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# STABILITY OF OSCILLATORY GRAVITY WAVE TRAINS WITH ENERGY DISSIPATION AND BENJAMIN-FEIR INSTABILITY

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Abstract. The Benjamin-Feir instability describes the instability of a uniform oscillatory wave train in an irrotational flow subject to small perturbation of wave number, amplitude and frequency. Their instability analysis is based on the perturbation around the second order Stokes wave which satisfies the dynamic and kinematic free-surface boundary conditions up to the second order. In the same irrotational flow and perturbation framework of the Benjamin-Feir analysis, the perturbation in the present paper is around a nonlinear oscillatory wave train which solves exactly the dynamic free-surface boundary condition and satisfies the kinematic free-surface boundary condition up to the third order. It is shown that the nonlinear oscillatory wave train is stable with respect to the perturbation when the irrotational flow involves small Rayleigh energy dissipation.

1. **Introduction.** The present investigation focuses on the stability of a uniform oscillatory wave train in an irrotational flow. Under the absence of energy dissipation, Benjamin and Feir [7] announced that a unform wave train in a two-dimensional fluid of infinite depth is unstable to a small perturbation of wave frequency, wave number and wave amplitude of the wave train. The deep water wave train $z = \eta(x,t)$ discussed by Benjamin and Feir [7] is defined by the dynamic free-surface boundary condition (see Whitham [17])

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + gz\Big|_{z=\eta} = 0, \quad \nabla = (\partial_x, \partial_z) \tag{1}$$

and kinematic free-surface boundary condition

$$\phi_t + \phi_x\eta_x - \phi_z|_{z=\eta} = 0 \tag{2}$$

with $g$ the gravitational acceleration and the velocity potential $\phi$ subject to the Laplace equation

$$\nabla^2\phi = 0 \quad \text{for} \quad -\infty < z < \eta, \ \nabla^2 = \nabla\cdot\nabla. \tag{3}$$

As there is no explicit analytical oscillatory solution to Equations (1)-(3), Benjamin and Feir [7, Equations (11)-(13)] referred the Stokes wave solution [14] from Lamb

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[9, §250] to take the following wave elevation, velocity potential and dispersion relation

$$\eta = a\cos(kx-\omega t)+\frac{1}{2}ka^2\cos 2(kx-\omega t), \quad (4)$$

$$\phi = \frac{\omega a}{k}\mathrm{e}^{ky}\sin(kx-\omega t), \quad (5)$$

$$\omega^2 = gk(1+k^2a^2) \quad (6)$$

as a basic wave solution to approximate the exact wave train solution of Equations (1)-(3). Benjamin and Feir [7] confirmed the instability of the nonlinear wave train solution of Equations (1)-(3) through the examination of the instability of the approximated solution (4)-(6) with respect to the perturbation of the wave amplitude $a$, the wave number $k$ and the wave frequency $\omega$. The validity of this instability of the nonlinear wave $\eta$ lies on smallness assumption of $a$. Otherwise, the Stokes wave solution is not suitable to approximate the exact solution of Equations (1)-(3).

It should be mentioned that the Stokes wave solution discussed by Lamb [9, §250] is in the third order with respect to small steepness ratio $ak$. However, the wave elevation (4) is in the second order and the velocity potential (5)

$$\phi = \frac{ga}{\omega}\mathrm{e}^{ky}\sin(kx-\omega t)=\frac{\omega a}{k}\mathrm{e}^{ky}\sin(kx-\omega t),$$

which uses the linear dispersion relation [7, Equation (4)]

$$\omega^2 = kg, \quad (7)$$

is in the second order. Thus the associated dispersion relation should be the linear dispersion relation, covering the first and the second order approximations [9, §229], rather than the third order dispersion relation (6) given in [9, §250].

To support the instability finding, Benjamin [6] provided an experimental result showing a wave train, which was nearly uniform close to a wavemaker and was developed into a largely disintegrated form at a distance of 200ft from the wavemaker. In fact, the two-dimensional potential flow determined by the equations (1)-(3) is irrotational and the shear force is not applicable in the fluid motion. Hence large difference arises between the two-dimensional irrotational flow and the three-dimensional towing tank viscous flow in such a long distance (200ft) fluid motion. Actually, the stability was not answered in [7] as mentioned by Tanaka [15] that the basic gravity waves were restricted only to those with sufficiently small amplitudes and the perturbation superposed on them were also restricted to those with much larger space scale than that of the basic flow. The linear stability around a steep oscillatory wave with respect to the existence of zero eigenvalue was examined numerically by Chen and Saffman [4], Tanaka [15, 16] and Longuet-Higgins [10]. *However, any result obtained numerically contains an inevitable lack of certainty, and some analytical proof should be necessary* (see Tanaka [16, p.288]).

A sufficient condition of linear instability of the oscillatory wave train was later rigorously discussed by Bridges and Mielke [8]. They formulated the gravity wave equations into an infinite-dimensional Hamiltonian system, which was reduced to a four-dimensional Hamiltonian system by employing centre-manifold theory. The existence of an unstable manifold of the wave train was therefore examined under certain assumptions on the group wave speed of the wave train and the conditions that the four-dimensional Hamiltonian system is non-degenerate and the fluid depth is finite.

