extra dimension. We don't have to restrict ourselves to circles (and toroids). We can make use of orbifolds as our extra dimensional space. In order to describe an orbifold it is best to describe exactly what we mean by a circle.

2.3.1 The $\mathbb{S}^1$ circle

The circle $\mathbb{S}^1$ circle is formed from the quotient space $\mathbb{R}^1/\Lambda$ where Λ is a one dimensional lattice. As this is 1d there is only one lattice vector e so points $x \in \mathbb{R}^1$ are identified as $x \sim x + ne$ where $n \in \mathbb{Z}$ and $e = 2\pi R$ where R is the radius of the extra dimension.

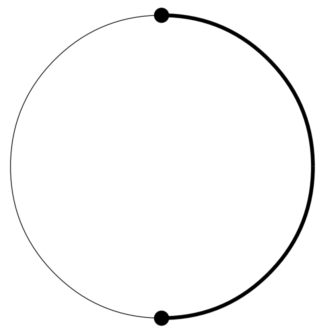

Figure 2.1: The $\mathbb{S}^1/\mathbb{Z}_2$ orbifold
The fundamental domain is shown in bold. It lies between the two fixed points at 0 and πR

2.3.2 Orbifolding

We define our first $\mathbb{Z}_2$ orbifolding by identifying $x \sim -x$. If our coordinate x on our circle is defined to be $-\pi R \leq x \leq \pi R$ then it is easy to see that the orbifolding maps the region with $x \geq 0$ to the region $x \leq 0$. There are two points which are mapped to themselves, these are the fixed points of the orbifold at $x = 0, \pi R$. The fundamental domain of the orbifold is now half of the original circle. The fixed points and fundamental domain are shown in figure 2.1. The orbifold is called $\mathbb{S}^1/\mathbb{Z}_2$.

2.3.3 A second orbifolding

We can create the orbifold $\frac{\mathbb{S}^1}{\mathbb{Z}_2 \times \mathbb{Z}'_2}$ by imposing another parity on the orbifold $\mathbb{S}^1/\mathbb{Z}_2$. We define a new set of coordinates x' on our orbifold by $x' = x + \frac{\pi R}{2}$ and then make the identification $x' \to -x'$. What we have done is apply a translation $T : x \to x + \pi R/2$ and then a parity $Z : x \to -x$ i.e. we have applied the operator ZT. We have again halved the size of our fundamental domain. Our orbifold now has a fundamental domain of $0 \leq x \leq \frac{\pi R}{2}$. The previous orbifold shown in figure 2.1 had two fixed points which were equivalent whereas now the two fixed points are inequivalent.

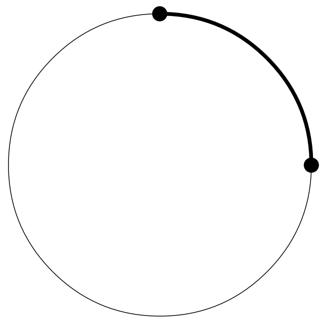

Figure 2.2: The $\mathbb{S}^1/(\mathbb{Z}_2 \times \mathbb{Z}'_2)$ orbifold.
The size of the fundamental domain has been halved and the fixed points are no longer equivalent.

2.4 Model building using orbifolds

In the context of model building the importance of orbifolds is that we can associate an automorphism with a reflection in the internal space, for our purposes this will be in a gauge or flavour space.

$$\mathbb{Z}_2 : x_5 \rightarrow \theta x_5 = -x_5 \tag{2.4.1}$$

$$\mathbb{Z}_2 : \mathbf{r} \rightarrow P_{\mathbf{r}} \mathbf{r} \tag{2.4.2}$$

where $\mathbf{r}$ is the representation of some gauge group G and $P_{\mathbf{r}}$ is the representation matrix of the automorphism. The requirement that the state be invariant under the orbifold action is given by:

$$\phi_{\mathbf{r}}(x_\mu, -x_5) = P_{\mathbf{r}} \phi_{\mathbf{r}}(x_\mu, x_5). \tag{2.4.3}$$

This condition must be satisfied by fields living in the orbifolded space, we can make use of it by choosing $P_{\mathbf{r}}$ such that fields we don't want to be light can transform non-trivially e.g. have a negative parity under a $\mathbb{Z}_2$ orbifolding. The fields with a negative parity therefore cannot be zero modes (which have an even parity under the orbifolding) and are heavy i.e. an odd KK mode of which the lightest has mass $\frac{1}{R}$. To illustrate this mechanism in the particular case of gauge fields we give a toy model breaking an SU(3) gauge symmetry using orbifold projection.

2.4.1 A simple example using an SU(3) gauge theory

The representation matrix of the automorphism for the fundamental representation is taken to be $P_{\mathbf{3}} = \mathrm{diag}(-1, -1, 1)$. The **3**-plets under SU(3) therefore have to satisfy:

$$\phi_{\mathbf{3}}(x_\mu, -x_5) = P_{\mathbf{3}}\phi_{\mathbf{3}}(x_\mu, x_5) \tag{2.4.4}$$

this condition implies that at the fixed point located at $x_5 = 0$ the fields must satisfy:

$$\phi_{\mathbf{3}}(x_\mu, x_5 = 0) = P_{\mathbf{3}}\phi_{\mathbf{3}}(x_\mu, x_5 = 0) \tag{2.4.5}$$

also since πR and $-\pi R$ coincide then the following must also be true:

$$\phi_{\mathbf{3}}(x_\mu, \pi R) = P_{\mathbf{3}}\phi_{\mathbf{3}}(x_\mu, \pi R). \tag{2.4.6}$$

It is easy to see that the eigenvalues of $P_{\mathbf{3}}$ are ± 1, of we denote the eigenstates of $P_{\mathbf{3}}$ as $\phi_\pm$

$$\phi_\pm(x_\mu, -x_5) = P_{\mathbf{3}}\phi_\pm(x_\mu, x_5) = \pm\phi_\pm(x_\mu, x_5). \tag{2.4.7}$$

As before we can expand the 5D states in the extra space as Fourier modes giving:

$$\phi_+(x_\mu, x_5) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2^{\delta_{n,0}}\pi R}}\phi_+^{(n)}(x_\mu)\cos\left(\frac{nx_5}{R}\right) \tag{2.4.8}$$

$$\phi_-(x_\mu, x_5) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi R}}\phi_-^{(n)}(x_\mu)\sin\left(\frac{nx_5}{R}\right). \tag{2.4.9}$$

At this point it should be noticed that only the ϕ_+ has a massless mode $\phi_+^{(0)}$ the other modes are heavy. As we shall see below the SU(3) gauge group has been broken to SU(2) $\times$ U(1), here we have seen that the triplet has been split into a SU(2) doublet and a singlet U(1) field. If we assign a positive parity to the triplet i.e:

$$\phi_{\mathbf{3}}(-x_5) = +P_{\mathbf{3}}\phi_{\mathbf{3}}(x_5) \tag{2.4.10}$$

then the SU(2) doublet gains a negative parity with the U(1) singlet having a positive parity, this leads to the doublet becoming heavy and the singlet remaining light.

However we have the alternative choice of assigning a negative parity to the triplet giving us the opposite case, a light doublet and heavy singlet. In chapter 4 we shall consider a model where we require the particle content of a complete multiplet at zero mode level in an orbifolded bulk space, we achieve this by introducing two multiplets with opposite parities. To use the above example this would be analogous to obtaining the particle content of a complete zero mode bulk triplet by using two bulk triplets one with positive parity and one with negative parity. The singlet would be derived from the positive parity triplet and the doublet from the negative parity triplet.

2.4.2 Gauge fields

We can perform the same analysis with the gauge fields. The boundary condition for the 4D vector fields are

$$A_\mu^a(x_\mu, -x_5)t_a = A_\mu^a(x_\mu, x_5)Pt_aP^{-1}. \tag{2.4.11}$$

It is easy to verify that the only gauge bosons with + parity i.e. the only massless ones, are those of the SU(2) × U(1) subgroup of SU(3). In terms of parity the generators of SU(3) look like:

$$t_a \sim \begin{pmatrix} + & + & - \\ + & + & - \\ - & - & + \end{pmatrix} \tag{2.4.12}$$

If we do the explicit calculation then only the gauge bosons associated with λ_1 for $i \in \{1, 2, 3, 8\}$ survive:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.4.13}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{2.4.14}$$

This shows the two key features of orbifolded extra dimensions namely that 1) Gauge symmetry can be reduced by orbifolding the extra dimensions and 2) Bulk multiplets under the larger gauge symmetry are split. The multiplets only survive the projection only partially. The appearance of split multiplets is a natural feature of an orbifold model.

We will now review some existing models from the literature which make use of orbifolded extra dimensions.

2.5 Recent models

In this section we shall give a brief overview of recent models using orbifolded extra dimensions.

2.5.1 The Kawamura model [35]

The model proposed by Kawamura [35] makes use of an orbifolded extra dimension and is based on the gauge group SU(5). As described above the orbifold has two fixed points located at $x_5 = 0$ and $x_5 = \frac{\pi R}{2}$. The parity of a bulk field under the two parities $\mathbb{Z}_2$ and $\mathbb{Z}_2'$ is described by:

$$\phi(x^\mu, x_5) \to \phi(x^\mu, -x_5) = P\phi(x^\mu, x_5) \tag{2.5.1}$$

$$\phi(x^\mu, x_5') \to \phi(x^\mu, -x_5') = P'\phi(x^\mu, x_5') \tag{2.5.2}$$

where $x_5' = x_5 + \frac{\pi R}{2}$. The Lagrangian is invariant under the two $\mathbb{Z}_2$ transformations and by definition the eigenvalues of P and P' are ± 1. The eigenstates are labelled ϕ_{++},ϕ_{+-},ϕ_{-+} and ϕ_{--} according to their eigenvalues under P and P' respectively.

We can Fourier expand the eigenstates as:

$$\phi_{++}(x_\mu, x_5) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{2^{\delta_{n,0}}\pi R}} \phi_{++}^{(2n)}(x_\mu) \cos\left(\frac{2nx_5}{R}\right) \tag{2.5.3}$$

$$\phi_{+-}(x_\mu, x_5) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{+-}^{(2n+1)}(x_\mu) \cos\left(\frac{(2n+1)x_5}{R}\right) \tag{2.5.4}$$

$$\phi_{-+}(x_\mu, x_5) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{-+}^{(2n+1)}(x_\mu) \sin\left(\frac{(2n+1)x_5}{R}\right) \tag{2.5.5}$$

$$\phi_{--}(x_\mu, x_5) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi R}} \phi_{--}^{(2n+2)}(x_\mu) \sin\left(\frac{(2n+2)x_5}{R}\right) \tag{2.5.6}$$

where n is an integer. At this point it is important to note that the fields $\phi_{++}^{(2n)}$, $\phi_{+-}^{(2n+1)}$, $\phi_{-+}^{(2n+1)}$, $\phi_{--}^{(2n+2)}$ acquire masses $\frac{2n}{R}$, $\frac{2n+1}{R}$, $\frac{2n+1}{R}$ and $\frac{2n+2}{R}$ respectively. A consequence of this is that only the fields with all positive parities have a massless state. Also some fields will vanish entirely at the fixed points for example: the fields $\phi_{-+}(x^\mu, x_5 = 0) = \phi_{--}(x^\mu, x_5 = 0) = 0$ at the fixed point located at $x_5 = 0$. The model assumes that the visible world is located at the $x_5 = 0$ fixed point referred to as a "wall". The matter content of the theory consisting of the three families quark and lepton chiral supermultiplets ($\Phi_{\bar{\mathbf{5}}} + \Phi_{\mathbf{10}}$) is placed on the wall at $x_5 = 0$. The gauge and Higgs bosons live in the 5D bulk and as such have parity assignments under the orbifoldings $\mathbb{Z}_2 \times \mathbb{Z}_2'$. The parity assignment is such that the SU(5) gauge group is broken to the Standard Model gauge group, a natural consequence of this gauge breaking is that the Higgs pentaplets are split into doublet and triplet representations of the SU$(2)_L$ and SU$(3)_C$ groups respectively. The parity assignments are such that the coloured triplets acquire a negative parity and as such are not present at he zero mode level i.e. they are heavy and doublet-triplet splitting has occurred. By including the Standard Model matter at the $x_5 = 0$ fixed point in a complete multiplet unaffected by the orbifolding and placing the gauge and Higgs bosons in the bulk, the model accounts for the appearance of both complete and split multiplets.

2.5.2 The Asaka-Buchmüller-Covi model [36]

We will now describe a model by Asaka, Buchmüller and Covi [36] using 2 extra compact dimensions which are again compactified on an orbifold. The model is based on the larger GUT group SO(10), however the extension from SU(5) to SO(10) is not trivial since G_{SM} is not a symmetric subgroup of SO(10). The two symmetric subgroups of SO(10) are the Pati-Salam and extended Georgi-Glashow gauge groups, SU(4) × SU(2) × SU(2) and SU(5) × U(1) respectively. However it is interesting to note that the maximal common subgroup of these groups is the extended Standard Model gauge group SM′ = SM × U(1).

The starting point for the model is $\mathcal{N} = 1$ supersymmetric Yang-Mills in 6D. The extra dimensions are compactified on the torus i.e. $\mathcal{M} = \mathbb{R}^4 \times \mathbb{T}^2$. The goal is to obtain a 4D N=1 Yang-Mills theory with extended standard model symmetry. The breaking of the extended SUSY in 4D and the breaking of the gauge group leads to the theory on the orbifold $M = T^2/(\mathbb{Z}_2 \times \mathbb{Z}_2^{PS} \times \mathbb{Z}_2^{GG})$. The authors consider the N=1 Yang-Mills theory in 6 dimensions, the Lagrangian is

$$\mathcal{L}_{6d}^{YM} = \mathrm{Tr}(-\frac{1}{2}V_{MN}V^{MN} + i\overline{\Lambda}\Gamma^M D_M \Lambda). \tag{2.5.7}$$

Where $V_M = t^a V_M^a$ and $\Lambda = t^a \Lambda^a$, here t^a are the generators of SO(10). $D_M\Lambda = \partial_m \Lambda - ig[V_M, \Lambda]$ and $V_{MN} = [D_M, D_N]/(ig)$. The Γ matrices are given by

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu & 0 \\ 0 & \gamma^\mu \end{pmatrix}, \qquad \Gamma^5 = \begin{pmatrix} 0 & i\gamma^5 \\ i\gamma^5 & 0 \end{pmatrix}, \qquad \Gamma^6 = \begin{pmatrix} 0 & \gamma^5 \\ -\gamma^5 & 0 \end{pmatrix} \tag{2.5.8}$$

with $\gamma^5 = I$ and $\{\Gamma_M, \Gamma_N\} = 2\eta_{MN} = \mathrm{diag}(1, -1, -1, -1, -1, -1)$. The gaugino Λ is composed of two Weyl fermions of opposite chirality in 4d,

$$\Lambda = (\lambda_1, -i\lambda_2),\ \gamma_5\lambda_1 = -\lambda_1,\ \gamma_5\lambda_2 = \lambda_2. \tag{2.5.9}$$

Overall the gaugino has negative 6d chirality $\Gamma_7\Lambda = -\Lambda$, where $\Gamma_7 = \mathrm{diag}(\gamma_5, -\gamma_5)$.

2.5.3 Compactification

The model compactifies the two extra dimensions on a torus $\mathbb{T}^2$ so that the theory lives on $\mathcal{M} = \mathbb{R}^4 \times \mathbb{T}^2$. The fields $\Phi = (V_M, \Lambda)$ can then be expanded in using the

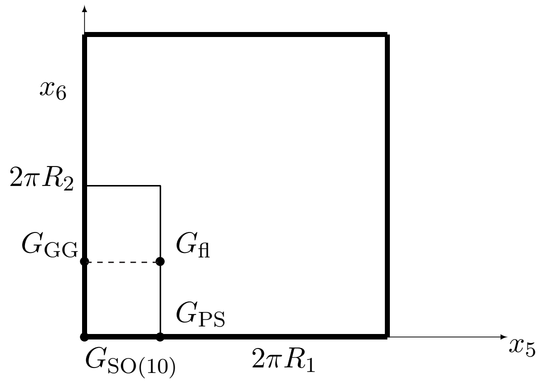

Figure 2.3: The $\mathbb{T}^2/(\mathbb{Z}\times\mathbb{Z}_{PS}\times\mathbb{Z}_{GG})$ orbifold from [36]

The orbifold used in [36], R_1 and R_2 are the radii of the torus. There are 3 orbifoldings: 1 breaking the extended supersymmetry and 2 breaking the gauge group. The orbifoldings leave fixed points with different gauge groups associated with them and are labelled. $G_{\rm fl}$ is flipped SU(5) and results from the combination of both the Pati-Salam and extended Georgi-Glashow gauge breaking.

Fourier expansion:

$$\Phi(x,x_5,x_6)=\frac{1}{2\pi\sqrt{R_1R_2}}\sum_{m,n}\Phi^{(m,n)}(x)\exp\left\{i\left(\frac{mx_5}{R_1}+\frac{nx_6}{R_2}\right)\right\} \tag{2.5.10}$$

here R_1 and R_2 are the two radii of the torus as shown in figure 2.3.

The orbifold shown in figure 2.3 is formed by using three orbifoldings each of which reduce the size of the fundamental domain by a factor of 2. The first orbifolding identifies the right and left halves of the torus leaving a fundamental domain half the size of the original torus which can be viewed as a pillow since the upper and lower edges are identified. The procedure is repeated twice more to leave a fundamental domain one eighth the size of the original torus. This is the fundamental domain shown in figure 2.3. The small rectangle shown is folded along the dotted line to form a pillow and the upper and lower edges are glued along with the left and right edges. The orbifold is left as a pillow with the fixed points located at the corners. The fixed points are labelled according to the gauge group which survives at that particular fixed point. This is because the some gauge bosons have been assigned a negative parity which makes them heavy, but also causes their wavefunctions to be vanishing at particular fixed points. For example, at the Pati-Salam fixed point only the wavefunctions of the Pati-Salam gauge bosons (some of which are heavy) are present, the remaining gauge bosons from the SO(10)/$G_{\rm PS}$ group are vanishing at this fixed point.

