K-theory for group C^*-algebras

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Introduction

These notes are based on a lecture course given by the first author in the Sedano Winter School on K-theory held in Sedano, Spain, on January 22-27th of 2007. They aim at introducing K-theory of C^*-algebras, equivariant K-homology and KK-theory in the context of the Baum-Connes conjecture.

We start by giving the main definitions, examples and properties of C^*-algebras in Section 1. A central construction is the reduced C^*-algebra of a locally compact, Hausdorff, second countable group G. In Section 2 we define K-theory for C^*-algebras, state the Bott periodicity theorem and establish the connection with Atiyah-Hirzebruch topological K-theory.

Our main motivation will be to study the K-theory of the reduced C^*-algebra of a group G as above. The Baum-Connes conjecture asserts that these K-theory groups are isomorphic to the equivariant K-homology groups of a certain G-space, by means of the index map. The G-space is the universal example for proper actions of G, written $\underline{E}G$. Hence we procceed by discussing proper actions in Section 3 and the universal space $\underline{E}G$ in Section 4.

Equivariant K-homology is explained in Section 5. This is an equivariant version of the dual of Atiyah-Hirzebruch K-theory. Explicitly, we define the groups $K_j^G(X)$ for $j = 0, 1$ and X a proper G-space with compact, second countable quotient $G \backslash X$. These are quotients of certain equivariant K-cycles by homotopy, although the precise definition of homotopy is postponed. We then address the problem of extending the definition to $\underline{E}G$, whose quotient by the G-action may not be compact.

In Section 6 we concentrate on the case when G is a discrete group, and in Section 7 on the case G compact. In Section 8 we introduce KK-theory for the first time. This theory, due to Kasparov, is a generalization of both K-theory of C^*-algebras and K-homology. Here we define $KK^j_G(A, \mathbb{C})$ for a separable C^*-algebra A and $j = 0, 1$, although we again postpone the exact definition of homotopy. The already defined $K^G_j(X)$ coincides with this group when $A = C_0(X)$.

At this point we introduce a generalization of the conjecture called the Baum-Connes conjecture with coefficients, which consists in adding coefficients in a G-C^*-algebra (Section 9). To fully describe the generalized conjecture we need to introduce Hilbert modules and the reduced crossed-product (Section 10), and to define KK-theory for pairs of C^*-algebras. This is done in the non-equivariant situation in Section 11 and in the equivariant setting in Section 12. In addition we give at this point the missing definition of homotopy. Finally, using equivariant KK-theory, we can insert coefficients in equivariant K-homology, and then extend it again to $\underline{E}G$.

The only ingredient of the conjecture not yet accounted for is the index map. It is defined in Section 13 via the Kasparov product and descent maps in KK-theory. We finish with a brief exposition of the history of K-theory and a discussion of Karoubi's conjecture, which symbolizes the unity of K-theory, in Section 14.

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1 C^*-algebras

We start with some definitions and basic properties of C^*-algebras. Good references for C^*-algebra theory are [1], [15], [40] or [42].

1.1 Definitions

Definition 1. *A* Banach algebra *is an (associative, not necessarily unital) algebra A over $\mathbb{C}$ with a given norm $\| \ \|$*

$$\| \ \| : A \longrightarrow [0, \infty)$$

such that A is a complete normed algebra, that is, for all $a, b \in A$, $\lambda \in \mathbb{C}$,

1. $\|\lambda a\| = |\lambda| \|a\|$,
2. $\|a + b\| \leq \|a\| + \|b\|$,
3. $\|a\| = 0 \Leftrightarrow a = 0$,
4. $\|ab\| \leq \|a\| \|b\|$,

5. *every Cauchy sequence is convergent in* A *(with respect to the metric* $d(a,b) = \|a - b\|$*).*

A C^*-algebra is a Banach algebra with an involution satisfying the *C^*-algebra identity*.

Definition 2. *A* C^*-algebra $A = (A, \| \ \|, *)$ *is a Banach algebra* $(A, \| \ \|)$ *with a map* $* : A \to A, a \mapsto a^*$ *such that for all* $a, b \in A$, $\lambda \in \mathbb{C}$

1. $(a+b)^* = a^* + b^*$,
2. $(\lambda a)^* = \overline{\lambda} a^*$,
3. $(ab)^* = b^* a^*$,
4. $(a^*)^* = a$,
5. $\|aa^*\| = \|a\|^2$ *(*C^*-algebra identity*).*

Note that in particular $\|a\| = \|a^*\|$ for all $a \in A$: for $a = 0$ this is clear; if $a \neq 0$ then $\|a\| \neq 0$ and $\|a\|^2 = \|aa^*\| \leq \|a\|\|a^*\|$ implies $\|a\| \leq \|a^*\|$, and similarly $\|a^*\| \leq \|a\|$.

A C^*-algebra is *unital* if it has a multiplicative unit $1 \in A$. A *sub-C^*-algebra* is a non-empty subset of A which is a C^*-algebra with the operations and norm given on A.

Definition 3. *A* $*$-homomorphism *is an algebra homomorphism* $\varphi : A \to B$ *such that* $\varphi(a^*) = (\varphi(a))^*$*, for all* $a \in A$.

Proposition 1. *If* $\varphi : A \to B$ *is a* $*$*-homomorphism then* $\|\varphi(a)\| \leq \|a\|$ *for all* $a \in A$*. In particular,* φ *is a (uniformly) continuous map.*

For a proof see, for instance, [42, Thm. 1.5.7].

1.2 Examples

We give three examples of C^*-algebras.

Example 1. Let X be a Hausdorff, locally compact topological space. Let $X^+ = X \cup \{p_\infty\}$ be its one-point compactification. (Recall that X^+ is Hausdorff if and only if X is Hausdorff and locally compact.)

Define the C^*-algebra

$$C_0(X) = \left\{ \alpha : X^+ \to \mathbb{C} \,|\, \alpha \text{ continuous}, \, \alpha(p_\infty) = 0 \right\} ,$$

with operations: for all $\alpha, \beta \in C_0(X), p \in X^+, \lambda \in \mathbb{C}$

$$\begin{aligned}
(\alpha + \beta)(p) &= \alpha(p) + \beta(p), \\
(\lambda\alpha)(p) &= \lambda\alpha(p), \\
(\alpha\beta)(p) &= \alpha(p)\beta(p), \\
\alpha^*(p) &= \overline{\alpha(p)}, \\
\|\alpha\| &= \sup_{p \in X} |\alpha(p)| \,.
\end{aligned}$$

Note that if X is compact Hausdorff, then

$$C_0(X) = C(X) = \{\alpha : X \to \mathbb{C} \,|\, \alpha \text{ continuous}\}\,.$$

Example 2. Let H be a Hilbert space. A Hilbert space is *separable* if it admits a countable (or finite) orthonormal basis. (We shall deal with separable Hilbert spaces unless explicit mention is made to the contrary.)

Let $\mathcal{L}(H)$ be the set of bounded linear operators on H, that is, linear maps $T : H \to H$ such that

$$\|T\| = \sup_{\|u\|=1} \|Tu\| < \infty\,,$$

where $\|u\| = \langle u, u\rangle^{1/2}$. It is a complex algebra with

$$\begin{aligned}(T + S)u &= Tu + Su,\\ (\lambda T)u &= \lambda(Tu),\\ (TS)u &= T(Su),\end{aligned}$$

for all $T, S \in \mathcal{L}(H)$, $u \in H$, $\lambda \in \mathbb{C}$. The norm is the operator norm $\|T\|$ defined above, and T^* is the adjoint operator of T, that is, the unique bounded operator such that

$$\langle Tu, v\rangle = \langle u, T^*v\rangle$$

for all $u, v \in H$.

Example 3. Let $\mathcal{L}(H)$ be as above. A bounded operator is *compact* if it is a norm limit of operators with finite-dimensional image, that is,

$$\mathcal{K}(H) = \{T \in \mathcal{L}(H) \,|\, T \text{ compact operator}\} = \overline{\{T \in \mathcal{L}(H) \,|\, \dim_{\mathbb{C}} T(H) < \infty\}}\,,$$

where the overline denotes closure with respect to the operator norm. $\mathcal{K}(H)$ is a sub-C^*-algebra of $\mathcal{L}(H)$. Moreover, it is an ideal of $\mathcal{L}(H)$ and, in fact, the only norm-closed ideal except 0 and $\mathcal{L}(H)$.

1.3 The reduced C^*-algebra of a group

Let G be a topological group which is locally compact, Hausdorff and second countable (i.e. as a topological space it has a countable basis). There is a C^*-algebra associated to G, called the *reduced C^*-algebra of G*, defined as follows.

Remark 1. We need G to be locally compact and Hausdorff to guarantee the existence of a Haar measure. The countability assumption makes the Hilbert space $L^2(G)$ separable and also avoids some technical difficulties when later defining Kasparov's KK-theory.

Fix a left-invariant Haar measure dg on G. By left-invariant we mean that if $f: G \to \mathbb{C}$ is continuous with compact support then

$$\int_G f(\gamma g)dg = \int_G f(g)dg \qquad \text{for all } \gamma \in G\,.$$

Define the Hilbert space L^2G as

$$L^2G = \Big\{ u : G \to \mathbb{C} \mid \int_G |u(g)|^2 dg < \infty \Big\},$$

with scalar product

$$\langle u, v \rangle = \int_G \overline{u(g)}\, v(g) dg$$

for all $u, v \in L^2G$.

Let $\mathcal{L}(L^2G)$ be the C^*-algebra of all bounded linear operators $T : L^2G \to L^2G$. On the other hand, define

$$C_cG = \{ f : G \to \mathbb{C} \mid f \text{ continuous with compact support} \}\,.$$

It is an algebra with

$$(f+h)(g) = f(g) + h(g),$$
$$(\lambda f)(g) = \lambda f(g),$$

for all $f, h \in C_cG$, $\lambda \in \mathbb{C}$, $g \in G$, and multiplication given by *convolution*

$$(f * h)(g_0) = \int_G f(g)h(g^{-1}g_0)\, dg \quad \text{for all } g_0 \in G.$$

Remark 2. When G is discrete, $\int_G f(g)dg = \sum_G f(g)$ is a Haar measure, C_cG is the complex group algebra $\mathbb{C}[G]$ and $f * h$ is the usual product in $\mathbb{C}[G]$.

There is an injection of algebras

$$\begin{array}{ccccc} 0 \longrightarrow & C_cG & \longrightarrow & \mathcal{L}(L^2G) \\ & f & \mapsto & T_f \end{array}$$

where

$$T_f(u) = f * u \qquad\qquad u \in L^2G\,,$$
$$(f * u)(g_0) = \int_G f(g)u(g^{-1}g_0)dg \qquad\qquad g_0 \in G\,.$$

Note that C_cG is not necessarily a sub-C^*-algebra of $\mathcal{L}(L^2G)$ since it may not be complete. We define $C^*_r(G)$, the *reduced C^*-algebra of* G, as the norm closure of C_cG in $\mathcal{L}(L^2G)$:

$$C^*_r(G) = \overline{C_cG} \subset \mathcal{L}(L^2G).$$

Remark 3. There are other possible completions of C_cG. This particular one, i.e. $C^*_r(G)$, uses only the left regular representation of G (cf. [42, Chapter 7]).

1.4 Two classical theorems

We recall two classical theorems about C^*-algebras. The first one says that any C^*-algebra is (non-canonically) isomorphic to a C^*-algebra of operators, in the sense of the following definition.

Definition 4. *A subalgebra A of $\mathcal{L}(H)$ is a* C^*-algebra of operators *if*

1. *A is closed with respect to the operator norm;*
2. *if $T \in A$ then the adjoint operator $T^* \in A$.*

That is, A is a sub-C^*-algebra of $\mathcal{L}(H)$, for some Hilbert space H.

Theorem 1 (I. Gelfand and V. Naimark). *Any C^*-algebra is isomorphic, as a C^*-algebra, to a C^*-algebra of operators.*

The second result states that any commutative C^*-algebra is (canonically) isomorphic to $C_0(X)$, for some topological space X.

Theorem 2 (I. Gelfand). *Let A be a commutative C^*-algebra. Then A is (canonically) isomorphic to $C_0(X)$ for X the space of maximal ideals of A.*

Remark 4. The topology on X is the *Jacobson topology* or *hull-kernel topology* [40, p. 159].

Thus a non-commutative C^*-algebra can be viewed as a 'non-commutative, locally compact, Hausdorff topological space'.