Recently, Segur *et al.* [13] examined the following damped nonlinear Schrödinger equation,

$$\mathrm{i}\psi_t + \alpha\psi_{xx} + \kappa|\psi|^2\psi + \mathrm{i}\delta\psi = 0, \quad \mathrm{i} = \sqrt{-1} \tag{8}$$

to derive the stability of a free-surface oscillatory wave train. Here $\psi(x,t)$ denotes the envelop of the wave train and $\delta > 0$ is a viscosity parameter proportional to the Reynolds viscosity coefficient. If $\delta = 0$, this equation reduced to the nonlinear Schrödinger equation given by Zakharov [20]. Equation (8) admits an explicit analytic solution,

$$\psi_0 = A\mathrm{e}^{-\delta t} \exp\left(\frac{\mathrm{i}\kappa|A|^2(1-\mathrm{e}^{-2\delta t})}{2\delta}\right)$$

for a complex constant $A$. Segur *et al.* [13] provided a rigorous result showing that $\psi_0$ is stable to the small perturbation of the initial function $\psi(0)$. That is, for any $\varepsilon > 0$ there is a constant $\delta$ such that the following inequality

$$\sup_{t>0} \int_D |\psi(x,t) - \psi_0(t)|^2 dx < \varepsilon$$

holds true, provided that the initial condition

$$\int_D |\psi(x,0) - \psi_0(0)|^2 dx < \delta,$$

is valid for an interval $D$. Based on this analytical result, Segur *et al.* [13] concluded that the viscosity stabilizes the Benjamin-Feir instability. Additionally, many experiments were presented by Segur *et al.* [13] and the reference therein. None of them show actual instability.

The rigorous analytic conclusion of [13] about long-time stabilizing of the Benjamin-Feir instability in the presence of dissipation is also confirmed by the numerical simulations of [19] on the problem of the nonlinear evolution of long-crested water waves in finite depth including the effects of dissipation and surface tension. One may also refer to Canney and Carter [2] for a related discussion on the stability of the damped nonlinear Schrödinger equation.

However, the wave number $k$ and wave frequency $\omega$ in the perturbation of Benjamin-Feir instability are not properly controlled in the nonlinear Schödinger equations. In the present paper we consider the original problem introduced by Benjamin and Feir [7] with respect to the stability of the basic oscillatory wave solution subject to the wave equation system

$$0 = \nabla^2\phi, \quad z < \eta, \tag{9}$$

$$0 = \phi_t + \frac{1}{2}|\nabla\phi|^2 + gz\Big|_{z=\eta}, \tag{10}$$

$$0 = \left(\frac{\partial}{\partial t} + \nabla\phi\cdot\nabla\right)p\Big|_{z=\eta}, \tag{11}$$

$$0 = \lim_{z\to-\infty} \phi_z \tag{12}$$

with $p$ the pressure.

The complete mathematical equations (2), (9), (10) and (12) of a potential flow bounded between a free-surface wave $z = \eta$ and the infinite fluid bottom can be derived by Luke's variational formulation [11, 17]. The combination of the dynamic boundary condition (10) and the kinematic boundary condition (2) is equivalently to the combination of the condition (10) and the free-surface boundary condition

(11), which is subject to the Bernoulli equation defining the pressure $p$ and the kinematic property that a particle remains on the free surface whenever it is on the free surface initially.

If the potential flow motion involves Rayleigh energy dissipation, the dynamic free-surface boundary condition (10) is replaced by the dissipative counterpart

$$0 = \phi_t + \frac{1}{2}|\nabla\phi|^2 + gz + \mu\phi\Big|_{z=\eta}, \tag{13}$$

for $\mu \geq 0$ the dissipative coefficient.

Instead of the perturbation around the second order Stokes wave solution in [7] to approximate the exact basic wave solution, the perturbation is now around a much more accurate wave solution, which satisfies exactly the dynamic free-surface boundary condition (13) and satisfies Equation (11) or the kinematic free-surface boundary condition (2) up to the third order. More precisely, the approximate basic oscillatory wave solution to the dissipative wave equation system is in the implicit formulation

$$\eta = -\frac{1}{g}\left(\phi_t + \frac{1}{2}|\nabla\phi|^2 + \mu\phi\right)\Big|_{z=\eta(x,t)}, \tag{14}$$

$$\phi = \frac{ga}{\omega}\mathrm{e}^{-\frac{1}{2}\mu t + kz}\sin(kx - \omega t), \quad z \leq \eta, \tag{15}$$

$$\omega^2 + \frac{1}{4}\mu^2 = kg\frac{1 + \sqrt{1 + 4(\omega^2 + \frac{1}{4}\mu^2)\frac{k^2a^2}{\omega^2}}}{2}. \tag{16}$$

For $\mu = 0$, the dispersion relation (12) reduces to the following non-dissipative form

$$\omega^2 = kg(1 + k^2a^2 - \frac{1}{4}k^4a^4 + \frac{1}{8}k^6a^6 - \cdots).$$

It is proved that the nonlinear oscillatory wave solution expressed by Equations (14)-(16) is stable under the perturbation of wave number $k$, wave frequency $\omega$ and wave amplitude $a$, when $\mu > 0$.

The energy dissipation also gives rise to a dissipative free-surface Green function method (see Chen [5]) for the numerical simulation of free-surface waves produced by fluid-structure interactions.

For the understanding of the potential flow involving the energy dissipation, we consider a fluid motion governed by the two-dimensional incompressible Navier-Stokes equations,

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} + \frac{1}{\rho}\nabla p + \mu\mathbf{v} - \nu\nabla^2\mathbf{v} = \mathbf{f} \tag{17}$$

and the continuity equation,

$$\nabla\cdot\mathbf{v} = 0 \tag{18}$$

for the fluid velocity $\mathbf{v} = (u_1, u_2)$ and an external body force $\mathbf{f}$. Here, $\nu$ denotes the Reynolds viscocity coefficient and $\mu > 0$ is the parameter measuring the strength of the Ekman like energy dissipation described by the linear friction force $\mu\mathbf{v}$ applied in the two-dimensional fluid domain.