The vector field is hermitian so the coefficients satisfy the relation $V_M^{(-m,-n)} = V_M^{(m,n)\dagger}$. By integrating over the extra dimensions we can obtain the 4d effective Lagrangian. Note that we are only including terms below $\mathcal{O}(1/R)$ so there are only bilinear terms in the 4d Lagrangian. We make a convenient choice of variables for the 4d scalars by rearranging into the mass eigenstate basis given by:

$$\Pi_1^{(m,n)}(x) = \frac{i}{M(m,n)}\left(\frac{m}{R_1}V_5^{(m,n)}(x) + \frac{n}{R_2}V_6^{(m,n)}(x)\right) \tag{2.5.11}$$

$$\Pi_2^{(m,n)}(x) = \frac{i}{M(m,n)}\left(-\frac{n}{R_2}V_5^{(m,n)}(x) + \frac{m}{R_1}V_6^{(m,n)}(x)\right) \tag{2.5.12}$$

where $M(m,n) = \sqrt{\left(\frac{m}{R_1}\right)^2 + \left(\frac{n}{R_2}\right)^2}$. The 4d Lagrangian for the gauge and scalar fields is then given by:

$$\begin{aligned}\mathcal{L}_{4d}^{(1)} = \sum_{m,n} \mathrm{Tr}(&-\frac{1}{2}\tilde{V}_{\mu\nu}^{(m,n)\dagger}\tilde{V}^{(m,n)\mu\nu} + M(m,n)^2 V_\mu^{(m,n)\dagger}V^{(m,n)\mu} \\ &+ \partial_\mu \Pi_2^{(m,n)\dagger}\partial^\mu \Pi_2^{(m,n)\dagger} + M(m,n)^2 \Pi_2^{(m,n)\dagger}\Pi_2^{(m,n)} \\ &+ \partial_\mu \Pi_1^{(m,n)\dagger}\partial^\mu \Pi_1^{(m,n)} \\ &- M(m,n)(V_\mu^{(m,n)\dagger}\partial^\mu \Pi_1^{(m,n)} + \partial^\mu \Pi_1^{(m,n)\dagger}V_\mu^{(m,n)}))\end{aligned} \tag{2.5.13}$$

where $\tilde{V}_{\mu\nu}^{(m,n)} = \partial_\mu V_\nu^{(m,n)} - \partial_\nu V_\mu^{(m,n)}$. The massless states are the zero modes, the higher modes in the Kaluza-Klein expansion are massive with the mass given by $M(m,n)$. The basis for the scalars $\Pi_{1,2}$ is chosen such that they are in the mass eigenstates with $\Pi_1^{(m,n)}$ being the Goldstone bosons from the broken higher dimensional Lorentz symmetry. The Goldstone bosons $\Pi_1^{(m,n)}$ are not observed as they are eaten by the higher KK modes which then acquire a mass. From the higher dimensional viewpoint a gauge transformation corresponds to an infinite number of gauge transformations which mix up the KK modes of different levels. After the mode expansion is made the theory has an infinite number of gauge transformations parametrised by the KK numbers m and n. However later on we shall be compactifying on an orbifold where m and n can no longer assume arbitrary values, due to the non-trivial orbifolding conditions. From the 4d perspective the possible gauge transformations are reduced breaking the higher dimensional gauge symmetry to a smaller symmetry in 4 dimensions.

The gaugino part of the Lagrangian integrates to:

$$\mathcal{L}_4^{(2)} = \sum_{m,n} \mathrm{Tr}(i\overline{\lambda}_1^{(m,n)}\gamma^\mu\partial_\mu\lambda_1^{(m,n)} + i\overline{\lambda}_2^{(m,n)}\gamma^\mu\partial_\mu\lambda_2^{(m,n)} \\ - \left(\frac{m}{R_1} - i\frac{n}{R_2}\right)\overline{\lambda}_1^{(m,n)}\lambda_2^{(m,n)} + \text{c.c.}). \tag{2.5.14}$$

This is the kinetic term for a Dirac fermion $\lambda_D = (\lambda_1, \lambda_2)$ with a mass $M(m,n)$. To summarise in total there is the vector $V_\mu^{(m,n)}$, scalars $\Pi_{1,2}^{(m,n)}$ and λ_D forming a massive $\mathcal{N} = 1$ vector multiplet in 4d. However when we look at the massless sector of the theory we have unwanted $\mathcal{N} = 2$ symmetry, this extended supersymmetry is removed by orbifolding to obtain the effective $\mathcal{N} = 1$ theory in 4 dimensions.

When we look at the particle content of the theory we have a massive $\mathcal{N} = 1$ vector multiplet consisting of the gauge bosons V_μ, the scalars $\Pi_{1,2}$ and and massive Dirac fermion λ_D. However this massive gauge boson $\mathcal{N} = 1$ vector multiplet may be represented by a $\mathcal{N} = 1$ massless vector multiplet $\mathcal{V} = (V_\mu, \lambda_1)$ together with a chiral supermultiplet $\mathcal{V}' = (\Pi_{1,2}, \lambda_2)$. These two multiplets form a massive $\mathcal{N} = 2$ vector supermultiplet:

$$V = \begin{pmatrix} V_\mu & \Pi_{1,2} \\ \lambda_1 & \lambda_2 \end{pmatrix}. \tag{2.5.15}$$

The scalar field Π_1 from the chiral multiplet $\mathcal{V}'$ becomes the longitudinal component of the massive gauge boson. The other scalar Π_2 remains in the particle spectrum at the massive level along with the Weyl fermion λ_2.

2.5.4 SUSY orbifold breaking

Rather than compactifying on the torus the authors compactify the extra dimensions on the orbifold $\mathbb{T}^2/\mathbb{Z}_2$ where parities are assigned under the reflection $(x_5, x_6) \rightarrow$

$(-x_5, -x_6)$ to the vectors and scalars.

$$PV_\mu(x, -x_5, -x_6)P^{-1} = +V_\mu(x, x_5, x_6) \tag{2.5.16}$$

$$PV_{5,6}(x, -x_5, -x_6)P^{-1} = -V_{5,6}(x, x_5, x_6) \tag{2.5.17}$$

where the choice $P = I$ is made, so for the Fourier modes we are left with

$$V_\mu^{(-m,-n)} = +V_\mu^{(m,n)} = +\, V_\mu^{(m,n)\dagger}, \tag{2.5.18}$$

$$V_{5,6}^{(-m,-n)} = -V_{5,6}^{(m,n)} = +\, V_{5,6}^{(m,n)\dagger}. \tag{2.5.19}$$

This eliminates the scalar zero modes, also the number of massive modes is halved. Because the derivatives $\partial_{5,6}$ are odd under the reflection the two Weyl fermions (λ_1, λ_2) must have opposite parities,

$$P\lambda_1(x, -x_5, -x_6)P^{-1} = +\lambda_1(x, x_5, x_6), \tag{2.5.20}$$

$$P\lambda_2(x, -x_5, -x_6)P^{-1} = -\lambda_2(x, x_5, x_6), \tag{2.5.21}$$

(V_μ, λ_1) and $(V_{5,6}, \lambda_2)$ form vector and chiral multiplets respectively, only the vector multiplets have zero modes. The orbifolding has therefore broken the extended $\mathcal{N} = 2$ symmetry to $\mathcal{N} = 1$ in 4D. The gauge bosons and gauginos form a gauge superfield which is the special case of a vector superfield where the condition $V = V^\dagger$ is preserved by the gauge transformation. The general form of the gauge superfield can be given in the Wess-Zumino gauge as:

$$\begin{aligned} V(x, \theta, \bar{\theta}) =& \bar{\theta}\sigma^\mu\theta V_\mu(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) \\ & - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \end{aligned} \tag{2.5.22}$$

where V_μ and λ are the gauge bosons and gauginos respectively with the field D being an auxiliary field.

2.5.5 Gauge breaking by orbifolding

Here breaking of the SO(10) gauge group must be done by using two parities P_{PS} and P_{GG} which define the symmetric subgroups of SO(10), Pati-Salam $G_{PS} = SU(4) \times SU(2) \times SU(2)$ and Georgi-Glashow $G_{GG} = SU(5) \times U(1)$. In the vector representation these parities are

$$P_{GG} = \begin{pmatrix} \sigma_2 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_2 & 0 & 0 \\ 0 & 0 & 0 & \sigma_2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_2 \end{pmatrix}, P_{PS} = \begin{pmatrix} -\sigma_0 & 0 & 0 & 0 & 0 \\ 0 & -\sigma_0 & 0 & 0 & 0 \\ 0 & 0 & -\sigma_0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_0 \end{pmatrix} \quad (2.5.23)$$

where σ_0, σ_2 are the familiar Pauli matrices. We require for the vector fields and gauginos λ_1:

$$P_{GG} V_\mu(x, -x_5, -x_6 + \pi R_2/2) P_{GG}^{-1} = +V_\mu(x, x_5, x_6 + \pi R_2/2), \quad (2.5.24)$$

$$P_{PS} V_\mu(x, -x_5 + \pi R_1/2, -x_6) P_{PS}^{-1} = +V_\mu(x, x_5 + \pi R_1/2, x_6). \quad (2.5.25)$$

Thus fields belonging to the symmetric subgroup G_s have positive parity and those of $SO(10)/G_s$ have negative parity. The $\mathbb{Z}_2$ parity requires the scalars and gauginos λ_2 to have negative parity. Because of the $\mathbb{Z}_2$ parity we also require:

$$P_{GG} V_{5,6}(x, -x_5, -x_6 + \pi R_2/2) P_{GG}^{-1} = -V_{5,6}(x, x_5, x_6 + \pi R_2/2), \quad (2.5.26)$$

$$P_{PS} V_{5,6}(x, -x_5 + \pi R_1/2, -x_6) P_{PS}^{-1} = -V_{5,6}(x, x_5 + \pi R_1/2, x_6). \quad (2.5.27)$$

The mode expansions of the fields $\Phi(x, x_5, x_6)$ is explicitly:

$$\Phi_{+++} = \frac{1}{\pi\sqrt{R_1 R_2}} \sum_{m,n} \frac{1}{2^{\delta_{m,0}\delta_{n,0}}} \phi_{+++}^{(2m,2n)}(x) \cos\left(\frac{2m x_5}{R_1} + \frac{2n x_6}{R_2}\right), \quad (2.5.28)$$

$$\Phi_{++-} = \frac{1}{\pi\sqrt{R_1 R_2}} \sum_{m,n} \phi_{++-}^{(2m,2n+1)}(x) \cos\left(\frac{2m x_5}{R_1} + \frac{(2n+1) x_6}{R_2}\right), \quad (2.5.29)$$

$$\Phi_{+-+} = \frac{1}{\pi\sqrt{R_1 R_2}} \sum_{m,n} \phi_{+-+}^{(2m+1,2n)}(x) \cos\left(\frac{(2m+1)x_5}{R_1} + \frac{2nx_6}{R_2}\right), \tag{2.5.30}$$

$$\Phi_{+--} = \frac{1}{\pi\sqrt{R_1 R_2}} \sum_{m,n} \phi_{+--}^{(2m+1,2n+1)}(x) \cos\left(\frac{(2m+1)x_5}{R_1} + \frac{(2n+1)x_6}{R_2}\right), \tag{2.5.31}$$

$$\Phi_{-++} = \frac{1}{\pi\sqrt{R_1 R_2}} \sum_{m,n} \phi_{-++}^{(2m+1,2n+1)}(x) \sin\left(\frac{(2m+1)x_5}{R_1} + \frac{(2n+1)x_6}{R_2}\right), \tag{2.5.32}$$

$$\Phi_{-+-} = \frac{1}{\pi\sqrt{R_1 R_2}} \sum_{m,n} \phi_{-+-}^{(2m+1,2n)}(x) \sin\left(\frac{(2m+1)x_5}{R_1} + \frac{2nx_6}{R_2}\right), \tag{2.5.33}$$

$$\Phi_{--+} = \frac{1}{\pi\sqrt{R_1 R_2}} \sum_{m,n} \phi_{--+}^{(2m,2n+1)}(x) \sin\left(\frac{2mx_5}{R_1} + \frac{(2n+1)x_6}{R_2}\right), \tag{2.5.34}$$

$$\Phi_{---} = \frac{1}{\pi\sqrt{R_1 R_2}} \sum_{m,n} \phi_{---}^{2m,2n)}(x) \sin\left(\frac{2mx_5}{R_1} + \frac{2nx_6}{R_2}\right), \tag{2.5.35}$$

where the subscripts, + and −, on the fields refer to the parities under the Supersymmetry breaking, Pati-Salam and Georgi-Glashow orbifoldings respectively. Again the only fields with zero modes are those with parities all positive, they form a $\mathcal{N} = 1$ massless vector multiplet in the adjoint representation of the unbroken extended standard model group. All the other fields with one or more negative parities combine to form massive vector multiplets.

limiting cases: We can take the limiting cases of $R_1 \to 0$ with R_2 fixed and $R_2 \to 0$ with R_1 fixed. In both these cases we are effectively dealing with a 5 dimensional theory. In the first case the dependence on R_1 disappears and we are dealing with a 5 dimensional theory with the extra dimension compactified onto a one dimensional orbifold with two fixed points. In this case there will be SO(10) and Pati-Salam fixed points with the effective four dimensional theory broken to the Pati-Salam group ($SU(4) \times SU(2) \times SU(2)$). In the second case the dependence on R_2 disappears and the one dimensional orbifold has SO(10) and Georgi-Glashow fixed points with the effective four dimensional theory broken to Georgi-Glashow ($SU(5) \times U(1)$). It is only when R_1 and R_2 are finite is the gauge group broken to the extended standard model. If we take one of the compact radii to be large then the Fourier series expansion becomes a Fourier transform and we would no longer be left with the extended standard model in 4 dimensions, we would have either Georgi-Glashow or Pati-Salam in 5d depending on which extra dimension was taken to be large.

2.5.6 Adding matter to the theory

Adding matter to the 6d SUSY theory is easy, consider the case of the **10**-plet of Higgs fields. It contains two complex scalars H and H', and a fermion $h = (h, h')$. The chiralities are $\gamma_5 h = h, \gamma_5 h' = -h'$ in 4d with an overall positive 6d chirality $\Gamma_7 h = h$.

The Lagrangian reads:

$$\mathcal{L}_{6d}^{\text{higgs}} = |D_M H|^2 + |D_M H'|^2 - \frac{1}{2} g^2 (H^\dagger t^a H + H'^\dagger t^a H')^2 + i\overline{h}\Gamma^M D_M h - i\sqrt{2} g(\overline{h}\Lambda H + \overline{h}\Lambda^c H' + c.c). \tag{2.5.36}$$

Again we integrate over the compact dimensions to get:

$$\begin{aligned}\mathcal{L}_{4d}^{\text{higgs}} = \sum_{m,n} & i\overline{h}^{(m,n)}\gamma^\mu \partial_\mu h^{(m,n)} + i\overline{h'}^{(m,n)}\gamma^\mu \partial_\mu h'^{(m,n)} \\ & + (\frac{m}{R_1} - i\frac{n}{R_2})\overline{h}^{(m,n)} h'^{(m,n)} + c.c. \\ & + \partial_\mu H^{(m,n)\dagger} \partial^\mu H^{(m,n)} + M(m,n)^2 H^{(m,n)\dagger} H^{(m,n)} \\ & + \partial_\mu H'^{(m,n)\dagger} \partial^\mu H'^{(m,n)} + M(m,n)^2 H'^{(m,n)\dagger} H'^{(m,n)}. \end{aligned} \tag{2.5.37}$$

2.5.7 Higgs parities

We can now define the action of the parities on the Higgs multiplets $H = (H, h)$ and $H' = (H', h')$. For the $\mathbb{Z}_2$ we can choose

$$PH(x, -x_5, -x_6) = +H(x, x_5, x_6) \tag{2.5.38}$$

$$PH'(x, -x_5, -x_6) = -H'(x, x_5, x_6) \tag{2.5.39}$$

with $P = I$. As is the case with the **45**-plet this breaks the extended supersymmetry present in 4d. For $\mathbb{Z}_2^{GG}$ we choose

$$P_{GG}H(x, -x_5, -x_6 + \pi R_2/2) = +H(x, x_5, x_6 + \pi R_2/2) \tag{2.5.40}$$

$$P_{GG}H'(x, -x_5, -x_6 + \pi R_2/2) = -H'(x, x_5, x_6 + \pi R_2/2) \tag{2.5.41}$$

The parity P_{PS} gives us the desired doublet-triplet splitting, again the same mechanism is used to break the gauge symmetry as well as providing the doublet-triplet splitting. The action of P_{PS} is given by:

$$P_{PS}H(x, -x_5 + \pi R_1/2, -x_6) = +H(x, x_5 + \pi R_1/2, x_6), \tag{2.5.42}$$

$$P_{PS}H'(x, -x_5 + \pi R_1/2, -x_6) = -H'(x, x_5 + \pi R_1/2, x_6). \tag{2.5.43}$$

Again we have a SU(2) $\mathcal{N} = 1$ supermultiplet as the zero modes and the SU(3) triplet is heavy. If we were to chose the signs the other way round we would get a massless colour triplet and heavy weak doublet. In order to obtain the two Higgs doublets as zero modes we have to introduce two **10**-plets of Higgs with parities different with respect to $\mathbb{Z}_2^{GG}$.

2.6 Family symmetry from extra dimensions

In this section we review [3] which introduces the possibility that discrete symmetries can arise from orbifold compactifications. In this particular case the group A_4 which we will then extend to a GUT model in chapter 3. The model described in section 2.5.2 is built around a non-twisted torus and the orbifold forms a square "pillow". This "pillow" can be seen as having the symmetry of a square in the same manner as the orbifold with twist angle $\theta = \pi/3$ has the symmetry of a tetrahedron. The symmetry group of the square is the Dihedral group D_4 and has been considered as a possible family symmetry for example see [37]. In order to modify the model given in [36] to incorporate the D_4 symmetry, the 3 families of the standard model would need to be arranged into the 4 inequivalent singlet and 1 doublet representations of the D_4 group.

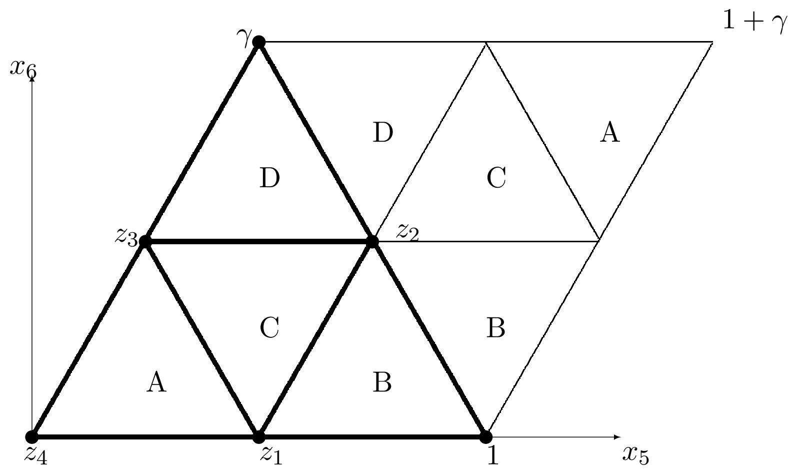

Figure 2.4: The Orbifold $\mathbb{T}^2/\mathbb{Z}_2$. The fundamental domain is outlined in bold and forms a tetrahedron. Regions labelled by A,B,C and D are identified. The fixed points are labelled z_i and are symmetrically permuted under the symmetry group A_4.