1.5 The categorical viewpoint

Example 1 gives a functor between the category of locally compact, Hausdorff, topological spaces and the category of C^*-algebras, given by $X \mapsto C_0(X)$. Theorem 2 tells us that its restriction to commutative C^*-algebras is an equivalence of categories,

$$\begin{pmatrix} \text{commutative} \\ C^*\text{-algebras} \end{pmatrix} \simeq \begin{pmatrix} \text{locally compact, Hausdorff,} \\ \text{topological spaces} \end{pmatrix}^{op}$$
$$C_0(X) \longleftarrow X$$

On one side we have C^*-algebras and $*$-homorphisms, and on the other locally compact, Hausdorff topological spaces with morphisms from Y to X being continuous maps $f\colon X^+ \to Y^+$ such that $f(p_\infty) = q_\infty$. (The symbol *op* means the opposite or dual category, in other words, the functor is contravariant.)

Remark 5. This is not the same as continuous proper maps $f\colon X \to Y$ since we do not require that the map $f\colon X^+ \to Y^+$ maps X to Y.

2 K-theory of C^*-algebras

In this section we define the K-theory groups of an arbitrary C^*-algebra. We first give the definition for a C^*-algebra with unit and then extend it to the non-unital case. We also discuss Bott periodicity and the connection with topological K-theory of spaces. More details on K-theory of C^*-algebras is given in Section 3 of Cortiñas' notes [12], including a proof of Bott periodicity. Other references are [40], [43] and [50].

Our main motivation is to study the K-theory of $C_r^*(G)$, the reduced C^*-algebra of G. From Bott periodicity, it suffices to compute $K_j\left(C_r^*(G)\right)$ for $j = 0, 1$. In 1980, Paul Baum and Alain Connes conjectured that these K-theory groups are isomorphic to the *equivariant K-homology* (Section 5) of a certain G-space. This G-space is the *universal example for proper actions of* G (Sections 3 and 4), written $\underline{E}G$. Moreover, the conjecture states that the isomorphism is given by a particular map called the *index map* (Section 13).

Conjecture 1 (P. Baum and A. Connes, 1980). Let G be a locally compact, Hausdorff, second countable, topological group. Then the index map

$$\mu : K_j^G\left(\underline{E}G\right) \longrightarrow K_j\left(C_r^*(G)\right) \quad j = 0, 1$$

is an isomorphism.

2.1 Definition for unital C^*-algebras

Let A be a C^*-algebra with unit 1_A. Consider $GL(n, A)$, the group of invertible n by n matrices with coefficients in A. It is a topological group, with topology inherited from A. We have a standard inclusion

$$GL(n, A) \hookrightarrow GL(n+1, A)$$
$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \mapsto \begin{pmatrix} a_{11} & \dots & a_{1n} & 0 \\ \vdots & \dots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & 0 \\ 0 & \dots & 0 & 1_A \end{pmatrix} .$$

Define $GL(A)$ as the direct limit with respect to these inclusions

$$GL(A) = \bigcup_{n=1}^{\infty} GL(n, A) .$$

It is a topological group with the *direct limit topology*: a subset θ is open if and only if $\theta \cap GL(n, A)$ is open for every $n \geq 1$. In particular, $GL(A)$ is a topological space, and hence we can consider its homotopy groups.

Definition 5 (K-theory of a unital C^*-algebra).

$$K_j(A) = \pi_{j-1}\left(GL(A)\right) \quad j = 1, 2, 3, \dots$$

Finally, we define $K_0(A)$ as the *algebraic* K-theory group of the ring A, that is, the Grothendieck group of finitely generated (left) projective A-modules (cf. [12, Remark 2.1.9]),

$$K_0(A) = K_0^{\mathrm{alg}}(A)\,.$$

Remark 6. Note that $K_0(A)$ only depends on the ring structure of A and so we can 'forget' the norm and the involution. The definition of $K_1(A)$ does require the norm but not the involution, so in fact we are defining K-theory of Banach algebras with unit. Everything we say in 2.2 below, including Bott periodicity, is true for Banach algebras.

2.2 Bott periodicity

The fundamental result is *Bott periodicity.* It says that the homotopy groups of $GL(A)$ are periodic modulo 2 or, more precisely, that the double loop space of $GL(A)$ is homotopy equivalent to itself,

$$\Omega^2 GL(A) \simeq GL(A)\,.$$

As a consequence, the K-theory of the C^*-algebra A is periodic modulo 2

$$K_j(A) = K_{j+2}(A) \quad j \geq 0.$$

Hence from now on we will only consider $K_0(A)$ and $K_1(A)$.

2.3 Definition for non-unital C^*-algebras

If A is a C^*-algebra without a unit, we formally adjoin one. Define $\widetilde{A} = A \oplus \mathbb{C}$ as a complex algebra with multiplication, involution and norm given by

$$\begin{aligned} (a,\lambda)\cdot(b,\mu) &= (ab + \mu a + \lambda b, \lambda\mu), \\ (a,\lambda)^* &= (a^*, \overline{\lambda}), \\ \|(a,\lambda)\| &= \sup_{\|b\|=1} \|ab + \lambda b\|\,. \end{aligned}$$

This makes $\widetilde{A}$ a unital C^*-algebra with unit $(0,1)$. We have an exact sequence

$$0 \longrightarrow A \longrightarrow \widetilde{A} \longrightarrow \mathbb{C} \longrightarrow 0.$$

Definition 6. *Let A be a non-unital C^*-algebra. Define $K_0(A)$ and $K_1(A)$ as*

$$\begin{aligned} K_0(A) &= \ker\left(K_0(\widetilde{A}) \to K_0(\mathbb{C})\right) \\ K_1(A) &= K_1(\widetilde{A}). \end{aligned}$$

This definition agrees with the previous one when A has a unit. It also satisfies Bott periodicity (see Cortiñas' notes [12, 3.2]).

Remark 7. Note that the C^*-algebra $C^*_r(G)$ is unital if and only if G is discrete, with unit the Dirac function on 1_G.

Remark 8. There is algebraic K-theory of rings (see [12]). Althought a C^*-algebra is in particular a ring, the two K-theories are different; algebraic K-theory does not satisfy Bott periodicity and K_1 is in general a quotient of K_1^{alg}. We shall compare both definitions in Section 14.3 (see also [12, Section 7]).

2.4 Functoriality

Let A, B be C^*-algebras (with or without units), and $\varphi : A \to B$ a $*$-homomorphism. Then φ induces a homomorphism of abelian groups

$$\varphi_* : K_j(A) \longrightarrow K_j(B) \quad j = 0, 1.$$

This makes $A \mapsto K_j(A)$, $j = 0, 1$, covariant functors from C^*-algebras to abelian groups [43, Sections 4.1 and 8.2].

Remark 9. When A and B are unital and $\varphi(1_A) = 1_B$, the map φ_* is the one induced by $GL(A) \to GL(B)$, $(a_{ij}) \mapsto (\varphi(a_{ij}))$ on homotopy groups.

2.5 More on Bott periodicity

In the original article [9], Bott computed the stable homotopy of the classical groups and, in particular, the homotopy groups $\pi_j(GL(n, \mathbb{C})$ when $n \gg j$.

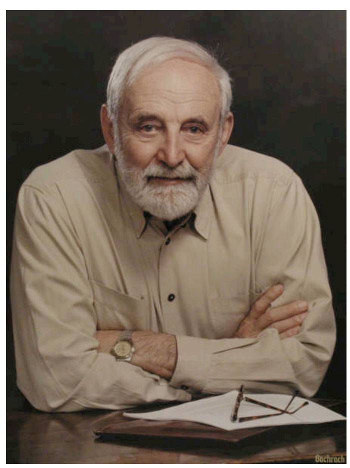


Fig. 1. Raoul Bott

Theorem 3 (R. Bott [9]). *The homotopy groups of $GL(n,\mathbb{C})$ are*

$$\pi_j\left(GL(n,\mathbb{C})\right) = \begin{cases} 0 & j \text{ even} \\ \mathbb{Z} & j \text{ odd} \end{cases}$$

for all $j = 0, 1, 2, \ldots, 2n-1$.

As a corollary of the previous theorem, we obtain the K-theory of $\mathbb{C}$, considered as a C^*-algebra.

Theorem 4 (R. Bott).

$$K_j(\mathbb{C}) = \begin{cases} \mathbb{Z} & j \text{ even,} \\ 0 & j \text{ odd.} \end{cases}$$

Sketch of proof. Since $\mathbb{C}$ is a field, $K_0(\mathbb{C}) = K_0^{alg}(\mathbb{C}) = \mathbb{Z}$. By the polar decomposition, $GL(n,\mathbb{C})$ is homotopy equivalent to $U(n)$. The homotopy long exact sequence of the fibration $U(n) \to U(n+1) \to S^{2n+1}$ gives $\pi_j(U(n)) = \pi_j(U(n+1))$ for all $j \leq 2n+1$. Hence $K_j(\mathbb{C}) = \pi_{j-1}(GL(\mathbb{C})) = \pi_{j-1}(GL(2j-1,\mathbb{C}))$ and apply the previous theorem.

Remark 10. Compare this result with $K_1^{alg}(\mathbb{C}) = \mathbb{C}^*$ (since $\mathbb{C}$ is a field, see [12, Ex. 3.1.6]). Higher algebraic K-theory groups for $\mathbb{C}$ are only partially understood.

2.6 Topological *K*-theory

There is a close connection between K-theory of C^*-algebras and topological K-theory of spaces.

Let X be a locally compact, Hausdorff, topological space. Atiyah and Hirzebruch [3] defined abelian groups $K^0(X)$ and $K^1(X)$ called *topological K-theory with compact supports*. For instance, if X is compact, $K^0(X)$ is the Grothendieck group of complex vector bundles on X.

Theorem 5. *Let X be a locally compact, Hausdorff, topological space. Then*

$$K^j(X) = K_j\left(C_0(X)\right), \ j = 0, 1.$$

Remark 11. This is known as Swan's theorem when $j = 0$ and X compact.

In turn, topological K-theory can be computed up to torsion via a Chern character. Let X be as above. There is a *Chern character* from topological K-theory to rational cohomology with compact supports

$$ch : K^j(X) \longrightarrow \bigoplus_{l \geq 0} H_c^{j+2l}(X;\mathbb{Q}), \quad j = 0, 1.$$

Here the target cohomology theory $H_c^*(-;\mathbb{Q})$ can be Čech cohomology with compact supports, Alexander-Spanier cohomology with compact supports or representable Eilenberg-MacLane cohomology with compact supports.

This map becomes an isomorphism when tensored with the rationals.

Theorem 6. *Let X be a locally compact, Hausdorff, topological space. The Chern character is a rational isomorphism, that is,*

$$K^j(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \bigoplus_{l \geq 0} H_c^{j+2l}(X;\mathbb{Q})\,, \quad j = 0, 1$$

is an isomorphism.

Remark 12. This theorem is still true for singular cohomology when X is a locally finite CW-complex.

3 Proper G-spaces

In the following three sections, we will describe the left-hand side of the Baum-Connes conjecture (Conjecture 1). The space $\underline{E}G$ appearing on the topological side of the conjecture is the *universal example for proper actions for G*. Hence we will start by studying proper G-spaces.

Recall the definition of G-space, G-map and G-homotopy.

Definition 7. *A G-*space *is a topological space X with a given continuous action of G*

$$G \times X \longrightarrow X.$$

*A G-*map *is a continuous map $f : X \to Y$ between G-spaces such that*

$$f(gp) = gf(p) \text{ for all } (g, p) \in G \times X.$$

*Two G-maps $f_0, f_1 : X \to Y$ are G-*homotopic *if they are homotopic through G-maps, that is, there exists a homotopy $\{f_t\}_{0 \leq t \leq 1}$ with each f_t a G-map.*

We will require proper G-spaces to be Hausdorff and paracompact. Recall that a space X is *paracompact* if every open cover of X has a locally finite open refinement or, alternatively, a locally finite partition of unity subordinate to any given open cover.

Remark 13. Any metrizable space (i.e. there is a metric with the same underlying topology) or any CW-complex (in its usual CW-topology) is Hausdorff and paracompact.

Definition 8. *A G-space X is* proper *if*

- *X is Hausdorff and paracompact;*

- *the quotient space $G\backslash X$ (with the quotient topology) is Hausdorff and paracompact;*
- *for each $p \in X$ there exists a triple (U, H, ρ) such that*
  1. *U is an open neighborhood of p in X with $gu \in U$ for all $(g, u) \in G \times U$;*
  2. *H is a compact subgroup of G;*
  3. *$\rho : U \to G/H$ is a G-map.*

Note that, in particular, the stabilizer stab(p) is a closed subgroup of a conjugate of H and hence compact.

Remark 14. The converse is not true in general; the action of $\mathbb{Z}$ on S^1 by an irrational rotation is free but it is not a proper $\mathbb{Z}$-space.

Remark 15. If X is a *G-CW-complex* then it is a proper G-space (even in the weaker definition below) if and only if all the cell stabilizers are compact, see Thm. 1.23 in [31].

