Group and Total Dissipativity and Stability of Multi-Equilibria Hybrid Automata

Eva M. Navarro-López, *Member, IEEE*, and Dina. S. Laila *Senior Member, IEEE*

***Abstract*—Complex systems, which consist of different interdependent and interlocking subsystems, typically have multiple equilibrium points associated with different set points of each operation mode. These systems are usually interpreted as hybrid systems. This paper studies the conditions for dissipativity and some stability properties of a class of hybrid systems with multiple co-existing equilibrium points, modelled as nonlinear hybrid automata. A classification of equilibria for hybrid automata is proposed. The objective is to identify dissipative components as groups of discrete locations within the hybrid automaton, formed according to existing equilibria. An example is provided.**

***Index Terms*—Dissipativity theory, energy control, hybrid automata, control systems, computational methods.**

I. Motivation

Many questions still remain unanswered in the modelling and analysis of switched and hybrid systems with myriad interdependent and interlocking subsystems. These subsystems are entire systems in themselves, not only different operation modes from the whole system. In this scenario, the hybrid system has many different equilibria and some subsystems probably have no equilibrium point. Ignoring these details may lead to oversimplification. The real potential of hybrid automata lies in the capability to capture the dynamics of these kinds of systems: this is the motivation behind this work. More general than switched systems, hybrid automata explicitly consider the influence of the transition from one subsystem to another through guards, as well as impulses in the states represented by reset functions. We here define a framework to deal with multiple isolated equilibria in nonlinear hybrid automata and characterize some stability and dissipativity properties. The conditions proposed in this paper for stability and dissipativity can be automatically checked using recent formal verification techniques for hybrid systems [1].

Dissipativity in switched systems has been studied by means of common storage functions [2] and, with less restriction, multiple storage functions [3]. The expanded results of these are given in [4], [5], [6], and within the framework of differential inclusions [7]. There are also studies of feedback passivity of continuous and discrete-time switched systems [8], [9]. Dissipativity in hybrid automata has not attracted as much attention. Within hybrid systems, dissipativity has been successfully applied to study the asymptotic stability of compact sets in a general class of jump systems (see [10], [11] and references therein), the control of interconnected impulsive systems [12], or the control of impact mechanical systems [13]. The analysis of switched and hybrid systems with multiple equilibria is less common [14], [15], [16]. Our approach differs because we provide an alternative framework for hybrid automata, with reference to complex large-scale systems with different types of discontinuities, multiple isolated equilibria, and non-identical subsystem dynamic structures – which allows having different continuous state space for every subsystem. In this work, we do not consider Zeno equilibria as in [15].

In brief, the contribution of this paper is three-fold. First, we establish a framework within nonlinear hybrid automata to define different types of co-existing equilibrium points. Second, pre-existing stability conditions are adapted to illuminate the co-existence of different types of equilibria by combining common and multiple Lyapunov-like functions. Finally, we identify dissipative parts within a hybrid automaton and give the definition of *group dissipativity* for groups of locations of the hybrid automaton, and *total dissipativity* for the whole hybrid automaton. Dissipativity of the groups of discrete locations will not imply the dissipativity of the whole hybrid automaton. Additional cross-group-coupling conditions are established, and common and multiple storage-like functions are used.

II. Preliminaries

Following [17], a hybrid automaton with inputs and outputs

$$H = (Q, E, \mathcal{X}, \mathcal{U}, \mathcal{Y}, Dom, \mathcal{F}, Init, G, R, h)$$

is a model for a hybrid system with:

- **Discrete locations**: $Q = \{q_1, q_2, \ldots, q_{\mathrm{N_q}}\}$.
- **Continuous state, input and output spaces:** $\mathcal{X} \subseteq \mathbb{R}^{\mathrm{n}}, \mathcal{U} \subseteq \mathbb{R}^{\mathrm{m}}$ and $\mathcal{Y} \subseteq \mathbb{R}^{\mathrm{p}}$.
- **Continuous inputs**: for each $q_{\mathrm{i}} \in Q$, there is one input space $\mathcal{U}_{q_i} \subseteq \mathcal{U}$, and $\mathcal{U} = \bigcup_{q_i \in Q} \mathcal{U}_{q_i}$.
- **Transitions**: $E \subseteq Q \times Q$, with E a finite set of edges.
- **Location domains**: for each $q_{\mathrm{i}} \in Q$, there is one continuous state space $\mathcal{X}_{q_i} \subseteq \mathcal{X}$, with $\bigcup_{q_i \in Q} \mathcal{X}_{q_{\mathrm{i}}} = \mathcal{X}$, and $Dom : Q \to 2^{\mathcal{X}_{q_i}}$. $Dom(q_i) \subseteq \mathcal{X}_{q_i}$.
- **Continuous dynamics**: $\mathcal{F} = \{\mathbf{f}_{\mathrm{q_i}}(\mathbf{x}, \mathbf{u}) \; : \; q_{\mathrm{i}} \in Q\}$ is a collection of vector fields such that $f_{\mathrm{q_i}} : \mathcal{X}_{q_i} \times \mathcal{U}_{q_i} \to \mathcal{X}_{q_i}$. Each $\mathbf{f}_{\mathrm{q_i}}(\mathbf{x}, \cdot)$ is Lipschitz continuous on $\mathcal{X}_{q_i}$ in order to ensure that in each q_i the solution exists and is unique.
- **Set of initial states:** $Init \subseteq \bigcup_{q_i \in Q} q_i \times \mathcal{X}_{\mathrm{q_i}} \subseteq Q \times \mathcal{X}$.
- **Guard maps**: $G : E \to 2^{\mathcal{X}}$.
- **Reset maps**: $R : E \times \mathcal{X} \times \mathcal{U} \to 2^{\mathcal{X}}$. For each $e = (q_{\mathrm{i}}, q_{\mathrm{j}}) \in E$, $\mathbf{x} \in G(e)$ and $\mathbf{u} \in \mathcal{U}_{\mathrm{q_i}}$, $R(e, \mathbf{x}, \mathbf{u}) \subset \mathcal{X}_{\mathrm{q_j}}$.
- **Continuous outputs**: $\mathbf{y} = \mathbf{h}(q_{\mathrm{i}}, \mathbf{x}, \mathbf{u})$, $h : Q \times \mathcal{X}_{q_i} \times \mathcal{U}_{q_i} \to \mathcal{Y}_{q_i}$. For each $q_{\mathrm{i}} \in Q$, there is one output space $\mathcal{Y}_{q_i} \subseteq \mathcal{Y}$, and $\mathcal{Y} = \bigcup_{q_i \in Q} \mathcal{Y}_{q_i}$.

Consider the execution of H, $\phi = (\tau, q, \mathbf{x})$, with hybrid time trajectory $\tau = \{[t_i, t_i']\}_{i=0}^{N} \in \mathcal{T}$, and $\mathcal{T}$ the set of all hybrid time trajectories [18]. We highlight that for all $0 \leq i < N$, $t_i \leq t_i' = t_{i+1}$.

Definition 1. *An* **input sequence of** *H is a collection $\phi_u = (\tau, \mathbf{u})$ with hybrid time trajectory $\tau = \{[t_i, t_i']\}_{i=0}^{N} \in \mathcal{T}$, and the mapping $\mathbf{u} : \tau \to \mathcal{U}$, satisfying*

1) **Initial condition**. $\mathbf{u}(t_0) \in \mathcal{U}_{q(t_0)}$ *with* $(q(t_0), \mathbf{x}(t_0)) \in Init$ *and* $\mathbf{x}(t_0) \in Dom(q(t_0))$.
2) **Continuous evolution**. *For all i: $\forall t \in [t_i, t_i']$, $q(t)$ is constant and $\forall t \in [t_i, t_i')$, $\mathbf{u}(t) \in \mathcal{U}_{q(t)}$ is continuous.*
3) **Discrete transitions**. *For all* $e = (q(t_i'), q(t_{i+1})) \in E$, $i \in \{0, 1, \ldots, N-1\}$*:* $\exists \mathbf{u}(t_{i+1}) \in \mathcal{U}_{q(t_{i+1})}$. ■

Definition 2. *An* **output sequence of** *H is a collection $\phi_y = (\tau, \mathbf{y})$ with hybrid time trajectory $\tau = \{[t_i, t_i']\}_{i=0}^{N} \in \mathcal{T}$, and the mapping $\mathbf{y} : \tau \to \mathcal{Y}$, satisfying*