2.6.1 The A_4 orbifold $\mathbb{T}^2/\mathbb{Z}_2$

The Orbifold introduced in [3] is based on the twisted torus with the twist angle $\theta = 60°$. We set $R_1 = R_2$, as shown in figure 2.4. We then perform the $\mathbb{Z}_2$ orbifolding which folds the rhombus into a tetrahedron giving rise to the A_4 symmetry, this symmetry will later be exploited as a family symmetry.

2.6.2 The orbifold with $\theta = \pi/3$

We are working with a quantum field theory in 6 dimensions with the 2 extra dimensions compactified onto an orbifold $\mathbb{T}^2/\mathbb{Z}_2$. The extra dimensions are complexified such that $z = x_5 + ix_6$ are the coordinates on the extra space. The torus $\mathbb{T}^2$ is defined by identifying the points:

$$z \to z + 1, \tag{2.6.1}$$

$$z \to z + \gamma \quad \gamma = e^{i\frac{\pi}{3}}. \tag{2.6.2}$$

We have set the length $2\pi R_{1,2}$ to unity for clarity. The orbifolding is defined by the parity $\mathbb{Z}_2$ identifying:

$$z \to -z, \tag{2.6.3}$$
$$(x_5, x_6) \to (-x_5, -x_6),$$

leaving the orbifold to be represented by the bold triangular region shown in figure 2.4. The orbifold has 4 fixed points which are unchanged under the symmetries of the orbifold, equations (2.6.3),(2.6.1),(2.6.2). The orbifold can be described as a regular tetrahedron with the fixed points as the vertices. The 6d spacetime symmetry is broken by the orbifolding, previously the symmetry consisted of 6d translations and proper Lorentz transformations*. We are now left with a 4d space-time symmetry and a discrete symmetry of rotations and translations due to the special geometry of the orbifold. We can generate this group with the transformations:

$$\mathcal{S} : z \to z + \tfrac{1}{2} \ , \tag{2.6.4}$$
$$\mathcal{T} : z \to \quad \omega z \quad , \qquad\qquad \omega \equiv \gamma^2. \tag{2.6.5}$$

These two generators are even permutations of the four fixed points:

$$\mathcal{S} : (z_1, z_2, z_3, z_4) \to (z_4, z_3, z_2, z_1), \tag{2.6.6}$$
$$\mathcal{T} : (z_1, z_2, z_3, z_4) \to (z_2, z_3, z_1, z_4). \tag{2.6.7}$$

The above two transformations generate the group A_4 which is the symmetry of the tetrahedron (see section 1.6.2 for an introduction to A_4). This can be verified by showing that S and T obey the characteristic relations, the presentation, of the generators of A_4,

$$\mathcal{S}^2 = \mathcal{T}^3 = (\mathcal{S}\mathcal{T})^3 = 1. \tag{2.6.8}$$

*if we had allowed improper Lorentz transformations,i.e. reflections, then rather than A_4 we would have S_4 the group of permutations of 4 objects

2.6.3 Irreducible representations of A_4

The 4d representations of the A_4 generators can be block diagonalised to give the irreducible representations of the A_4 group

$$S = \begin{pmatrix} 0\,0\,0\,1 \\ 0\,0\,1\,0 \\ 0\,1\,0\,0 \\ 1\,0\,0\,0 \end{pmatrix}, \quad T = \begin{pmatrix} 0\,0\,1\,0 \\ 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \end{pmatrix}$$

which satisfy the presentation of the group, equation (2.6.8). Since the only irreducible representations of A_4 are a triplet and 3 singlet representations (see section 1.6.2) then the 4d representation is not irreducible.

Since this 4d representation is reducible then we can block diagonalise the generators using a matrix U given by:

$$U = \frac{1}{2}\begin{pmatrix} +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & -1 \\ +1 & -1 & +1 & -1 \\ +1 & +1 & -1 & -1 \end{pmatrix}$$

and we find that:

$$S_{\text{block diagonal}} = \begin{pmatrix} 1 & \cdots & 0 & \cdots \\ \vdots & \ddots & & \\ 0 & & S_3 & \\ \vdots & & & \ddots \end{pmatrix}, T_{\text{block diagonal}} = \begin{pmatrix} 1 & \cdots & 0 & \cdots \\ \vdots & \ddots & & \\ 0 & & T_3 & \\ \vdots & & & \ddots \end{pmatrix}$$

where T_3 and S_3 are the generators of A_4 in the 3D irreducible representation given by:

$$S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, T_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{2.6.9}$$

2.6.4 Parametrising multiplets

If we are to place fields at the fixed points of the orbifold then we will need to parametrise a 4 dimensional representation in terms of singlet and triplet representations as in [3]. We now briefly summarise the results of [3] to build a dictionary from a 6d orbifolded theory to an effective 4d one. If we consider a multiplet $u = (u_1, u_2, u_3, u_4)^T$ transforming as:

$$\mathcal{S} : u \to Su$$
$$\mathcal{T} : u \to Tu,$$

we can now make a change of basis defining $v = (v_0, v_1, v_2, v_3)^T = Uu$ transforming as:

$$\mathcal{S} : v \to (USU^\dagger)v$$
$$\mathcal{T} : v \to (UTU^\dagger)v$$

with the v_0 component transforming as a singlet and the $v_{1,2,3}$ components transforming with T_3 and S_3 as triplets. This gives us a parametrisation for a multiplet $u = (u_1, u_2, u_3, u_4)^T$. We can decompose the reducible quadruplet into a triplet and invariant singlet irreducible representations:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} v_0 \\ v_0 \\ v_0 \\ v_0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -v_1 + v_2 + v_3 \\ +v_1 - v_2 + v_3 \\ +v_1 + v_2 - v_3 \\ -v_1 - v_2 - v_3 \end{pmatrix}.$$

As noted in [3] this parametrisation is not unique, this is a result of a property of the A_4 generators. We can generalise the transformations of the brane multiplet in the following way:

$$\mathcal{S} : a \to Sa \tag{2.6.10}$$

$$\mathcal{T} : a \to \omega^{r_a} Ta \equiv T_{r_a} a \tag{2.6.11}$$

where ω is the first cubic root of unity and $r_a = 0, \pm 1$. This clearly still satisfies the presentation of the group equation (2.6.8) and we can repeat the block diagonalising procedure to find the parametrisation if $r_a \neq 0$. If we take the case where $r_a = +1$, we again block diagonalise the generators S and T_{r_a}:

$$S = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad T_{r_a} = \begin{pmatrix} 0 & 0 & \omega & 0 \\ \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix} \tag{2.6.12}$$

with a matrix U_ω which was not explicitly given in [3]:

$$U_\omega = \frac{1}{2} \begin{pmatrix} +\omega^2 & +\omega^2 & +\omega^2 & +\omega^2 \\ -1 & +1 & +1 & -1 \\ +\omega^2 & -\omega^2 & +\omega^2 & -\omega^2 \\ +\omega & +\omega & -\omega & -\omega \end{pmatrix}. \tag{2.6.13}$$

This splits the four dimensional representation into the irreducible triplet and singlet parts:

$$U_\omega S U_\omega^\dagger = \begin{pmatrix} 1 & \cdots & 0 & \cdots \\ \vdots & \ddots & & \\ 0 & & S_3 & \\ \vdots & & & \ddots \end{pmatrix}, \quad U_\omega T_{r_a} U_\omega^\dagger = \begin{pmatrix} \omega & \cdots & 0 & \cdots \\ \vdots & \ddots & & \\ 0 & & T_3 & \\ \vdots & & & \ddots \end{pmatrix}. \tag{2.6.14}$$

This leaves us with a different parametrisation of a brane multiplet:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \frac{\omega}{2} \begin{pmatrix} v_0 \\ v_0 \\ v_0 \\ v_0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -v_1 + \omega v_2 + \omega^2 v_3 \\ +v_1 - \omega v_2 + \omega^2 v_3 \\ +v_1 + \omega v_2 - \omega^2 v_3 \\ -v_1 - \omega v_2 - \omega^2 v_3 \end{pmatrix}. \tag{2.6.15}$$

We can repeat the process for the $r_a = -1$ case or we can simply take the complex conjugate of equation (2.6.15).

It should be noted that the 1 dimensional representation of S and T found from equation (2.6.14) is that of the $1'$ representation of A_4 ($S = 1, T = \omega$). This is because we have decomposed the quadruplet $\mathbf{4}$ into irreducible representations as $\mathbf{4} = \mathbf{1'} \oplus \mathbf{3}$. In the first case ($r_a = 0$) we decomposed the quadruplet as $\mathbf{4} = \mathbf{1} \oplus \mathbf{3}$, the 1 dimensional representation is $S = 1, T = 1$, simply read from the block diagonal forms of S and T. As in [3] we label the 4 dimensional reducible representations $\mathcal{R}_{0,-1,+1}$, $\mathcal{R}_0$ decomposes into a triplet plus an invariant singlet $\mathcal{R}_{+1}$ decomposes into a triplet plus an non-invariant singlet $1'$ and finally $\mathcal{R}_{-1}$ decomposes into a triplet plus a non-invariant singlet $1''$. Brane singlets are given by a vector of the form $a_{singlet} = (a_c/2, a_c/2, a_c/2, a_c/2)^T$, i.e. brane fields having the same value at each fixed point. Brane Triplets $a = (a_1, a_2, a_3)$are in one of three representations $\mathcal{R}_{0,\pm1}$ given by

$$a^{\mathcal{R}_1} = a^{\mathcal{R}_{-1}*} = \frac{1}{2} \begin{pmatrix} -a_1 + \omega a_2 + \omega^2 a_3 \\ +a_1 - \omega a_2 + \omega^2 a_3 \\ +a_1 + \omega a_2 - \omega^2 a_3 \\ -a_1 - \omega a_2 - \omega^2 a_3 \end{pmatrix}, \quad a^{\mathcal{R}_0} = \frac{1}{2} \begin{pmatrix} -a_1 + a_2 + a_3 \\ +a_1 - a_2 + a_3 \\ +a_1 + a_2 - a_3 \\ -a_1 - a_2 - a_3 \end{pmatrix} \tag{2.6.16}$$

depending on which singlet the triplets are forming in the superpotential. Bulk singlets depend on the extra coordinates and transform as $S\xi(z) = \xi(z + 1/2)$ and $T\xi(z) = \xi(\omega z)$. We require these decompositions because we will want to construct non-invariant singlets from products of triplets and if we were to restrict ourselves to the first parametrisation we would be unable to do so.

2.6.5 Bulk and brane Fields

Following [3] we now look at the coupling of a bulk multiplet: $\mathbf{B}(z) = (\mathbf{B}_1(z), \mathbf{B}_2(z), \mathbf{B}_3(z))$, transforming as a triplet of A_4 and the brane triplet $a = (a_1, a_2, a_3)$ transforming as $\mathcal{R}_0$, as in equation (2.6.16). The transformations of $\mathbf{B}$ are:

$$\mathcal{S} : \mathbf{B}'(z_S) = S_3\mathbf{B}(z) \quad z_S = z + \frac{1}{2} \tag{2.6.17}$$

$$\mathcal{S} : \mathbf{B}'(z_T) = T_3\mathbf{B}(z) \quad z_T = \omega z. \tag{2.6.18}$$

We can write a bilinear in a and $\mathbf{B}$ given by:

$$J = \sum_{iK} \alpha_{iK} a_i^{\mathcal{R}_0} \mathbf{B}_K(z)\delta_i \tag{2.6.19}$$

where α_{iK} is a four by three matrix of constant coefficients, and $\delta_i = \delta(z - z_i)$ where z_i are the fixed points. We want J to be invariant under A_4 then we choose:

$$\alpha_{iK} = \frac{1}{2}\begin{pmatrix} -1 & +1 & +1 \\ +1 & -1 & +1 \\ +1 & +1 & -1 \\ -1 & -1 & -1 \end{pmatrix}.$$

Since a is in the $\mathcal{R}_0$ representation after integration over extra dimensions:

$$\begin{aligned} J = & \frac{1}{4}(-v_1 + v_2 + v_3)(-\mathbf{B}_1(z_1) + \mathbf{B}_2(z_1) + \mathbf{B}_3(z_1)) \\ + & \frac{1}{4}(+v_1 - v_2 + v_3)(+\mathbf{B}_1(z_2) - \mathbf{B}_2(z_2) + \mathbf{B}_3(z_2)) \\ + & \frac{1}{4}(+v_1 + v_2 - v_3)(+\mathbf{B}_1(z_3) + \mathbf{B}_2(z_3) - \mathbf{B}_3(z_3)) \\ + & \frac{1}{4}(+v_1 + v_2 + v_3)(+\mathbf{B}_1(z_4) + \mathbf{B}_2(z_4) + \mathbf{B}_3(z_4)). \end{aligned} \tag{2.6.20}$$

If the triplet $\mathbf{B}(z)$ acquires a constant VEV $\langle \mathbf{B}(z) \rangle = (\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3)$ then J becomes:

$$J = v_1 \mathbf{B}_1 + v_2 \mathbf{B}_2 + v_3 \mathbf{B}_3.$$

We can do the same for a bilinear J' given by:

$$J' = \sum_{iK} \alpha'_{iK} a_i \mathbf{B}_K(z) \delta_i$$

which transforms as a $1'$ with the matrix α'_{iK} given by:

$$\alpha'_{iK} = \frac{1}{2} \begin{pmatrix} -1 & +\omega & +\omega^2 \\ +1 & -\omega & +\omega^2 \\ +1 & +\omega & -\omega^2 \\ -1 & -\omega & -\omega^2 \end{pmatrix}.$$

After integrating over z and after $\mathbf{B}$ has acquired a constant VEV we find that:

$$J' = v_1 \mathbf{B}_1 + \omega v_2 \mathbf{B}_2 + \omega^2 v_3 \mathbf{B}_3.$$

We can obtain the $1''$ singlet by simply substituting α'_{iK} by its complex conjugate to get α''_{iK}.

2.7 Other discrete symmetries from orbifolding

As noted in [38] A_4 is not the only discrete symmetry that can be exploited from the geometry of the orbifold compactification. We noted above that if we had allowed reflections then the group generated by the compactification would not have been A_4, the group of even permutations of 4 objects, but the group generated would be S_4, the group of all permutations of 4 objects. We shall simply list a number of $\mathbb{T}^2/\mathbb{Z}_N$ orbifolds and the associated discrete symmetry in table 2.1. Such orbifolds may be used to form a theory of family symmetry similar to [3] and [1].

Orbifold	Symmetry
$\mathbb{T}/\mathbb{Z}_2$	A_4, S_4, D_4
$\mathbb{T}/\mathbb{Z}_3$	D_3, S_3
$\mathbb{T}/\mathbb{Z}_4$	D_4
$\mathbb{T}/\mathbb{Z}_6$	$D_6 \cong D_3 \times \mathbb{Z}_2 \cong S_3 \times \mathbb{Z}_2$

Table 2.1: Orbifolds and their symmetries

A list of 2 dimensional orbifolds and the discrete symmetries that may be associated with them. Orbifolds can have different symmetries depending on the twist angle of the torus and the symmetry that they are orbifolded by, see [38] for details.

2.8 GUT models with family symmetry and orbifolding

Recently a model [39] has been proposed that incorporates a GUT group with a family symmetry while also making use of orbifolding extra compact dimensions. The model is based on the SU(5) GUT group and has a single extra dimension compactified on the orbifold. The GUT group is broken by giving a negative parity to those gauge bosons not belonging to the Standard Model gauge group, this mechanism also solves the doublet-triplet splitting problem by rendering the coloured Higgs triplets heavy. In addition to the model proposed by Kawamura [35] the model also has an A_4 family symmetry and also makes use of a Froggatt-Nielsen mechanism. The $\bar{\mathbf{5}}$-plet of matter transforms as a triplet of A_4 and the three families of **10**-plets transform as the three singlet representations of A_4. The third family of **10**'s is placed at a fixed point and the first two families are placed in the bulk, this leads to a suppression of the yukawa coupling in these bulk fields as a bulk field and its zero mode are related by:

$$B = \frac{1}{\sqrt{\pi R}} B^0 + \ldots. \tag{2.8.1}$$

This is made use of alongside the Froggatt-Nielsen mechanism to obtain realistic masses and mixings. A complication of placing the first two families in the bulk is that the same GUT breaking mechanism leads to the splitting of the multiplets. This is rectified by introducing an extra copy of the first families into the bulk which transform with opposite parities thus leaving a complete particle content. The doubling of the first two families also allows too rigid GUT relations (eqns. (1.5.5),1.5.6),(1.5.7)) to be avoided so the introduction of a Georgi-Jarlskog mechanism is not required.

Chapter 3

A_4 family symmetry from $\mathrm{SU}(5)$ GUTs in 6d

3.1 Introduction

The pattern of quark and lepton masses and mixing angles remains central to any attempt to construct a theory of physics beyond the Standard Model. As discussed in section 1.6 the most obvious extension to the Standard Model is to introduce some symmetry between the families, a so-called family or flavour symmetry. A particular difficulty is reconciling the large mixing angles in the lepton sector with the relatively small mixing angles in the quark sector.

If we restrict ourselves to the lepton sector then it is comparatively straightforward to build models that are compatible with data. As discussed in section 1.3.2 the so-called "Tri-Bimaximal mixing" scheme of Harrison, Perkins and Scott [12] is compatible with data, such a mixing scheme results from a MNS matrix of a particular form:

$$U_{TB} = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{3.1.1}$$

The ansatz of TBM lepton mixing matrix is interesting due to its symmetry properties which seem to call for a possibly discrete non-Abelian Family Symmetry in nature [40]. There has been a considerable amount of theoretical work in which the observed TBM neutrino flavour symmetry may be related to some Family Symmetry [3,26,41–53,53–89]. These models may be classified according to the way that TBM mixing is achieved, namely either directly or indirectly [90]. The direct models are based on A_4 or S_4, or a larger group that contains these groups as a subgroup, and in these models some of the generators of the Family Symmetry survive to form at least part of the neutrino flavour symmetry. In the indirect models, typically based on $\Delta(3n^2)$ or $\Delta(6n^2)$, none of the generators of the Family Symmetry appear in the neutrino flavour symmetry [90].