Our definition is stronger than the usual definition of proper G-space, which requires the map $G \times X \to X \times X$, $(g, x) \mapsto (gx, x)$ to be proper, in the sense that the pre-image of a compact set is compact. Nevertheless, both definitions agree for locally compact, Hausdorff, second countable G-spaces.

Proposition 2 (J. Chabert, S. Echterhoff, R. Meyer [11]). *If X is a locally compact, Hausdorff, second countable G-space, then X is proper if and only if the map*

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (gx, x) \end{aligned}$$

is proper.

Remark 16. For a more general comparison among these and other definitions of proper actions see [7].

4 Classifying space for proper actions

Now we are ready for the definition of the space $\underline{E}G$ appearing in the statement of the Baum-Connes Conjecture. Most of the material in this section is based on Sections 1 and 2 of [5].

Definition 9. *A* universal example for proper actions of G, *denoted $\underline{E}G$, is a proper G-space such that:*

- *if X is any proper G-space, then there exists a G-map $f : X \to \underline{E}G$ and any two G-maps from X to $\underline{E}G$ are G-homotopic.*

$\underline{E}G$ exists for every topological group G [5, Appendix 1] and it is unique up to G-homotopy, as follows. Suppose that $\underline{E}G$ and $(\underline{E}G)'$ are both universal examples for proper actions of G. Then there exist G-maps

$$f : \underline{E}G \longrightarrow (\underline{E}G)'$$
$$f' : (\underline{E}G)' \longrightarrow \underline{E}G$$

and $f' \circ f$ and $f \circ f'$ must be G-homotopic to the identity maps of $\underline{E}G$ and $(\underline{E}G)$' respectively.

The following are equivalent axioms for a space Y to be $\underline{E}G$ [5, Appendix 2].

1. Y is a proper G-space.
2. If H is any compact subgroup of G, then there exists $p \in Y$ with $hp = p$ for all $h \in H$.
3. Consider $Y \times Y$ as a G-space via $g(y_0, y_1) = (gy_0, gy_1)$, and the maps

$$\rho_0, \rho_1 \colon Y \times Y \longrightarrow Y$$
$$\rho_0(y_0, y_1) = y_0\,, \quad \rho_1(y_0, y_1) = y_1\,.$$

   Then ρ_0 and ρ_1 are G-homotopic.

Remark 17. It is possible to define a universal space for any family of (closed) subgroups of G closed under conjugation and finite intersections [33]. Then $\underline{E}G$ is the universal space for the family of compact subgroups of G.

Remark 18. The space $\underline{E}G$ can always be assumed to be a G-CW-complex. Then there is a homotopy characterization: a proper G-CW-complex X is an $\underline{E}G$ if and only if for each compact subgroup H of G the fixed point subcomplex X^H is contractible (see [33]).

Examples

1. If G is compact, $\underline{E}G$ is just a one-point space.
2. If G is a Lie group with finitely many connected components then $\underline{E}G = G/H$, where H is a maximal compact subgroup (i.e. maximal among compact subgroups).
3. If G is a p-adic group then $\underline{E}G = \beta G$ the affine Bruhat-Tits building for G. For example, $\beta SL(2, \mathbb{Q}_p)$ is the $(p+1)$-regular tree, that is, the unique tree with exactly $p+1$ edges at each vertex (see Figure 3) (cf. [47]).
4. If Γ is an arbitrary (countable) discrete group, there is an explicit construction,

$$\underline{E}\Gamma = \Big\{ f \colon \Gamma \to [0,1] \,\Big|\, f \text{ finite support }, \sum_{\gamma \in \Gamma} f(\gamma) = 1 \Big\}\,,$$

   that is, the space of all finite probability measures on Γ, topologized by the metric $d(f,h) = \sqrt{\sum_{\gamma \in \Gamma} |f(\gamma) - h(\gamma)|^2}$.

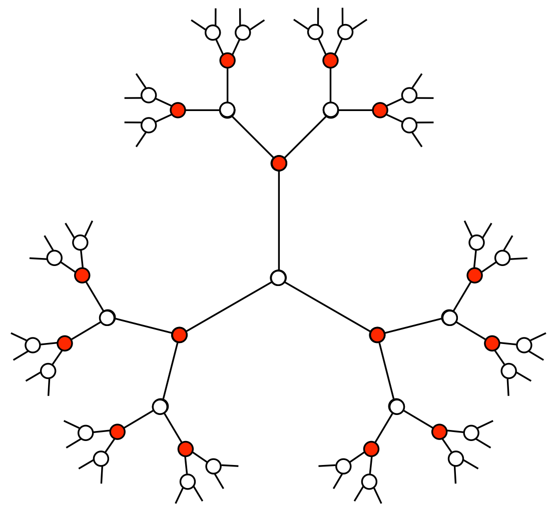


Fig. 2. The $(p+1)$-regular tree is $\beta SL(2,\mathbb{Q}_p)$

5 Equivariant *K*-homology

K-homology is the dual theory to Atiyah-Hirzebruch K-theory (Section 2.6). Here we define an equivariant generalization due to Kasparov [25, 26]. If X is a proper G-space with compact, second countable quotient then $K_i^G(X)$, $i = 0, 1$, are abelian groups defined as homotopy classes of K-cycles for X. These K-cycles can be viewed as G-equivariant abstract elliptic operators on X.

Remark 19. For a discrete group G, there is a topological definition of equivariant K-homology and the index map via equivariant spectra [14]. This and other constructions of the index map are shown to be equivalent in [19].

5.1 Definitions

Let G be a locally compact, Hausdorff, second countable, topological group.

Let H be a separable Hilbert space. Write $\mathcal{U}(H)$ for the set of unitary operators

$$\mathcal{U}(H) = \{U \in \mathcal{L}(H) \,|\, UU^* = U^*U = I\}\,.$$

Definition 10. *A* unitary representation *of G on H is a group homomorphism $\pi\colon G \to \mathcal{U}(H)$ such that for each $v \in H$ the map $\pi_v\colon G \to H$, $g \mapsto \pi(g)v$ is a continuous map from G to H.*

Definition 11. *A G-C^*-algebra is a C^*-algebra A with a given continuous action of G*

$$G \times A \longrightarrow A$$

such that G acts by C^*-algebra automorphisms.

The continuity condition is that, for each $a \in A$, the map $G \to A$, $g \mapsto ga$ is a continuous map. We also have that, for each $g \in G$, the map $A \to A$, $a \mapsto ga$ is a C^*-algebra automorphism.

Example 4. Let X be a locally compact, Hausdorff G-space. The action of G on X gives an action of G on $C_0(X)$,

$$(g\alpha)(x) = \alpha(g^{-1}x),$$

where $g \in G$, $\alpha \in C_0(X)$ and $x \in X$. This action makes $C_0(X)$ into a G-C^*-algebra.

Recall that a C^*-algebra is *separable* if it has a countable dense subset.

Definition 12. *Let A be a separable G-C^*-algebra. A* representation *of A is a triple (H, ψ, π) with:*

- *H is a separable Hilbert space,*
- *$\psi\colon A \to \mathcal{L}(H)$ is a $*$-homomorphism,*
- *$\pi\colon G \to \mathcal{U}(H)$ is a unitary representation of G on H,*
- *$\psi(ga) = \pi(g)\psi(a)\pi(g^{-1})$ for all $(g, a) \in G \times A$.*

Remark 20. We are using a slightly non-standard notation; in the literature this is usually called a *covariant representation*.

Definition 13. *Let X be a proper G-space with compact, second countable quotient space $G\backslash X$. An* equivariant odd K-cycle *for X is a 4-tuple (H, ψ, π, T) such that:*

- *(H, ψ, π) is a representation of the G-C^*-algebra $C_0(X)$,*
- *$T \in \mathcal{L}(H)$,*
- *$T = T^*$,*
- *$\pi(g)T - T\pi(g) = 0$ for all $g \in G$,*
- *$\psi(\alpha)T - T\psi(\alpha) \in \mathcal{K}(H)$ for all $\alpha \in C_0(X)$,*
- *$\psi(\alpha)(I - T^2) \in \mathcal{K}(H)$ for all $\alpha \in C_0(X)$.*

Remark 21. If G is a locally compact, Hausdorff, second countable topological group and X a proper G-space with locally compact quotient then X is also locally compact and hence $C_0(X)$ is well-defined.

Write $\mathcal{E}_1^G(X)$ for the set of equivariant odd K-cycles for X. This concept was introduced by Kasparov as an abstraction an equivariant self-adjoint elliptic operator and goes back to Atiyah's theory of elliptic operators [2].

Example 5. Let $G=\mathbb{Z}$, $X=\mathbb{R}$ with the action $\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$, $(n,t)\mapsto n+t$. The quotient space is S^1, which is compact. Consider $H=L^2(\mathbb{R})$ the Hilbert space of complex-valued square integrable functions with the usual Lebesgue measure. Let $\psi\colon C_0(\mathbb{R})\to\mathcal{L}(L^2(\mathbb{R}))$ be defined as $\psi(\alpha)u=\alpha u$, where $\alpha u(t)=\alpha(t)u(t)$, for all $\alpha\in C_0(\mathbb{R})$, $u\in L^2(\mathbb{R})$ and $t\in\mathbb{R}$. Finally, let $\pi\colon\mathbb{Z}\to\mathcal{U}(L^2(\mathbb{R}))$ be the map $(\pi(n)u)(t)=u(t-n)$ and consider the operator $\left(-i\frac{d}{dt}\right)$. This operator is self-adjoint but *not* bounded on $L^2(\mathbb{R})$. We "normalize" it to obtain a bounded operator

$$T=\left(\frac{x}{\sqrt{1+x^2}}\right)\left(-i\frac{d}{dt}\right).$$

This notation means that the function $\frac{x}{\sqrt{1+x^2}}$ is applied using functional calculus to the operator $\left(-i\frac{d}{dt}\right)$. Note that the operator $\left(-i\frac{d}{dt}\right)$ is essentially self adjoint. Thus the function $\frac{x}{\sqrt{1+x^2}}$ can be applied to the unique self-adjoint extension of $\left(-i\frac{d}{dt}\right)$.

Equivalently, T can be constructed using Fourier transform. Let $\mathcal{M}_x$ be the operator "multiplication by x"

$$\mathcal{M}_x(f(x))=xf(x).$$

The Fourier transform $\mathcal{F}$ converts $-i\frac{d}{dt}$ to $\mathcal{M}_x$, i.e. there is a commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}) \\ {\scriptstyle -i\frac{d}{dt}}\downarrow & & \downarrow{\scriptstyle \mathcal{M}_x} \\ L^2(\mathbb{R}) & \xrightarrow[\mathcal{F}]{} & L^2(\mathbb{R}). \end{array}$$

Let $\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}$ be the operator "multiplication by $\frac{x}{\sqrt{1+x^2}}$"

$$\mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}(f(x))=\frac{x}{\sqrt{1+x^2}}f(x).$$

T is the unique bounded operator on $L^2(\mathbb{R})$ such that the following diagram is commutative

$$\begin{array}{ccc} L^2(\mathbb{R}) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \mathcal{M}_{\frac{x}{\sqrt{1+x^2}}}} \\ L^2(\mathbb{R}) & \xrightarrow[\mathcal{F}]{} & L^2(\mathbb{R}). \end{array}$$

Then we have an equivariant odd K-cycle $(L^2(\mathbb{R}),\psi,\pi,T)\in\mathcal{E}_1^{\mathbb{Z}}(\mathbb{R})$.

Let X be a proper G-space with compact, second countable quotient $G\backslash X$ and $\mathcal{E}_1^G(X)$ defined as above. The *equivariant K-homology group* $K_1^G(X)$ is defined as the quotient

$$K_1^G(X)=\mathcal{E}_1^G(X)/\sim,$$

where $\sim$ represents *homotopy*, in a sense that will be made precise later (Section 11). It is an abelian group with addition and inverse given by

$$(H, \psi, \pi, T) + (H', \psi', \pi', T') = (H \oplus H', \psi \oplus \psi', \pi \oplus \pi', T \oplus T'),$$
$$-(H, \psi, \pi, T) = (H, \psi, \pi, -T).$$

Remark 22. The K-cycles defined above differ slightly from the K-cycles used by Kasparov [26]. However, the abelian group $K_1^G(X)$ is isomorphic to the Kasparov group $KK_G^1(C_0(X), \mathbb{C})$, where the isomorphism is given by the evident map which views one of our K-cycles as one of Kasparov's K-cycles. In other words, the K-cycles we are using are more special than the K-cycles used by Kasparov, however the obvious map of abelian groups is an isomorphism.

We define *even* K-cycles in a similar way, just dropping the condition of T being self-adjoint.