1) **Initial condition**. $\mathbf{y}(t_0) \in \mathcal{Y}_{q(t_0)}$ *with* $\mathbf{y}(t_0) = \mathbf{h}(q(t_0), \mathbf{x}(t_0), \mathbf{u}(t_0))$, *and* $(q(t_0), \mathbf{x}(t_0)) \in Init$, $\mathbf{x}(t_0) \in Dom(q(t_0))$, $\mathbf{u}(t_0) \in \mathcal{U}_{q(t_0)}$.
2) **Continuous evolution**. *For all i: $\forall t \in [t_i, t_i']$, $q(t)$ is constant, and $\forall t \in [t_i, t_i')$ we have that $\mathbf{y}(t) = \mathbf{h}(q(t), \mathbf{x}(t), \mathbf{u}(t))$, $\mathbf{h}$ is smooth, $\mathbf{y}(t) \in \mathcal{Y}_{q(t)}$, $\mathbf{x}(t) \in Dom(q(t))$, and $\mathbf{u}(t) \in \mathcal{U}_{q(t)}$.*
3) **Discrete transitions**. *For all* $e = (q(t_i'), q(t_{i+1})) \in E$, $i \in \{0, 1, \ldots, N-1\}$*:* $\mathbf{y}(t_i') = \mathbf{h}(q(t_i'), \mathbf{x}(t_i'), \mathbf{u}(t_i'))$ *with* $\mathbf{y}(t_i') \in \mathcal{Y}_{q(t_i')}$, $\mathbf{x}(t_i') = G(e)$ *and* $\mathbf{u}(t_i') \in \mathcal{U}_{q(t_i')}$, *and* $\exists \mathbf{y}(t_{i+1}) \in \mathcal{Y}_{q(t_{i+1})}$ *obtained by using* $\mathbf{x}(t_{i+1}) \in R(e, \mathbf{x}(t_i'), \mathbf{u}(t_i'))$. ■

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E.M. Navarro is with the School of Computer Science, The University of Manchester, Oxford Road, Kilburn Building, Manchester M13 9PL, UK. eva.navarro@cs.man.ac.uk. D.S. Laila is with the Faculty of Engineering and the Environment, The University of Southampton, Highfield, Southampton SO17 1BJ, UK. d.laila@soton.ac.uk

An execution ϕ, an input sequence ϕ_u or an output sequence ϕ_y is *finite* if τ is a finite sequence ending with a closed interval, that is $N < \infty$, $I_{\rm N} = [t_{\rm N}, t'_{\rm N}]$ with $t'_N < \infty$, and is *infinite* if τ is (i) a finite sequence ending with an infinite interval ($N < \infty, I_{\rm N} = [t_{\rm N}, t'_{\rm N}), t'_N = \infty$) or (ii) an infinite sequence ($N = \infty$). The set of executions with initial condition $(q(t_0), \mathbf{x}(t_0))$ is $\mathcal{E}_{(q(t_0),\mathbf{x}(t_0))}$. It is $\mathcal{E}^F_{(q(t_0),\mathbf{x}(t_0))}$ for finite executions or $\mathcal{E}^\infty_{(q(t_0),\mathbf{x}(t_0))}$ for infinite executions.

For any $q_{\rm i} \in Q$, we consider $T|q_{\rm i} = \{t_{\rm qi_1}, t_{\rm qi_2}, \ldots, t_{\rm qi_k}, \ldots, t_{\rm qi_{N_{q_i}}};\ q(t_{\rm qi_k}) = q_{\rm i}, k \in \mathbb{N}\}$, as the sequence of times when the location $q_{\rm i}$ becomes ACTIVE, and $T'|q_{\rm i} = \{t'_{\rm qi_1}, t'_{\rm qi_2}, \ldots, t'_{\rm qi_k}, \ldots, t'_{\rm qi_{M_{q_i}}};\ q(t_{\rm qi_k}) = q_{\rm i}, k \in \mathbb{N}\}$, as the sequence of times when the location $q_{\rm i}$ becomes INACTIVE, with $N_{\rm q_i}$ and $M_{\rm q_i}$ the number of entrances to and exits from $q_{\rm i}$, respectively. For instance, if $t \in [t_{\rm qi_k}, t'_{\rm qi_k}] \in \tau$, $q_{\rm i}$ is active, for the kth time. We also use $T'|^{\rm q_j}_{\rm q_i}$ to denote the sequence of times when $\rm q_i$ becomes inactive to change to another location $\rm q_j$. We define $\mathcal{I}(T|q_{\rm i})$ as the set of time intervals during which location $q_{\rm i}$ is active: that is, $\mathcal{I}(T|q_{\rm i}) = \bigcup_{k=1}^{N_{q_i}} [t_{\rm qi_k}, t'_{\rm qi_k}]$.

Consider the following systems:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)),\ \mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)), &(1)\\ \mathbf{x}(k+1) &= \mathbf{F}(\mathbf{x}(k), \mathbf{u}(k)),\ \mathbf{y}(k) = \mathbf{H}(\mathbf{x}(k), \mathbf{u}(k)), &(2)\end{aligned}$$

with $\mathbf{x} \in \mathbb{R}^{\rm n}$, $\mathbf{u} \in \mathbb{R}^{\rm m}$, $\mathbf{y} \in \mathbb{R}^{\rm p}$ and $\mathbf{f}, \mathbf{h}, \mathbf{F}, \mathbf{H}$ are smooth mappings and maps. The system (1) is dissipative w.r.t. the supply rate function $s(\mathbf{y}, \mathbf{u})$, with $\int_0^t |(s(\mathbf{y}(\sigma), \mathbf{u}(\sigma))| \, d\sigma < \infty, \forall t \geq 0$, if there exists a positive definite storage function $V : \mathbb{R}^{\rm n} \to \mathbb{R}$, such that for any t_0 and any $t_{\rm f} > t_0$, the following relation is satisfied for all $\mathbf{x}(t_0)$ [19]:

$$V(\mathbf{x}(t_{\rm f})) - V(\mathbf{x}(t_0)) \leq \int_{\rm t_0}^{\rm t_f} s(\mathbf{y}(\sigma), \mathbf{u}(\sigma))\, d\sigma,\ \forall (\mathbf{x}, \mathbf{u}). \quad (3)$$

For $V \in \mathcal{C}^1$, inequality (3) is equivalent to [19],

$$\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}) \leq s(h(\mathbf{x}, \mathbf{u}), \mathbf{u}),\ \forall \mathbf{x} \in \mathbb{R}^{\rm n}, \forall \mathbf{u} \in \mathbb{R}^{\rm m}. \quad (4)$$

The system (2) is dissipative w.r.t. the supply rate function s if there exists a positive definite storage function V, such that $\forall x(0)$, $\forall k \in \{0, 1, 2, \ldots\}$ [20]:

$$V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \leq s(\mathbf{y}(k), \mathbf{u}(k)),\ \forall (\mathbf{x}(k), \mathbf{u}(k)). \quad (5)$$

III. A Motivating Example

To illustrate the results in this paper, we consider a simplified model of the torsional behaviour of a conventional vertical oilwell drillstring that has multiple equilibria and is given in [17]. The system may exhibit self-excited stick-slip oscillations depending on the values of the control input to the system, u, and the weight on the bit, $W_{\rm ob}$, which is a varying parameter. The drillstring with u a constant can be modelled as a 5-location hybrid automaton [17].