In the approach in [39] the A_4 is simply assumed to exist in the 5d theory. However it has been shown how an A_4 Family Symmetry could have a dynamical origin as a result of the compactification of a 6d theory down to 4d [3]. Similar considerations have been applied to other discrete family symmetries [38], and the connection to string theory of these and other orbifold compactifications has been discussed in [91]. According to [3], the A_4 appears as a symmetry of the orbifold fixed points on which 4d branes, which accommodate the matter fields, reside, while the flavons which break A_4 are in the bulk. The formulation of a theory in 6d is also closer in spirit to string theories which are formulated in 10d where such theories are often compactified in terms of three complex compact dimensions. The 6d theory here will involve one complex compact dimension z.

In this chapter we formulate a realistic direct model in which an A_4 Family Symmetry arises dynamically from an $SU(5)$ SUSY GUT in 6d. The A_4 Family Symmetry emerges as a result of the compactification of the extra complex compact dimension z, assuming a particular orbifolding. $SO(10)$ in 6d has been considered in [36], with the extra dimensions compactified on a rectangular torus. In order to realize an A_4 Family Symmetry upon compactification, we shall generalise the formalism of 6d GUTs in [36] to the case of compactification on a twisted torus. Then, starting from an $SU(5)$ SUSY GUT in 6d, we shall show how the A_4 Family Symmetry can result from the symmetry of the orbifold fixed points after compactification, assuming a particular twist angle $\theta = 60°$ and a particular orbifold $\mathbb{T}^2/(\mathbb{Z}_2 \times \mathbb{Z}_2^{\rm SM})$. Unlike the model in [39], the resulting model has all three ten-plets T_i, as well as the pentaplet F, located on the 3-branes at the fixed points. However, as in [39], we shall assume an additional $U(1)$ Froggatt-Nielsen Family Symmetry to account for inter-family mass hierarchies. We emphasise that this model is the first which combines the idea of orbifold GUTs with A_4 family symmetry resulting from the orbifolding.

The layout of the remainder of the chapter is as follows. In Section 3.2 we generalise the formulation of 6d GUTs (usually compactified on a rectangular torus) to the general case of compactification on a twisted torus with a general twist angle θ. Then we show how compactification of the $SU(5)$ SUSY GUT in 6d on an orbifold $\mathbb{T}^2/(\mathbb{Z}_2 \times \mathbb{Z}_2^{\rm SM})$ leads to an effective 4d theory with $\mathcal{N} = 1$ SUSY preserved but the $SU(5)$ GUT broken to the Standard Model (SM) gauge group. We also show how Higgs doublet-triplet splitting emerges if the Higgs fields are in the bulk. In Section 3.3 we present the $SU(5)$ SUSY GUT model in 6d in which the A_4 Family Symmetry emerges after the above compactification. We specify the superfield content and symmetries of the model and provide a dictionary for the realization of the 4d effective superpotential in terms of the 6d A_4 invariants. From the effective 4d superpotential we show how a successful pattern of quark and lepton masses and mixing, including Tri-Bimaximal neutrino mixing, can emerge. In Section 3.4 we comment on the vacuum alignment and subleading corrections expected in the model. Section 3.5 concludes the chapter.

3.2 $SU(5)$ GUTs in six dimensions on a twisted torus

We are considering a $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in 6 dimensions, the Lagrangian is given by equation (2.5.7). The gaugino Λ is composed of two Weyl fermions of opposite chirality in 4d as given in equation (2.5.9).

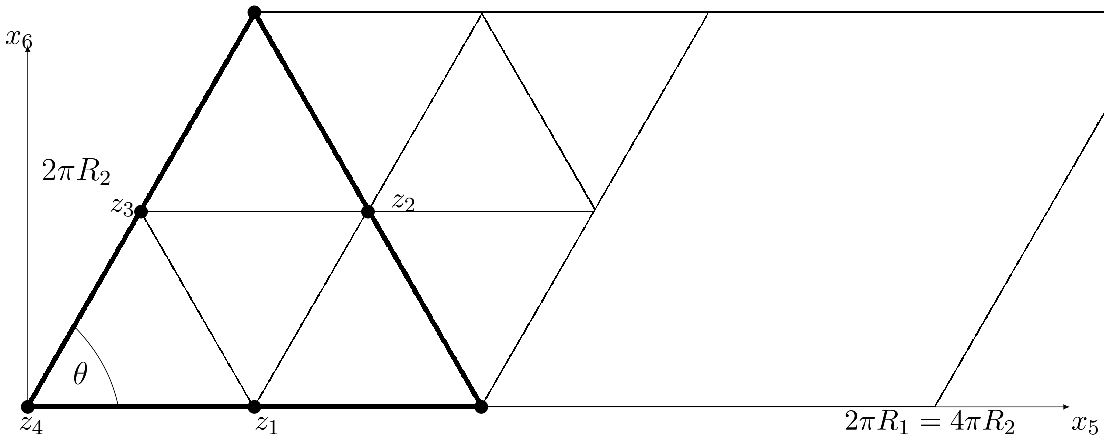

Figure 3.1: The Twisted Torus

The twisted torus, R_1 and R_2 are the radii and θ is the twist angle (later we shall specify $\theta = \pi/3$, $R_1 = 2R_2$ and orbifold to leave a fundamental domain shown in bold above).

3.2.1 Compactification on a twisted torus

We compactify the two extra dimensions on a twisted torus $\mathbb{T}^2$ so that the theory lives on $M = \mathcal{R}^4 \times \mathbb{T}^2$. The torus is defined by:

$$(x_5, x_6) \to (x_5 + 2\pi R_1, x_6) \tag{3.2.1}$$

$$(x_5, x_6) \to (x_5 + 2\pi R_2 \cos\theta, x_6 + 2\pi R_2 \sin\theta). \tag{3.2.2}$$

We can expand the $SU(5)$ gauge multiplet fields $\Phi = (V_M, \Lambda)$ using the mode expansion:

$$\Phi(x, x_5, x_6) = \frac{1}{2\pi\sqrt{R_1 R_2 \sin\theta}} \sum_{m,n=0}^{\infty} \Phi^{(m,n)}(x) \exp\left\{ i \left(\frac{m}{R_1}\{x_5 - \frac{x_6}{\tan\theta}\} + \frac{n x_6}{R_2 \sin\theta} \right) \right\}, \tag{3.2.3}$$

where R_1 and R_2 are the two radii of the torus and θ is the angle of twist as shown in figure 3.1. The limit $\theta \to 0$ represents an unphysical limit where the coordinates of the two extra dimensions coincide. To visualise this we can think of constructing the torus from a cylinder by gluing the two ends together, the limit $\theta \to 0$ would be equivalent to putting an infinite number of twists on the cylinder before gluing the ends together. Such a torus would be unphysical as travelling any length along the cylinder requires travelling an infinite number of turns around the cylinder. Later the radii will be set such that $R_1 = 2R_2$ and $\theta = \pi/3$. The first orbifolding in the x_5 direction halves the area of the torus to give the rhombus shown in figure 3.1. A further orbifolding identifies the three corners of the bold triangle leaving the fundamental domain one quarter of the original size which is shown in bold. This fundamental domain has a tetrahedral symmetry which will later be exploited as a family symmetry. The compactification proceeds as described earlier in section 2.5.3 Our choice of variables for the 4d scalars is modified from equations (2.5.11, 2.5.12) due to the twisted torus:

$$\Pi_1^{(m,n)}(x) = \frac{i}{M(m,n)} \left(\frac{m}{R_1} V_5^{(m,n)}(x) + \left(\frac{m}{R_1 \tan\theta} - \frac{n}{R_2 \sin\theta} \right) V_6^{(m,n)}(x) \right) \tag{3.2.4}$$

$$\Pi_2^{(m,n)}(x) = \frac{i}{M(m,n)} \left(- \left(\frac{m}{R_1 \tan\theta} - \frac{n}{R_2 \sin\theta} \right) V_5^{(m,n)}(x) + \frac{m}{R_1} V_6^{(m,n)}(x) \right) \tag{3.2.5}$$

where $M(m,n) = \frac{1}{\sin\theta}\sqrt{\left(\frac{m}{R_1}\right)^2 + \left(\frac{n}{R_2}\right)^2 - \frac{2mn\cos\theta}{R_1 R_2}}$. The 4d Lagrangian for the gauge and scalar fields is then given by equation (2.5.13).

The gaugino part of the Lagrangian integrates to equation (2.5.14). As before in section 2.5.3 this is the kinetic term for a Dirac fermion $\lambda_D = (\lambda_1, \lambda_2)$ with a mass $M(m,n)$. Our particle content consists of the vector $V_\mu^{(m,n)}$, scalars $\Pi_{1,2}^{(m,n)}$ and λ_D forming a massive $\mathcal{N} = 1$ vector multiplet in 4d. Again when we look at the massless sector of the theory we have unwanted $\mathcal{N} = 2$ symmetry which can be removed by orbifolding, as we now discuss.

Instead of compactifying on the torus we can compactify on the orbifold $\mathbb{T}^2/\mathbb{Z}_2$ where we assign parities, equations (2.5.16-2.5.21), under the reflection $(x_5, x_6) \to (-x_5, -x_6)$ to the vectors and scalars as given in section 2.5.4. Only the vector multiplet, (V_μ, λ_1), has zero modes whereas the chiral multiplet, $(V_{5,6}, \lambda_2)$, has none. The orbifolding breaks the extended $\mathcal{N} = 2$ SUSY in 4d down to $\mathcal{N} = 1$ SUSY.

3.2.2 Gauge symmetry breaking using the orbifold $\mathbb{T}^2/(\mathbb{Z}_2 \times \mathbb{Z}_2^{\rm SM})$

The zero modes obtained from the compactification on $\mathbb{T}^2/\mathbb{Z}_2$ form a $\mathcal{N} = 1$ SUSY SU(5) theory in 4d. The breaking of the $SU(5)$ gauge group down to that of the Standard Model can be achieved by another orbifolding. We make a coordinate shift to a new set of coordinates:

$$(x_5', x_6') = (x_5 + \pi R_1, x_6) \tag{3.2.6}$$

and introduce a second parity $\mathbb{Z}_2^{\rm SM}$ on these new coordinates

$$\mathbb{Z}_2^{\rm SM} : (x_5', x_6') \to (-x_5', -x_6'). \tag{3.2.7}$$

By using a single parity P_{SM},

$$P_{SM} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 \end{pmatrix} \tag{3.2.8}$$

we shall require that:

$$P_{SM} V_\mu(x, -x_5 + \pi R_1/2, -x_6) P_{SM}^{-1} = +V_\mu((x, x_5 + \pi R_1/2, x_6). \tag{3.2.9}$$

Gauge boson fields of the Standard Model thus have positive parity and fields belonging to $SU(5)/G_{SM}$ have negative parity. The orbifold is now $\mathbb{T}^2/(\mathbb{Z}_2 \times \mathbb{Z}_2^{SM})$.

Explicitly the expansion for the fields with any combination of parities is:

$$
\begin{aligned}
\Phi_{++}(x, x_5, x_6) &= \frac{1}{\pi\sqrt{R_1 R_2 \sin\theta}} \sum_{m\geq 0} \frac{1}{2^{\delta_{m,0}\delta_{n,0}}} \phi_{++}^{(2m,n)}(x) \\
&\quad \times \cos\left(\frac{2m}{R_1}\{x_5 - \frac{x_6}{\tan\theta}\} + \frac{n x_6}{R_2 \sin\theta}\right) \qquad (3.2.10) \\
\Phi_{+-}(x, x_5, x_6) &= \frac{1}{\pi\sqrt{R_1 R_2 \sin\theta}} \sum_{m\geq 0} \phi_{+-}^{(2m+1,n)}(x) \\
&\quad \times \cos\left(\frac{(2m+1)}{R_1}\{x_5 - \frac{x_6}{\tan\theta}\} + \frac{n x_6}{R_2 \sin\theta}\right) \qquad (3.2.11) \\
\Phi_{--}(x, x_5, x_6) &= \frac{1}{\pi\sqrt{R_1 R_2 \sin\theta}} \sum_{m\geq 0} \phi_{--}^{(2m,n)}(x) \\
&\quad \times \sin\left(\frac{2m}{R_1}\{x_5 - \frac{x_6}{\tan\theta}\} + \frac{n x_6}{R_2 \sin\theta}\right) \qquad (3.2.12) \\
\Phi_{-+}(x, x_5, x_6) &= \frac{1}{\pi\sqrt{R_1 R_2 \sin\theta}} \sum_{m\geq 0} \phi_{-+}^{(2m+1,n)}(x) \\
&\quad \times \sin\left(\frac{(2m+1)}{R_1}\{x_5 - \frac{x_6}{\tan\theta}\} + \frac{n x_6}{R_2 \sin\theta}\right). \qquad (3.2.13)
\end{aligned}
$$

Only fields with both parities positive have zero modes.

3.2.3 Higgs and doublet-triplet splitting

So far we have just considered the gauge sector of SUSY $SU(5)$. Adding the MSSM Higgs to the 6d SUSY theory is straightforward. In the $SU(5)$ GUT theory these are contained in the **5**-plet and $\overline{\mathbf{5}}$-plet of Higgs fields. These are two complex scalars H and H', and a fermion $h = (h, h')$. The chiralities are $\gamma_5 h = h, \gamma_5 h' = -h'$ in 4d with an overall positive 6d chirality $\Gamma_7 h = h$.

The Lagrangian is given by equation (2.5.37). Again we integrate over the compact dimensions to get,

$$
\begin{aligned}
\mathcal{L}_{4d}^{\text{higgs}} = \sum_{m,n} & i\overline{h}^{(m,n)} \gamma^\mu \partial_\mu h^{(m,n)} + i\overline{h'}^{(m,n)} \gamma^\mu \partial_\mu h'^{(m,n)} \\
& + (\frac{m}{R_1} - i\left(\frac{n}{R_2 \sin\theta} - \frac{m}{R_1 \tan\theta}\right))\overline{h}^{(m,n)} h'^{(m,n)} + c.c. \\
& + \partial_\mu H^{(m,n)\dagger} \partial^\mu H^{(m,n)} + M(m,n)^2 H^{(m,n)\dagger} H^{(m,n)} \\
& + \partial_\mu H'^{(m,n)\dagger} \partial^\mu H'^{(m,n)} + M(m,n)^2 H'^{(m,n)\dagger} H'^{(m,n)}. \qquad (3.2.14)
\end{aligned}
$$

For the first orbifolding parity we choose

$$PH(x, -x_5, -x_6) = +H(x, x_5, x_6) \tag{3.2.15}$$

$$PH'(x, -x_5, -x_6) = +H'(x, x_5, x_6) \tag{3.2.16}$$

with $P = I$.

For the gauge breaking orbifold we choose:

$$P_{SM}H(x, -x_5 + \pi R_1/2, -x_6) = H(x, x_5 + \pi R_1/2, x_6) \tag{3.2.17}$$

$$P_{SM}H'(x, -x_5 + \pi R_1/2, -x_6) = H'(x, x_5 + \pi R_1/2, x_6) \tag{3.2.18}$$

It is easy to see with the form of P_{SM} that the first three entries gain a minus sign which makes them heavy whereas the last two entries are left unchanged leaving them light, resulting in a light doublet and a heavy coloured triplet.

3.3 A_4 family symmetry from 6d SU(5) SUSY GUTs

The model will involve an A_4 family symmetry which is not assumed to exist in the 6d theory, but which originates after the compactification down to 4d. The way this happens is quite similar to the discussion in [3] based on the orbifold $\mathbb{T}^2/(\mathbb{Z}_2)$ but differs somewhat due to the different orbifold considered here, namely $\mathbb{T}^2/(\mathbb{Z}_2 \times \mathbb{Z}_2^{\rm SM})$. This is discussed in section 2.6, where we also briefly summarise all the results required in order to formulate our model, as necessary in order to make this thesis self-contained. Using the formalism of the previous section and section 2.6, we now present the model.

The basic set-up of the model is depicted in figure 3.2 and the essential features may be summarised as follows. The model assumes a 6d gauge $\mathcal{N} = 1$ SUSY $SU(5)$ Yang-Mills theory compactified down to 4d Minkowski space with two extra dimensions compactified on a twisted torus with a twist angle of $\theta = 60°$ and $R_1 = 2R_2$. Upon compactification, without orbifolding, the 6d supersymmetry would become extended to $\mathcal{N} = 2$ SUSY in 4d. However the $\mathcal{N} = 2$ SUSY is reduced to $\mathcal{N} = 1$

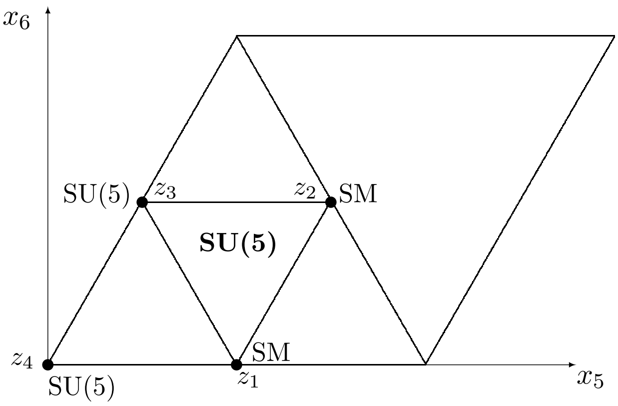

Figure 3.2: The orbifold giving rise to A_4 symmetry
The orbifold compactification of a 6d $\mathcal{N} = 1$ SUSY $SU(5)$ GUT which gives rise to an effective 4d theory with the $\mathcal{N} = 1$ SUSY SM gauge group together with A_4 Family Symmetry after compactification. The gauge symmetry at the four fixed points is explicitly labelled. Matter fields are localised at the fixed points as discussed in section 2.6 and in [3].

SUSY by use of a particular orbifolding and a further orbifolding is used to break the gauge symmetry to the SM, as discussed in Section 2. Due to the tetrahedral pattern of fixed points on the torus, the compactified extra dimensions have some additional symmetry left over from the 6d Poincaré spacetime symmetry, which is identified as a Family Symmetry corresponding to the A_4 symmetry group of the tetrahedron. The particular gauge breaking orbifolding also leads to the **5**-plets of Higgs splitting into a light doublet and heavy coloured triplet. It should be noted that the four fixed points of the tetrahedral orbifold are inequivalent in that they have different gauge groups associated with them. The A_4 symmetry is a symmetry of the standard model gauge bosons only and not the full SU(5) gauge group. The gauge bosons belonging to $SU(5)/G_{SM}$ have negative parity under the second gauge breaking orbifolding so these fields do not transform as trivial singlets under the A_4 as the standard model gauge bosons do. The model is therefore $A_4 \times$ SM not $A_4 \times$ SU(5).

