A biased approach to nonlinear robust stability and performance with applications to adaptive control

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Abstract

The nonlinear robust stability theory of Georgiou and Smith (IEEE Trans. Auto. Control, 42(9):1200–1229, 1997) is generalized to the case of notions of stability with bias terms. An example from adaptive control illustrates non trivial robust stability certificates for systems which the previous unbiased theory could not establish a non-zero robust stability margin. This treatment also shows that the BIBO robust stability results for adaptive controllers in French (IEEE Trans. Auto. Control, 53(2):461–478, 2008) can be refined to show preservation of biased forms of stability under gap perturbations. In the nonlinear setting, it also is shown that, in contrast to LTI systems, the problem of optimizing nominal performance is not equivalent to maximizing the robust stability margin.

1 Introduction

The fundamental nonlinear robust stability framework developed by Georgiou and Smith [9] considers the classical closed loop system:

$$[P,C] \;:\quad y_1 = Pu_1,\quad u_2 = Cy_2,\quad u_0 = u_1 + u_2,\quad y_0 = y_1 + y_2, \tag{1.1}$$

as depicted in Figure 1, and develops a generalisation of the linear gap metric and associated

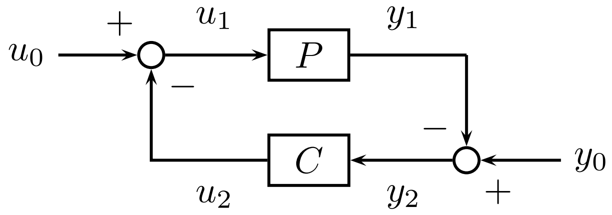

Figure 1: The closed-loop $[P,C]$.

robust stability results on the basis of a robust stability margin, $b_{P,C}$, which is taken to be the inverse of the induced norm of the closed loop operator $\Pi_{P//C}$:

$$\Pi_{P//C} \colon \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ y_1 \end{pmatrix}. \tag{1.2}$$

Under appropriate well posedness assumptions, the main robust stability theorem states that if $[P,C]$ is gain stable (that is $\|\Pi_{P//C}\| < \infty$), and if

$$\delta(P,P_1) < b_{P,C} := \|\Pi_{P//C}\|^{-1} \tag{1.3}$$

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then $[P_1, C]$ is gain stable ($\|\Pi_{P_1//C}\| < \infty$). Here δ is denotes the nonlinear gap metric, as described later in Section 3, and is a notion of distance between plants which renders typical unmodelled dynamics small: e.g. for linear plants, small multiplicative, inverse multiplicative, and co-prime factor perturbations are small in this sense, as are small time delays to proper continuous time plants (here we think of P as the model, and P_1 as the 'real' system).

To account for nonlinear gains, a regional version and a gain function version of the robust stability theorem were also given. All versions of the robust stability theorem assume that the plant and the controller map zero inputs to zero outputs ($P(0) = 0$, $C(0) = 0$) and that the closed loop operator $\Pi_{P//C}$ has an induced norm or a gain function.

However, there are important instances in which these sufficient conditions for robust stability generically fail; and yet for which robustness results should apply and for which, to date, either relatively ad-hoc methods have been utilized to establish robust stability, or no such such robust stability certificates have been established. Many such systems can be handled by developing a robust stability theory based on an underlying notion of stability which includes bias terms; for such notions of stability see [3, 15]. The first important class of examples are systems whose response depends on a non-zero initial condition, and which do not start at an equilibrium, see [8] for an alternative biased approach to such examples. The second class of systems are those for which $P(0) = 0$, $C(0) = 0$ but whose closed loop operator $\Pi_{P,C}$ is discontinuous at 0, thus precluding the existence of a (local) finite gain. Most adaptive controllers fall within this category [4]. A third class of examples includes systems which include inherent offsets, arising e.g. from quantization errors, sensors biases etc. Another such class of feedback systems include nonlinear high gain controller designs which attenuate the effects of unknown nonlinearities by high gain feedback, and which do not cancel the effect of the nonlinearities.

In this paper, we take an important class of examples from adaptive control to motivate the approach. In this setting, the need for a bias does not arise from an offset from a single specified trajectory, so the approach of [8] is not applicable. For this class of adaptive controllers we interpret the known BIBO robust stability results of [5] to provide an interpretation based on biased stability. This approach provides a more satisfactory robustness theory than the relatively ad-hoc techniques developed in [5], and shows the stronger result that stability with bias is preserved under sufficiently small gap perturbations.