Definition 14. *Let X be a proper G-space with compact, second countable quotient space $G\backslash X$. An* equivariant even K-cycle *for X is a 4-tuple (H, ψ, π, T) such that:*

- *(H, ψ, π) is a representation of the G-C^*-algebra $C_0(X)$,*
- *$T \in \mathcal{L}(H)$,*
- *$\pi(g)T - T\pi(g) = 0$ for all $g \in G$,*
- *$\psi(\alpha)T - T\psi(\alpha) \in \mathcal{K}(H)$ for all $\alpha \in C_0(X)$,*
- *$\psi(\alpha)(I - T^*T) \in \mathcal{K}(H)$ for all $\alpha \in C_0(X)$,*
- *$\psi(\alpha)(I - TT^*) \in \mathcal{K}(H)$ for all $\alpha \in C_0(X)$.*

Write $\mathcal{E}_0^G(X)$ for the set of such equivariant even K-cycles.

Remark 23. In the literature the definition is somewhat more complicated. In particular, the Hilbert space H is required to be $\mathbb{Z}/2$-graded. However, at the level of abelian groups, the abelian group $K_0^G(X)$ obtained from the equivariant even K-cycles defined here will be isomorphic to the Kasparov group $KK_G^0(C_0(X), \mathbb{C})$ [26]. More precisely, let $(H, \psi, \pi, T, \omega)$ be a K-cycle in Kasparov's sense, where ω is a $\mathbb{Z}/2$-grading of the Hilbert space $H = H_0 \oplus H_1$, $\psi = \psi_0 \oplus \psi_1$, $\pi = \pi_0 \oplus \pi_1$ and T is self-adjoint but off-diagonal

$$T = \begin{pmatrix} 0 & T_- \\ T_+ & 0 \end{pmatrix}.$$

To define the isomorphism from $KK_G^0(C_0(X), \mathbb{C})$ to $K_0^G(X)$, we map a Kasparov cycle $(H, \psi, \pi, T, \omega)$ to (H', ψ', π', T') where

$$H' = \ldots H_0 \oplus H_0 \oplus H_0 \oplus H_1 \oplus H_1 \oplus H_1 \ldots$$
$$\psi' = \ldots \psi_0 \oplus \psi_0 \oplus \psi_0 \oplus \psi_1 \oplus \psi_1 \oplus \psi_1 \ldots$$
$$\pi' = \ldots \pi_0 \oplus \pi_0 \oplus \pi_0 \oplus \pi_1 \oplus \pi_1 \oplus \pi_1 \ldots$$

and T' is the obvious right-shift operator, where we use T_+ to map the last copy of H_0 to the first copy of H_1. The isomorphism from $\mathcal{E}_0^G(X)$ to $KK_G^0(C_0(X),\mathbb{C})$ is given by

$$(H,\psi,\pi,T) \mapsto (H\oplus H, \psi\oplus\psi, \pi\oplus\pi, \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}).$$

Let X be a proper G-space with compact, second countable quotient $G\backslash X$ and $\mathcal{E}_0^G(X)$ as above. The *equivariant K-homology group* $K_0^G(X)$ is defined as the quotient

$$K_0^G(X) = \mathcal{E}_0^G(X)/\sim,$$

where $\sim$ is *homotopy*, in a sense that will be made precise later. It is an abelian group with addition and inverse given by

$$(H,\psi,\pi,T) + (H',\psi',\pi',T') = (H\oplus H', \psi\oplus\psi', \pi\oplus\pi', T\oplus T'),$$
$$-(H,\psi,\pi,T) = (H,\psi,\pi,T^*).$$

Remark 24. Since the even K-cycles are more general, we have $\mathcal{E}_1^G(X) \subset \mathcal{E}_0^G(X)$. However, this inclusion induces the zero map from $K_1^G(X)$ to $K_0^G(X)$.

5.2 Functoriality

Equivariant K-homology gives a (covariant) functor between the category proper G-spaces with compact quotient and the category of abelian groups. Indeed, given a continuous G-map $f\colon X \to Y$ between proper G-spaces with compact quotient, it induces a map $\widetilde{f}\colon C_0(Y) \to C_0(X)$ by $\widetilde{f}(\alpha) = \alpha \circ f$ for all $\alpha \in C_0(Y)$. Then, we obtain homomorphisms of abelian groups

$$K_j^G(X) \longrightarrow K_j^G(Y) \quad j = 0,1$$

by defining, for each $(H,\psi,\pi,T) \in \mathcal{E}_j^G(X)$,

$$(H,\psi,\pi,T) \mapsto (H,\psi\circ\widetilde{f},\pi,T)\,.$$

5.3 The index map

Let X be a proper second countable G-space with compact quotient $G\backslash X$. There is a map of abelian groups

$$\begin{array}{rcl} K_j^G(X) & \longrightarrow & K_j\left(C_r^*(G)\right) \\ (H,\psi,\pi,T) & \mapsto & \mathrm{Index}(T) \end{array}$$

for $j = 0,1$. It is called the *index map* and will be defined in Section 13.

This map is natural, that is, if X and Y are proper second countable G-spaces with compact quotient and if $f\colon X \to Y$ is a continuous G-equivariant map, then the following diagram commutes:

$$\begin{array}{ccc} K_j^G(X) & \xrightarrow{f_*} & K_j^G(Y) \\ \downarrow{\scriptstyle \text{Index}} & & \downarrow{\scriptstyle \text{Index}} \\ K_j^G(C_r^*(G)) & \xrightarrow{=} & K_j^G(C_r^*(G)). \end{array}$$

We would like to define equivariant K-homology and the index map for $\underline{E}G$. However, the quotient of $\underline{E}G$ by the G-action might not be compact. The solution will be to consider all proper second countable G-subspaces with compact quotient.

Definition 15. *Let Z be a proper G-space. We call $\Delta \subseteq Z$ G-*compact *if*

1. *$gx \in \Delta$ for all $g \in G$, $x \in \Delta$,*
2. *Δ is a proper G-space,*
3. *the quotient space $G \backslash \Delta$ is compact.*

That is, Δ is a G-subspace which is proper as a G-space and has compact quotient $G \backslash \Delta$.

Remark 25. Since we are always assuming that G is locally compact, Hausdorff and second countable, we may also assume without loss of generality that any G-compact subset of $\underline{E}G$ is second countable. From now on we shall assume that $\underline{E}G$ has this property.

We define the *equivariant K-homology of $\underline{E}G$ with G-compact supports* as the direct limit

$$K_j^G(\underline{E}G) = \varinjlim_{\substack{\Delta \subseteq \underline{E}G \\ G\text{-compact}}} K_j^G(\Delta)\,.$$

There is then a well-defined index map on the direct limit

$$\begin{aligned} \mu\colon K_j^G(\underline{E}G) &\longrightarrow K_j(C_r^*G) \\ (H, \psi, \pi, T) &\mapsto \text{Index}(T), \end{aligned} \tag{1}$$

as follows. Suppose that $\Delta \subset \Omega$ are G-compact. By the naturality of the functor $K_j^G(-)$, there is a commutative diagram

$$\begin{array}{ccc} K_j^G(\Delta) & \longrightarrow & K_j^G(\Omega) \\ \downarrow{\scriptstyle \text{Index}} & & \downarrow{\scriptstyle \text{Index}} \\ K_j(C_r^*G) & \xrightarrow{=} & K_j(C_r^*G)\,, \end{array}$$

and thus the index map is defined on the direct limit.

6 The discrete case

We discuss several aspects of the Baum-Connes conjecture when the group is discrete.

6.1 Equivariant K-homology

For a discrete group Γ, there is a simple description of $K_j^\Gamma(\underline{E}\Gamma)$ up to torsion, in purely algebraic terms, given by a Chern character. Here we follow section 7 in [5].

Let Γ be a (countable) discrete group. Define $F\Gamma$ as the set of finite formal sums

$$F\Gamma = \left\{ \sum_{\text{finite}} \lambda_\gamma[\gamma] \text{ where } \gamma \in \Gamma, \text{order}(\gamma) < \infty,\ \lambda_\gamma \in \mathbb{C} \right\}.$$

$F\Gamma$ is a complex vector space and also a Γ-module with Γ-action:

$$g \cdot \left(\sum_{\lambda \in \Gamma} \lambda_\gamma[\gamma] \right) = \sum_{\lambda \in \Gamma} \lambda_\gamma [g\gamma g^{-1}]\,.$$

Note that the identity element of the group has order 1 and therefore $F\Gamma \neq 0$.

Consider $H_j(\Gamma; F\Gamma)$, $j \geq 0$, the homology groups of Γ with coefficients in the Γ-module $F\Gamma$.

Remark 26. This is standard homological algebra, with no topology involved (Γ is a discrete group and $F\Gamma$ is a non-topologized module over Γ). They are classical homology groups and have a purely algebraic description (cf. [10]). In general, if M is a Γ-module then $H_*(\Gamma; M)$ is isomorphic to $H_*(B\Gamma; \underline{M})$, where $\underline{M}$ means the local system on $B\Gamma$ obtained from the Γ-module M.

Let us write $K_j^{\text{top}}(\Gamma)$ for $K_j^\Gamma(\underline{E}\Gamma)$, $j = 0, 1$. There is a Chern character $\text{ch}\colon K_*^{\text{top}}(\Gamma) \to H_*(\Gamma; F\Gamma)$ which maps into odd, respectively even, homology

$$\text{ch}\colon K_j^{\text{top}}(\Gamma) \to \bigoplus_{l \geq 0} H_{j+2l}(\Gamma; F\Gamma) \quad j = 0, 1.$$

This map becomes an isomorphism when tensored with $\mathbb{C}$(cf. [4] or [32]).

Proposition 3. *The map*

$$\text{ch} \otimes_{\mathbb{Z}} \mathbb{C} : \ K_j^{\text{top}}(\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \longrightarrow \bigoplus_{l \geq 0} H_{j+2l}(\Gamma; F\Gamma) \quad j = 0, 1$$

is an isomorphism of vector spaces over $\mathbb{C}$.

Remark 27. If G is finite, the rationalized Chern character becomes the character map from $R(G)$, the complex representation ring of G, to class functions, given by $\rho \mapsto \chi(\rho)$ in the even case, and the zero map in the odd case.

If the Baum-Connes conjecture is true for Γ, then Proposition 3 computes the tensored topological K-theory of the reduced C^*-algebra of Γ.

Corollary 1. *If the Baum-Connes conjecture is true for* Γ *then*

$$K_j(C_r^*\Gamma) \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_{l \geq 0} H_{j+2l}(\Gamma; F\Gamma) \quad j = 0, 1\,.$$

6.2 Some results on discrete groups

We recollect some results on discrete groups which satisfy the Baum-Connes conjecture.

Theorem 7 (N. Higson, G. Kasparov [22]). *If Γ is a discrete group which is* amenable *(or, more generally,* a-T-menable*) then the Baum-Connes conjecture is true for Γ.*

Theorem 8 (I.Mineyev, G. Yu [38]; independently V. Lafforgue [29]). *If Γ is a discrete group which is* hyperbolic *(in Gromov's sense) then the Baum-Connes conjecture is true for Γ.*

Theorem 9 (Schick [46]). *Let B_n be the* braid group *on n strands, for any positive integer n. Then the Baum-Connes conjecture is true for B_n.*

Theorem 10 (Matthey, Oyono-Oyono, Pitsch [36]). *Let M be a connected orientable 3-dimensional manifold (possibly with boundary). Let Γ be the fundamental group of M. Then the Baum-Connes conjecture is true for Γ.*

The Baum-Connes index map has been shown to be injective or rationally injective for some classes of groups. For example, it is injective for countable subgroups of $GL(n, K)$, K any field [18], and injective for

- closed subgroups of connected Lie groups [27];
- closed subgroups of reductive p-adic groups [28].

More results on groups satisfying the Baum-Connes conjecture can be found in [35].

The Baum-Connes conjecture remains a widely open problem. For example, it is not known for $SL(n, \mathbb{Z})$, $n \geq 3$. These infinite discrete groups have Kazhdan's property (T) and hence they are not a-T-menable. On the other hand, it is known that the index map is injective for $SL(n, \mathbb{Z})$ (see above) and the groups $K_j^G(\underline{E}G)$ for $G = SL(3, \mathbb{Z})$ have been calculated [45].

Remark 28. The conjecture might be too general to be true for all groups. Nevertheless, we expect it to be true for a large family of groups, in particular for all exact groups (a groups G is *exact* if the functor $C_r^*(G, -)$, as defined in 10.2, is exact).

6.3 Corollaries of the Baum-Connes Conjecture

The Baum-Connes conjecture is related to a great number of conjectures in functional analysis, algebra, geometry and topology. Most of these conjectures follow from either the injectivity or the surjectivity of the index map. A significant example is the Novikov conjecture on the homotopy invariance of higher signatures of closed, connected, oriented, smooth manifolds. This conjecture follows from the injectivity of the rationalized index map [5]. For more information on conjectures related to Baum-Connes, see the appendix in [39].

