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The Limiting Power of Autocorrelation Tests in Regression Models with Linear Restrictions

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Abstract

It is well known that the Durbin-Watson and several other tests for first-order autocorrelation have limiting power of either zero or one in a linear regression model without an intercept, and tend to a constant lying strictly between these values when an intercept term is present. This paper considers the limiting power of these tests in models with restricted coefficients. Surprisingly, it is found that with linear restrictions on the coefficients, the limiting power can still drop to zero even with the inclusion of an intercept in the regression. It is also shown that for regressions with valid restrictions, these test statistics have algebraic forms equivalent to the corresponding statistics in the unrestricted model.

I. Introduction

The Durbin-Watson (DW) test is by far the most common test for first-order autocorrelated (AR(1)) errors. Over the past three decades there has been a sizable literature on the power properties of the DW test and its power comparisons with various other testing procedures under differing circumstances. Several of these papers have appeared in this *Review*. See, for example, Blattberg (1973), Schmidt & Guilkey (1975), Bartels & Goodhew (1981), White (1992), among others. For a survey of the literature on testing for autocorrelation, see King (1987). A number of authors have shown that the power of the DW test can drop to zero when the autocorrelation coefficient in the AR(1) process is close to one. Krämer (1985) found that in a linear regression without an intercept, the limiting power of the DW test is either zero or one, and whether the limit is zero or one is determined by the sign of an eigenvalue that depends on the underlying regressor matrix. Krämer & Zeisel (1990) provided further analytical evidence suggesting that in addition to the DW test, the alternative DW (ADW) (King, 1981), the Berenblut-Webb (BW) (Berenblut & Webb, 1973), and King's point optimal (King, 1985) tests can all have zero power when the correlation between neighbouring errors is close to one. The latter two tests are most powerful invariant in a given neighbourhood of the alternative hypothesis' parameter space. So it may very well happen that all of these tests

do not reject the null hypothesis of no autocorrelation, but that nevertheless the neighbouring disturbances are nearly perfectly correlated. On the other hand, Zeisel (1989) proved that in the model with an intercept, the DW test has limiting power that never reaches unity or drops to zero. Small (1993) showed that Zeisel's (1989) results apply also to the ADW, BW and King's point optimal tests. It appears therefore that, by adding a constant to the model, one can avoid the worst scenario of the autocorrelation tests having no power at all as one approaches the unit root.

There is surprisingly very little written on the DW and other autocorrelation tests in models with restricted coefficients. On the other hand, there is some indirect evidence which suggests that the DW test applied to the restricted regression has somewhat similar properties to that applied to the unrestricted model. Giles and Lieberman (1992) studied the size and power of the DW test when its application is preceded by a preliminary t-test on the significance of a coefficient. Their Monte-Carlo results indicate no size distortion for the DW test applied without pre-testing to the restricted model when the restriction is true; but if the false restriction is imposed without pre-testing, then the subsequent DW test generally yields sizes far greater than the nominal size. The restricted version of the DW test is generally more powerful than the corresponding unrestricted and pre-test versions of the test when the restriction is true; if the restriction is false then, the power of the restricted DW test can be vastly inferior to the other two. Giles and Lieberman (1992) also reported the possibility of very poor power properties of the pre-test DW test when highly correlated errors are combined with a grossly incorrect restriction. Dufour (1990), on the other hand, suggested a joint procedure for testing linear restrictions on the coefficients and autocorrelated errors. This procedure, however, fails when applied to the testing of a zero intercept at the unit-root.

In this paper we present analytical results on the limiting power of the DW, ADW, BW and King's point optimal tests when applied to a model with linear restrictions on the coefficients. For a restricted regression with no intercept, it is found that the limiting power of all four tests tends to either one or zero as in the case of an unrestricted model; but with restrictions on the coefficients, these limiting power characteristics can still apply even with the inclusion of an intercept. The latter finding provides an interesting contrast with the existing results on the power properties of these tests in the unrestricted model. Whether or not the restrictions are in fact valid has no effect on the limiting power of the tests. On the other hand, provided that the restrictions are correct, there is an algebraic equivalence between these test statistics and their unrestricted model counterparts.