The results are cast in a framework which seeks to make minimal restrictive assumptions on the system class, and minimizes the conditions to be verified in applications. For example, the standard setting of extended spaces is widened to a wider space, termed the ambient space, which permits the systematic consideration of systems with the potential for finite escape times. Further the typical assumption of global well posedness of the perturbed system is reduced to a uniqueness and causality requirement only. This substantially eases the application of the robustness result, since typically it is far easier to ensure uniqueness of solutions than to guarantee their existence a-priori, and causality of solutions usually is physically apparent.

The remainder of the paper is structured as follows. In Section 2 we introduce the system theoretic setting and notation. In Section 3 we introduce the gap metric and the notions of the robust stability margin and nominal and robust performance. Section 4 considers the motivating example from adaptive control whereby it is shown that biases are present. Section 5 establishes the main regional robust stability result. In Section 6 we consider the result specialised to the case of linear plants, before revisiting the adaptive control theory in Section 7. In Section 8 we consider the special case of global robust stability and the implications of the robust stability result for the formulation of appropriate optimization problems in nonlinear control design. We draw conclusions in Section 10.

2 Systems

Let $\mathcal{T}$ denote either the discrete half-axis time set $\mathbb{Z}_+$ or the continuous time counterpart, $\mathbb{R}_+$. In both cases $\mathcal{T} \cup \{\infty\}$ is totally ordered in the natural manner. For $\omega \in \mathcal{T} \cup \{\infty\}$, let $\mathcal{S}_\omega$ denote the set of all measurable maps (in the discrete case, simply maps) $[0,\omega) \to \mathcal{X}$ where $\mathcal{X}$ is a normed vector space, for example for finite dimensional systems it would be $\mathcal{X} = \mathbb{R}^n$ and for distributed parameter systems it would be a function space, for example $L^2[0,1]$. For ease of notation define $\mathcal{S} := \mathcal{S}_\infty$. For $\tau \in \mathcal{T}$, $\omega \in \mathcal{T} \cup \{\infty\}$, $0 < \tau < \omega$ define a truncation operator T_τ and a restriction operator R_τ as follows:

$$T_\tau : \mathcal{S}_\tau \to \mathcal{S}\,, \quad v \mapsto T_\tau v := \left(t \mapsto \begin{cases} v(t), & t \in [0,\tau) \\ 0, & \text{otherwise} \end{cases} \right),$$
$$R_\tau : \mathcal{S}_\omega \to \mathcal{S}_\tau\,, \quad v \mapsto R_\tau v := (t \mapsto v(t), \quad t \in [0,\tau))\,.$$

We define $\mathcal{V} \subset \mathcal{S}$ to be a *signal space* if, and only if, it is a vector space. Suppose additionally that $\mathcal{V}$ is a normed vector space and that the norm on $\mathcal{V}$, $\|\cdot\| = \|\cdot\|_\mathcal{V}$ is (also) defined for signals of the form $T_\tau v$, $v \in \mathcal{V}_\tau$, $\tau > 0$. We can define a norm $\|\cdot\|_\tau$ on $\mathcal{S}_\tau$ by $\|v\|_\tau = \|T_\tau v\|$, for $v \in \mathcal{S}_\tau$. For notational simplicity, we utilize the same notation to define a functional on $\mathcal{S}$, namely $\|v\|_\tau = \|R_\tau v\|_\tau$, for $v \in \mathcal{S}$. The systems given in this paper have the potential for finite escape times, thus requiring the consideration of signals which are only defined on finite intervals. Within the classical approach to input-output analysis, all signals are considered to lie within extended spaces ($\mathcal{V}_e$ below). This forces signals to be globally defined and hence precludes finite escape times. We overcome this deficiency by defining a larger space, the ambient space ($\mathcal{V}_a$ below), which contains signals which are defined on finite as well as infinite intervals.

We associate spaces as follows:

- $\mathcal{V}[0,\tau) = \{v \in \mathcal{S}_\tau \mid v = R_\tau w,\ w \in \mathcal{V},\ \|v\|_\tau < \infty\}$, *the interval space*;
- $\mathcal{V}_e = \{v \in \mathcal{S} \mid \forall\, \tau > 0\ :\ R_\tau v \in \mathcal{V}[0,\tau)\}$, *the extended space*;
- $\mathcal{V}_\omega = \{v \in \mathcal{S}_\omega \mid \forall\, \tau \in (0,\omega)\ :\ R_\tau v \in \mathcal{V}[0,\tau)\}$, for $0 < \omega \leq \infty$; and
- $\mathcal{V}_a = \bigcup_{\omega \in (0,\infty]} \mathcal{V}_\omega$, *the ambient space.*