The model is further specified by matter fields located on the 3-branes in various configurations, at the fixed points shown in figure 3.2. These matter fields are 4d fields with components at the 4 fixed points as described in [3]. Matter fields carry an extra U(1) family dependent charge which is in turn broken by two A_4 singlet Froggatt-Nielsen flavons θ, θ' which live on the fixed points. Realistic charged fermion masses and mixings are produced using these Froggatt-Nielsen flavons θ, θ' together with the bulk flavon φ_T which breaks A_4 but preserves the T generator. Tri-Bimaximal mixing of the neutrinos is achieved using further bulk flavons φ_S which breaks A_4 but preserves the S generator, and the singlet bulk flavon ξ. A full list of the particle content of the model minus the gauge fields is given in Table 3.1 and we shall briefly describe here. The three $\bar{\mathbf{5}}$ (F) are grouped into an A_4 triplet as are the three right-handed neutrinos (N). The ten-plets $(T_{1,2,3})$ are assigned to the three different singlet representations of A_4. The $\mathbf{5}$-plet transforms as a trivial A_4 singlet and the $\bar{\mathbf{5}}$-plet transforms in the $\mathbf{1}'$ representation. The A_4 family symmetry is broken via the use of two A_4 triplet flavons φ_T and φ_S which obtain VEVs in the $(1,0,0)$ and $(1,1,1)$ directions respectively. There are also two singlet flavons transforming in the trivial singlet representation of A_4. In this scheme, at the leading order, the φ_T give mass to the charged leptons and to the down quarks while the $a\varphi_S, \xi\tilde{\xi}$ give mass to the neutrinos. In order to enforce this separation there is also a $\mathbb{Z}_3$ charge under which the ten-plets, pentaplets, right-handed neutrinos, $\varphi_S, \xi, \tilde{\xi}$ flavons and higgs bosons carry a charge of ω. The φ_T flavon is left invariant under this $\mathbb{Z}_3$ symmetry as are the Froggatt-Nielsen flavons. The ten-plets also carry positive U(1) Froggatt-Nielsen charge which is broken by two flavons (θ, θ') both carrying negative charge. The Froggatt-Nielsen flavons transform in the A_4 singlet representations, θ transforms as a $\mathbf{1}$ while θ' transforms as a $\mathbf{1}'$. In addition to the gauge, A_4, U(1) Froggatt-Nielsen, and $\mathbb{Z}_3$ symmetries there is also a $\mathrm{U}(1)_R$ symmetry. The effective $\mathcal{N} = 1$ superpotential carries a $\mathrm{U}(1)_R$ charge of +2 since the integration measure $\mathrm{d}^2\theta$ carries a charge of -2. This symmetry also has the feature of forbidding certain unwanted terms, in particular the proton decay operator $FTTT$ has an R-charge of +4. The R-symmetry also contains the discrete R-parity so baryon and lepton number violating operators are also forbidden. The superpotential of the theory is a sum of a bulk term depending on bulk fields, plus terms localised at the four fixed points. The 4D superpotential is produced from the 6D theory by integrating over the extra dimensions and assuming a constant background value for the bulk supermultiplets $\varphi_S(z), \varphi_T(z)$ and $\xi_S(z)$ as in ref [3].

3.3.1 Superfield content

Superfield	N	F	T_1	T_2	T_3	H_5	$H_{\overline{5}}$	φ_T	φ_S	$\xi, \tilde{\xi}$	θ	θ'
SU(5)	1	$\overline{5}$	10	10	10	5	$\overline{5}$	1	1	1	1	1
SM	1	(d^c, l)	$(u''^c_1, q''_1, e''^c_1)$	(u'^c_2, q'_2, e'^c_2)	(u^c_3, q_3, e^c_3)	H_u	H'_d	φ_T	φ_S	$\xi, \tilde{\xi}$	θ	θ'
A_4	3	3	$1''$	$1'$	1	1	$1'$	3	3	1	1	$1'$
$U(1)$	0	0	4	2	0	0	0	0	0	0	-1	-1
$\mathbb{Z}_3$	ω	ω	ω	ω	ω	ω	ω	1	ω	ω	1	1
$U(1)_R$	1	1	1	1	1	0	0	0	0	0	0	0
Brane/bulk	brane	brane	brane	brane	brane	bulk	bulk	bulk	bulk	bulk	brane	brane

Table 3.1: Superfield content of the model

Superfield content and their transformation properties under the symmetries of the model. Note that the $SU(5)$ GUT symmetry is broken by the compactification, while the A_4 Family Symmetry is only realized after the compactification. The matter fields are located at the fixed points on 3-branes, while the Higgs fields live in the 6d bulk. The Froggatt-Nielsen flavons are all located at the fixed point 3-branes while the A_4 flavons all live in the bulk.

After compactification, an effective 4d superpotential may be written down, using the dictionary for the realisation of the 4d terms in terms of the local 6d A_4 invariants given in Table 3.2. Using this dictionary, we decompose the effective 4d superpotential into several parts:

$$w = w_{up} + w_{down} + w_{\text{charged lepton}} + w_{\nu} + w_d + \ldots \tag{3.3.1}$$

The term w_d is concerned with vacuum alignment whose effect will be discussed later. The first three terms give rise to the fermion masses after A_4, $U(1)$ and electroweak symmetry breaking and they are:

$$\begin{aligned} w_{\text{up}} \sim\ & \frac{1}{\Lambda} H_u q_3 u_3^c + \frac{\theta'^2}{\Lambda^3} H_u (q_2' u_3^c + q_3 u'^c_2) + \frac{\theta'^4 + \theta'\theta^3}{\Lambda^5} H_u q_2' u'^c_2 \\ & + \frac{\theta'^4 + \theta'\theta^3}{\Lambda^5} H_u (q_1'' u_3^c + q_3 u''^c_1) + \frac{\theta'^6 + \theta'^3\theta^3 + \theta^6}{\Lambda^7} H_u (q_2' u''^c_1 + q_1'' u'^c_2) \\ & + \frac{\theta'^8 + \theta'^5\theta^3 + \theta'^2\theta^6}{\Lambda^9} H_u q_1'' u''^c_1, \end{aligned} \tag{3.3.2}$$

$$\begin{aligned} w_{\text{down}} \sim\ & \frac{1}{\Lambda^3} H_d' (d^c \varphi_T)'' q_3 + \frac{\theta'^2}{\Lambda^5} H_d' (d^c \varphi_T)'' q_2' + \frac{\theta^2}{\Lambda^5} H_d' (d^c \varphi_T)' q_2' \\ & + \frac{\theta'\theta}{\Lambda^5} H_d' (d^c \varphi_T) q_2' + \frac{\theta'^4 + \theta'\theta^3}{\Lambda^7} H_d' (d^c \varphi_T)'' q_1'' \\ & + \frac{\theta'^2\theta^2}{\Lambda^7} H_d' (d^c \varphi_T)' q_1'' + \frac{\theta'^3\theta + \theta^4}{\Lambda^7} H_d' (d^c \varphi_T) q_1'', \end{aligned} \tag{3.3.3}$$

$$\begin{aligned} w_{\text{charged lepton}} \sim\ & \frac{1}{\Lambda^3} H_d' (l \varphi_T)'' e_3^c + \frac{\theta'^2}{\Lambda^5} H_d' (l \varphi_T)'' e^{c'}_2 + \frac{\theta^2}{\Lambda^5} H_d' (l \varphi_T)' e^{c'}_2 \\ & + \frac{\theta'\theta}{\Lambda^5} H_d' (l \varphi_T) e^{c'}_2 + \frac{\theta'^4 + \theta'\theta^3}{\Lambda^7} H_d' (l \varphi_T)'' e^{c''}_1 \\ & + \frac{\theta'^2\theta^2}{\Lambda^7} H_d' (l \varphi_T)' e^{c''}_1 + \frac{\theta'^3\theta + \theta^4}{\Lambda^7} H_d' (l \varphi_T) e^{c''}_1. \end{aligned} \tag{3.3.4}$$

Terms in contained within w_{up} originate from two A_4 singlet ten-plets of SU(5) together with the trivial A_4 singlet of the Higgs pentaplet, each field carries a $\mathbb{Z}_3$ charge of ω and the ten-plets may also carry a U(1) Froggatt-Nielsen charge. The Froggatt-Nielsen charge is also carried by the gauge singlet and A_4 singlet flavons

θ, θ' which allow an invariant term to be written down. Terms in both w_{down} and $w_{\text{charged lepton}}$ originate from terms of the form $H_{\bar{\mathbf{5}}}(F\varphi_T)_{\mathbf{1},\mathbf{1}',\mathbf{1}''}T_i$ where the term $(F\varphi_T)_{\mathbf{1},\mathbf{1}',\mathbf{1}''}$ is a singlet component of the product of the two A_4 triplet fields F (the $\bar{\mathbf{5}}$ of SU(5)) and φ_T (the A_4 flavon). Since the ten-plet T_i may also carry a Froggatt-Nielsen charge then the fields θ, θ' may also be included. In both w_{down} and $w_{\text{charged lepton}}$ certain entries are forbidden at first order e.g. the term $H_d'(l\varphi_T)e_3^c$ which would fill out the 13 entry of the mass matrix is not a trivial A_4 singlet.

The dimensionless coefficients of each term in the superpotential have been omitted and they aren't predicted by the flavour symmetry, though they are all expected to be of the same order. It should be noted that the up mass matrix m_u is not symmetric since the Lagrangian is invariant under the standard model and not $SU(5)$. The powers of the cut-off Λ are determined by the dimensionality of the various fields, recalling that brane fields have mass dimension 1 and bulk fields have mass dimension 2 in 6d.

The neutrinos have both Dirac and Majorana masses:

$$w_\nu \sim \frac{y^D}{\Lambda} H_u(Nl) + \frac{1}{\Lambda}(x_a\xi + \tilde{x}_a\tilde{\xi})(NN) + \frac{x_b}{\Lambda}(\varphi_S NN) \tag{3.3.5}$$

where $\tilde{\xi}$ is a linear combination of two independent ξ type fields which has a vanishing VEV and therefore doesn't contribute to the neutrino masses.

Using the alignment mechanism in [39] and described in section 3.4, the scalar components of the supermultiplets will be assumed to obtain VEVs according to the following scheme:

$$\frac{\langle\varphi_T\rangle}{\Lambda} = \frac{1}{\sqrt{\pi^2 R_1 R_2 \sin\theta}}(v_T, 0, 0), \tag{3.3.6}$$

$$\frac{\langle\varphi_S\rangle}{\Lambda} = \frac{1}{\sqrt{\pi^2 R_1 R_2 \sin\theta}}(v_S, v_S, v_S), \tag{3.3.7}$$

$$\frac{\langle\xi\rangle}{\Lambda} = \frac{1}{\sqrt{\pi^2 R_1 R_2 \sin\theta}}u, \tag{3.3.8}$$

$$\frac{\langle\theta\rangle}{\Lambda_i} = t_i, \tag{3.3.9}$$

$$\frac{\langle\theta'\rangle}{\Lambda_i} = t_i' \tag{3.3.10}$$

4d	6d
$H_u q_3 u_3^c$	$\sum_i q_{3i} u_{3i}^c \mathbf{H}_u(z)\delta_i$
$\theta^6\theta'^2 H_u q_1'' u^{c''}_1$	$\sum_i \theta_i^6\theta_i'^2 \mathbf{H}_u(z) q''_{1\,i} u^{c''}_{1\,i}\delta_i$
$\theta'^4 H_u q_2' u^{c'}_2$	$\sum_i \theta_i'^4 \mathbf{H}_u(z) q'_{2i} u^{c'}_{2i}\delta_i$
$\theta'^8 H_u q_1'' u^{c''}_1$	$\sum_i \theta_i'^8 \mathbf{H}_u(z) q''_{1\,i} u^{c''}_{1\,i}\delta_i$
$\theta^3\theta'^3 H_u q_2' u^{c''}_1$	$\sum_i \theta_i^3\theta_i'^3 \mathbf{H}_u(z) q'_{2i} u^{c''}_{1\,i}\delta_i$
$\theta'^4 H_u q_1'' u^c{}_3$	$\sum_i \theta_i'^4 \mathbf{H}_u(z) q_1{}_i'' u^c{}_{3i}\delta_i$
$\theta^4 H_d'(d^c\varphi_T) q_1''$	$\sum_{iK} \theta_i^4 \mathbf{H}_d'(z)(d^{c\mathcal{R}_0}{}_i \alpha_{iK}\varphi_{T\,K}(z)) q''_{1\,i}$
$\theta^2\theta'^2 H_d'(d^c\varphi_T)' q_1''$	$\sum_{iK} \theta_i^2\theta_i'^2 \mathbf{H}_d'(z)(d^{c\mathcal{R}_0}{}_i \alpha'_{iK}\varphi_{T\,K}(z))' q''_{1\,i}\delta_i$
$\theta\theta' H_d'(d^c\varphi_T) q_2'$	$\sum_{iK} \theta_i\theta_i' \mathbf{H}_d'(z)(d^{c\mathcal{R}_0}{}_i \alpha_{iK}\varphi_{T\,K}(z)) q'_{2i}\delta_i$
$H_d'(d^c\varphi_T)'' q_3$	$\sum_{iK} \mathbf{H}_d'(z)(d^{c\mathcal{R}_0}{}_i \alpha''_{iK}\varphi_{T\,K}(z))'' q_{3i}\delta_i$
$H_u(Nl)$	$\sum_i \mathbf{H}_u(z)(N_i^{\mathcal{R}_0} l_i^{\mathcal{R}_0})\delta_i$
$\xi(NN)$	$\sum_i \xi(z)(N_i^{\mathcal{R}_0} N_i^{\mathcal{R}_0})\delta_i$
$\varphi_S(NN)$	$\sum_{iK} \varphi_{S\,K}(z)\alpha_{iK} N_i^{\mathcal{R}_0} N_i^{\mathcal{R}_0}\delta_i$

Table 3.2: Dictionary of terms

A dictionary for the realisation of the 4d terms in the superpotential in terms of the local 6d A_4 invariants. The 4d terms are obtained by integrating out the extra dimensions and assuming a constant background value for the bulk multiplets, as discussed in section 1.6.2 where the notation is defined. The delta function, $\delta_i = \delta(z - z_i)$ where z_i are the fixed points, restricts the couplings to the fixed points.

where $i = u, d, e$ allowing for different messenger masses [43]. Since the brane fields live in 4 dimensions the messengers will also be 4 dimensional particles so that the mechanism in [43], allowing different messenger masses, can be applied in this scenario. Also recall that the dimensions of the torus are now fixed

$$R_1 = 2R_2 \quad \text{and} \quad \sin\theta = \sqrt{3}/2. \tag{3.3.11}$$

In the remainder of this thesis we shall give results in terms of R_1,R_2 and $\sin\theta$. It should be noted that they are however fixed to the values in Eqn. (3.3.11). Note that the flavon VEVs v_T, v_S and u are defined to be dimensionless since the bulk fields have mass dimension of 2.

3.3.2 Higgs VEVs

The Higgs multiplets live in the bulk this gives the required doublet-triplet splitting. The value of the Higgs VEVs at the fixed points is what will enter in the Yukawa couplings, so the values of we are interested in will be averages over the fixed points z_i:

$$\langle\sum_i H_u(z_i)\rangle = \frac{v_u}{\sqrt{\pi^2 R_1 R_2 \sin\theta}}, \langle\sum_i H'_d(z_i)\rangle = \frac{v_d}{\sqrt{\pi^2 R_1 R_2 \sin\theta}} \tag{3.3.12}$$

where v_u and v_d have mass dimension 1. The electroweak scale will be determined by:

$$\mathbf{v}_u^2 + \mathbf{v}_d^2 \approx (174GeV)^2, \tag{3.3.13}$$

$$\mathbf{v}_u^2 \equiv \int d^2z |\langle H_u(z)\rangle|^2, \tag{3.3.14}$$

$$\mathbf{v}_d^2 \equiv \int d^2z |\langle H'_d(z)\rangle|^2. \tag{3.3.15}$$

Because we are using an extra dimensional setup a suppression factor s will enter into our mass matrices since a bulk field and its zero mode are given by:

$$\mathbf{B} = \frac{1}{\sqrt{\pi^2 R_1 R_2 \sin\theta}} B^0 + \{\text{higher order contributions}\} \tag{3.3.16}$$

which results in the suppression factor:

$$s = \frac{1}{\sqrt{\pi^2 R_1 R_2 \sin\theta \Lambda^2}} < 1. \tag{3.3.17}$$

R_1,R_2 and $\sin\theta$ are given by equation (3.3.11). The size of s is discussed below in section 3.3.3.

3.3.3 Quark and lepton mass matrices

We can now calculate the fermion mass matrices from the effective 4d superpotential, using the flavon and Higgs VEVs and expansion parameters above, (using a left-right convention throughout):

$$m_u \sim \begin{pmatrix} t_u^6 t_u'^2 + t_u'^8 + t_u^3 t_u'^5 & t_u^6 + t_u^3 t_u'^3 + t_u'^6 & t_u' t_u^3 + t_u'^4 \\ t_u^6 + t_u^3 t_u'^3 + t_u'^6 & t_u^3 t_u' + t_u'^4 & t_u'^2 \\ t_u' t_u^3 + t_u'^4 & t_u'^2 & 1 \end{pmatrix} s v_u, \tag{3.3.18}$$

$$m_d \sim \begin{pmatrix} t_d^4 + t_d'^3 t_d & t_d^2 t_d'^2 & t_d^3 t_d' + t_d'^4 \\ t_d t_d' & t_d^2 & t_d'^2 \\ \dots & \dots & 1 \end{pmatrix} s^2 v_T v_d, \tag{3.3.19}$$

$$m_e \sim \begin{pmatrix} t_e^4 + t_e'^3 t_e & t_e t_e' & \dots \\ t_e^2 t_e'^2 & t_e^2 & \dots \\ t_e^3 t_e' + t_e'^4 & t_e'^2 & 1 \end{pmatrix} s^2 v_T v_d, \tag{3.3.20}$$

where we have achieved different values for t_u, t_d and t_e via different messenger masses Λ_u, Λ_d and Λ_e and the dots represent contributions from subleading operators as discussed in section 3.4.