The rest of this paper is organized as follows. Section 2 below introduces the model and test statistics, while Section 3 presents the main results. Section 4 concludes the paper.

II. Model framework, test statistics and notations

We consider the following linear regression with possibly AR(1) errors:

$$\begin{aligned}
y &= X\beta + u, \\
u_t &= \rho u_{t-1} + \epsilon_t, \quad |\rho| < 1, \quad \epsilon_t \sim NID(0, \sigma^2)
\end{aligned} \tag{1}$$

where y and u are $n \times 1$ random vectors, X is an $n \times k$ non-stochastic matrix of full column rank and β is a $k \times 1$ vector of unknown coefficients. In addition to sample information, there exists prior information in the form of the following set of m independent linear restrictions on the coefficients:

$$R\beta = r, \tag{2}$$

where R is an $m \times k$ known prior information design matrix of rank m and r is a $m \times 1$ vector of known elements. We are interested in testing the hypothesis of $H_0: \rho = 0$ of no first-order autocorrelation versus $H_1: \rho \neq 0$. It is assumed that tests of H_0 are preceded by fitting the regression (1) with the linear restrictions (2) incorporated. Let $E(uu') = \sigma^2 V(\rho)$. Clearly, $V(0) = I$, an identity matrix. Imposing the restrictions in (1) gives rise to the restricted least squares (RLS) estimator

$$\hat{\beta}(\rho) = \hat{\beta}(\rho) + S^{-1}(\rho)R'RS^{-1}(\rho)R'\hat{\beta}(\rho) - R\hat{\beta}(\rho), \tag{3}$$

where $\hat{\beta}(\rho) = S^{-1}(\rho)X'V^{-1}(\rho)y$ and $S(\rho) = X'V^{-1}(\rho)X$. In practice, prior to testing H_0 , the estimator used is the one in (3) with $\rho = 0$, i.e.,

$$\hat{\beta}(0) = \hat{\beta}(0) + S^{-1}(0)R'RS^{-1}(0)R'\hat{\beta}(0) - R\hat{\beta}(0). \tag{4}$$

Let

$$M(\rho) = I - XS^{-1}(\rho)X'V^{-1}(\rho), \tag{5}$$

$$B(\rho) = XS^{-1}(\rho)R'RS^{-1}(\rho)R' - RS^{-1}(\rho)X'V^{-1}(\rho), \tag{6}$$

and

$$\hat{\beta}(\rho) = XS^{-1}(\rho)R'RS^{-1}(\rho)R'\hat{\beta}(\rho) + (R\hat{\beta}(\rho) - r)/\rho. \tag{7}$$

If the restrictions are correct then $\hat{\beta}(\rho) = 0$, for which

$$y = X\hat{\beta}(\rho) + M(\rho)u + B(\rho)u + \bar{M}(\rho)u. \tag{8}$$

On the other hand, if the restrictions are incorrect, then $\hat{\beta}(\rho) \neq 0$. Thus,

$$y = X\hat{\beta}(\rho) + \bar{M}(\rho)u + \epsilon(\rho). \tag{9}$$

Equation (9) is the vector of residuals corresponding to the RLS estimator given in (4) for $\rho = 0$.

We consider the test statistics which can be expressed as ratios of the following quadratic form:

$$G(\rho_o, Q) = \frac{y' X \rho(\rho_o)' Q y' X \rho(\rho_o)'}{y' X \rho(0)' y' X \rho(0)'} = \frac{u' \bar{M}(\rho_o) Q \bar{M}(\rho_o) u' \bar{M}(\rho_o)'}{u' \bar{M}(0) u' \bar{M}(0)'}, \quad (10)$$

where ρ_o is a known constant. Note that (10) is in fact the generalized likelihood ratio test for testing the hypothesis of the autoregressive parameter being zero or ρ_o if we take $\rho = \rho_o$ and $Q = V^{-1}(\rho_o)$ under the assumption that $R = r$.