As also shown in [17], the oscillations in the system can be eliminated using a switching controller that drives the angular velocity of the top-rotary system to a desired value $x_{3\rm r} > 0$. The switching control mechanism is driven by the changing sign of a function $s^{\rm r}(\mathbf{x}, t)$, which is an integral function of the angular velocities. Based on this model, the closed-loop system, given in [17], can be represented by the 15-location hybrid automaton of Fig. 1, with:

$$\begin{aligned}&q_1 = \{slip^+_{\rm b}, slip^+_{\rm r}\},\ q_2 = \{slip^+_{\rm b}, slip^-_{\rm r}\},\ q_3 = \{slip^-_{\rm b}, slip^+_{\rm r}\},\\ &q_4 = \{slip^-_{\rm b}, slip^-_{\rm r}\},\ q_5 = \{slip^+_{\rm b}, stick_{\rm r}\},\ q_6 = \{stick_{\rm b}, stick_{\rm r}\},\\ &q_7 = \{slip^-_{\rm b}, stick_{\rm r}\},\ q_8 = \{stick_{\rm b}, slip^+_{\rm r}\},\ q_9 = \{stick_{\rm b}, slip^-_{\rm r}\},\\ &q_{10} = \{tr^+, slip^+_{\rm r}\},\ q_{11} = \{tr^+, stick_{\rm r}\},\ q_{12} = \{tr^+, slip^-_{\rm r}\},\\ &q_{13} = \{tr^-, slip^+_{\rm r}\},\ q_{14} = \{tr^-, stick_{\rm r}\},\ q_{15} = \{tr^-, slip^-_{\rm r}\}.\end{aligned}$$

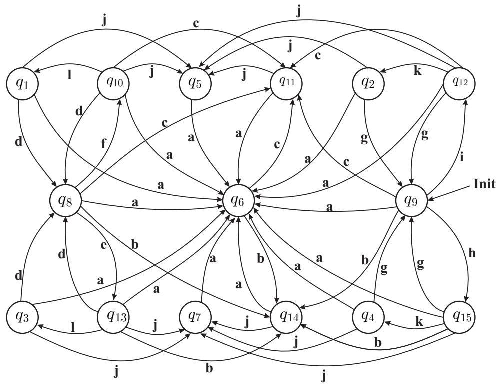

Fig. 1. 15-location hybrid automaton of the closed-loop drillstring.

$stick_{\rm r}$ stands for $S^{\rm r}_0 \equiv \{|s^{\rm r}| \leq \delta\}$, $stick_{\rm b}$ for $G^\delta_0 \equiv \{|x_3| \leq \delta, |u_{\rm eq}(\mathbf{x})| \leq T_{\rm s_b}\}$, $slip^+_{\rm r}$ for $S^{\rm r}_+ \equiv \{s^{\rm r} > \delta\}$, $slip^+_{\rm b}$ for $G_+\{x_3 > \delta\}$, $slip^-_{\rm r}$ for $S^{\rm r}_- \equiv \{s^{\rm r} < -\delta\}$, and $slip^-_{\rm b}$ for $\{x_3 < -\delta\}$; tr^+ denotes $G^\delta_+ \equiv \{|x_3| \leq \delta, u_{\rm eq}(\mathbf{x}) > T_{\rm s_b}\}$, and tr^- denotes $G^\delta_- \equiv \{|x_3| \leq \delta, u_{\rm eq}(\mathbf{x}) < -T_{\rm s_b}\}$.

Note that in the specification of the domains, to avoid numerical problems with zero detection in the simulation, we define a neighbourhood around zero with a small $\delta > 0$.

The letters on the edges represent the 12 guards of H: $\mathbf{a} \Leftrightarrow G^\delta_0 \cap S^{\rm r}_0$, $\mathbf{b} \Leftrightarrow G^\delta_- \cap S^{\rm r}_0$, $\mathbf{c} \Leftrightarrow G^\delta_+ \cap S^{\rm r}_0$, $\mathbf{d} \Leftrightarrow G^\delta_0 \cap S^{\rm r}_+$, $\mathbf{e} \Leftrightarrow G^\delta_- \cap S^{\rm r}_+$, $\mathbf{f} \Leftrightarrow G^\delta_+ \cap S^{\rm r}_+$, $\mathbf{g} \Leftrightarrow G^\delta_0 \cap S^{\rm r}_-$, $\mathbf{h} \Leftrightarrow G^\delta_- \cap S^{\rm r}_-$, $\mathbf{i} \Leftrightarrow G^\delta_+ \cap S^{\rm r}_-$, $\mathbf{j} \Leftrightarrow \{\mathbf{x} \in \mathbb{R}^3 : |x_3| > \delta\} \cap S^{\rm r}_0$, $\mathbf{k} \Leftrightarrow \{\mathbf{x} \in \mathbb{R}^3 : |x_3| > \delta\} \cap S^{\rm r}_-$, $\mathbf{l} \Leftrightarrow \{\mathbf{x} \in \mathbb{R}^3 : |x_3| > \delta\} \cap S^{\rm r}_+$.

IV. Classification of Equilibria

Inspired by [21], we propose several types of equilibria, and split the discrete locations of H into groups, depending on these equilibria (see Fig. 2).

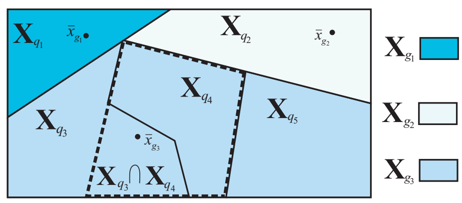

Fig. 2. An example of the division of the state space $\mathcal{X}$ of a hybrid automaton with 5 discrete locations, 3 groups of locations and 3 group equilibria. The unique equilibrium point within group g_3 belongs to q_3 and q_4. Moreover, $\mathcal{X}_{\rm q_3} \bigcap \mathcal{X}_{\rm q_4} \neq \emptyset$.

Definition 3. $\overline{\mathbf{x}}_{\rm q_i} \in \mathbb{R}^n$ *is a* **non-virtual equilibrium** *of a discrete location* $q_i \in Q$ *if:* **(i)** $\exists \overline{\mathbf{u}}_{\rm q_i} \in \mathcal{U}_{\rm q_i}$ *such that* $\mathbf{f}_{\rm q_i}(\overline{\mathbf{x}}_{\rm q_i}, \overline{\mathbf{u}}_{\rm q_i}) = 0$ *and* $\overline{\mathbf{x}}_{\rm q_i} \in {\rm cl}(\mathcal{X}_{\rm q_i})$; **(ii)** $\forall e \in (q_i \times Q) \cap E$ *with* $\overline{\mathbf{x}}_{\rm q_i} \in G(e)$, $R(e, \overline{\mathbf{x}}_{\rm q_i}, \overline{\mathbf{u}}_{\rm q_i}) = \{\overline{\mathbf{x}}_{\rm q_i}\}$. $\overline{\mathbf{x}}_{\rm q_i}$ *is* isolated *if it has a neighbourhood in* $\mathcal{X}_{\rm q_i}$ *which contains no other equilibria. The equilibrium output for* $q_{\rm i}$ *is* $\overline{\mathbf{y}}_{\rm q_i} = \mathbf{h}(q_{\rm i}, \overline{\mathbf{x}}_{\rm q_i}, \overline{\mathbf{u}}_{\rm q_i})$. ■

Definition 4. $\overline{\mathbf{x}}_{\rm q_i} \in \mathbb{R}^n$ *is a* **virtual equilibrium** *of location* $q_i \in Q$ *if* $\exists \overline{\mathbf{u}}_{\rm q_i} \in \mathcal{U}_{\rm q_i}$ *such that* $\mathbf{f}_{\rm q_i}(\overline{\mathbf{x}}_{\rm q_i}, \overline{\mathbf{u}}_{\rm q_i}) = 0$ *and* $\overline{\mathbf{x}}_{\rm q_i} \notin {\rm cl}(\mathcal{X}_{\rm q_i})$, *but* $\overline{\mathbf{x}}_{\rm q_i} \in {\rm cl}(\mathcal{X}_{\rm q_j})$ *for some* $q_j \in Q$, $q_j \neq q_i$. ■

Definition 5. *Let* $N_{\rm q}$ *be the number of discrete locations of the hybrid automaton* H. *Consider a partition* $P \subset Q$, *with* $P = \{g_1, g_2, \ldots, g_{N_g}\}$ *and* $N_{\rm g} \leq N_{\rm q}$, *such that* $\bigcup_{i=1}^{N_g} g_i = Q$ *and* $\bigcap_{i=1}^{N_g} g_i = \emptyset$. *Let* N_{g_i} *be the number of locations within each group* g_i, *with* $1 \leq N_{g_i} \leq N_{\rm q}$ *for all* i. *We associate with each group*

g_i *a subset of the state space* $\mathcal{X}_{\mathrm{g_i}}$ *such that* $\bigcup_{q_j \in g_i} \mathcal{X}_{\mathrm{q_j}} = \mathcal{X}_{\mathrm{g_i}}$, $\bigcup_{i=1}^{N_g} \mathcal{X}_{\mathrm{g_i}} = \mathcal{X}$ *and* $\bigcap_{i=1}^{N_g} \mathcal{X}_{\mathrm{g_i}} = \emptyset$. *Then,* $\overline{\mathbf{x}}_{\mathrm{g_i}} \in \mathcal{X}_{\mathrm{g_i}}$ *is a* **group equilibrium** *for* H *if:*
(i) *There exists at least one* $q_{\mathrm{i}} \in g_{\mathrm{i}}$ *for which* $\overline{\mathbf{x}}_{\mathrm{g_i}}$ *is an isolated non-virtual equilibrium for* q_{i}*;*
(ii) $\overline{\mathbf{x}}_{\mathrm{g_i}}$ *is the unique non-virtual equilibrium point for the discrete locations of the group* g_{i}*;*
(iii) $\overline{\mathbf{x}}_{\mathrm{g_i}}$ *is not a non-virtual equilibrium for any discrete location outside group* g_{i}*;*
(iv) *for* $e \in (g_{\mathrm{i}} \times g_{\mathrm{i}}) \cap E$ *with* $\overline{\mathbf{x}}_{g_i} \in G(e)$, $R(e, \overline{\mathbf{x}}_{g_i}, \cdot) = \{\overline{\mathbf{x}}_{g_i}\}$*;*
(v) *for all* $e \in (g_{\mathrm{i}} \times (Q \setminus g_{\mathrm{i}})) \cap E$, $\overline{\mathbf{x}}_{\mathrm{g_i}} \notin G(e)$. ■