Remark 29. By a "corollary" of the Baum-Connes conjecture we mean: if the Baum-Connes conjecture is true for a group G then the corollary is true for that group G. (For instance, in the Novikov conjecture G is the fundamental group of the manifold.)

7 The compact case

If G is compact, we can take $\underline{E}G$ to be a one-point space. On the other hand, $K_0(C_r^*G) = R(G)$ the (complex) representation ring of G, and $K_1(C_r^*G) = 0$ (see Remark below). Recall that $R(G)$ is the Grothendieck group of the category of finite dimensional (complex) representations of G. It is a free abelian group with one generator for each distinct (i.e. non-equivalent) irreducible representation of G.

Remark 30. When G is compact, the reduced C^*-algebra of G is a direct sum (in the C^*-algebra sense) over the irreducible representations of G, of matrix algebras of dimension equal to the dimension of the representation. The K-theory functor commutes with direct sums and $K_j(M_n(\mathbb{C})) \cong K_j(\mathbb{C})$, which is $\mathbb{Z}$ for j even and 0 otherwise (Theorem 4).

Hence the index map takes the form

$$\mu\colon K_G^0(point) \longrightarrow R(G)\,,$$

for $j = 0$ and is the zero map for $j = 1$.

Given $(H, \psi, T, \pi) \in \mathcal{E}_G^0(point)$, we may assume within the equivalence relation on $\mathcal{E}_G^0(point)$ that

$$\psi(\lambda) = \lambda I \quad \text{for all } \lambda \in C_0(point) = \mathbb{C}\,,$$

where I is the identity operator of the Hilbert space H. Hence the non-triviality of (H, ψ, T, π) is coming from

$$I - TT^* \in \mathcal{K}(H)\,, \quad \text{and} \quad I - T^*T \in \mathcal{K}(H)\,,$$

that is, T is a Fredholm operator. Therefore

$$\dim_{\mathbb{C}}(\ker(T)) < \infty,$$
$$\dim_{\mathbb{C}}(\operatorname{coker}(T)) < \infty,$$

hence $\ker(T)$ and $\operatorname{coker}(T)$ are finite dimensional representations of G (recall that G is acting via $\pi\colon G \to \mathcal{L}(H)$). Then

$$\mu(H, \psi, T, \pi) = \operatorname{Index}(T) = \ker(T) - \operatorname{coker}(T) \in R(G)\,.$$

Remark 31. The assembly map for G compact just described is an isomorphism (exercise).

Remark 32. In general, for G non-compact, the elements of $K_0^G(X)$ can be viewed as generalized elliptic operators on $\underline{E}G$, and the index map μ assigns to such an operator its 'index', $\ker(T) - \operatorname{coker}(T)$, in some suitable sense [5]. This should be made precise later using Kasparov's descent map and an appropriate Kasparov product (Section 13).

8 Equivariant K-homology for G-C^*-algebras

We have defined equivariant K-homology for G-spaces in Section 5. Now we define equivariant K-homology for a separable G-C^*-algebra A as the KK-theory groups $K_G^j(A, \mathbb{C})$, $j = 0, 1$. This generalises the previous construction since $K_j^G(X) = KK_G^j(C_0(X), \mathbb{C})$. Later on we shall define KK-theory groups in full generality (Sections 11 and 12).

Definition 16. *Let A be a separable G-C^*-algebra. Define $\mathcal{E}_G^1(A)$ to be the set of 4-tuples*

$$\{(H, \psi, \pi, T)\}$$

such that (H, ψ, π) is a representation of the G-C^*-algebra A, $T \in \mathcal{L}(H)$, and the following conditions are satisfied:

- $T = T^*$,
- $\pi(g)T - T\pi(g) \in \mathcal{K}(H)$,
- $\psi(a)T - T\psi(a) \in \mathcal{K}(H)$,
- $\psi(a)(I - T^2) \in \mathcal{K}(H)$,

for all $g \in G$, $a \in A$.

Remark 33. Note that this is not quite $\mathcal{E}_1^G(X)$ when $A = C_0(X)$ and X is a proper G-space with compact quotient, since the third condition is more general than before. However, the inclusion $\mathcal{E}_1^G(X) \subset \mathcal{E}_G^1(C_0(X))$ gives an isomorphism of abelian groups so that $K_1^G(X) = KK_G^1(C_0(X), \mathbb{C})$ (as defined below). The point is that, for a proper G-space with compact quotient, an averaging argument using a cut-off function and the Haar measure of the group G allows us to assume that the operator T is G-equivariant.

Given a separable G-C^*-algebra A, we define the KK-group $KK_G^1(A, \mathbb{C})$ as $\mathcal{E}_G^1(A)$ modulo an equivalence relation called *homotopy*, which will be made precise later. Addition in $KK_G^1(A, \mathbb{C})$ is given by direct sum

$$(H, \psi, \pi, T) + (H', \psi', \pi', T') = (H \oplus H', \psi \oplus \psi', \pi \oplus \pi', T \oplus T')$$

and the negative of an element by

$$-(H, \psi, \pi, T) = (H, \psi, \pi, -T)\,.$$

Remark 34. We shall later define $KK^1_G(A,B)$ for a separable G-C^*-algebras A and an arbitrary G-C^*-algebra B (Section 11).

Let A, B be separable G-C^*-algebras. A G-equivariant $*$-homomorphism $\phi\colon A \to B$ gives a map $\mathcal{E}^1_G(B) \to \mathcal{E}^1_G(A)$ by

$$(H, \psi, \pi, T) \mapsto (H, \psi \circ \phi, \pi, T)\,,$$

and this induces a map $KK^1_G(B,\mathbb{C}) \to KK^1_G(A,\mathbb{C})$. That is, $KK^1_G(A,\mathbb{C})$ is a contravariant functor in A.

For the even case, the operator T is not required to be self-adjoint.

Definition 17. *Let A be a separable G-C^*-algebra. Define $\mathcal{E}^0_G(A)$ as the set of 4-tuples*

$$\{(H, \psi, \pi, T)\}$$

such that (H, ψ, π) is a representation of the G-C^*-algebra A, $T \in \mathcal{L}(H)$ and the following conditions are satisfied:

- $\pi(g)T - T\pi(g) \in \mathcal{K}(H)$,
- $\psi(a)T - T\psi(a) \in \mathcal{K}(H)$,
- $\psi(a)(I - T^*T) \in \mathcal{K}(H)$,
- $\psi(a)(I - TT^*) \in \mathcal{K}(H)$,

for all $g \in G$, $a \in A$.

Remark 35. Again, if X is a proper G-space with compact quotient, the inclusion $\mathcal{E}^G_0(X) \subset \mathcal{E}^0_G(C_0(X))$ gives an isomorphism in K-homology, so we can write $K^G_0(X) = KK^0(C_0(X),\mathbb{C})$ (as defined below). The issue of the $\mathbb{Z}/2$-grading (which is present in the Kasparov definition but not in our definition) is dealt with as in Remark 23.

We define the KK-groups $KK^0_G(A,\mathbb{C})$ as $\mathcal{E}^0_G(A)$ modulo an equivalence relation called *homotopy*, which will be made precise later. Addition in $KK^1_G(A,\mathbb{C})$ is given by direct sum

$$(H, \psi, \pi, T) + (H', \psi', \pi', T') = (H \oplus H', \psi \oplus \psi', \pi \oplus \pi', T \oplus T')$$

and the negative of an element by

$$-(H, \psi, \pi, T) = (H, \psi, \pi, T^*)\,.$$

Remark 36. We shall later define in general $KK^0_G(A,B)$ for a separable G-C^*-algebras A and an arbitrary G-C^*-algebra B (Section 12).

Let A, B be separable G-C^*-algebras. A G-equivariant $*$-homomorphism $\phi\colon A \to B$ gives a map $\mathcal{E}^0_G(B) \to \mathcal{E}^0_G(A)$ by

$$(H, \psi, \pi, T) \mapsto (H, \psi \circ \phi, \pi, T)\,,$$

and this induces a map $KK^0_G(B,\mathbb{C}) \to KK^0_G(A,\mathbb{C})$. That is, $KK^0_G(A,\mathbb{C})$ is a contravariant functor in A.

9 The conjecture with coefficients

There is a generalized version of the Baum-Connes conjecture, known as the *Baum-Connes conjecture with coefficients*, which adds coefficients in a G-C^*-algebra. We recall the definition of G-C^*-algebra.

Definition 18. *A G-C^*-algebra is a C^*-algebra A with a given continuous action of G*

$$G \times A \longrightarrow A$$

such that G acts by C^*-algebra automorphisms. Continuity means that, for each $a \in A$, the map $G \to A$, $g \mapsto ga$ is a continuous map.

Remark 37. Observe that the only $*$-homomorphism of $\mathbb{C}$ as a C^*-algebra is the identity. Hence the only G-C^*-algebra structure on $\mathbb{C}$ is the one with trivial G-action.

Let A be a G-C^*-algebra. Later we shall define the reduced crossed-product C^*-algebra $C_r^*(G, A)$, and the equivariant K-homology group with coefficients $K_j^G(\underline{E}G, A)$. These constructions reduce to $C_r^*(G)$, respectively $K_j^G(\underline{E}G)$, when $A = \mathbb{C}$. Moreover, the index map extends to this general setting and is also conjectured to be an isomorphism.

Conjecture 2 (P. Baum, A. Connes, 1980). Let G be a locally compact, Hausdorff, second countable, topological group, and let A be any G-C^*-algebra. Then

$$\mu \colon K_j^G(\underline{E}G, A) \longrightarrow K_j(C_r^*(G, A)) \quad j = 0, 1$$

is an isomorphism.

Conjecture 1 follows as a particular case when $A = \mathbb{C}$. A fundamental difference is that the conjecture with coefficients is subgroup closed, that is, if it is true for a group G for *any* coefficients then it is true, for *any* coefficients, for any closed subgroup of G.

The conjecture with coefficients has been proved for:

- compact groups,
- abelian groups,
- groups acting simplicially on a tree with all vertex stabilizers satisfying the conjecture with coefficients [41],
- amenable groups and, more generally, a-T-menable groups (groups with the Haagerup property) [23],
- the Lie group $Sp(n, 1)$ [24],
- 3-manifold groups [36].

For more examples of groups satisfying the conjecture with coefficients see [35].

Expander graphs

Suppose that Γ is a finitely generated, discrete group which contains an expander family [13] in its Cayley graph as a subgraph. Such a Γ is a counter-example to the conjecture with coefficients [20]. M. Gromov outlined a proof that such Γ exists. A number of mathematicians are now filling in the details. It seems quite likely that this group exists.

10 Hilbert modules

In this section we introduce the concept of Hilbert module over a C^*-algebra. It generalises the definition of Hilbert space by allowing the inner product to take values in a C^*-algebra. Our main application will be the definition of the reduced crossed-product C^*-algebra in Section 10.2. For a concise reference on Hilbert modules see [30].

10.1 Definitions and examples

Let A be a C^*-algebra.

Definition 19. *An element $a \in A$ is* positive *(notation: $a \geq 0$) if there exists $b \in A$ with $bb^* = a$.*

The subset of positive elements, A^+, is a convex cone (closed under positive linear combinations) and $A^+ \cap (-A^+) = \{0\}$ [15, 1.6.1]. Hence we have a well-defined partial ordering in A given by $x \geq y \iff x - y \geq 0$.

Definition 20. *A* pre-Hilbert A-module *is a right A-module $\mathcal{H}$ with a given A-valued inner product $\langle\ ,\ \rangle$ such that:*

- $\langle u, v_1 + v_2 \rangle = \langle u, v_1 \rangle + \langle u, v_2 \rangle$,
- $\langle u, va \rangle = \langle u, v \rangle a$,
- $\langle u, v \rangle = \langle v, u \rangle^*$,
- $\langle u, u \rangle \geq 0$,
- $\langle u, u \rangle = 0 \Leftrightarrow u = 0$,

for all $u, v, v_1, v_2 \in \mathcal{H}$, $a \in A$.

Definition 21. *A* Hilbert A-module *is a pre-Hilbert A-module which is complete with respect to the norm*

$$\|u\| = \|\langle u, u \rangle\|^{1/2} .$$

Remark 38. If $\mathcal{H}$ is a Hilbert A-module and A has a unit 1_A then $\mathcal{H}$ is a complex vector space with

$$u\lambda = u(\lambda 1_A) \qquad u \in \mathcal{H}, \lambda \in \mathbb{C} .$$

If A does not have a unit, then by using an *approximate identity* [42] in A, it is also a complex vector space.