Below we give the restricted regression analogues of four of the autocorrelation test statistics considered in Krämer & Zeisel (1990):

i) The DW test: The restricted DW test statistic is given by $G(0, A)$, where

$$A = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 2 & 1 \\ 0 & 0 & \dots & 1 & 1 \end{pmatrix} \quad (11)$$

ii) The ADW test: For this test, the test statistic has the form $G(0, A_1)$, where $A_1 = A + C$, and C is an $n \times n$ matrix whose top left and bottom right elements are one and all remaining elements are zero. For regressions without restrictions, this test is found to be more powerful than the DW test against negative autocorrelation, with the reverse being the case against positive autocorrelation. See King (1981).

iii) The BW test: Here $\rho_o = 1$ and $Q = W$ in (10) (i.e., $V^{-1}(1)$ is replaced by W), where

$$W = \begin{pmatrix} \lambda^2 & \lambda\beta & 0 & \dots & 0 \\ \lambda\beta & \beta^2 & \beta & \dots & 0 \\ \lambda & \beta & \cdot & \cdot & \cdot \\ \lambda & \beta & \cdot & \cdot & \cdot \\ \lambda & \beta & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 2 & \lambda\beta \\ 0 & 0 & \dots & \lambda & 1 \end{pmatrix} \quad (12)$$

Empirical evidence has suggested that this test has a very clear power advantage over the DW test for large values of λ in the unrestricted regression.

- iv) King's point optimal ($s(\lambda_1)$) test: This test corresponds to setting $\lambda = \lambda_1$ and $Q = V^{-1}(\lambda_1)$ in (10) for some known constant λ_1 . For the unrestricted model, King (1985) showed that this test is most powerful invariant in a given region of the alternative hypothesis parameter space. By construction, this test yields a different test statistic for each choice of λ_1 . The BW and the $s(\lambda_1)$ tests are approximately identical as $\lambda_1 \rightarrow 1$.

III. Main results

In this section, we consider the limiting power of the test statistics based on the quadratic form $G(\lambda, Q)$ given in (10). Using results from Banerjee and Magnus (1999), we can write

$$V(\lambda) = \frac{1}{\lambda^2} LL' \quad (13)$$

where $L = L_0 + L_1 = O(\lambda^2)$, $\lambda = \sqrt{1 + \lambda^2}$, $L_0 = (i, 0_{n \times (n-1)})'$, $L_1 = \begin{pmatrix} 0 & 0 \\ \lambda & \lambda \end{pmatrix}$, and λ is an $(n-1) \times (n-1)$ lower triangular matrix with ones on and below the diagonal and zeros elsewhere. Write $u = \frac{\lambda}{\lambda} L \begin{pmatrix} \lambda \\ \lambda \end{pmatrix}$, where λ is a scalar, λ is an $(n-1) \times 1$ vector, and

$\begin{pmatrix} \lambda \\ \lambda \end{pmatrix} \sim N(0, I_n)$. Then we have

$$u = \frac{\lambda}{\lambda} \lambda \bar{P} = O_p(\lambda^2) \quad (14)$$

where $\bar{P} = J'$ and $J = (0, I_{n-1})$. Upon substituting (14) in (10), we obtain

be either one or zero, holds for both regressions with and without an intercept, and in the case of the former, at least one restriction must involve the intercept¹².