Remark 1. *Condition (i) of Definition 5 allows a shared non-virtual equilibrium for several discrete locations* $q_i \in g_i$*. This also allows locations with no equilibrium within the same group. Note that* $\bigcap_{q_j \in g_i} \mathcal{X}_{\mathrm{q_j}}$ *can be a non-empty set, allowing the situation shown in Fig. 2.*

For instance, for the 15-location hybrid automaton shown in Fig. 1, we have:

- **Virtual equilibrium** for q_1 and q_3 (for any value of $x_{3\mathrm{r}}$, η and λ), and for q_8, q_{10}, q_{13} (only if $x_{3\mathrm{r}} = \frac{\eta}{\lambda}$): $\overline{\mathbf{v}}_1 = (x_{3\mathrm{r}} - \frac{\eta}{\lambda},\ \frac{c_{\mathrm{b}}(x_{3\mathrm{r}} - \eta/\lambda)}{k_{\mathrm{t}}} + \frac{T_{\mathrm{f_b}}(x_{3\mathrm{r}} - \eta/\lambda)}{k_{\mathrm{t}}},\ x_{3\mathrm{r}} - \frac{\eta}{\lambda})^{\mathrm{T}}$.
- **Virtual equilibrium** for q_2: $\overline{\mathbf{v}}_2 = (x_{3\mathrm{r}} - \frac{\eta}{\lambda},\ \frac{c_{\mathrm{b}}(x_{3\mathrm{r}} + \eta/\lambda)}{k_{\mathrm{t}}} + \frac{T_{\mathrm{f_b}}(x_{3\mathrm{r}} + \eta/\lambda)}{k_{\mathrm{t}}},\ x_{3\mathrm{r}} + \frac{\eta}{\lambda})^{\mathrm{T}}$.
- **Group equilibrium** within location q_5: $\overline{\mathbf{x}}_{\mathrm{g_1}} = (x_{3\mathrm{r}},\ \frac{c_{\mathrm{b}} x_{3\mathrm{r}} + T^{+}_{\mathrm{f_b}}(x_{3\mathrm{r}}, W_{\mathrm{ob}})}{k_{\mathrm{t}}},\ x_{3\mathrm{r}})^{\mathrm{T}}$.

Locations $q_6, q_7, q_{11}, q_{14}, q_4, q_9, q_{12}$ and q_{15} have no equilibrium point. All the discrete locations of the 15-location hybrid automaton are grouped together in g_1.

V. Total stability of group equilibria in hybrid automata

The stability conditions presented in this section are adapted from [22], [14], [23], [24] for nonlinear hybrid automata. Whilst in these works, a different Lyapunov function is considered for each subsystem, we have a different common Lyapunov function for each group of locations. We define a *ball* of radius $r > 0$ around a point $p \in \mathbb{R}^n$ as $B(r, \mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{p}\| < r\}$, with $\|\cdot\|$ the Euclidean 2-norm.

Definition 6. *Given* $\overline{\mathbf{x}}_{\mathrm{g_j}}$ *a group equilibrium of* H. $\overline{\mathbf{x}}_{\mathrm{g_j}} \in \mathcal{X}_{\mathrm{q_i}}$*, for some* $q_{\mathrm{i}} \in g_{\mathrm{j}} \subseteq Q$*, is:*
(i) stable *iff for all* $\epsilon > 0$ *there exists* $\delta(\epsilon) > 0$ *such that* $\forall \phi = (\tau, q, \mathbf{x}) \in \mathcal{E}_{(q(t_0), \mathbf{x}(t_0))}$,

$$\mathbf{x}(t_0) \in B(\delta, \overline{\mathbf{x}}_{\mathrm{g_j}}) \cap \mathcal{X}_{\mathrm{q}(t_0)} \ \Rightarrow\ \mathbf{x}(t) \in B(\epsilon, \overline{\mathbf{x}}_{\mathrm{g_j}}),\ \forall\, t \in \tau.$$

(ii) attractive *iff there exists* $\delta_1 > 0$ *such that* $\forall \phi = (\tau, q, \mathbf{x}) \in \mathcal{E}^{\infty}_{(q(t_0), \mathbf{x}(t_0))}$, $t_\infty = \sum_i (t'_i - t_i)$,

$$\mathbf{x}(t_0) \in B(\delta_1, \overline{\mathbf{x}}_{\mathrm{g_j}}) \cap \mathcal{X}_{\mathrm{q}(t_0)} \ \Rightarrow\ \lim_{t \to t_\infty} \mathbf{x}(t) = \overline{\mathbf{x}}_{\mathrm{g_j}}.$$

(iii) asymptotically stable *if it is stable and attractive.* ■

Stability is defined for any executions, whether finite or infinite, but attractivity is defined for infinite executions only since it is a property of convergence to a certain value.

Definition 7. *Consider any group of locations* g_{j} *within* H*, and its associated group equilibrium* $\overline{\mathbf{x}}_{\mathrm{g_j}} \in \mathcal{X}_{\mathrm{g_j}}$*, with* $\mathcal{X}_{\mathrm{g_j}} = \bigcup_{q_i \in g_j} \mathcal{X}_{\mathrm{q_i}}$*. A function* $V_{\mathrm{g_j}} : \mathcal{X}_{\mathrm{g_j}} \to \mathbb{R}$ *such that: (i)* $V_{\mathrm{g_j}}$ *is continuously differentiable within every* $q_{\mathrm{i}} \in g_{\mathrm{j}}$*; (ii)* $V_{\mathrm{g_j}}(\mathbf{x}) > 0, \forall \mathbf{x} \in \mathcal{X}_{\mathrm{g_j}} \setminus \{\overline{\mathbf{x}}_{\mathrm{g_j}}\}$*; (iii)* $V_{\mathrm{g_j}}(\mathbf{x}) = 0 \Leftrightarrow \mathbf{x} = \overline{\mathbf{x}}_{\mathrm{g_j}}$*, is referred to as* **group candidate Lyapunov function** *for the group* g_{j} *of* H. ■

Assumption 1. H *switches from one location to another a finite number of times* S_H *on any finite time interval. For any finite time* T*, with* $t_0 < T \leq t_{\mathrm{N}}$*, and* $T \in I_{\mathrm{i}}$ *for some time interval* $I_{\mathrm{i}} \in \tau$*, there exists* $K_{\mathrm{T}} \in \mathbb{Z}^+$*, such that during the time interval* $[t_0, T]$*,* $S_H \leq K_{\mathrm{T}}$.