It is important to note that in general $\mathcal{V}[0,\tau) \neq \mathcal{V}_\tau$, since in contrast to $\mathcal{V}[0,\tau)$, elements of $\mathcal{V}_\tau$ may be unbounded on the whole interval $[0,\tau)$ – this is the feature that allows consideration of finite escape times. For example, if $\mathcal{V} = L^\infty(\mathbb{R}_+,\mathbb{R})$, then the function $\tan\colon [0,\pi/2) \to \mathbb{R}$ has the property that $\tan \notin L^\infty([0,\pi/2),\mathbb{R}) = \mathcal{V}[0,\pi/2)$, but $\tan \in L^\infty([0,\tau),\mathbb{R}) = \mathcal{V}[0,\tau)$ for all $\tau < \pi/2$, hence $\tan \in \mathcal{V}_{\pi/2}$. Hence the ambient space consists of signals defined on intervals of both finite length (as $\mathcal{V}_\tau \subset \mathcal{V}_a$) and infinite length (as $\mathcal{V}_\infty \subset \mathcal{V}_a$), and thus for example $\mathcal{V}_a$ is an appropriate space to capture the behaviour of e.g. solutions to differential equations where the presence of finite escape times may lead to solutions which are only defined on finite intervals.

A signal space $\mathcal{V}$ is said to be truncation complete if $V[0,\tau)$ is complete for all $0 < \tau < \infty$.

The notion of a closed loop solution is formally defined as follows. Given signal spaces $\mathcal{U}$ and $\mathcal{Y}$, a mapping $Q\colon \mathcal{U}_a \to \mathcal{Y}_a$ is said to be *causal* if

$$\forall\ u,v \in \mathcal{U}_a\ \forall\ \tau \in \mathrm{dom}(u) \cap \mathrm{dom}(v)\ :\ [R_\tau u = R_\tau v\ \Rightarrow\ R_\tau(Qu) = R_\tau(Qv)]\,.$$

We remark that causality is often defined by the property that $T_\tau Q T_\tau = T_\tau Q$. However this equality is only defined if $\mathcal{U}_a$ is closed under the operation of truncation (for example not if $\mathcal{U} = C(\mathbb{R}_+,\mathbb{R})$).

Given signal spaces $\mathcal{U}$ and $\mathcal{Y}$, let $P : \mathcal{U}_a \to \mathcal{Y}_a$ and $C : \mathcal{Y}_a \to \mathcal{U}_a$ be causal mappings representing the plant and the controller, respectively and consider the system of equations (1.1) corresponding to Figure 1. Given external signals (or disturbances) $w_0 = (u_0, y_0)^T \in \mathcal{W} := \mathcal{U} \times \mathcal{Y}$ then a pair of internal signals $(w_1, w_2) = ((u_1, y_1)^T, (u_2, y_2)^T) \in \mathcal{W}_a \times \mathcal{W}_a$, $\mathcal{W}_a := \mathcal{U}_a \times \mathcal{Y}_a$, is a solution if, and only if, (1.1) holds on $\mathrm{dom}(w_1, w_2) := \mathrm{dom}(w_1) \cap \mathrm{dom}(w_2)$.

The twin properties of existence of solutions and uniqueness of closed loop solutions coupled with a notion of causality (introduced below) define the notion of well posedness. It is of critical importance when dealing with systems with the potential for finite escape times to distinguish between the local and the global version of these properties, and a particular contribution of this formalism is to provide results that allow the guarantee of existence of bounded global solutions of a perturbed system from the minimal assumptions of bounded global solutions of the nominal system together with a uniqueness and causality assumption on the perturbed system. This is considerably more applicable than the typical assumption in the input-output literature whereby global well posedness is assumed a-priori for the perturbed system, since a common feature of nonlinear analysis is the relative ease in which uniqueness of solutions of differential equations can be guaranteed vs. the difficulty of establishing existence of solutions. Indeed a typical result establishing the existence of solutions of differential equations operates by establishing boundedness and existence simultaneously, hence here we are providing the setting so that our robust stability theorems can be established on the same basis, thus greatly increasing their formal applicability.