Example 6. Let A be a C^*-algebra and $n \geq 1$. Then $A^n = A \oplus \ldots \oplus A$ is a Hilbert A-module with operations

$$\begin{gathered}(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = (a_1 + b_1, \ldots, a_n + b_n), \\ (a_1, \ldots, a_n)a = (a_1 a, \ldots, a_n a), \\ \langle (a_1, \ldots, a_n), (b_1, \ldots, b_n) \rangle = a_1^* b_1 + \ldots + a_n^* b_n,\end{gathered}$$

for all $a_j, b_j, a \in A$.

Example 7. Let A be a C^*-algebra. Define

$$\mathcal{H} = \left\{ (a_1, a_2, \ldots) \,\middle|\, a_j \in A,\ \sum_{j=1}^{\infty} a_j^* a_j \text{ is norm-convergent in } A \right\},$$

with operations

$$\begin{gathered}(a_1, a_2, \ldots) + (b_1, b_2 \ldots) = (a_1 + b_1, a_2 + b_2, \ldots), \\ (a_1, a_2, \ldots)a = (a_1 a, a_2 a, \ldots), \\ \langle (a_1, a_2, \ldots), (b_1, b_2 \ldots) \rangle = \sum_{j=1}^{\infty} a_j^* b_j .\end{gathered}$$

The previous examples can be generalized. Note that a C^*-algebra A is a Hilbert module over itself with inner product $\langle a, b \rangle = a^* b$.

Example 8. If $\mathcal{H}_1, \ldots, \mathcal{H}_n$ are Hilbert A-modules then the direct sum $\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_n$ is a Hilbert A-module with

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = \sum_i x_i^* y_i \,.$$

We write $\mathcal{H}^n$ for the direct sum of n copies of a Hilbert A-module $\mathcal{H}$.

Example 9. If $\{\mathcal{H}_i\}_{i \in \mathbb{N}}$ is a countable family of Hilbert A-modules then

$$\mathcal{H} = \left\{ (x_1, x_2, \ldots) \,\middle|\, x_i \in \mathcal{H}_i,\ \sum_{j=1}^{\infty} \langle x_j, x_j \rangle \text{ is norm-convergent in } A \right\}$$

is a Hilbert A-module with inner product $\langle x, y \rangle = \sum_{j=1}^{\infty} \langle x_j, y_j \rangle$.

The following is our key example.

Example 10. Let G be a locally compact, Hausdorff, second countable, topological group. Fix a left-invariant Haar measure dg for G. Let A be a G-C^*-algebra. Then $L^2(G, A)$ is a Hilbert A-module defined as follows. Denote by

$C_c(G,A)$ the set of all continuous compactly supported functions from G to A. On $C_c(G,A)$ consider the norm

$$\|f\| = \left\| \int_G g^{-1}\left(f(g)^* f(g)\right) dg \right\| .$$

$L^2(G,A)$ is the completion of $C_c(G,A)$ in this norm. It is a Hilbert A-module with operations

$$\begin{aligned} (f+h)g &= f(g)+h(g), \\ (fa)g &= f(g)(ga), \\ \langle f,h\rangle &= \int_G g^{-1}\left(f(g)^* h(g)\right) dg \,. \end{aligned}$$

Note that when $A=\mathbb{C}$ the group action is trivial and we get $L^2(G)$ (cf. Remark 37).

Definition 22. *Let $\mathcal{H}$ be a Hilbert A-module. An A-module map $T\colon \mathcal{H}\to\mathcal{H}$ is* adjointable *if there exists an A-module map $T^*\colon \mathcal{H}\to\mathcal{H}$ with*

$$\langle Tu,v\rangle = \langle u,T^*v\rangle \quad \text{for all } u,v\in\mathcal{H} \,.$$

If T^* exists, it is unique, and $\sup_{\|u\|=1}\|Tu\|<\infty$. Set

$$\mathcal{L}(\mathcal{H}) = \{T\colon \mathcal{H}\to\mathcal{H} \,|\, T \text{ is adjointable}\} \,.$$

Then $\mathcal{L}(\mathcal{H})$ is a C^*-algebra with operations

$$\begin{aligned} (T+S)u &= Tu+Su, \\ (ST)u &= S(Tu), \\ (T\lambda)u &= (Tu)\lambda \\ T^* & \text{ as above}, \\ \|T\| &= \sup_{\|u\|=1}\|Tu\|. \end{aligned}$$

10.2 The reduced crossed-product $C_r^*(G,A)$

Let A be a G-C^*-algebra. Define

$$C_c(G,A) = \{f\colon G\to A \,|\, f \text{ continuous with compact support}\} \,.$$

Then $C_c(G,A)$ is a complex algebra with

$$\begin{aligned} (f+h)g &= f(g)+h(g), \\ (f\lambda)g &= f(g)\lambda, \\ (f*h)(g_0) &= \int_G f(g)\left(gh(g^{-1}g_0)\right) dg, \end{aligned}$$

for all $g, g_0 \in G$, $\lambda \in \mathbb{C}$, $f, h \in C_c(G, A)$. The product $*$ is called *twisted convolution*.

Consider the Hilbert A-module $L^2(G, A)$. There is an injection of algebras

$$\begin{aligned} C_c(G, A) &\hookrightarrow \mathcal{L}(L^2(G, A)) \\ f &\mapsto T_f \end{aligned}$$

where $T_f(u) = f * u$ is twisted convolution as above. We define $C_r^*(G, A)$ as the C^*-algebra obtained by completing $C_c(G, A)$ with respect to the norm $\|f\| = \|T_f\|$. When $A = \mathbb{C}$, the G-action must be trivial and $C_r^*(G, A) = C_r^*(G)$.

Example 11. Let G be a finite group, and A a G-C^*-algebra. Let dg be the Haar measure such that each $g \in G$ has mass 1. Then

$$C_r^*(G, A) = \left\{ \sum_{\gamma \in G} a_\gamma[\gamma] \,\middle|\, a_\gamma \in A \right\}$$

with operations

$$\begin{gathered} \left(\sum_{\gamma \in G} a_\gamma[\gamma] \right) + \left(\sum_{\gamma \in G} b_\gamma[\gamma] \right) = \sum_{\gamma \in G} (a_\gamma + b_\gamma)[\gamma], \\ \left(\sum_{\gamma \in G} a_\gamma[\gamma] \right) \lambda = \sum_{\gamma \in G} (a_\gamma \lambda)[\gamma], \\ (a_\alpha[\alpha])(b_\beta[\beta]) = a_\alpha(\alpha b_\beta)[\alpha\beta] \quad \text{(twisted convolution)}, \\ \left(\sum_{\gamma \in G} a_\gamma[\gamma] \right)^* = \sum_{\gamma \in G} (\gamma^{-1} a_\gamma^*)[\gamma^{-1}]. \end{gathered}$$

Here $a_\gamma[\gamma]$ denotes the function from G to A which has the value a_γ at γ and 0 at $g \neq \gamma$.

Let X be a Hausdorff, locally compact G-space. We know that $C_0(X)$ becomes a G-C^*-algebra with G-action

$$(gf)(x) = f(g^{-1}x),$$

for $g \in G$, $f \in C_0(X)$ and $x \in X$. The reduced crossed-product $C_r^*(G, C_0(X))$ will be denoted $C_r^*(G, X)$.

A natural question is to calculate the K-theory of this C^*-algebra. If G is compact, this is the Atiyah-Segal group $K_G^j(X)$, $j = 0, 1$. Hence for G non-compact, $K_j(C_r^*(G, X))$ is the natural extension of the Atiyah-Segal theory to the case when G is non-compact.

Definition 23. *We call a G-space G-*compact *if the quotient space $G \backslash X$ (with the quotient topology) is compact.*

Let X be a proper, G-compact G-space. Then a G-equivariant $\mathbb{C}$-vector bundle E on X determines an element

$$[E] \in K_0(C_r^*(G, X))\,.$$

Remark 39. From E, a Hilbert module over $C_r^*(G, X)$ is constructed. This Hilbert $C_r^*(G, X)$-module determines an element in $KK_0(\mathbb{C}, C_r^*(G, X)) \cong K_0(C_r^*(G, X))$. Note that, quite generally, a Hilbert A-module determines an element in $KK_0(A)$ if and only if it is finitely generated.

Recall that a G-equivariant vector bundle E over X is a (complex) vector bundle $\pi\colon E \to X$ together with a G-action on E such that π is G-equivariant and, for each $p \in X$, the map on the fibers $E_p \to E_{gp}$ induced by multiplication by g is linear.

Theorem 11 (W. Lück and B. Oliver [34]). *If Γ is a (countable) discrete group and X is a proper Γ-compact Γ-space, then*

$K_0(C_r^*(\Gamma, X)) =$ Grothendieck group of Γ-equivariant $\mathbb{C}$-vector bundles on X.

Remark 40. In [34] this theorem is not explicitly stated. However, it follows from their results. For clarification see [6] or [16].

Remark 41. Let X be a proper G-compact G-space. Let $\mathbb{I}$ be the trivial G-equivariant complex vector bundle on X,

$$\mathbb{I} = X \times \mathbb{C}\,, \quad g(x, \lambda) = (gx, \lambda),$$

for all $g \in G$, $x \in X$ and $\lambda \in \mathbb{C}$. Then $[\,\mathbb{I}\,] \in K_0(C_r^*(G, X))$.

10.3 Push-forward of Hilbert modules

Let A, B be C^*-algebras, $\varphi\colon A \to B$ a $*$-homomorphism and $\mathcal{H}$ a Hilbert A-module. We shall define a Hilbert B-module $\mathcal{H} \otimes_A B$, called the *push-forward of $\mathcal{H}$ with respect to φ* or *interior tensor product* ([30, Chapter 4]). First, form the algebraic tensor product $\mathcal{H} \odot_A B = \mathcal{H} \otimes_A^{alg} B$ (B is an A-module via φ). This is an abelian group and also a (right) B-module

$$(h \otimes b)b' = h \otimes bb' \quad \text{for all } h \in \mathcal{H}, b, b' \in B\,.$$

Define a B-valued inner product on $\mathcal{H} \odot_A B$ by

$$\langle h \otimes b, h' \otimes b' \rangle = b^* \varphi(\langle h, h' \rangle) b'.$$

Set

$$\mathcal{N} = \{\xi \in \mathcal{H} \odot_A B \mid \langle \xi, \xi \rangle = 0\}\,.$$

$\mathcal{N}$ is a B-sub-module of $\mathcal{H} \odot_A B$ and $(\mathcal{H} \odot_A B)/\mathcal{N}$ is a pre-Hilbert B-module.

Definition 24. *Define $\mathcal{H} \otimes_A B$ to be the Hilbert B-module obtained by completing $(\mathcal{H} \odot_A B)/\mathcal{N}$.*

Example 12. Let X be a locally compact, Hausdorff space. Let $A = C_0(X)$, $B = \mathbb{C}$ and $ev_p\colon C_0(X) \to \mathbb{C}$ the evaluation map at a point $p \in X$. Then we can consider the push-forward of a Hilbert $C_0(X)$-module $\mathcal{H}$. This gives a Hilbert space $\mathcal{H}_p$. These Hilbert spaces do not form a vector bundle but something more general (not necessarily locally trivial), sometimes called *continuous field of Hilbert spaces* [15, chapter 10].

11 Homotopy made precise and KK-theory

We first define homotopy and Kasparov's KK-theory in the non-equivariant setting, for pairs of separable C^*-algebras. A first introduction to KK-theory and further references can be found in [21].

Let A be a C^*-algebra and let $\mathcal{H}$ be a Hilbert A-module. Consider $\mathcal{L}(\mathcal{H})$ the bounded operators on $\mathcal{H}$. For each $u, v \in \mathcal{H}$ we have a bounded operator $\theta_{u,v}$ defined as

$$\theta_{u,v}(\xi) = u\langle v, \xi\rangle \,.$$

It is clear that $\theta^*_{u,v} = \theta_{v,u}$. The $\theta_{u,v}$ are called *rank one operators* on $\mathcal{H}$. A *finite rank operator* on $\mathcal{H}$ is any $T \in \mathcal{L}(\mathcal{H})$ such that T is a finite sum of rank one operators,

$$T = \theta_{u_1,v_1} + \ldots + \theta_{u_n,v_n} \,.$$

Let $\mathcal{K}(\mathcal{H})$ be the closure (in $\mathcal{L}(\mathcal{H})$) of the set of finite rank operators. $\mathcal{K}(\mathcal{H})$ is an ideal in $\mathcal{L}(\mathcal{H})$. When $A = \mathbb{C}$, $\mathcal{H}$ is a Hilbert space and $\mathcal{K}(\mathcal{H})$ coincides with the usual compact operators on $\mathcal{H}$.