Let define $Q_{\mathrm{i}} := \{q_{\mathrm{i}} \in g_{\mathrm{j}} : \overline{\mathbf{x}}_{\mathrm{g_j}}$ is a group equilibrium and non-virtual equilibrium of $q_{\mathrm{i}}\}$, $T|Q_{\mathrm{i}} = \{t_{\mathrm{Qi_1}}, t_{\mathrm{Qi_2}},\ \ldots, t_{\mathrm{Qi_k}}, \ldots\}$ as the sequence of times when any location within Q_{i} becomes active and $|T|Q_{\mathrm{i}}| = N_{\mathrm{in,Qi}}$. Consider $\Omega_{\mathrm{g_j}}$ as the set within $\mathcal{X}_{\mathrm{g_j}}$ where $V_{\mathrm{g_j}}$ is a Lyapunov function:

$$\begin{aligned} &\Omega_{\mathrm{q_i}} = \left\{ \mathbf{x} \in \mathcal{X}_{\mathrm{q_i}} : q_{\mathrm{i}} \in Q_{\mathrm{i}},\ \frac{\partial V_{\mathrm{g_j}}(\mathbf{x})}{\partial \mathbf{x}} f_{\mathrm{q_i}}(\mathbf{x}, \cdot) \leq 0 \right\}, \\ &\Omega_{\mathrm{g_j}} = \{\overline{\mathbf{x}}_{\mathrm{g_j}}\} \bigcup_{\mathrm{q_i} \in \mathrm{Q_i}} \Omega_{\mathrm{q_i}}. \end{aligned} \qquad (6)$$

Now, we state a result on total stability of a group equilibrium against all co-existing equilibria in H for a particular case of hybrid automata, in which executions start at a location whose domain does not contain the domain of attraction of other group equilibrium different from $\overline{\mathbf{x}}_{\mathrm{g_j}}$.

Definition 8. *Consider a hybrid automaton* H*. Assume there is a group equilibrium of* H*,* $\overline{\mathbf{x}}_{\mathrm{g_j}}$*, associated with the group* g_{j}*, with* $\overline{\mathbf{x}}_{\mathrm{g_j}}$ *a non-virtual equilibrium of* q_{i}*, for at least one* $q_{\mathrm{i}} \in g_{\mathrm{j}}$*. Then,* H *is an* **Init-constrained hybrid automaton** *if*

$$(q(t_0), \mathbf{x}(t_0)) \in \left((Q_{\mathrm{i}} \times \Omega_{\mathrm{g_j}}) \bigcap \mathit{Init}\right),\ \mathbf{u}(t_0) \in \bigcup_{q_{\mathrm{i}} \in Q_{\mathrm{i}}} \mathcal{U}_{\mathrm{q_i}}.$$

for all executions $\phi = (\tau, q, \mathbf{x}) \in \mathcal{E}_{(q(t_0), \mathbf{x}(t_0))}$*, and all input sequences* $\phi_u = (\tau, \mathbf{u})$. ■

Theorem 1. *(Total stability of a group equilibrium of* H*) Consider an Init-constrained hybrid automaton* H*. Let* N_{g} *be the number of groups of locations in* H*. Consider* $\mathcal{I}(T|q_{\mathrm{i}})$ *as the set of time intervals during which location* q_{i} *is active. Let define* $T|g_{\mathrm{j}} = \{t_{\mathrm{gj_1}}, t_{\mathrm{gj_2}}, \ldots, t_{\mathrm{gj_k}}, \ldots\}$ *as the sequence of times when any location of group* g_{j} *becomes active, and* $T|g_{\mathrm{s}} = \{t_{\mathrm{gs_1}}, t_{\mathrm{gs_2}}, \ldots, t_{\mathrm{gs_k}}, \ldots\}$ *as the sequence of times when any location* q_{s}*, that does not belong to group* g_{j} *becomes active, with* $q_{\mathrm{s}} \in g_{\mathrm{s}}$*,* $g_{\mathrm{s}} \subset Q \setminus g_{\mathrm{j}}$*. Let Assumption 1 hold.* $\overline{\mathbf{x}}_{\mathrm{g_j}}$ *is* **totally stable** *if there exist* N_{g} *group candidate Lyapunov functions* $\{V_{\mathrm{g_1}}, \ldots, V_{\mathrm{g_{N_g}}}\}$ *such that* $\forall \phi = (\tau, q, \mathbf{x}) \in \mathcal{E}_{(q(t_0), \mathbf{x}(t_0))}$ *and* $\forall \phi_u = (\tau, \mathbf{u})$*, the following conditions hold:*
(i) Condition related to locations within g_{j} **for which** $\overline{\mathbf{x}}_{\mathrm{g_j}}$ **is a non-virtual equilibrium**. $\forall t \in \bigcup_{\mathrm{q_i} \in \mathrm{Q_i}} \mathcal{I}(T|q_{\mathrm{i}})$*,* $\forall q_{\mathrm{i}} \in Q_{\mathrm{i}}$*, and for* $\mathbf{x}(t) \in \mathit{Dom}(q_{\mathrm{i}})$*,* $\mathbf{u}(t) \in \mathcal{U}_{\mathrm{q_i}}$ *:*

$$\frac{\partial V_{\mathrm{g_j}}(\mathbf{x}(t))}{\partial \mathbf{x}} f_{\mathrm{q_i}}(\mathbf{x}(t), \mathbf{u}(t)) \leq 0. \qquad (7)$$

(ii) Condition related to the entrances to any q_{i} **for which** $\overline{\mathbf{x}}_{\mathrm{g_j}}$ **is a non-virtual equilibrium**. $\forall k \in \{1, \ldots, N_{\mathrm{in,Qi}} - 1\}$*:*

$$V_{\mathrm{g_j}}(\mathbf{x}(t_{\mathrm{Qi_{k+1}}})) - V_{\mathrm{g_j}}(\mathbf{x}(t_{\mathrm{Qi_k}})) \leq 0,\ \textit{with}\ t_{\mathrm{Qi_k}}, t_{\mathrm{Qi_{k+1}}} \in T|Q_{\mathrm{i}}. \quad (8)$$

(iii) Conditions related to locations of H **for which** $\overline{\mathbf{x}}_{\mathrm{g_j}}$ **is not a non-virtual equilibrium**. *For every* $q_{\mathrm{r}} \in Q \setminus Q_{\mathrm{i}}$*:*

a. *There exists* $\lambda > 0$ *such that for every time* q_{r} *is active:*

$$V_{\mathrm{g_r}}(\mathbf{x}(t)) \leq \lambda V_{\mathrm{g_j}}(\mathbf{x}(t^*_{\mathrm{Qi_k}})), \quad \textit{for}\ t \in [t_{\mathrm{qr_k}}, t'_{\mathrm{qr_k}}], \qquad (9)$$

with $q_{\mathrm{r}} \in g_{\mathrm{r}}$*, and* $t^*_{\mathrm{Qi_k}} = \max_k \{t_{\mathrm{Qi_k}} \in T|Q_{\mathrm{i}} : t_{\mathrm{Qi_k}} \leq t_{\mathrm{qr_k}}\}$ *the last switch-on time before entering* q_{r} *of any location* $q_{\mathrm{i}} \in Q_{\mathrm{i}}$*, with* $t_0 \leq t^*_{\mathrm{Qi_k}} \leq t_{\mathrm{qr_k}}$*. Note that if* $q_{\mathrm{r}} \in g_{\mathrm{j}} \setminus Q_{\mathrm{i}}$*, we substitute* $V_{\mathrm{g_r}}$ *by* $V_{\mathrm{g_j}}$ *in* (9)*;*

b. $\forall t \in \mathcal{I}(T|q_{\mathrm{r}}), \mathbf{x}(t)$ *does not exhibit finite escape times, i.e.,* $\nexists t, \|\mathbf{x}(t)\| \to \infty$ *as* $t \to t_{\mathrm{e}} < \infty$.

(iv) Cross-group-coupling conditions when entering g_j from any other group.

a. *For every group $g_s \subset Q \setminus g_j$, if $\exists t_{gs_k}$, with $t_{gs_k} \in T|g_s$:*

$$V_{g_j}(\mathbf{x}(t_{gj_k})) \leq V_{g_s}(\mathbf{x}(t^*_{gs_k})), \tag{10}$$

$\forall t_{gj_k} \in T|g_j$ for which any location of g_j becomes active coming from any location of group g_s, with $t^*_{gs_k} = \max_k\{t_{gs_k} \in T|g_s : t_{gs_k} \leq t_{gj_k}\}$ the last time when a location within g_s became active before entering any location of g_j.

b. *Condition on resets. For every $(q_s, q_j) \in E$, with $q_s \in g_s \subset Q \setminus g_j$ and $q_j \in g_j$:*

$$V_{g_j}(\mathbf{x}^+) - V_{g_s}(\mathbf{x}(t'_{qs_k})) \leq 0, \tag{11}$$

for all $t'_{qs_k} \in T'|^{q_j}_{q_s}$ such that $\mathbf{x}(t'_{qs_k}) \in G(q_s, q_j)$, $\mathbf{x}^+ = R(q_s, q_j, \mathbf{x}(t'_{qs_k}), \mathbf{u}(t'_{qs_k}))$, $\mathbf{u}(t'_{qs_k}) \in \mathcal{U}_{q_s}$, with t'_{qs_k} the time when q_s becomes inactive to change to any $q_j \in g_j$. ■

The total stability conditions can be strengthened to total asymptotic stability as stated next.