If $\mathbf{f}$ is a Lorentz forcing, Equations (17)-(18) describe a liquid-metal flow in an electromagnetic field (see Bondarenko *et al.* [1]). If $\mathbf{f}$ is a wind jet forcing in the mid latitude atmosphere, Equations (17)-(18) form a simplified model of an wind-driven atmospheric circulation problem in a mid-latitude troposphere (see Charney and DeVore [3], Pedlosky [12] and Wolansky [18]).

In the present paper, $\mathbf{f}$ is the gravity forcing $\mathbf{f} = (0, -g)$. Consider the flow to be irrotational or the velocity to be the gradient of the velocity potential $\phi$:

$$\mathbf{v} = \nabla\phi.$$

This removes the Reynolds viscous forcing

$$\nu\nabla^2\mathbf{v} = \nu\nabla\nabla^2\phi = 0.$$

The substitution of the velocity expression $\mathbf{v} = \nabla\phi$ into the Navier-Stokes equations (17) produces the dissipative Bernoulli equation

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + \frac{1}{\rho}p + \mu\phi + gz = 0. \tag{19}$$

By assuming the free-surface atmospheric pressure to be zero, the equation (19) gives the dissipative free-surface wave equation (13).

As in [7], we consider the gravity fluid of infinite depth. Thus, the continuity equation (18) reduces to the Laplace equation (9) and the velocity potential $\phi$ is subject to the rigid wall boundary condition (12).

The purpose of the present paper is to prove the stability of the dissipative oscillatory wave train defined by Equations (9), (12), (13) and the third order approximation of the equation (11)

$$0 \quad = \quad \left[\left(\frac{\partial}{\partial t} + \nabla\phi\cdot\nabla\right)p + O(\phi^4)\right]_{z=\eta}, \tag{20}$$

where the pressure $p$ is defined by the dissipative Bernoulli equation (19).

The main result of this paper reads as follows.

**Theorem 1.1.** *(a): Let the dissipative parameter $\mu \geq 0$ and a constant $T > 0$ such that $\mu T$ is sufficiently small. Assume that the positive wave amplitude $a$, the wave number $k$ and the wave frequency $\omega$ satisfy the bound*

$$ka\sqrt{1 + \frac{\mu^2}{4\omega^2}} < \frac{\sqrt{5}-1}{2}e^{-\frac{\sqrt{5}-1}{2}} \approx 0.333, \tag{21}$$

*Then Equations (9), (12), (13) and (20) for $0 \leq t \leq T$ admit a nontrivial oscillatory wave solution expressed implicitly in the form of Equations (14)-(16) .*

*(b): For small $\mu > 0$, the oscillatory wave solution defined by Equations (14)-(16) and (21) is stable in the following sense:*

*Let the oscillatory wave solutions $(\eta, a, k, \omega, \mu)$ and $(\hat{\eta}, \hat{a}, \hat{k}, \hat{\omega}, \mu)$ be defined by Equations (14)-(16) and (21) so that $(\hat{a}, \hat{k}, \hat{\omega})$ is perturbed from $(a, k, \omega)$. Then for any constants $\varepsilon > 0$ and $M > 0$, there is a small constant $\delta > 0$ such that the following inequality*

$$\sup_{0<t<\infty,\,|x|<M} \mathrm{e}^{\frac{1}{4}\mu t}|\eta(x,t) - \hat{\eta}(x,t)| < \varepsilon \tag{22}$$

*holds true, provided that the following condition*

$$|a - \hat{a}| + |k - \hat{k}| + |\omega - \hat{\omega}| < \delta \tag{23}$$

*is valid.*

The oscillatory wave solution is obtainable numerically for large values of the steepness ratio $ak$, though the analytical existence in Theorem 1.1 is limited for the moderate values of $ak$ bounded in (21). When $ak$ is large ($ak = 0.8$), a comparison between the second order Stokes wave expressed by Equations (4), (7) and the