The limiting power in part 1) is determined by the non-zero eigenvalues of the matrix $\{\bar{M}(\beta_0)Q\bar{M}(\beta_0) - c\bar{M}(0)\beta_0\beta_0'$ which has rank of at most one. When $\bar{M}(0)\beta_0 = 0$ and $g^* = c$, $\{\bar{M}(\beta_0)Q\bar{M}(\beta_0) - c\bar{M}(0)\beta_0\beta_0'$ has a unique non-zero eigenvalue whose sign determines the limiting power of $G(\beta_0, Q)$; if $g^* = c$, this eigenvalue is positive and the limiting power takes the value of 1; alternatively, if $g^* < c$, this eigenvalue is negative and the limiting power is 0. In part 2) of Theorem 1, the limiting power is a constant lying strictly between zero and one. This occurs when $\bar{M}(0)\beta_0 \neq 0$, or equivalently, $M(0)\beta_0 \neq 0$ and $B(0)\beta_0 \neq 0$, i.e., the model contains an intercept but the restrictions do not involve the intercept. Thus the existing results, which in particular have shown that the limiting power of test cannot drop to zero or reach one when the (unrestricted) regression contains an intercept, do *not* apply to regression models with linear restrictions. It is possible for the tests to have zero power even with the inclusion of an intercept in the model if at least one of the linear restrictions involves the intercept, irrespective of whether the restrictions are in fact valid. Theorem 1 yields the corresponding results for the unrestricted model when $m = 0$, then $\bar{M}(0) = M(0)$ and the two parts of Theorem 1 give the limiting power characteristics of the tests for the regressions with no intercept and with an intercept, respectively.

So, it is clear from the foregoing discussion that the four test statistics in restricted regressions do not possess the same properties as their unrestricted counterparts. Nevertheless, provided that the restrictions are correct, there is an interesting correspondence between the unrestricted and restricted versions of the statistics as one moves from the unrestricted model to the restricted model:

Theorem 2: Let the restrictions $R = \beta_0 = r$ hold. Then with a suitable data transformation, $G(\beta_0, Q)$ and the corresponding quadratic form in the unrestricted regression are algebraically equivalent.

Proof: Our proof requires the following Lemma from Rao (1973, p.77):

¹ Note that zero restriction on the intercept would imply, the restricted model is same as an unrestricted model with no intercept, and the corresponding results are well-known from the work of Krämer (1985) and Krämer and Zeisel (1990).

² The case of models testing for non-zero restrictions involving the intercept often comes up in the context of the Solow growth models. These models look at the Solow residuals between countries and test for the differences in productivity growth represented by the intercept term, and then compare them with the productivity growth of the benchmark region (see Bernard and Jones, 1996). Our results in Theorem 1 suggest that diagnostic checking for AR(1) errors in such models may be prone to errors because the power of the autocorrelation tests may be low.

Lemma 1: If V is any $n \times n$ positive definite matrix, and U and T are $n \times k$ and $n \times (n - k)$ matrices, respectively, such that $(U | T)$ is orthogonal, then we have

$$V^{-1} V^{-1} U (U' V^{-1} U)^{-1} U' V^{-1} + T (T' V T)^{-1} T'. \quad (20)$$

Now, since the matrix R is of rank m , we can write $R = [I_{m \times m} | 0_{m \times (k-m)}] P$, where P is an $m \times m$ orthogonal matrix and P is a $k \times k$ non-singular matrix. Denote $X^* = X P^{-1}$ and $R^* = (I | 0)$. When $R = r$, the test statistic $G(\theta_0, Q)$ can be written as

$$G(\theta_0, Q) = \frac{u' \bar{M}^*(\theta_0) Q \bar{M}^*(\theta_0) u}{u' \bar{M}^*(0) u}, \quad (21)$$

where $\bar{M}^*(\theta_0) = M^*(\theta_0) + B^*(\theta_0)$, $M^*(\theta_0) = I + X^* (X^{*'} V^{-1}(\theta_0) X^*)^{-1} X^{*'} V^{-1}(\theta_0)$ and $B^*(\theta_0) = X^* (X^{*'} V^{-1}(\theta_0) X^*)^{-1} R^{*'} [R^* (X^{*'} V^{-1}(\theta_0) X^*)^{-1} R^{*'}]^{-1} R^* X^{*'} V^{-1}(\theta_0) X^* [X^{*'} V^{-1}(\theta_0) X^*]^{-1}$.