Definition 25. *$\mathcal{H}$ is* countably generated *if in $\mathcal{H}$ there is a countable (or finite) set such that the A-module generated by this set is dense in $\mathcal{H}$.*

Definition 26. *Let $\mathcal{H}_0$, $\mathcal{H}_1$ be two Hilbert A-modules. We say that $\mathcal{H}_0$ and $\mathcal{H}_1$ are* isomorphic *if there exists an A-module isomorphism $\Phi\colon \mathcal{H}_0 \to \mathcal{H}_1$ with*

$$\langle u, v\rangle_0 = \langle \Phi u, \Phi v\rangle_1 \quad \text{for all } u, v \in \mathcal{H}_0 \,.$$

We want to define non-equivariant KK-theory for pairs of C^*-algebras. Let A and B be C^*-algebras where A is also separable. Define the set

$$\mathcal{E}^1(A, B) = \{(\mathcal{H}, \psi, T)\}$$

such that $\mathcal{H}$ is a countably generated Hilbert B-module, $\psi\colon A \to \mathcal{L}(\mathcal{H})$ is a $*$-homomorphism, $T \in \mathcal{L}(\mathcal{H})$, and the following conditions are satisfied:

- $T = T^*$,
- $\psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H})$,

- $\psi(a)(I - T^2) \in \mathcal{K}(\mathcal{H})$,

for all $a \in A$. We call such triples *odd bivariant K-cycles.*

Remark 42. In the Kasparov definition of $KK^1(A, B)$ [25], the conditions of the K-cycles are the same as our conditions except that the requirement $T = T^*$ is replaced by $\psi(a)(T - T^*) \in \mathcal{K}(H)$ for all $a \in A$. The isomorphism of abelian groups from the group defined using these bivariant K-cycles to the group defined using our bivariant K-cycles is obtained by sending a Kasparov cycle (H, ψ, T) to $(H, \psi, \frac{T+T^*}{2})$.

We say that two such triples $(\mathcal{H}_0, \psi_0, T_0)$ and $(\mathcal{H}_1, \psi_1, T_1)$ in $\mathcal{E}^1(A, B)$ are *isomorphic* if there is an isomorphism of Hilbert B-modules $\Phi\colon \mathcal{H}_0 \to \mathcal{H}_1$ with

$$\Phi\psi_0(a) = \psi_1(a)\Phi,$$
$$\Phi T_0 = T_1\Phi,$$

for all $a \in A$. That is, the following diagrams commute

$$\begin{array}{ccc} \mathcal{H}_0 & \xrightarrow{\psi_0(a)} & \mathcal{H}_0 \\ \Phi\downarrow & & \downarrow\Phi \\ \mathcal{H}_1 & \xrightarrow[\psi_1(a)]{} & \mathcal{H}_1 \end{array} \qquad \begin{array}{ccc} \mathcal{H}_0 & \xrightarrow{T_0} & \mathcal{H}_0 \\ \Phi\downarrow & & \downarrow\Phi \\ \mathcal{H}_1 & \xrightarrow[T_1]{} & \mathcal{H}_1 \end{array}$$

Let A, B, D be C^*-algebras where A is also separable. A $*$-homomorphism $\varphi\colon B \to D$ induces a map $\varphi_*\colon \mathcal{E}^1(A, B) \to \mathcal{E}^1(A, D)$ by

$$\varphi_*(\mathcal{H}, \psi, T) = (\mathcal{H} \otimes_B D, \psi \otimes_B I, T \otimes_B I)$$

where I is the identity operator on D, that is, $I(\alpha) = \alpha$ for all $\alpha \in D$.

We can now make the definition of homotopy precise. Consider the C^*-algebra of continuous functions $C([0, 1], B)$, and set ρ_0, ρ_1 to be the $*$-homomorphisms

$$C([0,1], B) \overset{\rho_0}{\underset{\rho_1}{\rightrightarrows}} B$$

defined by $\rho_0(f) = f(0)$ and $\rho_1(f) = f(1)$. In particular, we have induced maps

$$(\rho_j)_*\colon \mathcal{E}^1(A, C([0, 1], B)) \longrightarrow \mathcal{E}^1(A, B) \qquad j = 0, 1$$

for any separable C^*-algebra A.

Definition 27. *Two triples* $(\mathcal{H}_0, \psi_0, T_0)$ *and* $(\mathcal{H}_1, \psi_1, T_1)$ *in* $\mathcal{E}^1(A, B)$ *are* homotopic *if there exists* $(\mathcal{H}, \psi, T)$ *in* $\mathcal{E}^1(A, C([0, 1], B))$ *with*

$$(\rho_j)_*(\mathcal{H}, \psi, T) \cong (\mathcal{H}_j, \psi_j, T_j) \quad j = 0, 1\,.$$

The even case is analogous, removing the self-adjoint condition $T = T^*$.

Remark 43. As above, we do not require the Hilbert B-module $\mathcal{H}$ to be $\mathbb{Z}/2$-graded. The isomorphism between the abelian group we are defining and the group $KK_0(A, B)$ as defined by Kasparov [25] is dealt with as before (see Remark 23).

Hence we have the set of *even bivariant K-cycles*

$$\mathcal{E}^0(A, B) = \{(\mathcal{H}, \psi, T)\}$$

where $\mathcal{H}$ is a countably generated Hilbert B-module, $\psi\colon A \to \mathcal{L}(\mathcal{H})$ a $*$-homomorphism, $T \in \mathcal{L}(\mathcal{H})$, and the following conditions are satisfied:

- $\psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H})$,
- $\psi(a)(I - T^*T) \in \mathcal{K}(\mathcal{H})$,
- $\psi(a)(I - TT^*) \in \mathcal{K}(\mathcal{H})$,

for all $a \in A$. The remaining definitions carry over, in particular the definition of *homotopy* in $\mathcal{E}^0(A, B)$.

We define the *(non-equivariant) Kasparov KK-theory groups* of the pair (A, B) as

$$KK^1(A, B) = \mathcal{E}^1(A, B)/(\text{homotopy}),$$
$$KK^0(A, B) = \mathcal{E}^0(A, B)/(\text{homotopy}).$$

A key property is that KK-theory incorporates K-theory of C^*-algebras: for any C^*-algebra B, $KK^j(\mathbb{C}, B)$ is isomorphic to $K_j(B)$ (see Theorem 25 in [37]).

12 Equivariant *KK*-theory

We generalize KK-theory to the equivariant setting. An alternative definition to ours, by means of a universal property, is described in Section 2 of Meyer's notes [37].

All through this section, let A be a G-C^*-algebra.

Definition 28. *A G-Hilbert A-module is a Hilbert A-module $\mathcal{H}$ with a given continuous action*

$$G \times \mathcal{H} \to \mathcal{H}$$
$$(g, v) \mapsto gv$$

such that

1. $g(u + v) = gu + gv$,
2. $g(ua) = (gu)(ga)$,
3. $\langle gu, gv\rangle = g\langle u, v\rangle$,

for all $g \in G$, $u, v \in \mathcal{H}$, $a \in A$.

Here 'continuous' means that for each $u \in \mathcal{H}$, the map $G \to \mathcal{H}$, $g \mapsto gu$ is continuous.

Example 13. If $A = \mathbb{C}$, a G-Hilbert $\mathbb{C}$-module is just a unitary representation of G (the action of G on $\mathbb{C}$ must be trivial).

Remark 44. Let $\mathcal{H}$ be a G-Hilbert A-module. For each $g \in G$, denote by L_g the map

$$L_g \colon \mathcal{H} \to \mathcal{H}, \quad L_g(v) = gv\,.$$

Note that L_g might not be in $\mathcal{L}(\mathcal{H})$. But if $T \in \mathcal{L}(\mathcal{H})$, then $L_g T L_g^{-1} \in \mathcal{L}(\mathcal{H})$.

Hence G acts on the C^*-algebra $\mathcal{L}(\mathcal{H})$ by

$$gT = L_g T L_g^{-1}\,.$$

Example 14. Let A be a G-C^*-algebra. Set $n \geq 1$. Then A^n is a G-Hilbert A-module (cf. Example 6) with

$$g(a_1, \ldots, a_n) = (ga_1, \ldots, ga_n).$$

Let A and B be G-C^*-algebras, where A is also separable. Define the set

$$\mathcal{E}^0_G(A, B) = \{(\mathcal{H}, \psi, T)\}$$

such that $\mathcal{H}$ is a countably generated G-Hilbert B-module, $\psi \colon A \to \mathcal{L}(\mathcal{H})$ is a $*$-homomorphism with

$$\psi(ga) = g\psi(a) \quad \text{for all } g \in G, a \in A\,,$$

and $T \in \mathcal{L}(\mathcal{H})$, and so that the following conditions are satisfied:

- $gT - T \in \mathcal{K}(\mathcal{H})$,
- $\psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H})$,
- $\psi(a)(I - T^*T) \in \mathcal{K}(\mathcal{H})$,
- $\psi(a)(I - TT^*) \in \mathcal{K}(\mathcal{H})$,

for all $g \in G$, $a \in A$. We define

$$KK^0_G(A, B) = \mathcal{E}^0_G(A, B)/(\text{homotopy})\,.$$

The definition of *homotopy* in Section 11 can be defined in a straightforward way in this setting.

$KK^0_G(A, B)$ is an abelian group with addition and negative

$$(\mathcal{H}, \psi, T) + (\mathcal{H}', \psi', T') = (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', T \oplus T'),$$
$$-(\mathcal{H}, \psi, T) = (\mathcal{H}, \psi, T^*)\,.$$

The odd case is similar, just restricting to self-adjoint operators. Define the set

$$\mathcal{E}^1_G(A,B) = \{(\mathcal{H},\psi,T)\}$$

such that $\mathcal{H}$ is a countably generated G-Hilbert B-module, $\psi\colon A \to \mathcal{L}(\mathcal{H})$ is a $*$-homomorphism with

$$\psi(ga) = g\psi(a) \quad \text{for all } g \in G, a \in A\,,$$

and $T \in \mathcal{L}(\mathcal{H})$, and so that the following conditions are satisfied:

- $T = T^*$,
- $gT - T \in \mathcal{K}(\mathcal{H})$,
- $\psi(a)T - T\psi(a) \in \mathcal{K}(\mathcal{H})$,
- $\psi(a)(I - T^2) \in \mathcal{K}(\mathcal{H})$,

for all $g \in G$, $a \in A$.

We define

$$KK^1_G(A,B) = \mathcal{E}^1_G(A,B)/(\text{homotopy})\,.$$

$KK^1_G(A,B)$ is an abelian group with addition and inverse given by

$$(\mathcal{H},\psi,T) + (\mathcal{H}',\psi',T') = (\mathcal{H}\oplus\mathcal{H}',\psi\oplus\psi',T\oplus T'),$$
$$-(\mathcal{H},\psi,T) = (\mathcal{H},\psi,-T).$$

Remark 45. In the even case we are not requiring a $\mathbb{Z}/2$-grading. The isomorphism to the abelian group defined by Kasparov [26] is given as in Remark 23. Our general principle is that the even and odd cases are identical except that in the odd case the operator T is required to be self-adjoint but not in the even case.

Using equivariant KK-theory, we can introduce coefficients for equivariant K-homology. Let X be a proper G-space with compact quotient. Recall that

$$K^G_j(X) = KK^j_G(C_0(X),\mathbb{C}) \quad \text{and}$$
$$K^G_j(\underline{E}G) = \varinjlim_{\substack{\Delta\subseteq\underline{E}G\\ G\text{-compact}}} K^G_j(\Delta)\,.$$

We define the *equivariant K-homology of X*, respectively *of $\underline{E}G$*, with coefficients in a G-C^*-algebra A as

$$K^G_j(X,A) = KK^j_G(C_0(X),A),$$
$$K^G_j(\underline{E}G,A) = \varinjlim_{\substack{\Delta\subseteq\underline{E}G\\ G\text{-compact}}} K^G_j(\Delta,A)\,.$$

13 The index map

Our definition of the index map uses the Kasparov product and the descent map.

13.1 The Kasparov product

Let A, B, D be (separable) G-C^*-algebras. There is a product

$$KK_G^i(A,B) \otimes_{\mathbb{Z}} KK_G^j(B,D) \longrightarrow KK_G^{i+j}(A,D)\,.$$

The definition is highly non-trivial. Some motivation and examples, in the non-equivariant case, can be found in [21, Section 5].

Remark 46. Equivariant KK-theory can be regarded as a category with objects separable G-C^*-algebras and morphisms $\mathrm{mor}(A,B) = KK_G^i(A,B)$ (as a $\mathbb{Z}/2$-graded abelian group), and composition given by the Kasparov product (cf. [37, Thm. 33]).