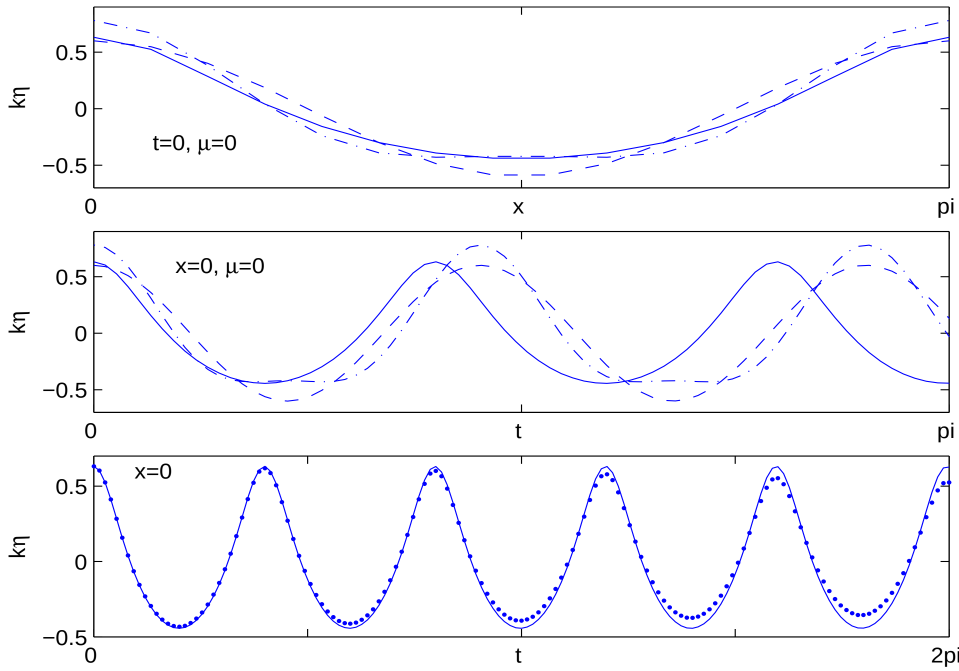


FIGURE 1. Comparisons of the linear Stokes wave (−−), the second order Stokes wave (− · −) defined by Equations (4) and (7), and the nonlinear oscillatory waves defined by Equations (14)-(16) with $\mu = 0$ (—) and $\mu = 0.1$ ($\cdots$) for $a = 0.4$ and $k = 2$.

nonlinear oscillatory wave defined by Equations (14)-(16) is illustrated in Figure 1, which shows the occurrence of the non-physical phenomenon of the multiple troughs in a single period for the second order Stokes wave, while the wave (14)-(16) still travels in a proper way. When $\mu = 0.1$ the time decay property of the wave (14)-(16) is also displayed in Figure 1.

The proof of this theorem is split into three sections. In Section 2, under the assumption of the existence of the oscillatory wave elevation (14) and the smallness assumption of $\mu T$, the nonlinear dispersion relation (16) is derived from solving the third order free-surface boundary condition (20). In Section 3, the existence of the wave elevation (14) is obtained and thus the demonstration of the assertion (a) is completed. Section 4 is contributed to the proof of the stability assertion.

2. **Derivation of the dispersion relation (16) from (20).** Now we solve the approximate free-surface boundary condition (20) for a given function $\eta = O(|\phi|)$.

The application of the pressure distribution defined by the Bernoulli equation (19) shows that

$$
\begin{aligned}
&-\frac{(\partial_t+\nabla\phi\cdot\nabla)p}{\rho}\bigg|_{z=\eta}\\
&= (\partial_t+\nabla\phi\cdot\nabla)\left(\phi_t+\frac{1}{2}|\nabla\phi|^2+\mu\phi+gz\right)\bigg|_{z=\eta}\\
&= \phi_{tt}+\mu\phi_t+g\phi_z+(\partial_t+\mu)|\nabla\phi|^2+\frac{1}{2}\nabla\phi\cdot\nabla|\nabla\phi|^2\bigg|_{z=\eta}.
\end{aligned}
$$

Thus, the equation (20) can be rewritten as

$$
0=\phi_{tt}+\mu\phi_t+g\phi_z+(\partial_t+\mu)|\nabla\phi|^2+\frac{1}{2}e^{\mu t-2kz}\nabla\phi\cdot\nabla|\nabla\phi|^2\bigg|_{z=\eta} \tag{24}
$$

based on the observation

$$
\frac{1}{2}(1-e^{\mu t-2kz})\nabla\phi\cdot\nabla|\nabla\phi|^2=O(\phi^4),
$$

which is valid due to the assumption of $\mu t\le\mu T$ being sufficiently small.

It is readily seen that the function $\phi$ given in (15) solves the Laplace equation (9). On the substitution of the function $\phi$ into the free-surface boundary conditions (24) and upon the observation

$$
(\partial_t+\mu)|\nabla\phi|^2=(\partial_t+\mu)\frac{g^2a^2}{\omega^2}e^{-\mu t+2kz}=0, \tag{25}
$$

we obtain the equation

$$
\begin{aligned}
0 &= \phi_{tt}+\mu\phi_t+g\phi_z+(\partial_t+\mu)|\nabla\phi|^2+\frac{1}{2}e^{\mu t-2kz}\nabla\phi\cdot\nabla|\nabla\phi|^2\bigg|_{z=\eta}\\
&= \left(-\omega^2-\frac{1}{4}\mu^2+kg\right)\frac{ga}{\omega}e^{-\frac{1}{2}\mu t+k\eta}\sin(kx-\omega t)\\
&\quad+\frac{k}{2}\frac{ga}{\omega}e^{-\frac{1}{2}\mu t+k\eta}\sin(kx-\omega t)2k^3\frac{g^2a^2}{\omega^2}\\
&= \left(-\omega^2-\frac{1}{4}\mu^2+kg+k^2a^2\frac{k^2g^2}{\omega^2}\right)\frac{ga}{\omega}e^{-\frac{1}{2}\mu t+k\eta}\sin(kx-\omega t)
\end{aligned}
$$

and hence the nonlinear dispersion relation (16) or

$$
\omega^2+\frac{1}{4}\mu^2=kg\left(1+k^2a^2\frac{kg}{\omega^2}\right).
$$

3. **Existence of the wave train satisfying (14)-(15).** In the previous section, for a given function $\eta$, the velocity potential (15) and the dispersion relation (16) solve Equations (9), (12) and (20). Thus it remains to show the existence of the function $\eta$ satisfying Equation (14) or

$$
\eta = -\frac{1}{g}\left(\phi_t+\frac{1}{2}|\nabla\phi|^2+\mu\phi\right)\bigg|_{z=\eta} \tag{26}
$$

with $\phi$ subject to (15).

For convenience, in the present section, we may let $t$ vary in the time domain $(0,\infty)$.

Define the nonlinear operator

$$A\eta = -\frac{1}{g}\left(\phi_t + \mu\phi + \frac{1}{2}|\nabla\phi|^2\right)\Big|_{z=\eta}. \tag{27}$$

Therefore, the solution of Equation (26) becomes a fixed point of the operator $A$.