Down sector

For the down quark mass matrix, m_d, we can choose $t_d \sim \epsilon$ and $t_d' \sim \epsilon^{2/3}$ to give:

$$m_d \sim \begin{pmatrix} \epsilon^3 & \epsilon^{10/3} & \epsilon^{8/3} \\ \epsilon^{5/3} & \epsilon^2 & \epsilon^{4/3} \\ \dots & \dots & 1 \end{pmatrix} v_T s^2 v_d. \tag{3.3.21}$$

For example, assuming a value $\epsilon \approx 0.15$ allows the order unity coefficients to be tuned to $\mathcal{O}(\epsilon)$ to give acceptable down-type quark mass ratios. The 11 element of the mass matrix is of order ϵ^3, which needs to be tuned to order ϵ^4 using the dimensionless coefficients we have omitted to write in the superpotential. The dots again represent subleading operators as discussed in section 3.4.

Up sector

The up quark matrix is given by:

$$m_u \sim \begin{pmatrix} \bar{\epsilon}^8 & \bar{\epsilon}^6 & \bar{\epsilon}^4 \\ \bar{\epsilon}^6 & \bar{\epsilon}^4 & \bar{\epsilon}^2 \\ \bar{\epsilon}^4 & \bar{\epsilon}^2 & 1 \end{pmatrix} s v_u \tag{3.3.22}$$

with $t_u \sim t'_u \sim \bar{\epsilon}$. Again we have left out the $\mathcal{O}(1)$ coefficients for each term, which for $\bar{\epsilon} \approx 0.22$, may be tuned to give acceptable up-type quark mass ratios. The CKM mixing angles will arise predominantly from the down-mixing angles, but with possibly significant corrections from the up-mixing angles, depending on the unspecified operators represented by dots. In general there will be corrections to all the Yukawa matrices as discussed later. Since the top mass is given by the size of s, we would expect a value around $s \sim 0.5$.

Charged lepton mass matrix

The mass matrix for the charged lepton sector is of the form:

$$m_e \sim \begin{pmatrix} t_e^4 + {t'_e}^3 t_e & t_e t'_e & \dots \\ t_e^2 {t'_e}^2 & t_e^2 & \dots \\ t_e^3 t'_e + {t'_e}^4 & {t'_e}^2 & 1 \end{pmatrix} s^2 v_T v_d = \begin{pmatrix} \epsilon^3 & \epsilon^{5/3} & \dots \\ \epsilon^{10/3} & \epsilon^2 & \dots \\ \epsilon^{8/3} & \epsilon^{4/3} & 1 \end{pmatrix} v_T s^2 v_d. \tag{3.3.23}$$

with $t_e \sim \epsilon$ and $t'_e \sim \epsilon^{2/3}$. The dots again represent subleading operators as discussed in section 3.4.

Neutrino sector

In the neutrino sector, after the fields develop VEVs and the gauge singlets N become heavy the seesaw mechanism takes place as discussed in detail in [41]. After the seesaw mechanism the effective mass matrix for the light neutrinos is given by:

$$m_\nu \sim \frac{1}{3a(a+b)} \begin{pmatrix} 3a+b & b & b \\ b & \frac{2ab+b^2}{b-a} & \frac{b^2-ab-3a^2}{b-a} \\ b & \frac{b^2-ab-3a^2}{b-a} & \frac{2ab+b^2}{b-a} \end{pmatrix} \frac{s(v_u)^2}{\Lambda}, \qquad (3.3.24)$$

where

$$a \equiv \frac{2x_a u}{(y^D)^2}, b \equiv \frac{2x_b v_S}{(y^D)^2}.$$

The neutrino mass matrix is diagonalised by the transformation

$$U_\nu^T m_\nu U_\nu = \mathrm{diag}(m_1, m_2, m_3)$$

with U_ν given by:

$$U_\nu = \begin{pmatrix} -\sqrt{2/3} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{pmatrix} \qquad (3.3.25)$$

which is of the TBM form in equation (1.3.8). However, although we have TBM neutrino mixing in this model we do not have exact TBM lepton mixing due to fact that the charged lepton mass matrix is not diagonal in this basis. Thus there will be charged lepton mixing corrections to TBM mixing resulting in mixing sum rules as discussed in [42, 92–98]. Due to the inexact TBM we can estimate the mixing angle θ_{13} from the from of the mass matrix m_{up}. The prediction is that $\theta_{13} \sim \bar{\epsilon}^4 \sim 0.002$ which is consistent with current data (table 1.2).

3.4 Vacuum alignment and subleading corrections

The resulting A_4 model is of the direct kind discussed in [90] in which the vacuum alignment is achieved via F-terms resulting in the A_4 generator S being preserved in the neutrino sector. The vacuum alignment is achieved by the superpotential w_d introduced in [39], where we have absorbed the mass dimension into the coefficients

Field	φ_T	φ_S	ξ	$\tilde{\xi}$	φ_0^T	φ_0^S	ξ_0
$\mathbb{Z}_3$	1	ω	ω	ω	1	ω	ω
$U(1)_R$	0	0	0	0	2	2	2
Brane/Bulk	Bulk	Bulk	Bulk	Bulk	Bulk	Bulk	Bulk

Table 3.3
The flavon fields and driving fields leading to the vacuum alignment.

g_i, f_i.

$$\begin{aligned} w_d =& M(\varphi_T\varphi_0^T) + g(\varphi_0^T\varphi_T\varphi_T) + g_1(\varphi_0^S\varphi_S\varphi_S) \\ &+ (f_1\xi + f_2\tilde{\xi})\varphi_0^S\varphi_S + f_3\xi_0(\varphi_S\varphi_S) \\ &+ f_4\xi_0\xi\tilde{\xi} + f_5\xi_0\xi^2 + f_6\xi_0\tilde{\xi}^2, \end{aligned} \tag{3.4.1}$$

involving additional gauge singlets, the driving fields φ_0^T, φ_0^S and ξ_0 in Table 3.3. The above form of the driving superpotential w_d and the vanishing of the F-terms,

$$\frac{\partial w}{\partial \varphi_0^T} = \frac{\partial w}{\partial \varphi_0^S} = \frac{\partial w}{\partial \xi_0} = 0, \tag{3.4.2}$$

yields the vacuum alignment anticipated in the previous section. For more details see [39]. Note that the FN flavons θ, θ' require no special vacuum alignment and their VEVs may be generated dynamically by a radiative symmetry breaking mechanism. The ratio of VEVs of θ, θ' will depend on the details of all the Yukawa couplings involving these flavons from which the desired VEVs can emerge. In general we do not address the question of the correlation of flavon VEVs here.

3.4.1 Subleading corrections

Subleading corrections in the mass matrices arise from shifts in the VEVs of the flavons, these corrections arise from higher order operators entering into the superpotential w_d. The shifted VEVs including such corrections are of the general form:

$$\langle\varphi_T\rangle/\Lambda = \frac{1}{\sqrt{\pi^2 R_1 R_2 \sin\theta}}(v_T + \delta v_T, \delta v_T, \delta v_T) \quad (3.4.3)$$

$$\langle\varphi_S\rangle/\Lambda = \frac{1}{\sqrt{\pi^2 R_1 R_2 \sin\theta}}(v_S + \delta v_{S1}, v_S + \delta v_{S2}, v_S + \delta v_{S3}) \quad (3.4.4)$$

$$\langle\xi\rangle/\Lambda = \frac{1}{\sqrt{\pi^2 R_1 R_2 \sin\theta}}u \quad (3.4.5)$$

$$\langle\tilde{\xi}\rangle/\Lambda = \frac{1}{\sqrt{\pi^2 R_1 R_2 \sin\theta}}\delta u' \quad (3.4.6)$$

as discussed in [39], [26]. φ_T obtains a correction proportional to the VEV of φ_S, where φ_S obtains a correction in an arbitrary direction. The VEV of $\tilde{\xi}$, which was zero at leading order, obtains a small correction. The shift in the VEV of ξ has been absorbed into a redefinition of u since at this stage u is a free parameter.

3.4.2 Corrections to m_{up}

The leading order terms in the up sector are of the form $\theta^m\theta'^n H_u q_i u_j$. Terms are gauge and A_4 singlets, to create higher order terms we need to introduce flavon fields. The most straightforward way to do this is to introduce terms that contain factors quadratic in φ_T relative to the leading order terms, since φ_T is an A_4 triplet we need two fields in order to construct a singlet. Such terms will lead to entries in the mass matrix suppressed by a factor of v_T^2. Because of the $\mathbb{Z}_3$ symmetry the flavon fields $\varphi_S, \xi, \tilde{\xi}$ must enter at the three flavon level so entries will be suppressed by a factor of $v_S^2 u, v_S^3$ and u^3 relative to the leading order term.

3.4.3 Corrections to m_{down} and $m_{\text{charged lepton}}$

In the down mass matrix subleading corrections fill in the entries indicated by dots. Entries in the matrix are generated by terms of the form $\theta^m\theta'^n H_d'((d^c\varphi_T)q_i + (l\varphi_T)e_i^c)$, higher order terms can come from replacing φ_T with a product of flavon fields or including the effect of the corrections to the VEV of φ_T. We can replace φ_T with $\varphi_T\varphi_T$, this is compatible with the $\mathbb{Z}_3$ charges and results in corrections with the same form as m_{down} but with an extra overall suppression of v_T. If we include the corrections to the VEV of φ_T then we fill in the entries indicated by dots in eqn. (3.3.19), the corrections are of the form:

$$m_d \sim \begin{pmatrix} \epsilon^{8/3}\delta v_T & \epsilon^{8/3}\delta v_T & \epsilon^{8/3}\delta v_T \\ \epsilon^{4/3}\delta v_T & \epsilon^{4/3}\delta v_T & \epsilon^{4/3}\delta v_T \\ \delta v_T & \delta v_T & \delta v_T \end{pmatrix} s^2 v_d. \tag{3.4.7}$$

The corrections to the charged lepton mass matrix are, up to $\mathcal{O}(1)$ coefficients, the transpose of the above matrix:

$$m_e \sim \begin{pmatrix} \epsilon^{8/3}\delta v_T & \epsilon^{4/3}\delta v_T & \delta v_T \\ \epsilon^{8/3}\delta v_T & \epsilon^{4/3}\delta v_T & \delta v_T \\ \epsilon^{8/3}\delta v_T & \epsilon^{4/3}\delta v_T & \delta v_T \end{pmatrix} s^2 v_d. \tag{3.4.8}$$

Following ref. [39], $\delta v/v \sim \mathcal{O}(\epsilon^2)$ leading to negligible corrections to the leading order m_d, m_e mass matrices.

3.4.4 Corrections to m_ν

The Dirac mass term ($H_u(Nl)$) can be modified with an insertion of the φ_T flavon, producing corrections suppressed by sv_T. The leading Dirac mass correction is the term $H_u(\varphi_T Nl)$. This leads to a correction to the Dirac mass matrix suppressed by a factor of sv_T relative to the leading order (LO) term.

$$m_{LR} = m_{LR}^{LO} + \Delta m_{LR} = y^D s v_u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + v_u s^2 v_T \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & 0 & 1/6 \\ 0 & -5/6 & 0 \end{pmatrix} \tag{3.4.9}$$

The Majorana mass term can receive corrections from a number of higher order terms since the (NN) term can be a $1, 1', 1''$ or 3. The higher order terms all consist of insertions of 2 flavon fields where the leading order terms have only one insertion e.g.

the term $(NN)'(\varphi_T\varphi_S)''$ obeys the $\mathbb{Z}_3$ symmetry, is an A_4 singlet and results in a higher order correction to the terms $(x_a\xi + \tilde{x}_a\tilde{\xi})(NN) + x_b(\varphi_S NN)$. If we call the correction to the Majorana mass matrix δm_{RR} then for this example the correction is given below,

$$m_{RR} = m_{RR}^{LO} + \delta m_{RR} \tag{3.4.10}$$

$$m_{RR}^{LO} = x_a su\Lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{x_b s v_S \Lambda}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{3.4.11}$$

$$\delta m_{RR} = s^2 \Lambda v_T v_S \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.4.12}$$

Such corrections have a relative suppression of $sv_{T,S}$ to the leading order term. After the seesaw mechanism this leads to an effective mass matrix with every entry suppressed by a factor of $sv_{T,S}$. This leads to corrections to the neutrino Tri-Bimaximal mixing angles of order $sv_{T,S}$:

$$\begin{aligned} m_\nu + \Delta m_\nu &= m_{LR} m_{RR}^{-1} m_{LR}^t = (m_{LR}^{LO} + \Delta m_{LR})(m_{RR}^{LO} + \Delta m_{RR})^{-1}(m_{LR}^{LO} + \Delta m_{LR})^t \\ \frac{(\Delta m_\nu)_{ij}}{(m_\nu)_{ij}} &\sim \mathcal{O}(sv_{T,S}). \end{aligned} \tag{3.4.13}$$

The magnitude of v_T depends on the ratio of the top and bottom quark Yukawa couplings, but may be roughly between $v_T \sim \mathcal{O}(\epsilon^2) - \mathcal{O}(\epsilon)$ leading to significant corrections to Tri-Bimaximal mixing. The flavon shifts δv_S also give corrections to the leading order term $(x_b(\varphi_S NN))$, however if $v_T \sim \mathcal{O}(\epsilon^2)$ these corrections are of $\mathcal{O}(\epsilon^2)$ they enter at the same order of magnitude as the corrections from higher order corrections. If however $v_T \sim \mathcal{O}(\epsilon)$ then the correction enters at the order of ϵ.The effect of the VEV of $\tilde{\xi}$, which was zero at leading order, and obtains a small correction, leads to a small shift in the overall scale of the right-handed neutrino masses. And, as already remarked, the shift in the VEV of ξ has been absorbed into a redefinition of u, which we are free to do since u is a free parameter.

3.5 Conclusion

We have proposed a model in which an A_4 Family Symmetry arises dynamically from an $\mathcal{N}=1$ $SU(5)$ SUSY GUT in 6d. The A_4 Family Symmetry emerges as a result of the compactification of the extra complex compact dimension z, assuming a particular twist angle $\theta = 60°$ and a particular orbifold $\mathbb{T}^2/(\mathbb{Z}_2 \times \mathbb{Z}_2^{\rm SM})$ which breaks the $\mathcal{N}=1$ $SU(5)$ SUSY GUT in 6d down to the effective 4d $\mathcal{N}=1$ SUSY SM gauge group. In this model the A_4 Family Symmetry emerges after compactification as a residual symmetry of the full 6d spacetime symmetry of 6d translations and proper Lorentz transformations. It should be noted that had improper Lorentz transformations been included then the residual symmetry would have been S_4 and not A_4. The model also involves other symmetries, in particular we assume a Froggatt-Nielsen $U(1)$ Family Symmetry and other $\mathbb{Z}_N$ symmetries in order to achieve a realistic model.

We emphasise that the $SU(5)$ GUT symmetry is broken by the compactification, while the A_4 Family Symmetry is only realized after the compactification. The matter fields are located at the fixed points on 3-branes, while the Higgs fields live in the 6d bulk. The Froggatt-Nielsen flavons are all located at the fixed point 3-branes while the A_4 flavons all live in the bulk. We have adopted an A_4 classification scheme of quarks and leptons compatible with the $SU(5)$ symmetry. We have also used a Froggatt-Nielsen mechanism for the inter-family mass hierarchies. By placing the $\mathbf{5}$ and $\overline{\mathbf{5}}$ of Higgs in the 6d bulk we have avoided the doublet-triplet splitting problem by making the coloured triplets heavy. The model naturally has TB mixing at the first approximation and reproduces the correct mass hierarchies for quarks and charged leptons and the CKM mixing pattern. The presence of $SU(5)$ GUTs means that the charged lepton mixing angles are non-zero resulting in predictions such as a lepton mixing sum rule of the kind discussed in [42, 92].

In conclusion, this chapter represents the first realistic 6d orbifold $SU(5)$ SUSY GUT model in the literature which leads to an A_4 Family Symmetry after compactification. We emphasise that the motivation for building such higher dimensional models is purely bottom-up, namely to make contact with high energy theories and to solve the conceptual problems with GUT theories such as Higgs doublet-triplet splitting and the origin of Family Symmetry in a higher dimensional setting. The hope is that 6d models such as the one presented here, based on one extra complex dimension z, may provide a useful stepping-stone towards a 10d fully unified string theory (including gravity, albeit perhaps decoupled in some limit) in which GUT breaking and the emergence of Family Symmetry can both be naturally explained as the result of the compactification of three extra complex dimensions.

Chapter 4

$A_4 \times$ SU(5) SUSY GUT of flavour in 8d

4.1 Introduction

In chapter 3 we described a model using an A_4 family symmetry derived from the geometry of an extra dimensional space. The A_4 symmetry is broken in the direct manner using two triplet flavons which acquire a particular alignment in their VEVs. In order to obtain the correct alignment further so-called driving fields are introduced. These driving fields have the required symmetries such that upon minimising the scalar potential the required VEVs emerge. However when breaking the GUT symmetry we didn't' have to resort to higgsing the GUT symmetry down to the Standard Model we are able to make unwanted particles heavy by giving them parity assignments under certain orbifoldings. A similar idea has been explored [99] where orbifolding is used to obtain a particular VEV alignment in family symmetry models.

The purpose of this chapter is to formulate the first realistic SU(5) SUSY GUT model with A_4 family symmetry in 8d where the vacuum alignment is straightforwardly achieved by the use of boundary conditions on orbifolds of the four compact dimensions. We emphasise that we are motivated to consider an 8d theory by the desire to achieve vacuum alignment in an elegant way using orbifold boundary conditions. It is not possible to implement this idea with lower dimensional models such as the the 5d model in [39] or the 6d model in chapter 3 since the desired alignment mechanism is not possible under a single orbifolding. This is due to the requirement that the two triplet flavons φ_T and φ_S have different boundary conditions in order to have the different alignments at the zero mode level. Working in 8d also brings additional benefits, for example the inter-family mass hierarchies will arise in part due to suppression factors arising from an asymmetric geometric dilution of the wavefunctions in the four compact dimensions, although a $U(1)$ Froggatt-Nielsen family symmetry will also be required. In the 8d model the 4 extra dimensions are compactified onto 2 complex directions which are each orbifolded with $\mathbb{Z}_2$ and $\mathbb{Z}_3$ symmetries. These orbifoldings are also used to specify non-trivial boundary conditions on the various multiplets which break the SU(5) gauge symmetry and the extended $\mathcal{N} = 4$ symmetry to leave an effective $\mathcal{N} = 1$ Standard Model theory in 4 dimensions. It is worth noting that due to the orbifoldings the first two families of **10**-plets are duplicated introducing new GUT scale mass particles to the theory, although such a feature removes any desirable GUT predictions it also removes some unwanted GUT mass relations.