13.2 The Kasparov descent map

Let A and B be (separable) G-C^*-algebras. There is a map between the equivariant KK-theory of (A,B) and the non-equivariant KK-theory of the corresponding reduced crossed-product C^*-algebras,

$$KK_G^j(A,B) \longrightarrow KK^j\left(C_r^*(G,A), C_r^*(G,B)\right) \quad j = 0,1\,.$$

The definition is also highly non-trivial and can be found in [26, Section 3]. Alternatively, see Proposition 26 in Meyer's notes [37].

13.3 Definition of the index map

We would like to define the index map

$$\mu\colon K_j^G(\underline{E}G) \longrightarrow K_j(C_r^*G)\,.$$

Let X be a proper G-compact G-space. First, we define a map

$$\mu\colon K_j^G(X) = KK_G^j(C_0(X),\mathbb{C}) \longrightarrow K_j(C_r^*G)$$

to be the composition of the Kasparov descent map

$$KK_G^j(C_0(X),\mathbb{C}) \longrightarrow KK^j\left(C_r^*(G,X), C_r^*(G)\right)$$

(the trivial action of G on $\mathbb{C}$ gives the crossed-product $C_r^*(G,\mathbb{C}) = C_r^*G$) and the Kasparov product with the trivial bundle

$$\mathbb{I} \in K_0(C_r^*(G,X)) = KK^0(\mathbb{C}, C_r^*(G,X)),$$

that is, the Kasparov product with the trivial vector bundle $\mathbb{I}$, when $A = \mathbb{C}$, $B = C_r^*(G,X)$, $D = C_r^*G$ and $i = 0$.

Recall that

$$K_j^G(\underline{E}G) = \varinjlim_{\substack{\Delta\subset\underline{E}G\\ G\text{-compact}}} KK_G^j\left(C_0(\Delta),\mathbb{C}\right) .$$

For each G-compact $\Delta \subset \underline{E}G$, we have a map as before

$$\mu\colon KK_G^j(C_0(\Delta),\mathbb{C}) \longrightarrow K_j(C_r^*G) .$$

If Δ and Ω are two G-compact subsets of $\underline{E}G$ with $\Delta \subset \Omega$, then by naturality the following diagram commutes:

$$\begin{array}{ccc} KK_G^j(C_0(\Delta),\mathbb{C}) & \longrightarrow & KK_G^j(C_0(\Omega),\mathbb{C}) \\ \downarrow & & \downarrow \\ K_jC_r^*G & \xrightarrow{=} & K_jC_r^*G. \end{array}$$

Thus we obtain a well-defined map on the direct limit $\mu\colon K_j^G(\underline{E}G) \to K_jC_r^*G$.

13.4 The index map with coefficients

The coefficients can be introduced in KK-theory at once. Let A be a G-C^*-algebra. We would like to define the index map

$$\mu\colon K_j^G(\underline{E}G;A) \longrightarrow K_jC_r^*(G,A) .$$

Let X be a proper G-compact G-space and A a G-C^*-algebra. First, we define a map

$$\mu\colon KK_G^j(C_0(X),A) \longrightarrow K_jC_r^*(G,A)$$

to be the composition of the Kasparov descent map

$$KK_G^j(C_0(X),A) \longrightarrow KK^j\left(C_r^*(G,X),C_r^*(G,A)\right)$$

and the Kasparov product with the trivial bundle

$$\mathbb{I} \in K_0C_r^*(G,X) = KK^0(\mathbb{C},C_r^*(G,X)).$$

For each G-compact $\Delta \subset \underline{E}G$, we have a map as above

$$\mu\colon KK_G^j(C_0(\Delta),A) \longrightarrow K_jC_r^*(G,A) .$$

If Δ and Ω are two G-compact subsets of $\underline{E}G$ with $\Delta \subset \Omega$, then by naturality the following diagram commutes:

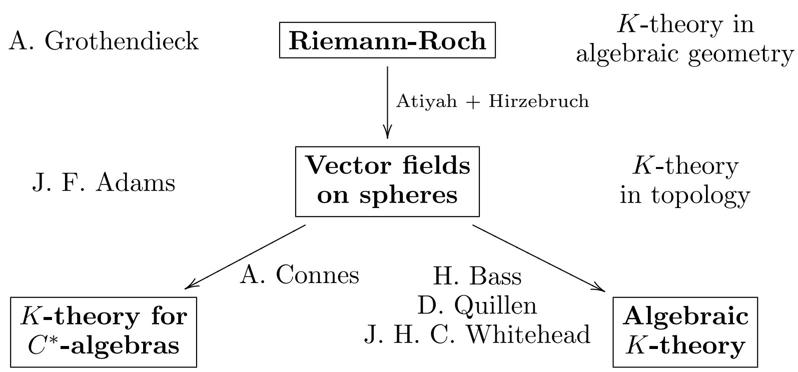
$$\begin{array}{ccc} KK_G^j(C_0(\Delta),A) & \longrightarrow & KK_G^j(C_0(\Omega),A) \\ \downarrow & & \downarrow \\ K_jC_r^*(G,A) & \xrightarrow{=} & K_jC_r^*(G,A). \end{array}$$

Thus we obtain a well-defined map on the direct limit $\mu\colon K_j^G(\underline{E}G;A) \to K_jC_r^*(G,A)$.

14 A brief history of *K*-theory

14.1 The *K*-theory genealogy tree

Grothendieck invented K-theory to give a conceptual proof of the Hirzebruch–Riemann–Roch theorem. The subject has since then evolved in different directions, as summarized by the following diagram.



Atiyah and Hirzebruch defined topological K-theory. J. F. Adams then used the Atiyah-Hirzebruch theory to solve the problem of vector fields on spheres. C^*-algebra K-theory developed quite directly out of Atiyah-Hirzebruch topological K-theory. From its inception, C^*-algebra K-theory has been closely linked to problems in geometry-topology (Novikov conjecture, Gromov-Lawson-Rosenberg conjecture, Atiyah-Singer index theorem) and to classification problems within C^*-algebras. More recently, C^*-algebra K-theory has played an essential role in the new subject of non-commutative geometry.

Algebraic K-theory was a little slower to develop [52]; much of the early development in the 1960s was due to H. Bass, who organized the theory on K_0 and K_1 and defined the negative K-groups. J. Milnor introduced K_2. Formulating an appropriate definition for higher algebraic K-theory proved to be a difficult and elusive problem. Definitions were proposed by several authors, including J. Milnor and Karoubi-Villamayor. A remarkable breakthrough was achieved by D. Quillen with his plus-construction. The resulting definition of higher algebraic K-theory (i.e. Quillen's algebraic K-theory) is perhaps the most widely accepted today. Many significant problems and results (e.g. the Lichtenbaum conjecture) have been stated within the context of Quillen algebraic K-theory. In some situations, however, a different definition is relevant. For example, in the recently proved Bloch-Kato conjecture, it is J. Milnor's definition of higher algebraic K-theory which is used.

Since the 1970s, K-theory has grown considerably, and its connections with other parts of mathematics have expanded. For the interested reader, we have included a number of current K-theory textbooks in our reference list ([8], [43], [44], [48], [50], [51]). For a taste of the current developments, it is useful to take a look at the *Handbook of K-theory* [17] or at the lectures in this volume. The *Journal of K-theory* (as well as its predecessor, *K-theory*) is dedicated to the subject, as is the website maintained by D. Grayson at `http://www.math.uiuc.edu/K-theory`. This site, started in 1993, includes a preprint archive which at the moment when this is being written contains 922 preprints. Additionally, see the *Journal of Non-Commutative Geometry* for current results involving C^*-algebra K-theory.

Finally, we have not in these notes emphasized cyclic homology. However, cyclic (co-)homology is an allied theory to K-theory and any state-of-the-art survey of K-theory would have to recognize this central fact.

14.2 The Hirzebruch–Riemann–Roch theorem

Let M be a non-singular projective algebraic variety over $\mathbb{C}$. Let E be an algebraic vector bundle on M. Write $\underline{E}$ for the sheaf (of germs) of algebraic sections of E. For each $j \geq 0$, consider $H^j(M, \underline{E})$ the j-th cohomology group of M using $\underline{E}$.

Lemma 1. *For all $j \geq 0$, $\dim_{\mathbb{C}} H^j(M, \underline{E}) < \infty$ and for $j > \dim_{\mathbb{C}}(M)$, $H^j(M, \underline{E}) = 0$.*

Define the *Euler characteristic* of M with respect to E as

$$\chi(M, E) = \sum_{j=0}^{n} (-1)^j \dim_{\mathbb{C}} H^j(M, \underline{E}), \quad \text{where } n = \dim_{\mathbb{C}}(M).$$

Theorem 12 (Hirzebruch–Riemann–Roch). *Let M be a non-singular projective algebraic variety over $\mathbb{C}$ and let E be an algebraic vector bundle on M. Then*

$$\chi(M, E) = (\mathrm{ch}(E) \cup \mathrm{Td}(M))[M]$$

where $\mathrm{ch}(E)$ *is the Chern character of E,* $\mathrm{Td}(M)$ *is the Todd class of M and $\cup$ stands for the cup product.*

14.3 The unity of K-theory

We explain how K-theory for C^*-algebras is a particular case of algebraic K-theory of rings.

Let A be a C^*-algebra. Consider the inclusion

$$
\begin{aligned}
M_n(A) &\hookrightarrow M_{n+1}(A) \\
\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} &\mapsto \begin{pmatrix} a_{11} & \dots & a_{1n} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix} .
\end{aligned}
\tag{2}
$$

This is a one-to-one $*$-homomorphism, and it is norm preserving. Define $M_\infty(A)$ as the limit of $M_n(A)$ with respect to these inclusions. That is, $M_\infty(A)$ is the set of infinite matrices where almost all a_{ij} are zero. Finally, define the *stabilization* of A (cf. [43, 6.4] or [50, 1.10]) as the closure

$$\dot{A} = \overline{M_\infty(A)} .$$

Here we mean the completion with respect to the norm on $M_\infty(A)$ and the main point is that the inclusions above are all norm-preserving. The result is a C^*-algebra without unit.

Remark 47. There is an equivalent definition of $\dot{A}$ as the tensor product $A \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of all compact operators on a separable infinite-dimensional Hilbert space, and the tensor product is in the sense of C^*-algebras.

Example 15. Let H be a separable, infinite-dimensional, Hilbert space. That is, H has a countable, but not finite, orthonormal basis. It can be shown that

$$\dot{\mathbb{C}} = \mathcal{K} \subset \mathcal{L}(H),$$

where $\mathcal{K}$ is the subset of compact operators on H. We have then

$$K_j(\mathbb{C}) = K_j(\dot{\mathbb{C}}) ,$$

where $K_j(-)$ is C^*-algebra K-theory. This is true in general for any C^*-algebra (Proposition 4 below).
On the other hand, the algebraic K-theory of $\dot{\mathbb{C}}$ is

$$K_j^{\mathrm{alg}}(\dot{\mathbb{C}}) = \begin{cases} \mathbb{Z} & j \text{ even,} \\ 0 & j \text{ odd,} \end{cases}$$

which therefore coincides with the C^*-algebra K-theory of $\mathbb{C}$. This is also true in general (Theorem 13 below). This answer is simple compared with the algebraic K-theory of $\mathbb{C}$, where only some partial results are known.

The stabilization of a C^*-algebra does not change its (C^*-algebra) K-theory.

Proposition 4. *Let A be a C^*-algebra and write $K_j(-)$ for K-theory of C^*-algebras. Then*

$$K_j(A) = K_j(\dot{A}) \quad j \geq 0 .$$

The proof is a consequence of the definition of C^*-algebra K-theory: the inclusions (2) induce isomorphisms in K-theory, and the direct limit (in the sense of C^*-algebras) commute with the K-theory functor (cf. [50, 6.2.11 and 7.1.9]).

Remark 48. In the terminology of Cortiñas' notes[12], Proposition 4 says that the functors K_0 and K_1 are $\mathcal{K}$-*stable*.

M. Karoubi conjectured that the algebraic K-theory of $\dot{A}$ is isomorphic to its C^*-algebra K-theory. The conjecture was proved by A. Suslin and M. Wodzicki.

Theorem 13 (A. Suslin and M. Wodzicki [49]). *Let A be a C^*-algebra. Then*

$$K_j(\dot{A}) = K_j^{\mathrm{alg}}(\dot{A}) \quad j \geq 0\,,$$

where the left-hand side is C^*-algebra K-theory and the right-hand side is (Quillen's) algebraic K-theory of rings.

A proof can be found in Cortiñas' notes [12, Thm. 7.1.3]. In these notes Cortiñas elaborates the isomorphism above into a long exact sequence which involves cyclic homology.

Theorem 13 is the unity of K-theory: It says that C^*-algebra K-theory is a pleasant subdiscipline of algebraic K-theory in which Bott periodicity is valid and certain basic examples are easy to calculate.

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