In order to prove the existence of the the fixed point, we define the complete metric space

$$X = \left\{\eta \in C(R\times[0,\infty));\ \|\eta\| < \infty, \sup_{t\geq 0,\, x\in R} \mathrm{e}^{\frac{\mu}{2}t}\eta(x,t) \leq \frac{\sqrt{5}-1}{2k}\right\}$$

associated with the norm

$$\|\eta\| = \sup_{t\geq 0,\, x\in R} \mathrm{e}^{\frac{\mu}{2}t}|\eta(x,t)|.$$

Here $C(R\times[0,\infty))$ denotes the space of continuous functions over the domain $R\times[0,\infty)$. Hence, by Banach contraction principle, it suffices to show that the operator $A$ maps $X$ into itself:

$$A:\ X \mapsto X \tag{28}$$

and satisfies the contraction property:

$$\|A\eta_1 - A\eta_2\| \leq \alpha\|\eta_1 - \eta_2\|$$

for a constant $\alpha < 1$.

To do so, we see that the assumption (21) or

$$kae^{\frac{\sqrt{5}-1}{2}}\sqrt{1+\frac{\mu^2}{4\omega^2}} < \frac{\sqrt{5}-1}{2}$$

implies the existence of a constant $\epsilon > 0$ such that

$$k\epsilon < \frac{\sqrt{5}-1}{2} \tag{29}$$

and

$$ae^{\frac{\sqrt{5}-1}{2}}\sqrt{1+\frac{\mu^2}{4\omega^2}} \leq \epsilon. \tag{30}$$

The expression of the velocity potential $\phi$ implies that

$$\begin{aligned} -\frac{1}{g}(\phi_t + \mu\phi) &= a\mathrm{e}^{kz-\frac{1}{2}\mu t}\left(\cos(kx-\omega t) - \frac{\mu}{2\omega}\sin(kx-\omega t)\right) \\ &= a\sqrt{1+\frac{\mu^2}{4\omega^2}}\mathrm{e}^{kz-\frac{1}{2}\mu t}\cos(kx-\omega t+\gamma) \end{aligned}$$

with the constant $\gamma \in [0, \frac{\pi}{2}]$ given by the functions

$$\cos\gamma = \frac{1}{\sqrt{1+\frac{\mu^2}{4\omega^2}}} \ \text{ and } \ \sin\gamma = \frac{\frac{\mu}{2\omega}}{\sqrt{1+\frac{\mu^2}{4\omega^2}}}. \tag{31}$$

By utilising Equations (29) and (30), we have, for $\eta \in X$,

$$\begin{aligned} A\eta &= -\frac{1}{g}\left(\phi_t + \mu\phi + \frac{1}{2}|\nabla\phi|^2\right)\Big|_{z=\eta} \\ &= a\mathrm{e}^{k\eta-\frac{1}{2}\mu t}\sqrt{1+\frac{\mu^2}{4\omega^2}}\cos(kx-\omega t+\gamma) - \frac{1}{2}\frac{a^2gk^2}{\omega^2}\mathrm{e}^{2k\eta-\mu t} \qquad (32) \\ &\le a\mathrm{e}^{\frac{\sqrt{5}-1}{2}-\frac{1}{2}\mu t}\sqrt{1+\frac{\mu^2}{4\omega^2}} \\ &\le \epsilon\mathrm{e}^{-\frac{1}{2}\mu t}. \end{aligned}$$

Thus, Equation (28) follows from Equations (29) and (32).

We now carry out the contraction property of $A$. For $\eta_1$, $\eta_2 \in X$, we have the result

$$\begin{aligned} |A\eta_1 - A\eta_2| &= \Bigg|a\mathrm{e}^{-\frac{1}{2}\mu t}\mathrm{e}^{k\eta_1}\sqrt{1+\frac{\mu^2}{4\omega^2}}\cos(kx-\omega t+\gamma) - \frac{a^2}{2}\frac{gk^2}{\omega^2}\mathrm{e}^{2k\eta_1-\mu t} \\ &\quad -a\mathrm{e}^{-\frac{1}{2}\mu t}\mathrm{e}^{k\eta_2}\sqrt{1+\frac{\mu^2}{4\omega^2}}\cos(kx-\omega t+\gamma) + \frac{a^2}{2}\frac{gk^2}{\omega^2}\mathrm{e}^{2k\eta_2-\mu t}\Bigg| \\ &= \chi\mathrm{e}^{-\frac{1}{2}\mu t}|\eta_1-\eta_2| \end{aligned}$$

with the function

$$\begin{aligned} \chi &= \Bigg|ka\sqrt{1+\frac{\mu^2}{4\omega^2}}\cos(kx-\omega t+\gamma)\int_0^1 \mathrm{e}^{k[s\eta_1+(1-s)\eta_2]}ds \\ &\quad -k^2a^2\frac{kg}{\omega^2}\mathrm{e}^{-\frac{1}{2}\mu t}\int_0^1 \mathrm{e}^{2k[s\eta_1+(1-s)\eta_2]}ds\Bigg|. \end{aligned}$$

Thus, it remains to show that $\chi < 1$ uniformly with respect to $t \ge 0$ and $x \in R$.