The layout of the remainder of the chapter is as follows. In Section 4.2 we introduce the model and show how the 8 dimensions are compactified upon two $\mathbb{T}^2/(\mathbb{Z}_2 \times \mathbb{Z}_3)$ orbifolds leading to gauge and SUSY breaking as above. We specify the superfield content and symmetries of the model. We describe the transformation of the fields under these orbifoldings which leads to an effective 4d Standard Model theory from the 8d SU(5) theory. We first show how the GUT group is broken and how this naturally leads to doublet-triplet splitting of the Higgs multiplets. We then discuss vacuum alignment in the 8d theory, and show how boundary conditions can lead to the desired alignment directions. We also discuss the values of the Higgs and flavon VEVs, including the effects of bulk suppression factors. In Section 4.3 we write down the effective 4d superpotential and the resulting mass matrices. We also analyse contributions from terms beyond the leading order to the mass matrices. Section 4.4 concludes the paper.

4.2 The model

We are considering a model in 8 dimensions with the extra dimensions compactified on two 2d orbifolds as described in sec. 2.3. The SU(5) gauge group lives in the full 8d bulk, with the 8d space compactified to 4d Minkowski space $\times$ 4d compact dimensions with the two complex compact dimensions described by the coordinates z_1 and z_2. We suppose that the 8d space is compactified by orbifolding. In the z_1 direction the $\mathbb{Z}_2$ orbifolding breaks the gauge symmetry and gives the alignment of the A_4 flavon φ_S, while the $\mathbb{Z}_3$ orbifold breaks the extended supersymmetry as described below. In the z_2 direction the $\mathbb{Z}_2$ orbifolding also breaks the gauge symmetry to the Standard Model in exactly the same way as in the z_1 direction, while the $\mathbb{Z}_3$ symmetry is used to give the alignment of the A_4 flavon φ_T as described in sec. 4.2.5 and [99].

We suppose that some of the matter and Higgs fields do not feel the full 8d but are restricted to live in a 6d subspace of the full 8d theory. The second family of $\mathbf{10}$'s, T_2, live in the z_1 direction along with both Higgs multiplets, H_5 and $H_{\overline{5}}$. The first family of $\mathbf{10}$'s, T_1, is placed in the z_2 direction. Similarly, the flavons φ_S, ξ and θ'' live in the z_1 direction, with φ_T and θ in the z_2 direction. We confine the other matter fields to live in a 4d subspace, with the three families of $\bar{\mathbf{5}}$ matter, F, and the third family of $\mathbf{10}$'s, T_3, along with the three families of right-handed neutrinos, N, located at the 4 dimensional fixed point $z_1 = z_2 = 0$, with the Yukawa couplings given by the overlap of the wavefunctions at this fixed point. The particle content of the model is summarised in table 4.1.

A schematic diagram of the model is shown in figure 4.1. As both the z_1 and z_2 directions have a $\mathbb{Z}_2$ orbifolding breaking the gauge symmetry, doublet-triplet splitting of the Higgs multiplets occurs. However this results in half the $\mathbf{10}$-plet becoming heavy. To overcome this, an extra copy of $\mathbf{10}$'s must be included in both directions with opposite parity under the $\mathbb{Z}_2$ symmetry. This results in the complete matter content and also allows us to escape unwanted GUT mass relations. In addition to the unwanted GUT mass relations the doubling of the first two families also prevents

Superfield	N	F	T_1	T_2	T_3	H_5	$H_{\overline{5}}$	φ_T	φ_S	ξ	θ	θ''
SU(5)	1	$\overline{5}$	10	10	10	5	$\overline{5}$	1	1	1	1	1
SM	1	(d^c,l)	(u_1^c,q_1,e_1^c)	(u_2^c,q_2,e_2^c)	(u_3^c,q_3,e_3^c)	H_u	H_d	φ_T	φ_S	ξ	θ	θ''
A_4	3	3	$1''$	$1'$	1	1	$1'$	3	3	1	1	$1''$
$U(1)$	0	0	2	1	0	0	0	0	0	0	-1	-1
$\mathbb{Z}_3$	ω	ω	ω	ω	ω	ω	ω	1	ω	ω	1	1
$U(1)_R$	1	1	1	1	1	0	0	0	0	0	0	0
Location	$z_1=z_2=0$	$z_1=z_2=0$	$z_1=0$	$z_2=0$	$z_1=z_2=0$	$z_2=0$	$z_2=0$	$z_1=0$	$z_2=0$	$z_2=0$	$z_1=0$	$z_2=0$

Table 4.1: The particle content and symmetries of the model.

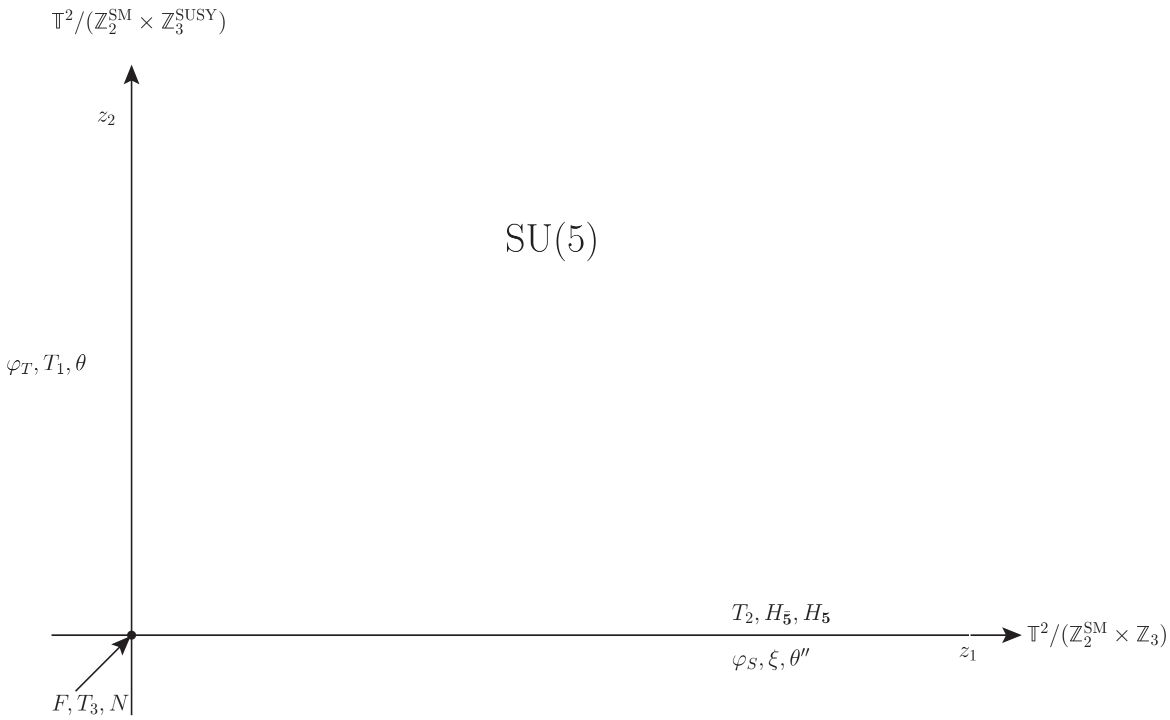

Figure 4.1: A Schematic diagram of the model.
The $SU(5)$ gauge group is in the 8d bulk, represented here by the entire (z_1, z_2) plane, while matter and Higgs fields are confined to 6d subspaces, represented by the complex coordinate directions z_1 and z_2, or to the 4d subspace, represented by the point at the origin. The First and Second families are placed in the z_2 and z_1 directions respectively. Because there is a gauge breaking orbifolding, in both directions, half of the **10**-plets become heavy so additional multiplets are introduced in both directions with opposite parity to obtain the full SM particle content.

good GUT predictions such as the Gatto-Sartori-Tonin [100] and Georgi-Jarlskog relations. The 8 dimensional theory has an A_4 family symmetry which is broken by three flavons φ_T, φ_S and ξ. The vacuum alignment of the flavons is achieved by imposing non-trivial boundary conditions on the flavons so that only the required alignment has a zero-mode. In addition to the A_4 flavour symmetry there is volume suppression for superpotential terms involving 6d fields. This suppression, however, turns out to be insufficient to account for realistic masses and mixings. To obtain a realistic pattern we also exploit the Froggatt-Nielsen mechanism [27] with a $U(1)$ symmetry and the two Froggatt-Nielsen flavons θ and θ'' living in the different orbifolded directions. We also make use of $U(1)_R$ and $\mathbb{Z}_3$ symmetries as shown in table 4.1.

4.2.1 The $\mathbb{T}^2/(\mathbb{Z}_2 \times \mathbb{Z}_3)$ orbifolds

The orbifolding can be used to break both the gauge symmetry and SUSY [36]. As discussed earlier in chapter 3 a model has also been proposed that combine these two ideas to give an extra dimensional GUT theory with a family symmetry arising from the compactification of the extra dimensions. In the present model we will not insist that the family symmetry is dynamically generated from the compactified geometry of extra dimensions, but merely suppose that it pre-exists in the 8d theory. However the part of the orbifold $\mathbb{T}^2/\mathbb{Z}_2$ described in this section is the same as that described in [1, 3] where the A_4 is dynamically generated. The new feature here is that we shall use orbifold boundary conditions to give the desired vacuum alignment for the flavons which break A_4, thereby yielding TB neutrino mixing. We complexify the extra dimensions x_5, x_6 so that they are described by one complex coordinate $z_1 = x_5 + ix_6$. The extra dimensions are compactified on the a twisted torus defined by identifying the following translations:

$$z_1 \rightarrow z_1 + 1 \tag{4.2.1}$$

$$z_1 \rightarrow z_1 + \gamma \tag{4.2.2}$$

where $\gamma = e^{i\pi/3}$ and we have set $2\pi R_{z_1}$, the length of the extra dimension, to unity. We then impose the following identification:

$$\mathbb{Z}_2 : z_1 \rightarrow -z_1. \tag{4.2.3}$$

This defines the orbifold $\mathbb{T}^2/\mathbb{Z}_2$ as in [1, 3]. We can also impose a $\mathbb{Z}_3$ symmetry in order to define the orbifold $\mathbb{T}^2/(\mathbb{Z}_2 \times \mathbb{Z}_3)$, we impose the following identification:

$$\mathbb{Z}_3 : z_1 \rightarrow \omega z_1. \tag{4.2.4}$$

Combining eqns. (4.2.1)-(4.2.4) gives the definition of the orbifold $\mathbb{T}^2/(\mathbb{Z}_2 \times \mathbb{Z}_3)$ which is the complex direction denoted by z_1 in figure 4.1. We follow an analogous procedure for the remaining 2 extra dimensions by defining $z_2 = x_7 + ix_8$ and imposing the above definitions substituting z_2 for z_1. In other words, we apply $\mathbb{T}^2/(\mathbb{Z}_2 \times \mathbb{Z}_3)$ orbifolding separately in each of the z_1 and z_2 spaces. The overall orbifold has a single fixed point invariant under both the $\mathbb{Z}_2$ and $\mathbb{Z}_3$ transformations which is located at $z_{1,2} = 0$. It is at this 4d point that the Yukawa interactions occur.

4.2.2 SUSY breaking

The full 8d theory is $\mathcal{N} = 1$ SU(5) and the 8d bulk of the theory contains the SU(5) gauge bosons. Because spinors in 8 dimensions contain a minimum of 16 real components then in 4 dimensions the effective theory must have $\mathcal{N} = 4$ supersymmetry [101]. In order to eliminate this extended supersymmetry we can impose boundary conditions on the multiplets so that they become heavy and play no part in the zero mode physics. The $\mathcal{N} = 4$ vector multiplet decomposes into 3 chiral ϕ_i and one vector V $\mathcal{N} = 1$ multiplets. We can use the $\mathbb{T}^2/\mathbb{Z}_3$ part of the orbifolding to eliminate the unwanted multiplets by imposing the boundary conditions:

$$V(x^\mu, z_1, z_2) = V(x^\mu, \omega z_1, z_2) \tag{4.2.5}$$

$$\phi_i(x^\mu, z_1, z_2) = \omega \phi_i(x^\mu, \omega z_1, z_2), \tag{4.2.6}$$

where ω are the cube roots of unity, leaving $\phi = 0$ at the fixed point at $z_{1,2} = 0$. We are therefore left with an effective $\mathcal{N} = 1$ theory in 4 dimensions.

4.2.3 Gauge breaking through orbifolding

The breaking of the $SU(5)$ gauge group down to that of the Standard Model can be achieved by the $\mathbb{Z}_2$ part of the orbifolding. By using a single parity P_{SM},

$$P_{SM} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 \end{pmatrix} \tag{4.2.7}$$

we shall require that:

$$P_{SM} V_\mu(x,-z) P_{SM}^{-1} = +V_\mu(x,z). \tag{4.2.8}$$

Gauge boson fields of the standard model thus have positive parity and fields belonging to $SU(5)/G_{SM}$ have negative parity. Only fields with a positive parity have zero modes and therefore gauge bosons not belonging to the standard model gauge group become heavy and the gauge symmetry is broken. In our model both the z_1 and z_2 directions are orbifolded in this way, this allows us to relax unwanted GUT relations between the down quark and charged lepton mass matrices.

4.2.4 Higgs and doublet-triplet splitting

So far we have just considered the gauge sector of $SU(5)$. Adding the Higgs to the 6d theory is straightforward. In the $SU(5)$ GUT theory these are contained in the **5**-plet and $\overline{\mathbf{5}}$-plet of Higgs fields. For the gauge breaking orbifold we choose:

$$P_{SM} H_{\mathbf{5}}(x,-z_1) = +H_{\mathbf{5}}(x,z_1) \tag{4.2.9}$$

$$P_{SM} H_{\overline{\mathbf{5}}}(x,-z_1) = +H_{\overline{\mathbf{5}}}(x,z_1) \tag{4.2.10}$$

It is easy to see with the form of P_{SM} that the last three entries gain a minus sign which makes them heavy whereas the first two entries are left unchanged leaving them light, resulting in a light doublet and a heavy coloured triplet. Similarly with the **10-**

plets living in the z_1 and z_2 directions half the multiplet becomes heavy, however by introducing extra multiplets with opposite parity the full particle content is restored at zero mode. This feature also allows us to evade unwanted GUT relations.

4.2.5 Vacuum alignment, VEVs and expansion parameters

In order to break the A_4 family symmetry we will impose non-trivial boundary conditions on flavons under the orbifoldings so that only a particular alignment survives to low energy. By imposing boundary conditions we are able to avoid introducing the driving fields and avoid having to write down a possibly complicated flavon potential. We will now describe the procedure for obtaining the alignment, closely following the procedure developed in [99] to which we refer the reader for more details. The first $\mathbb{Z}_2$ boundary condition,

$$\varphi_S(-z_1) = P_2\varphi_S(z_1), \tag{4.2.11}$$

requires the matrix P_2 to be of order 2. For A_4 we have the elements in the fourth conjugacy class to choose from. We can choose the matrix $P_2 = S$ where S is given by

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4.2.12}$$

in the basis of A_4 where S is diagonal. This makes it trivial to see which alignment is left as a zero mode. This choice leaves a single zero mode in the $(1,0,0)$ direction in this basis. To find what this alignment is in the T diagonal basis it is a simple matter to rotate the vector using (for example see chapter 3):

$$V = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \\ 1 & 1 & 1 \end{pmatrix}. \tag{4.2.13}$$

This leaves us with the alignment $\varphi_S \propto (1,1,1)$ in the T diagonal basis. For the $\mathbb{Z}_3$ orbifolding we can impose the boundary condition:

$$\varphi_T(\omega z_2) = P_3 \varphi_T(z_2) \tag{4.2.14}$$

and we can choose A_4 elements which have order 3. For P_3 we choose $P_3 = T$ where

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}. \tag{4.2.15}$$

This gives a single zero mode $\varphi_T \propto (1,0,0)$.

Turning to the VEVs themselves, for simplicity from now on we shall set the radii of the compact directions to $R_5 = R_6 = R_{z_1}$ and $R_7 = R_8 = R_{z_2}$, which implies that the Higgs VEVs are given by

$$\langle H_u(z_2)\rangle = \frac{v_u}{\sqrt{\pi^2 R_{z_1}^2 \sin\theta}}, \quad \langle H_d(z_1)\rangle = \frac{v_d}{\sqrt{\pi^2 R_{z_1}^2 \sin\theta}} \tag{4.2.16}$$

where we have included the effect of arbitrary twist angle θ on the torus [1]. For numerical estimates we will set the twist angle to $60°$ (by choosing $\gamma = e^{i\pi/3}$ in eqn. 4.2.2) as in [1] (although in the present model this is an arbitrary choice).

A useful feature of this setup is the suppression of the Yukawa couplings of fields living in the bulk. A field living in the 6d bulk of one of the orbifolded directions is related to its zero mode by

$$F(x^\mu, z) = \frac{1}{\sqrt{V}} F^0 + \ldots \tag{4.2.17}$$

where the dots represent the higher, heavy modes and V is the volume of the extra dimensional space. The above expansion produces a factor s:

$$s = \frac{1}{\sqrt{V\Lambda^2}}. \tag{4.2.18}$$

This feature will produce suppression for couplings involving these bulk fields. Since we are considering 6 dimensional fields that live in either the z_1 or z_2 direction we will have two not necessarily equal volume factors, s_1 and s_2:

$$s_1 = \frac{1}{\sqrt{\pi^2 R_{z_1}^2 \sin\theta \Lambda^2}} = \frac{1}{\sqrt{V_{z_1}\Lambda^2}} < 1 \tag{4.2.19}$$

and

$$s_2 = \frac{1}{\sqrt{\pi^2 R_{z_2}^2 \sin\theta \Lambda^2}} = \frac{1}{\sqrt{V_{z_2}\Lambda^2}} < 1. \tag{4.2.20}$$

Including volume suppression factors, we summarise the aligned flavon VEVs as follows,

$$\frac{\langle \varphi_T \rangle}{\Lambda} = \frac{1}{\sqrt{V_{z_2}}}(v_T, 0, 0), \tag{4.2.21}$$

$$\frac{\langle \varphi_S \rangle}{\Lambda} = \frac{1}{\sqrt{V_{z_1}}}(v_S, v_S, v_S), \tag{4.2.22}$$

$$\frac{\langle \xi \rangle}{\Lambda} = \frac{1}{V_{z_1}}u, \tag{4.2.23}$$

$$\frac{\langle \theta \rangle}{\Lambda} = \frac{1}{\sqrt{V_{z_2}}}t, \tag{4.2.24}$$

$$\frac{\langle \theta'' \rangle}{\Lambda} = \frac{1}{\sqrt{V_{z_1}}}t''. \tag{4.2.25}$$

We have defined the parameters v_T, v_S, t and t'' so that they are dimensionless recalling that 6d fields have mass dimension two. The Froggatt-Nielsen flavons θ, θ'' require no special vacuum alignment and are assumed to obtain VEVs t, t'' of $\mathcal{O}(1)$. Such VEVs can be obtained as in [39] by minimising the D-term scalar potential. Obtaining VEVs of $\mathcal{O}(1)$ can be found by assuming appropriate mass and coupling parameters.