Theorem 2. *(Total asymptotic stability of a group equilibrium of H) In addition to conditions of Theorem 1, if (7) is a strict inequality and one of the following conditions is satisfied for all $\phi = (\tau, q, \mathbf{x}) \in \mathcal{E}^\infty_{(q(t_0),\mathbf{x}(t_0))}$ and their associated infinite input sequences:* **(i)** *condition (8) is substituted by the fact that for all $q_i \in Q_i$ the sequence $\{V_{g_j}(\mathbf{x}(t_{Qi_k}))\}$ converges to zero as $k \to \infty$; or* **(ii)** *for some $q_i \in Q_i$, the set $T|q_i$ is finite and $q(t) = q_i$ for all $t \in [t_{qi_{N_{q_i}}}, \infty)$, with $t_{qi_{N_{q_i}}} \in T|q_i$ the last switch-on time for q_i, then $\overline{\mathbf{x}}_{g_j}$ is a* **totally asymptotically stable** *equilibrium of H in the sense of Lyapunov.* ■

Remark 2. *The case of having the same non-virtual equilibrium point for all the locations of H is a special case of our grouping of locations.*

VI. Dissipative groups within a hybrid automaton and total dissipativity

We introduce the notion of *group dissipativity* for each group of locations of H and *total dissipativity* for the whole hybrid automaton. Two key differences from previous works are: 1) multiple isolated equilibria are present in the system, and some locations might have no equilibrium, 2) due to the nature of hybrid automata, jumps between locations at switching times are considered. We use multiple storage functions, different for each group, whilst a group of locations will share a common storage function.

To study the dissipativity in hybrid automata, we can exploit the dissipativity of groups of locations to state the dissipativity of the whole hybrid automaton. This is done by establishing appropriate input and output relationships between the groups of locations.

Definition 9. *Let $T|g_j$ be the sequence of times when any location of group g_j becomes active and $N_{\mathrm{in},gj}$ the number of these entrances to any location $q_j \in g_j$. Under Assumption 1, a group of locations g_j of H is* **group dissipative** *w.r.t. the supply functions $s_{q_j}(\mathbf{y}, \mathbf{u})$ defined for each $q_j \in g_j$, if there exists a group storage-like function $V_{g_j}(\mathbf{x})$ satisfying the conditions of a group candidate Lyapunov function, such that for all executions $\phi = (\tau, q, \mathbf{x}) \in \mathcal{E}_{(q(t_0),\mathbf{x}(t_0))}$, and all input and output sequences $\phi_u = (\tau, \mathbf{u})$, $\phi_y = (\tau, \mathbf{y})$, the followings hold:*

(i) Condition on discrete locations. *$\forall q_j \in g_j$ and $\forall t \in \tau$ for which $q(t) = q_j$:*

$$\frac{\partial V_{g_j}(\mathbf{x}(t))}{\partial \mathbf{x}} f_{q_j}(\mathbf{x}(t), \mathbf{u}(t)) \leq s_{q_j}(\mathbf{y}(t), \mathbf{u}(t)), \tag{12}$$

with $\mathbf{x}(t) \in Dom(q_j)$, $\mathbf{u}(t) \in \mathcal{U}_{q_j}$, $\mathbf{y}(t) \in \mathcal{Y}_{q_j}$.

(ii) Condition related to the entrances to locations within g_j. *$\forall k \in \{1, \ldots, N_{\mathrm{in},gj} - 1\}$:*

$$V_{g_j}(\mathbf{x}(t_{gj_{k+1}})) - V_{g_j}(\mathbf{x}(t_{gj_k})) \leq s(\mathbf{y}(t_{gj_k}), \mathbf{u}(t_{gj_k})), \tag{13}$$

with $t_{gj_k}, t_{gj_{k+1}} \in T|g_j$, and $s = s_{q_j}(\mathbf{y}(t_{gj_k}), \mathbf{u}(t_{gj_k}))$ if $q_j \in g_j$ became active at t_{gj_k}. ■

Condition (12) is equivalent to the dissipation inequality (4), and must be verified for all time intervals that every discrete location in g_j is active. Furthermore, condition (13) generalizes for dissipative systems the passivity conditions given in [3]. This is an extra condition which guarantees that the switching sequence only adds a bounded amount of energy into the system. Since sequences of values of the group storage functions are considered in discrete time, it is more appropriate to use the dissipation inequality for discrete-time systems (5). The time gap between consecutive entrances to any q_j in g_j includes the time when q_j is active and inactive. Thus, (13) considers the energy stored by the location while inactive, and is bounded by the supplied energy calculated at the most recent entrance to q_j.

Inspired by the results of [4] and [5], condition (12) can be relaxed as follows.

Definition 10. *Let N_{q_j} and M_{q_j} be the number of entrances to and exits from q_j, respectively; and consider the set-up of Definition 9. A group of locations g_j of H is* **weakly group dissipative** *w.r.t. the supply functions $s_{q_j}(\mathbf{y}, \mathbf{u})$ of all $q_j \in g_j$, if:*

(i) Condition on discrete locations. *$\forall q_j \in g_j$, and $\forall t \in \tau$ for which $q(t) = q_j$, the followings hold instead of (12):*

a. *If $N_{q_j} = M_{q_j}$:*

$$\sum_{\forall q_j \in g_j} \sum_{k=1}^{N_{q_j}} \Bigg[V_{g_j}(\mathbf{x}(t'_{qj_k})) - V_{g_j}(\mathbf{x}(t_{qj_k})) - \\ - \int_{t_{qj_k}}^{t'_{qj_k}} s_{q_j}(\mathbf{y}(\tau), \mathbf{u}(\tau))\, d\tau \Bigg] \leq 0. \tag{14}$$

b. *If $N_{q_j} > M_{q_j}$, then the execution has entered q_j and remains there until terminal time t'_N, with $t'_N \geq t_{qj_{N_{q_j}}}$, $t_{qj_{N_{q_j}}} \in T|q_j$ is the last switch-on instant time of q_j, and:*

$$\sum_{\forall q_j \in g_j} \sum_{k=1}^{N_{q_j}} \Bigg[V_{g_j}(\mathbf{x}(t'_{qj_k})) - V_{g_j}(\mathbf{x}(t_{qj_k})) - \\ - \int_{t_{qj_k}}^{t'_{qj_k}} s_{q_j}(\mathbf{y}(\tau), \mathbf{u}(\tau))\, d\tau \Bigg] + \sum_{\forall q_j \in g_j} \mathcal{B}_{q_j} \leq 0, \tag{15}$$

with

$$\mathcal{B}_{q_j} = V_{g_j}(\mathbf{x}(t'_N)) - V_{g_j}(\mathbf{x}(t_{qj_{N_{q_j}}})) - \\ - \int_{t_{qj_{N_{q_j}}}}^{t'_N} s_{q_j}(\mathbf{y}(\tau), \mathbf{u}(\tau))\, d\tau.$$

(ii) Condition related to the entrances to any $q_j \in g_j$. *Condition (13) is satisfied.* ■

With conditions (14) and (15), during the time intervals any q_j is active, the balance of stored and supplied energy of g_j is allowed to grow for all $q_j \in g_j$, and the dissipativity of each group is obtained as the total balance of stored and supplied energy when each $q_j \in g_j$ is active.

To expand group dissipativity to the whole hybrid automaton, we define total dissipativity.