Indeed, by the definition of the function space $X$, we have the estimate

$$\begin{aligned} \chi &\le ka\sqrt{1+\frac{\mu^2}{4\omega^2}}\mathrm{e}^{k\epsilon\mathrm{e}^{-\frac{1}{2}\mu t}} + k^2a^2\frac{kg}{\omega^2}\mathrm{e}^{2k\epsilon\mathrm{e}^{-\frac{1}{2}\mu t}} \\ &\le ka\mathrm{e}^{k\epsilon}\sqrt{1+\frac{\mu^2}{4\omega^2}} + k^2a^2\mathrm{e}^{2k\epsilon}\frac{kg}{\omega^2}. \end{aligned}$$

This together with the inequality

$$\frac{kg}{\omega^2} \le 1 + \frac{\mu^2}{4\omega^2}$$

implied from the nonlinear dispersion relation (16) yields

$$\chi \le ka\mathrm{e}^{k\epsilon}\sqrt{1+\frac{\mu^2}{4\omega^2}} + k^2a^2\mathrm{e}^{2k\epsilon}\left(1+\frac{\mu^2}{4\omega^2}\right).$$

Thus the assumptions (29) and (30) yield

$$\begin{aligned} \chi &\le k\epsilon + k^2\epsilon^2 \qquad (33) \\ &< \frac{\sqrt{5}-1}{2} + \left(\frac{\sqrt{5}-1}{2}\right)^2 = 1. \end{aligned}$$

This gives the contraction property:

$$\|A\eta_1 - A\eta_2\| \le \alpha\|\eta_1-\eta_2\|$$

with the constant

$$\alpha = k\epsilon + k^2\epsilon^2 < 1.$$

Therefore, Banach contraction mapping principle implies the unique existence of a solution $\eta \in X$ to Equations (14) and (15).

It follows from Equations (27) and (32) that the desired oscillatory wave solution $\eta$ is expressed implicitly as

$$\eta \quad = \quad a\mathrm{e}^{k\eta - \frac{1}{2}\mu t}\sqrt{1 + \frac{\mu^2}{4\omega^2}}\cos(kx - \omega t + \gamma) - \frac{ka^2}{2}\frac{kg}{\omega^2}\mathrm{e}^{2k\eta - \mu t}. \tag{34}$$

4. **Proof of the stability of the oscillatory wave train.** Consider the wave train $\eta$ defined by the wave parameters $(a, k, \omega)$ and the wave train $\hat{\eta}$ defined by the wave parameters $(\hat{a}, \hat{k}, \hat{\omega})$ perturbed from $(a, k, \omega)$. From the proof of the existence assertion in the previous section, there exist constant $\epsilon > 0$ and $\hat{\epsilon} > 0$ such that Equations (29) and (30) and the equation

$$\hat{k}\hat{a}e^{\frac{\sqrt{5}-1}{2}}\sqrt{1 + \frac{\mu^2}{4\hat{\omega}^2}} \le \hat{k}\hat{\epsilon} < \frac{\sqrt{5}-1}{2} \tag{35}$$

hold true.

By the expression of the wave trains in Equation (34), we have the difference

$$\begin{aligned}\eta - \hat{\eta} \quad &= \quad a\sqrt{1 + \frac{\mu^2}{4\omega^2}}\mathrm{e}^{k\eta - \frac{1}{2}\mu t}\cos(kx - \omega t + \gamma) - \frac{ka^2}{2}\frac{kg}{\omega^2}\mathrm{e}^{2k\eta - \mu t} \\ &\quad -\hat{a}\sqrt{1 + \frac{\mu^2}{4\hat{\omega}^2}}\mathrm{e}^{\hat{k}\hat{\eta} + \frac{1}{2}\mu t}\cos(\hat{k}x - \hat{\omega}t + \hat{\gamma}) - \frac{\hat{k}\hat{a}^2}{2}\frac{\hat{k}g}{\hat{\omega}^2}\mathrm{e}^{2\hat{k}\hat{\eta} - \mu t} \\ &\equiv \quad I_1 - I_2 - \hat{I}_1 + \hat{I}_2. \end{aligned} \tag{36}$$

Especially, the difference $I_1 - \hat{I}_1$ can be expressed in the form

$$\begin{aligned} I_1 - \hat{I}_1 \quad = \quad & a\sqrt{1 + \frac{\mu^2}{4\omega^2}}\left(\mathrm{e}^{k\eta} - \mathrm{e}^{\hat{k}\hat{\eta}}\right)\mathrm{e}^{-\frac{1}{2}\mu t}\cos(kx - \omega t + \gamma) \\ & +(a - \hat{a})\sqrt{1 + \frac{\mu^2}{4\omega^2}}\mathrm{e}^{\hat{k}\hat{\eta} - \frac{1}{2}\mu t}\cos(kx - \omega t + \gamma) \\ & +\hat{a}\left(\sqrt{1 + \frac{\mu^2}{4\omega^2}} - \sqrt{1 + \frac{\mu^2}{4\hat{\omega}^2}}\right)\mathrm{e}^{\hat{k}\hat{\eta} - \frac{1}{2}\mu t}\cos(kx - \omega t + \gamma) \\ & +\hat{a}\sqrt{1 + \frac{\mu^2}{4\hat{\omega}^2}}\mathrm{e}^{\hat{k}\hat{\eta} - \frac{1}{2}\mu t}\left(\cos(kx - \omega t + \gamma) - \cos(\hat{k}x - \hat{\omega}t + \hat{\gamma})\right). \end{aligned}$$