Consider a $k \times (k - m)$ matrix Z such that $[R^* | Z]$ is orthogonal. Then from Lemma 1, we have

$$\begin{aligned} & (X^{*'} V^{-1}(\theta_0) X^*)^{-1} + (X^{*'} V^{-1}(\theta_0) X^*)^{-1} R^{*'} [R^* (X^{*'} V^{-1}(\theta_0) X^*)^{-1} R^{*'}]^{-1} R^* X^{*'} V^{-1}(\theta_0) X^* [X^{*'} V^{-1}(\theta_0) X^*]^{-1} \\ & = [Z' X^{*'} V^{-1}(\theta_0) X^* Z]^{-1} Z' \end{aligned} \quad (22)$$

Using (22) and the definition of $\bar{M}^*(\theta_0)$, we obtain

$$\bar{M}^*(\theta_0) = I + Z [Z' V^{-1}(\theta_0) Z]^{-1} Z' V^{-1}(\theta_0), \quad (23)$$

where $Z = X^*$. In particular,

$$\bar{M}^*(0) = I + Z' Z Z^{-1} Z. \quad (24)$$

Now, consider the linear regression $y = Z' u$, $u_t = ? u_{t-1} + ?_t$ without restrictions. Krämer & Zeisel (1990) show that the quadratic form

$$G_z^u(\theta_0, Q) = \frac{u' \bar{M}_z^*(\theta_0) Q \bar{M}_z^*(\theta_0) u}{u' \bar{M}_z^*(0) u} \quad (25)$$

encompasses the DW, ADW, BW and King's $s(\rho_1)$ statistics, where $M_z(0) = I - Z(Z'Z)^{-1}Z'$ and $M_z(\rho_0) = I - Z(Z'V^{-1}(\rho_0)Z)^{-1}Z'V^{-1}(\rho_0)$. Clearly, $\bar{M}^*(\rho_0) = M_z(\rho_0)$, $\bar{M}^*(0) = M_z(0)$, and hence $G(\rho_0, Q) = G_z^u(\rho_0, Q)$. This completes the proof of Theorem 2.

So, provided that the restrictions are correct, the algebra of $G(\rho_0, Q)$ in the X and R space is essentially the same as that of $G_z^u(\rho_0, Q)$ in the Z space. Accordingly, the existing results on the power properties of $G_z^u(\rho_0, Q)$, for a given Z matrix, will hold precisely for $G(\rho_0, Q)$ that corresponds to the data matrix X and the restriction matrix R such that $Z = X^*$. For example, the Monte-Carlo results of Bartels (1992) are invariant to this transformation; by transforming his design matrices from Z in the original (unrestricted) space to X and R in the transformed (restricted) space, his results hold exactly in the restricted model as in the unrestricted model. The algebra of the $G(\rho_0, Q)$ statistic facilitates this generalization of results. On the other hand, the arguments used to prove Theorem 2 cannot be extended to cover the situation of incorrect restrictions because $\bar{M}(\rho_0)(\rho_0) \neq 0$ when $r \neq R \neq 0$. It is also clear from the foregoing discussion that for $G(\rho_0, Q)$ and $G_z^u(\rho_0, Q)$ to be equivalent, there must exist a suitable data transformation that turns $G(\rho_0, Q)$ into $G_z^u(\rho_0, Q)$; in general when a *common* data set is applied to both the unrestricted and restricted regressions, properties of the tests considered can differ substantially across the two models. Furthermore, note that while $G_z^u(\rho_0, Q)$ possesses an invariance property (see King, 1981), but since $B(\rho_0)X \neq 0$ for any given ρ_0 , there exists no corresponding invariance property for $G(\rho_0, Q)$ to linear transformation of the form $y = y^* + \rho_0 y$, $X = X^*$, for any positive scalar ρ_0 and $k \times 1$ vector ρ .

IV. Conclusions

This paper has provided new insights into the limiting power of the four well-known tests for AR(1) errors. We demonstrate in particular that with restricted coefficients, the probability of detecting autocorrelation can fall to zero as the autocorrelation gets very strong, *even when the model includes an intercept term*. So the well-known results regarding the limiting power of these tests do not all apply to regressions with restricted coefficients. We have also noted that all four autocorrelation tests lack a common invariance property in restricted regressions, but algebraically, provided that the restrictions are true, the test statistics in the unrestricted and restricted regressions are equivalent. Altogether, given the widespread use of these tests, in particular the Durbin-Watson test, our observations should be of interest to both theorists and applied researchers.

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