Definition 11. *Let $T|g_{\rm s} = \{t_{{\rm gs}_1}, t_{{\rm gs}_2}, \ldots, t_{{\rm gs}_{\rm k}}, \ldots\}$ be the sequence of times when any location of group $g_{\rm s}$ becomes active. Under Assumption 1, the hybrid automaton H is* **totally dissipative** *w.r.t. a set of supply functions $\{s_{{\rm q}_1}(\mathbf{y},\mathbf{u}), \ldots, s_{{\rm q}_{\rm N_q}}(\mathbf{y},\mathbf{u})\}$, if there exists a set of group storage-like functions $\{V_{\rm g1}(\mathbf{x}), \ldots, V_{{\rm g}_{\rm N_g}}(\mathbf{x})\}$ satisfying the conditions of a group candidate Lyapunov function, such that for all $\phi = (\tau, q, \mathbf{x}) \in \mathcal{E}_{(q(t_0),\mathbf{x}(t_0))}$, and all $\phi_u = (\tau, \mathbf{u})$, $\phi_y = (\tau, \mathbf{y})$:*
(i) Condition on groups of locations. *All $g_{\rm j}$ are group dissipative, with $1 \leq j \leq N_{\rm g}$.*
(ii) Cross-group coupling when changing from one group $g_{\rm s}$ to another $g_{\rm j}$. *$\forall g_{\rm j}$ and $\forall g_{\rm s} \subset Q \setminus g_{\rm j}$, if $\exists t_{{\rm gs}_{\rm k}}$, with $t_{{\rm gs}_{\rm k}} \in T|g_{\rm s}$, such that:*

a. *$\forall t_{{\rm gj}_{\rm k}} \in T|g_{\rm j}$ for which any location of $g_{\rm j}$ becomes active coming from any location of group $g_{\rm s}$, with $t^*_{{\rm gs}_{\rm k}} = \max_k\{t_{{\rm gs}_{\rm k}} \in T|g_{\rm s} : t_{{\rm gs}_{\rm k}} \leq t_{{\rm gj}_{\rm k}}\}$ the last time when a location $q_{\rm s}$ within any group $g_{\rm s}$ was active before entering any location of $g_{\rm j}$:*

$$V_{\rm g_j}(\mathbf{x}(t_{{\rm gj}_{\rm k}})) - V_{\rm g_s}(\mathbf{x}(t^*_{{\rm gs}_{\rm k}})) \leq s_{\rm q_s}\big(\mathbf{y}(t^*_{{\rm gs}_{\rm k}}), \mathbf{u}(t^*_{{\rm gs}_{\rm k}})\big). \quad (16)$$

b. *Condition on resets. For every $q_{\rm j} \in g_{\rm j}$ and every $q_{\rm s} \in g_{\rm s}$ such that $(q_{\rm s}, q_{\rm j}) \in E$:*

$$V_{\rm g_j}(\mathbf{x}^+) - V_{\rm g_s}(\mathbf{x}(t'_{{\rm qs}_{\rm k}})) \leq s_{\rm q_s}\big(\mathbf{y}(t'_{{\rm qs}_{\rm k}}), \mathbf{u}(t'_{{\rm qs}_{\rm k}})\big), \quad (17)$$

for all $t'_{{\rm qs}_{\rm k}} \in T'|^{\rm q_j}_{\rm q_s}$ such that $\mathbf{x}(t'_{{\rm qs}_{\rm k}}) \in G(q_{\rm s}, q_{\rm j})$, $\mathbf{x}^+ = R(q_{\rm s}, q_{\rm j}, \mathbf{x}(t'_{{\rm qs}_{\rm k}}), \mathbf{u}(t'_{{\rm qs}_{\rm k}}))$, $\mathbf{y}(t'_{{\rm qs}_{\rm k}}) \in \mathcal{Y}_{\rm q_s}$, $\mathbf{u}(t'_{{\rm qs}_{\rm k}}) \in \mathcal{U}_{\rm q_s}$, with $t'_{{\rm qs}_{\rm k}}$ the time when $q_{\rm s}$ becomes inactive to change to any $q_{\rm j} \in g_{\rm j}$. ■

Note that conditions (16) and (17) are required to take into account the impact of the stored and supplied energy at one group in the past, on the stored energy in the most recently active group of locations. These conditions are only checked when we change group.

Definition 12. *The hybrid automaton H is* **weakly totally dissipative** *under all the assumptions considered in Definition 11 if one of the following conditions holds:*
(i) *At least one group location is weakly group dissipative, and the others are group dissipative.*
(ii) *All the group locations are group dissipative, and instead of (16) and/or (17) we have:*

- *$\forall g_{\rm j}, \forall g_{\rm s} \neq g_{\rm j}$, with $t^*_{{\rm gs}_{\rm k}}$ as in Definition 11, the following holds instead of (16):*

$$\sum_{\substack{\forall t_{{\rm gj}_{\rm k}}: g_{\rm j} \text{ becomes} \\ \text{active after } g_{\rm s}}} \Big[V_{\rm g_j}(\mathbf{x}(t_{{\rm gj}_{\rm k}})) - V_{\rm g_s}(\mathbf{x}(t^*_{{\rm gs}_{\rm k}})) - s_{\rm q_s}\big(\mathbf{y}(t^*_{{\rm gs}_{\rm k}}), \mathbf{u}(t^*_{{\rm gs}_{\rm k}})\big) \Big] \leq 0; \quad (18)$$

- *$\forall q_{\rm j} \in g_{\rm j}, \forall g_{\rm j}, \forall g_{\rm s} \neq g_{\rm j}$, $\forall q_{\rm s} \in g_{\rm s}$ such that $(q_{\rm s}, q_{\rm j}) \in E$, the following holds instead of (17):*

$$\sum_{(\forall q_{\rm s}:\,(q_s,q_j)\in E)} \sum_{\rm k=1}^{\rm M^{q_j}_{q_s}} \Big[V_{\rm g_j}(\mathbf{x}^+) - V_{\rm g_j}(\mathbf{x}(t'_{{\rm qs}_{\rm k}})) - s_{\rm q_s}\big(\mathbf{y}(t'_{{\rm qs}_{\rm k}}), \mathbf{u}(t'_{{\rm qs}_{\rm k}})\big) \Big] \leq 0, \quad (19)$$

$\forall t'_{{\rm qs}_{\rm k}} \in T'|^{\rm q_j}_{\rm q_s}$ such that $\mathbf{x}(t'_{{\rm qs}_{\rm k}}) \in G(q_{\rm s}, q_{\rm j})$, $\mathbf{x}^+ = R(q_{\rm s}, q_{\rm j}, \mathbf{x}(t'_{{\rm qs}_{\rm k}}), \mathbf{u}(t'_{{\rm qs}_{\rm k}}))$, $\mathbf{y} \in \mathcal{Y}_{\rm q_s}, \mathbf{u} \in \mathcal{U}_{\rm q_s}$, with $M^{\rm q_j}_{\rm q_s}$ the number of exits from $q_{\rm s}$ to $q_{\rm j}$, and $t'_{{\rm qs}_{\rm k}}$ the time when $q_{\rm s}$ changes to $q_{\rm j} \in g_{\rm j}$. ■

Definition 13. *A group of locations $g_{\rm j}$ of H is* **group passive** *if it is group dissipative w.r.t. the supply functions $s_{\rm q_i}(\mathbf{y},\mathbf{u}) = \mathbf{y}^{\rm T}\mathbf{u}$, $\forall q_{\rm i} \in g_{\rm j}$. The hybrid automaton H is* **totally passive** *if it is totally dissipative with $s_{\rm q_i}(\mathbf{y},\mathbf{u}) = \mathbf{y}^{\rm T}\mathbf{u}$, $\forall q_{\rm i} \in Q$.* ■

From the classical theory of dissipative systems [20], it is well known that dissipative systems exhibit some stability properties for some specific inputs, outputs and supply functions. Similarly, from our dissipativity definitions, we can conclude some of the stability properties given in Section V for particular classes of hybrid automata. For example, if a hybrid automaton H which is Init-constrained to some group $g_{\rm j}$ is totally dissipative with respect to supply functions $\{s_{{\rm q}_1}(\mathbf{y},\mathbf{u}), \ldots, s_{{\rm q}_{\rm N_q}}(\mathbf{y},\mathbf{u})\}$ which are zero for zero inputs (i.e., $\forall q_{\rm i}$, $s_{\rm q_i}(\mathbf{h}(q_{\rm i}, \mathbf{x}, 0), 0) = 0, \forall \mathbf{x} \in \mathcal{X}_{\rm q_i}$) then for some $q_{\rm i} \in g_{\rm j}$, the equilibrium point of the zero-input dynamics $\dot{\mathbf{x}} = f_{\rm q_i}(\mathbf{x}, 0)$ coincides with the group equilibrium point $\overline{\mathbf{x}}_{\rm g_j}$, and $\overline{\mathbf{x}}_{\rm g_j}$ is totally stable.