Hence

$$\begin{aligned} |I_1 - \hat{I}_1| \quad \le \quad & |k\eta - \hat{k}\hat{\eta}|a\sqrt{1 + \frac{\mu^2}{4\omega^2}}\mathrm{e}^{-\frac{1}{2}\mu t}\int_0^1 \mathrm{e}^{sk\eta + (1-s)\hat{k}\hat{\eta}}ds \\ & +|a - \hat{a}|\sqrt{1 + \frac{\mu^2}{4\omega^2}}\mathrm{e}^{\hat{k}\hat{\eta} - \frac{1}{2}\mu t} + |\omega - \hat{\omega}|\hat{a}\frac{\frac{\mu^2(\omega + \hat{\omega})}{4\omega^2\hat{\omega}^2}}{\sqrt{1 + \frac{\mu^2}{4\omega^2}} + \sqrt{1 + \frac{\mu^2}{4\hat{\omega}^2}}}\mathrm{e}^{\hat{k}\hat{\eta} - \frac{1}{2}\mu t} \\ & +\hat{a}\sqrt{1 + \frac{\mu^2}{4\hat{\omega}^2}}\mathrm{e}^{\hat{k}\hat{\eta} - \frac{1}{2}\mu t}\left(|k - \hat{k}||x| + |\omega - \hat{\omega}|t + |\gamma - \hat{\gamma}|\right). \end{aligned}$$

Observing

$$\frac{1}{\sqrt{1 + \frac{\mu^2}{4\omega^2}} + \sqrt{1 + \frac{\mu^2}{4\hat{\omega}^2}}} \le \frac{2\omega\hat{\omega}}{\mu(\omega + \hat{\omega})}$$

and employing Equations (29) and (35), we obtain that

$$
\begin{aligned}
|I_1-\hat{I}_1| \quad \leq \quad & |\eta-\hat{\eta}|kae^{\frac{\sqrt{5}-1}{2}}\sqrt{1+\frac{\mu^2}{4\omega^2}}\mathrm{e}^{-\frac{1}{2}\mu t}+|k-\hat{k}|\,|\hat{\eta}|\,ae^{\frac{\sqrt{5}-1}{2}}\sqrt{1+\frac{\mu^2}{4\omega^2}}\mathrm{e}^{-\frac{1}{2}\mu t} \\
& +|a-\hat{a}|e^{\frac{\sqrt{5}-1}{2}}\sqrt{1+\frac{\mu^2}{4\omega^2}}e^{-\frac{1}{2}\mu t}+|\omega-\hat{\omega}|\hat{a}e^{\frac{\sqrt{5}-1}{2}}\frac{\mu}{2\omega\hat{\omega}}e^{-\frac{1}{2}\mu t} \\
& +\hat{a}e^{\frac{\sqrt{5}-1}{2}}\sqrt{1+\frac{\mu^2}{4\hat{\omega}^2}}\mathrm{e}^{-\frac{1}{2}\mu t}\left(|k-\hat{k}||x|+|\omega-\hat{\omega}|t+|\gamma-\hat{\gamma}|\right). \qquad (37)
\end{aligned}
$$

On the other hand, the difference $I_2-\hat{I}_2$ is estimated similarly as

$$
\begin{aligned}
|I_2-\hat{I}_2| \quad \leq \quad & 2|k\eta-\hat{k}\hat{\eta}|\frac{k^2a^2}{2}\frac{g}{\omega^2}\mathrm{e}^{-\mu t}\int_0^1 \mathrm{e}^{2sk\eta+2(1-s)\hat{k}\hat{\eta}}ds \\
& +|k-\hat{k}|\frac{(k+\hat{k})a^2}{2}\frac{g}{\omega^2}\mathrm{e}^{2\hat{k}\hat{\eta}-\mu t}+|a-\hat{a}|\frac{(a+\hat{a})\hat{k}^2}{2}\frac{g}{\omega^2}\mathrm{e}^{2\hat{k}\hat{\eta}-\mu t} \\
& +|\omega-\hat{\omega}|\frac{\hat{k}^2\hat{a}^2}{2}\frac{g(\omega+\hat{\omega})}{\omega^2\hat{\omega}^2}\mathrm{e}^{2\hat{k}\hat{\eta}-\mu t}.
\end{aligned}
$$

By Equations (29) and (35), we have

$$
\begin{aligned}
|I_2-\hat{I}_2| \quad \leq \quad & \left[(|\eta-\hat{\eta}|k+|k-\hat{k}||\hat{\eta}|)k^2a^2\frac{g}{\omega^2}+|k-\hat{k}|\frac{(k+\hat{k})a^2}{2}\frac{g}{\omega^2}\right]\mathrm{e}^{\sqrt{5}-1}\mathrm{e}^{-\mu t} \\
& +\left[|a-\hat{a}|\frac{a+\hat{a}}{2}\frac{\hat{k}^2g}{\omega^2}+|\omega-\hat{\omega}|\frac{\hat{k}^2\hat{a}^2}{2}\frac{g(\omega+\hat{\omega})}{\omega^2\hat{\omega}^2}\right]\mathrm{e}^{\sqrt{5}-1}\mathrm{e}^{-\mu t}.
\end{aligned}
$$