4.3 Superpotentials and mass matrices

The couplings are localised at the single fixed point located at $z_1 = z_2 = 0$ in the extra dimensional space. The action reads:

$$\int \mathrm{d}^4x \int \mathrm{d}^{(4)}z \int \mathrm{d}^2\theta w(x)\delta(z_1)\delta(z_2) + h.c. = \int \mathrm{d}^4x \int \mathrm{d}^2\theta w(x) + h.c. \tag{4.3.1}$$

The effective superpotential w is expressed in terms of $\mathcal{N} = 1$ superfields can be decomposed into the following parts:

$$w = w_{\text{up}} + w_{\text{down}} + w_{\text{charged lepton}} + w_\nu + w_{\text{flavon}}. \tag{4.3.2}$$

The fermion masses and mixings are given by the first three parts after A_4, $U(1)$ Froggatt-Nielsen and electroweak symmetry breaking. The w_{flavon} part concerns the flavon fields, however since the A_4 flavon alignment is given by the non-trivial boundary conditions imposed by the orbifolding we can avoid writing down explicitly the (possibly complicated) flavon potential. However without explicitly writing the flavon potential we do lose the ability to make specific claims on relations between the A_4 flavon VEVs.

4.3.1 Superpotentials

We shall now write down the superpotentials of the model (excluding w_ν which is discussed in sec. 4.3.3). We shall use Standard Model notation since the theory is broken to the Standard Model gauge group by the compactification. We have suppressed the coefficients in each term of the superpotentials and we would expect such coefficients to be of $\mathcal{O}(1)$. We shall use the notation for fields $(f)'$ where the field transforms as a $\mathbf{1}'$ and similarly $(f)''$ for a $\mathbf{1}''$ of A_4.

$$\begin{aligned} w_{\text{up}} \sim\ & \frac{1}{\Lambda} H_u q_3 u_3^c + \frac{\theta''}{\Lambda^4} H_u\{(q_2)' u_3^c + q_3 (u_2^c)'\} + \frac{\theta''^2}{\Lambda^7} H_u\{(q_2)'(u_2^c)'\} \\ & + \frac{\theta''^2}{\Lambda^6} H_u\{(q_1)'' u_3^c + q_3 (u_1^c)''\} + \frac{\theta''^3 + \theta^3}{\Lambda^9} H_u\{(q_2)'(u_1^c)'' + (q_1)''(u_2^c)'\} \\ & + \frac{\theta''\theta^3 + \theta''^4}{\Lambda^{11}} H_u\{(q_1)''(u_1^c)''\}, \end{aligned} \tag{4.3.3}$$

$$w_{\rm down} \sim \frac{1}{\Lambda^3}(H_d)'(d^c\varphi_T)''q_3 + \frac{\theta''}{\Lambda^6}(H_d)'(d^c\varphi_T)''(q_2)' + \frac{\theta}{\Lambda^6}(H_d)'(d^c\varphi_T)'(q_2)' + \frac{\theta''^2}{\Lambda^8}(H_d)'(d^c\varphi_T)''(q_1)'' + \frac{\theta''\theta}{\Lambda^8}(H_d)'(d^c\varphi_T)'(q_1)'' + \frac{\theta^2}{\Lambda^8}(H_d)'(d^c\varphi_T)(q_1)'', \qquad (4.3.4)$$

$$w_{\rm charged\ lepton} \sim \frac{1}{\Lambda^3}(H_d)'(l\varphi_T)''e^c_3 + \frac{\theta''}{\Lambda^6}(H_d)'(l\varphi_T)''(e^c{}_2)' + \frac{\theta}{\Lambda^6}(H_d)'(l\varphi_T)'(e^c{}_2)' + \frac{\theta''^2}{\Lambda^8}(H_d)'(l\varphi_T)''(e^c{}_1)'' + \frac{\theta''\theta}{\Lambda^8}(H_d)'(l\varphi_T)'(e^c{}_1)'' + \frac{\theta^2}{\Lambda^8}(H_d)'(l\varphi_T)(e^c{}_1)''. \qquad (4.3.5)$$

4.3.2 Charged fermion mass matrices

The Higgs multiplets obtain their VEVs along with the A_4 and U(1) flavons $\varphi_T, \theta'', \theta$ as in equations (4.2.21-4.2.25) leading to mass matrices of the following form:

$$m_u \sim \begin{pmatrix} (s_1 s_2^3 t'' t^3 + s_1^4 t''^4)s_2^2 & (s_1^3 t''^3 + s_2^3 t^3)s_1 s_2 & s_1^2 t''^2 s_2 \\ (s_1^3 t''^3 + s_2^3 t^3)s_1 s_2 & s_1^2 t''^2 s_1^2 & s_1 t'' s_1 \\ s_1^2 t''^2 s_2 & s_1 t'' s_1 & 1 \end{pmatrix} s_1 v_u, \qquad (4.3.6)$$

$$m_d \sim \begin{pmatrix} s_2^3 t^2 & s_2^2 s_1 t'' t & s_2 s_1^2 t''^2 \\ \ldots & s_1 s_2 t & s_1^2 t'' \\ \ldots & \ldots & 1 \end{pmatrix} s_1 s_2 v_T v_d, \qquad (4.3.7)$$

$$m_e \sim \begin{pmatrix} s_2^3 t^2 & \ldots & \ldots \\ s_2^2 s_1 t'' t & s_2 s_1 t & \ldots \\ s_2 s_1^2 t''^2 & s_1^2 t'' & 1 \end{pmatrix} s_1 v_T v_d, \qquad (4.3.8)$$

The dots in m_d and m_e are from higher order corrections to the vev of the φ_T flavon alignment. Such corrections come from the heavier modes which have a higher mass through orbifolding and will alter the alignment of φ_T as discussed in section 4.2.5.

We set $s_1 = \lambda$ and $s_2 = \lambda^{3/2}$ with $\lambda = 0.22$, we choose for simplicity $t = t'' = \mathcal{O}(1)$. We should make clear that taking $t = t'' = \mathcal{O}(1)$ means that we are not using the Froggatt-Nielsen mechanism to provide the suppression. Instead the hierarchies originate from the bulk suppression factors s_i. The mass matrices are then given by:

$$m_u \sim \begin{pmatrix} \lambda^7 & \lambda^{5.5} & \lambda^{3.5} \\ \lambda^{5.5} & \lambda^4 & \lambda^2 \\ \lambda^{3.5} & \lambda^2 & 1 \end{pmatrix} \lambda v_u. \tag{4.3.9}$$

The down sector matrix is given by,

$$m_d \sim \begin{pmatrix} \lambda^{4.5} & \lambda^4 & \lambda^{3.5} \\ \dots & \lambda^{2.5} & \lambda^2 \\ \dots & \dots & 1 \end{pmatrix} \lambda^{2.5} v_T v_d, \tag{4.3.10}$$

where again the dots represent contributions from the corrections to the vacuum alignment. The charged lepton mass matrix is given by,

$$m_{charged\ lepton} \sim \begin{pmatrix} \lambda^{4.5} & \dots & \dots \\ \lambda^4 & \lambda^{2.5} & \dots \\ \lambda^{3.5} & \lambda^2 & 1 \end{pmatrix} \lambda^{2.5} v_T v_d. \tag{4.3.11}$$

In this model since the first two families are doubled, because the gauge breaking orbifolding makes half of the **10**-plets heavy the, GUT relation $m_{\text{down}} = m^T_{\text{charged lepton}}$ for the first two families is not valid.

These mass matrices give us approximate quark masses and mixing angles of the correct order of magnitude. For example the quark mixing angles are given roughly by,

$$\theta_{12} = \mathcal{O}(\lambda^{1.5}) \tag{4.3.12}$$

$$\theta_{23} = \mathcal{O}(\lambda^{2}) \tag{4.3.13}$$

$$\theta_{13} = \mathcal{O}(\lambda^{3.5}). \tag{4.3.14}$$

So far we have not specified the size of v_T and v_S, However from the ratio of the top and bottom quark masses we expect

$$\frac{m_b}{m_t} = \lambda^{3/2}\frac{v_d}{v_u}v_T \sim \lambda^2$$
$$\Rightarrow v_T \sim \frac{\lambda^{1/2}}{\tan\beta} \sim \frac{1}{2\tan\beta} \tag{4.3.15}$$

where $\frac{v_d}{v_u} = \tan\beta$.

4.3.3 Neutrino sector

In the neutrino sector the right-handed neutrino A_4 triplets live at the fixed point. The φ_S lives in the z_1 direction along with the A_4 singlet flavon ξ. After these flavons develop a vev the gauge singlets N become heavy and the seesaw mechanism takes place similar to [39], [1] with the alteration that a zero vev A_4 singlet flavon is no longer required as the vacuum alignment is determined by boundary conditions rather than by the use of driving fields. Thus we have,

$$w_\nu \sim \frac{y^D}{\Lambda}H_u(Nl) + \frac{1}{\Lambda}x_a\xi(NN) + \frac{x_b}{\Lambda}\varphi_S(NN). \tag{4.3.16}$$

After the fields develop VEVs, the gauge singlets N become heavy and the seesaw mechanism takes place as discussed in detail in [41], leading to the effective mass matrix for the light neutrinos:

$$m_\nu \sim \frac{1}{3a(a+b)} \begin{pmatrix} 3a+b & b & b \\ b & \frac{2ab+b^2}{b-a} & \frac{b^2-ab-3a^2}{b-a} \\ b & \frac{b^2-ab-3a^2}{b-a} & \frac{2ab+b^2}{b-a} \end{pmatrix} \frac{s_1 (v_u)^2}{\Lambda} \qquad (4.3.17)$$

where

$$a \equiv \frac{2x_a s_1 u}{(y^D)^2}, b \equiv \frac{2x_b s_1 v_S}{(y^D)^2}.$$

The neutrino mass matrix is diagonalised by the transformation

$$U_\nu^T m_\nu U_\nu = \mathrm{diag}(m_1, m_2, m_3)$$

with U_ν given by:

$$U_\nu = \begin{pmatrix} -\sqrt{2/3} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{pmatrix} \qquad (4.3.18)$$

which is of the TB form in Eq. (3.1.1). However, although we have TB neutrino mixing in this model we do not have exact TB lepton mixing due to fact that the charged lepton mass matrix is not diagonal in this basis. Thus there will be charged lepton mixing corrections to TB mixing resulting in mixing sum rules as discussed in [42, 92].

4.3.4 Higher order corrections

We will now discuss corrections to the mass matrices, such corrections come from additional flavon insertion of $\varphi_T, \varphi_S, \xi$ and θ, θ'', and also from corrections to the vacuum alignment of the A_4 triplet flavons φ_T and φ_S.

corrections to m_{up}

The leading order terms in the up sector are of the form $\theta^m \theta''^n H_u q_i u_j$. Terms are gauge and A_4 singlets, to create higher order terms we need to introduce flavon fields.

The most straightforward way to do this is to introduce two flavon fields $(\varphi_T\varphi_T)_{\mathbf{1}}$, since φ_T is an A_4 triplet we need the two triplet fields in order to construct an A_4 singlet. Such terms will lead to entries in the mass matrix suppressed by a factor of $s_2^2 v_T^2$. Due to the $\mathbb{Z}_3$ symmetry the flavon fields $\varphi_S, \xi, \tilde{\xi}$ must enter at the three flavon level so entries will be suppressed by a factor of $s_1^3 v_S^2 u, s_1^3 v_S^3$ and $s_1^3 u^3$ relative to the leading order term. Using the values assumed in sec. 4.3.2 the corrections enter at $\mathcal{O}(\lambda^3)$ relative to the leading order term.

corrections to m_d and m_e

In the down quark mass matrix sub-leading corrections fill in the entries indicated by dots in Eq. 4.3.7. Entries in the matrix are generated by terms of the form $\theta^m\theta''^n H_d'((d^c\varphi_T)q_i + (l\varphi_T)e_i^c)$, higher order terms can come from replacing φ_T with a product of flavon fields or including the effect of the corrections to the VEV of φ_T.

The obvious substitution is to replace φ_T with $\varphi_T\varphi_T$, this is compatible with the $\mathbb{Z}_3$ charges and results in corrections with the same form as m_{down} but with an extra overall suppression of $s_2 v_T$. Using the values assumed in sec. 4.3.2 this type of correction enters at the level of $\mathcal{O}(\lambda^{3/2})$.

If we include the corrections to the alignment of the VEV of φ_T then we fill in the entries indicated by dots in Eq. (4.3.7). Such corrections originate from higher,heavy modes of the flavon field φ_T, such corrections would be suppressed by an order of s_2 relative to the leading order term giving corrections to the mass matrix of the form:

$$\delta m_{\text{down}} \sim \begin{pmatrix} \lambda^5 & \lambda^5 & \lambda^5 \\ \lambda^{3.5} & \lambda^{3.5} & \lambda^{3.5} \\ \lambda^{1.5} & \lambda^{1.5} & \lambda^{1.5} \end{pmatrix} \lambda^{2.5} v_T v_d, \tag{4.3.19}$$

i.e. the corrections are suppressed by $\mathcal{O}(\lambda^{3/2})$ relative to the largest term in each row (or column for $m_{\text{charged lepton}}$).

As remarked, since the first two families are doubled, because the gauge breaking orbifolding makes half of the **10**-plets heavy the, GUT relation $m_{\text{down}} = m^T_{\text{charged lepton}}$

for the first two families is not valid. It does however hold up to orders of magnitude for the individual families so that the power of λ is the same for each family though the (suppressed) $\mathcal{O}(1)$ coefficient can be different for each family.

corrections to m_ν

The leading order Dirac mass term for the neutrinos is $H_u(Nl)$, sub-leading corrections to this term enter with a single flavon insertion of φ_T so the resulting term is $H_u(\varphi_T Nl)$ this results in the sub-leading corrections entering at the $s_2 v_T$ level. Using the values assumed in section 4.3.2 the corrections enter at the $\mathcal{O}(\lambda^{3/2})$ level.

Corrections to the Majorana mass matrix can arise from a number of terms. This is due to the term (NN) being a product of two triplets and can thus be a triplet or any of the singlet representations of A_4. Corrections to the Majorana mass matrix can have one extra flavon insertion relative to the leading order terms $\xi(NN), (\varphi_S NN)$. For example the term $(\varphi_S \varphi_T)(NN)$ is allowed by the $\mathbb{Z}_3$ symmetry and leads to corrections of order $s_2 v_T$. After the seesaw mechanism takes place corrections to the neutrino masses and Tri-Bimaximal mixing are of order $s_2 v_T$. Using the values assumed in section 4.3.2 these corrections are $\mathcal{O}(\lambda^{3/2})$ relative to the leading order term.

4.4 Conclusion

We have proposed the first realistic $\mathcal{N}=1$ SUSY SU(5) GUT model in 8 dimensions with an A_4 family symmetry where the vacuum alignment is straightforwardly achieved by the use of boundary conditions on orbifolds of the four compact dimensions. The low energy theory is the usual $\mathcal{N}=1$ SUSY Standard Model in 4 dimensions but with predictions for quark and lepton (including neutrino) masses and mixing angles. For example, the low energy 4d model naturally has TB mixing at the first approximation and reproduces the correct mass hierarchies for quarks and charged leptons and the CKM mixing pattern. The presence of $SU(5)$ GUTs means that the charged lepton mixing angles are non-zero resulting in predictions such as lepton mixing sum rules.

We were motivated to consider an 8d theory by the desire to achieve the A_4 flavon vacuum alignment in an elegant way using orbifold boundary conditions. Such boundary conditions result in the required alignment surviving at the zero mode level, and in relatively small corrections to the alignment resulting from heavy higher modes. However the extra dimensional set up also provides familiar added benefits such as orbifold gauge and SUSY breaking with doublet-triplet splitting of the **5** and $\bar{\mathbf{5}}$ Higgs multiplets, making the coloured triplets heavy. Because the first two generations of **10**-plets are doubled, both unwanted and desirable GUT relations are also avoided. The lack of such relations introduces more freedom into the theory. The specific model in in table 4.1 and figure 4.1 also includes a Froggatt-Nielsen $U(1)$ symmetry, which, together with the bulk suppression factors, leads to the desired inter-family hierarchies.

Finally we comment on the possible relation between the 8d orbifold GUT-Family model considered here and string theory. At first glance there is an intriguing similarity between the model here and the F-theory GUT recently discussed [102]. In both cases the $SU(5)$ GUT gauge group lives in the full 8d space, and also the matter and Higgs fields lie on matter curves in a 6d subspace, corresponding to two extra complex dimensions $z_{1,2}$, with Yukawa couplings occurring at a 4d point [102]. However any possible connection would be more subtle than this, since firstly one must uplift the 8d orbifold GUT-Family model here into full heterotic string theory, then one must identify duality relations between the heterotic string theory and F-theory as discussed in [103]. Nevertheless the 8d orbifold GUT-Family model presented here may provide a useful link towards some future unified string theory in which GUT breaking and the realisation of family symmetry, spontaneously broken with a particular vacuum alignment, can be explained as the result of the compactification of extra dimensions.

Chapter 5

Conclusion

Here we will simply provide a brief summary of the thesis. Chapters 1 and 2 serve as an introduction to the subject of the Standard Model and extensions to it, namely family symmetries and extra dimensions. The summaries of chapters 3 and 4 are contained within sections 3.5 and 4.4.

We have constructed models of fermion masses and mixings based in part on family symmetries. The model presented in chapter 3 is based upon the SU(5) GUT group. The family symmetry is given by the A_4 group which is derived from the geometry of an orbifolded complex extra dimension. Additionally a Froggatt-Nielsen mechanism is used to help generate the mass hierarchy and mixing angles.

In chapter 4 a similar model is presented, this time using 4 extra compact dimensions, again A_4 is used as a family symmetry however it is not assumed to be generated from the geometry. A feature of the model is that makes use of bulk suppression factors to generate a mass hierarchy and mixing scheme alongside a Froggatt-Nielsen mechanism. The flavons used to break the family symmetry also have a vacuum alignment determined by boundary conditions on the orbifold rather than by the introduction of additional "driving" fields as in chapter 3.

Both models predict realistic fermion mass and mixing patterns, in particular both exhibit near Tri-Bimaximal mixing in the lepton sector.

Final comments

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Figure 1.1 used with permission from Steve Martin.

Feynman diagrams created using JaxoDraw [104].

This thesis was written using LaTeX and slackware linux.

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