VII. Dissipativity properties in the example

We will check if the 15-location hybrid automaton is totally passive w.r.t. $s_{\rm q_i} = y_{\rm q_i} u$, with $y_{\rm q_i} = x_1 - x_3$ for $i \in \{1,2,3,4,5,7\}$, and $y_{\rm q_i} = x_1$ for all other locations. x_1 and x_3 are the angular velocities of the top-rotary system and the bit, respectively, and x_2 is the difference between the two angular displacements. For all the locations, $u = W_{\rm ob} R_{\rm b} \left[\mu_{\rm c_b} + (\mu_{\rm s_b} - \mu_{\rm c_b}) \exp^{-\frac{\gamma_{\rm b}}{v_{\rm f}} x_3}\right]$, with $R_b > 0$ the bit radius, $\mu_{s_b}, \mu_{c_b} \in (0,1)$ the static and Coloumb friction coefficients associated with the bit, $0 < \gamma_b < 1$ and $v_f > 0$. Note that there is only one group of locations, and we choose the following group storage-like function: $V_{\rm g_1} = \frac{1}{2}\left[(x_1 - \overline{x}_{\rm g1,1})^2 + (x_2 - \overline{x}_{\rm g1,2})^2 + \ +(x_3 - \overline{x}_{\rm g1,3})^2\right]$, with $\overline{\mathbf{x}}_{\rm g1}$ as given at the end of Section IV. The parameters used are: $J_r = 2122\,kg\,m^2$, $J_b = 471.9698\,kg\,m^2$, $R_b = 0.155\,m$, $k_t = 861.5336\,N\,m/rad$, $c_t = 172.3067\,N\,m\,s/rad$, $c_r = 425\,N\,m\,s/rad$, $c_b = 50\,N\,m\,s/rad$, $\mu_{c_b} = 0.5$, $\mu_{s_b} = 0.8$, $\delta = 10^{-6}$, $\gamma_b = 0.9$, $v_f = 1$. We will show how $x_{\rm 3r}$, λ and $W_{\rm ob}$ affect the passivity of H.

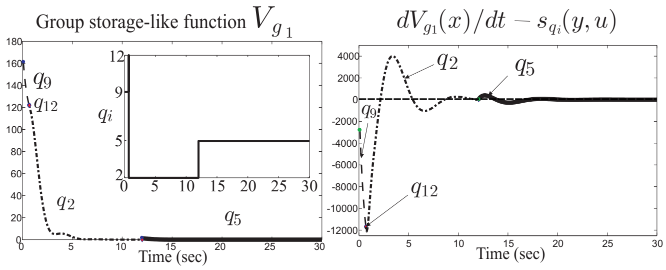

Fig. 3. The 15-location hybrid automaton is weakly totally passive for: $\lambda = 0.9$, $W_{\rm ob} = 20\,{\rm kN}$ and $x_{\rm 3r} = 12\,{\rm rad/s}$.

Fig. 3 shows the case where H is not totally passive but only weakly totally passive. In Fig 4, we show the case in which the trajectories of H converge to the group equilibrium point in q_5, although it is non-totally passive. For the non-passive locations, condition (12) does not hold; and for q_2 condition (14) of weak passivity also fails. Finally, for the stick-slip situation shown in Fig. 5, H is not totally passive because conditions (12) and (16) do not hold for q_5.

VIII. Conclusions

We propose a new classification of equilibria in hybrid automata and based on this, a partition of the continuous state space is given. Some stability properties of co-existing isolated equilibria for a type of hybrid automata are given, leading to what is called total stability. Finally, group and total dissipativity properties of hybrid automata are proposed. The example illustrates how the use of hybrid automata can be useful in the analysis of complex hybrid systems.

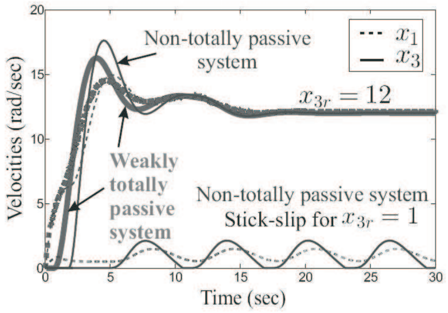

Fig. 4. The 15-location hybrid automaton is: 1) weakly totally passive with $\lambda = 0.9$, $W_{\rm ob} = 20\,{\rm kN}$ and $x_{3{\rm r}} = 12\,{\rm rad/s}$, in grey thick lines; 2) non-totally passive, but with trajectories converging to $\overline{x}_{{\rm g}_1}$ with $\lambda = 0.9$, $W_{\rm ob} = 65\,{\rm kN}$ and $x_{3{\rm r}} = 12\,{\rm rad/s}$; 3) non-totally passive with stick-slip oscillations with $\lambda = 0.9$, $W_{\rm ob} = 20\,{\rm kN}$ and $x_{3{\rm r}} = 1\,{\rm rad/s}$.

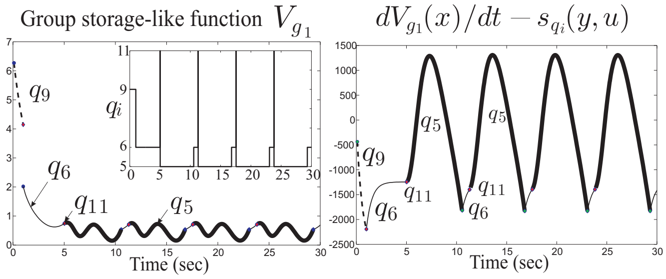

Fig. 5. Stick-slip situation for the 15-location hybrid automaton: H is not totally passive for small $x_{3{\rm r}}$'s.

APPENDIX

Proof of Theorem 1. If the conditions for total stability for $\overline{\mathbf{x}}_{\rm g_j}$ hold, the stability of $\overline{\mathbf{x}}_{\rm g_j}$, as given in Definition 6, is guaranteed. We divide the sketch of the proof into four cases.

Case 1. The executions only visit one location for which $\overline{\mathbf{x}}_{\rm g_j}$ is a non-virtual equilibrium. From condition (7), the proof corresponds to the well-known proof of stability for smooth systems.

Case 2. The executions travel along locations (all or some locations) within $g_{\rm j}$ for which $\overline{\mathbf{x}}_{\rm g_j}$ is a non-virtual equilibrium. With conditions (7) and (8), the proof follows the same arguments as Branicky's proof of Theorem 2.3 in [22] considering the common candidate Lyapunov function $V_{\rm g_j}(\mathbf{x})$ for all the locations within $g_{\rm j}$. In addition, (7) and (8) ensure that after a reset in every change of location, $V_{\rm g_j}$ is decreased/maintained – just as in [24]. Then, if $\mathbf{x}$ starts in $B(\delta, \overline{\mathbf{x}}_{\rm g_j})$ just before the reset, then $\mathbf{x}^+$ starts in $B(\delta, \overline{\mathbf{x}}_{\rm g_j})$, and hence, $\mathbf{x}^+$ stays in $B(\epsilon, \overline{\mathbf{x}}_{\rm g_j})$ at the time of the reset, with $\delta \in (0, \epsilon)$. In brief, $V_{\rm g_j}$ decreases or is maintained as time progresses.

Case 3. The executions switch between locations, within the same group, that do not contain $\overline{\mathbf{x}}_{\rm g_j}$ and the locations for which $\overline{\mathbf{x}}_{\rm g_j}$ is a non-virtual equilibrium. Since $\big(q(t_0), \mathbf{x}(t_0)\big) \in \big((Q_{\rm i} \times \Omega_{\rm g_j}) \bigcap Init\big)$, we will always start at $\Omega_{\rm g_j}$, a discrete location whose domain satisfies condition (7). Bearing in mind conditions of **Case 2** and conditions **(iii)**.a and **(iii)**.b of our Theorem 1, the proof follows the same arguments as given in Theorem 1 of [23] for the case of having a common Lyapunov function and switchings with resets.

Case 4. The executions travel along locations from different groups with different group equilibria. In addition to conditions of the three cases above, the cross-group-coupling conditions (10) and (11), one for each different group $g_{\rm s}$ visited, are considered. Notice that in this case, condition (9) is applied to any location in any group of H for which $\overline{\mathbf{x}}_{\rm g_j}$ is not a non-virtual equilibrium. Following similar arguments as those given in the proof of Theorem 1 and Corollary 1 of [14], we can prove that $\mathbf{x}$ does not move away from the union of the closed level sets for all the group candidate Lyapunov functions of H. In addition, from conditions of Theorem 1, it is ensured that $\exists t^* > 0$, such that $\forall t \geq t^*$, $\mathbf{x}(t)$ remains close to $\overline{\mathbf{x}}_{\rm g}$, if $\mathbf{x}(t)$ starts close to $\overline{\mathbf{x}}_{\rm g}$.

Proof of Theorem 2. It follows similar steps to the proof of Theorem 1.

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