This together with Equations (29), (35)-(37) gives the result

$$
\begin{aligned}
|\eta-\hat{\eta}| \quad \leq \quad & |\eta-\hat{\eta}|\mathrm{e}^{-\frac{1}{2}\mu t}\left(ka\mathrm{e}^{\frac{\sqrt{5}-1}{2}}\sqrt{1+\frac{\mu^2}{4\omega^2}}+\mathrm{e}^{\sqrt{5}-1}k^2a^2\frac{kg}{\omega^2}\right) \\
& +|k-\hat{k}|\,\frac{\sqrt{5}-1}{2\hat{k}}\mathrm{e}^{-\frac{1}{2}\mu t}\left(\mathrm{e}^{\frac{\sqrt{5}-1}{2}}a\sqrt{1+\frac{\mu^2}{4\omega^2}}+\mathrm{e}^{\sqrt{5}-1}ka^2\frac{kg}{\omega^2}\right) \\
& +|k-\hat{k}|\frac{(k+\hat{k})a^2}{2}\frac{g}{\omega^2}\mathrm{e}^{\sqrt{5}-1} \\
& +|a-\hat{a}|e^{-\frac{1}{2}\mu t}\left(\mathrm{e}^{\frac{\sqrt{5}-1}{2}}\sqrt{1+\frac{\mu^2}{4\omega^2}}+\mathrm{e}^{\sqrt{5}-1}\frac{(a+\hat{a})\hat{k}}{2}\frac{\hat{k}g}{\omega^2}\right) \\
& +|\omega-\hat{\omega}|e^{-\frac{1}{2}\mu t}\left(\mathrm{e}^{\frac{\sqrt{5}-1}{2}}\hat{a}\frac{\mu}{2\omega\hat{\omega}}+\mathrm{e}^{\sqrt{5}-1}\hat{k}^2\hat{a}^2\frac{g(\omega+\hat{\omega})}{2\omega^2\hat{\omega}^2}\right) \\
& +\mathrm{e}^{\frac{\sqrt{5}-1}{2}}\hat{a}\sqrt{1+\frac{\mu^2}{4\hat{\omega}^2}}\mathrm{e}^{-\frac{1}{2}\mu t}\left(|k-\hat{k}|M+|\omega-\hat{\omega}|t+|\gamma-\hat{\gamma}|\right).
\end{aligned}
$$

It follows from Equations (16), (30) and (33) that the first term on the right-hand side of the previous equation is bounded by the function

$$
\alpha\mathrm{e}^{-\frac{1}{2}\mu t}|\eta-\hat{\eta}|
$$

with a constant $\alpha < 1$. Thus, we have

$$
\begin{aligned}
&(1-\alpha)\mathrm{e}^{\frac{1}{4}\mu t}|\eta-\hat{\eta}| \\
\leq\;& |k-\hat{k}|\left[\frac{\sqrt{5}-1}{2\hat{k}}\left(\mathrm{e}^{\frac{\sqrt{5}-1}{2}}a\sqrt{1+\frac{\mu^2}{4\omega^2}}+\mathrm{e}^{\sqrt{5}-1}a^2\frac{kg}{\omega^2}\right)+\frac{(k+\hat{k})}{2}a^2\frac{g}{\omega^2}\mathrm{e}^{\sqrt{5}-1}\right] \\
&+|a-\hat{a}|\left(\mathrm{e}^{\frac{\sqrt{5}-1}{2}}\sqrt{1+\frac{\mu^2}{4\omega^2}}+\mathrm{e}^{\sqrt{5}-1}\frac{a+\hat{a}}{2}\frac{\hat{k}^2 g}{\omega^2}\right) \\
&+|\omega-\hat{\omega}|\left(\mathrm{e}^{\frac{\sqrt{5}-1}{2}}\hat{a}\frac{\mu}{2\omega\hat{\omega}}+\mathrm{e}^{\sqrt{5}-1}\hat{k}^2\hat{a}^2\frac{g(\omega+\hat{\omega})}{2\omega^2\hat{\omega}^2}\right) \\
&+\mathrm{e}^{\frac{\sqrt{5}-1}{2}}\hat{a}\sqrt{1+\frac{\mu^2}{4\hat{\omega}^2}}\mathrm{e}^{-\frac{1}{4}\mu t}\left(|k-\hat{k}|M+|\omega-\hat{\omega}|t+|\gamma-\hat{\gamma}|\right).
\end{aligned}
$$

This together with the finding

$$\sup_{t>0} e^{-\frac{1}{4}\mu t}t \leq \frac{4}{\mu}e^{-1} \tag{38}$$

implies the desired estimates

$$\mathrm{e}^{\frac{1}{4}\mu t}|\eta-\hat{\eta}| \leq c\left(|k-\hat{k}|+|a-\hat{a}|+|\omega-\hat{\omega}|+|\gamma-\hat{\gamma}|\right) \tag{39}$$

for $c$ a constant independent of $t \geq 0$. This and (31) give the stability assertion and the proof of Theorem 1.1 is thus completed.

If we consider the approximate basic solution in a finite time domain $[0,T]$ for a constant $T>0$, Equation (38) can be replaced as

$$\sup_{0<t<T} e^{-\frac{1}{4}\mu t}t \leq T \tag{40}$$

and so Equation (39) can be reformulated as

$$\mathrm{e}^{\frac{1}{4}\mu t}|\eta-\hat{\eta}| \leq c_T\left(|k-\hat{k}|+|a-\hat{a}|+|\omega-\hat{\omega}|\right) \tag{41}$$

for a constant $c_T$ independent of $t \in (0,T)$. Therefore the combination of Equations (40) and (41) and the stability proof for $\mu > 0$ gives the following.

**Corollary 1.** *For $\mu = 0$ and a constant $T > 0$, let the oscillatory wave solutions $(\eta, a, k, \omega)$ and $(\hat{\eta}, \hat{a}, \hat{k}, \hat{\omega})$ be given by Equations(14)-(16) so that $(\hat{a}, \hat{k}, \hat{\omega})$ is perturbed from $(a, k, \omega)$. Then for any $\varepsilon > 0$ and any constant $M > 0$, there is a constant $\delta > 0$ such that the following inequality*

$$\sup_{0<t<T,|x|<M} |\eta(x,t)-\hat{\eta}(x,t)| < \varepsilon$$

*holds true, provided that the following condition*

$$|a-\hat{a}|+|k-\hat{k}|+|\omega-\hat{\omega}| < \delta$$

*is valid.*

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