Let $\mathcal{X}_{w_0} := \{(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a \mid (w_1, w_2) \text{ solves (1.1)}\}$ be the set of all solutions, which may be empty. The closed loop system $[P, C]$ is said to have the *existence property*, if $\mathcal{X}_{w_0} \neq \emptyset$ for all $w_0 \in \mathcal{W}$, and the *uniqueness property*, if

$$\forall\, w_0 \in \mathcal{W} \;:\; (\hat{w}_1, \hat{w}_2), (\tilde{w}_1, \tilde{w}_2) \in \mathcal{X}_{w_0} \\ \implies (\hat{w}_1, \hat{w}_2) = (\tilde{w}_1, \tilde{w}_2) \quad \text{on} \quad \mathrm{dom}(\hat{w}_1, \hat{w}_2) \cap \mathrm{dom}(\tilde{w}_1, \tilde{w}_2)\,.$$

As discussed above, throughout the paper we will require that all closed loop systems under consideration satisfy the uniqueness property (but not necessarily the existence property).

For each $w_0 \in \mathcal{W}$, let $\omega_{w_0} \in \mathcal{T} \cup \{\infty\}$ define the maximal interval of existence $[0, \omega_{w_0})$ for the closed loop system, i.e. $[0, \omega_{w_0}) := \bigcup_{(\hat{w}_1, \hat{w}_2) \in \mathcal{X}_{w_0}} \mathrm{dom}(\hat{w}_1, \hat{w}_2)$ and define $(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a$, with $\mathrm{dom}(w_1, w_2) = [0, \omega_{w_0})$, by the property $R_t(w_1, w_2) \in \mathcal{X}_{w_0}$ for all $t \in [0, \omega_{w_0})$. This induces the operator

$$H_{P,C} : \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a, \quad w_0 \mapsto (w_1, w_2)\,.$$

Thus $H_{P,C}$ is the map from the external disturbances w_0 to the internal signals (w_1, w_2) defined on their maximal interval of existence.

For $\Omega \subset \mathcal{W}$ the closed loop system $[P, C]$ given by (1.1), is said to be:

- *locally well posed on* Ω if, and only if, it has the existence and uniqueness properties and the operator $H_{P,C}\big|_{\Omega} : \Omega \to \mathcal{W}_a \times \mathcal{W}_a$, $w_0 \mapsto (w_1, w_2)$, is causal;
- *globally well posed on* Ω if, and only if, it is locally well posed on Ω and $H_{P,C}(\Omega) \subset \mathcal{W}_e \times \mathcal{W}_e$;

Note that the local/global adjective are descriptions with respect to time. A locally well posed system is therefore one for which in $w_0 \in \Omega$ then the internal signals (w_1, w_2) are defined on some (potentially finite) time interval $[0, \omega_{w_0})$, in contrast to globally well posed systems which have internal signals (w_1, w_2) defined on the time interval $[0, \infty)$.

We next define both regional and global notions of stability both with and without biases. For normed signal spaces $\mathcal{X}$, $\mathcal{V}$ and $\Omega \subset \mathcal{X}$ define the following:

(i) A causal operator $Q\colon \mathcal{X} \to \mathcal{V}_a$ is called *gain stable on* Ω if, and only if, $Q(\Omega) \subset \mathcal{V}$, $Q(0) = 0$ and
$$\|Q|_{\Omega}\|_{\mathcal{X},\mathcal{V}} := \sup \left\{ \frac{\|R_\tau Qx\|_\tau}{\|R_\tau x\|_\tau} \;\middle|\; x \in \Omega\,,\ \tau > 0\,,\ R_\tau x \neq 0 \right\} < \infty\,.$$

(ii) A causal operator $Q\colon \mathcal{X} \to \mathcal{V}_a$ is called (γ, β) *gain stable with bias on* Ω if, and only if, $Q(\Omega) \subset \mathcal{V}$ and
$$\|R_\tau Qx\|_\tau \leq \gamma \|R_\tau x\|_\tau + \beta, \qquad \forall x \in \Omega\,,\ \tau > 0.$$

(iii) A causal operator $Q\colon \mathcal{X} \to \mathcal{V}_a$ is called *regionally gain stable with uniform bias on* Ω if, and only if, $Q(\Omega) \subset \mathcal{V}$ and there exists $\beta \geq 0$ such that for all $R > 0$ there exists $\gamma(R) > 0$ such that:
$$\|R_\tau Qx\|_\tau \leq \gamma(R) \|R_\tau x\|_\tau + \beta, \qquad \forall x \in \Omega \cap \mathcal{B}(R)\,,\ \tau > 0.$$
Hereafter, $\mathcal{B}(R)$ denotes the closed ball centred at the origin with radius R.

(iv) A causal operator $Q\colon \mathcal{X} \to \mathcal{V}_a$ is called *gain-function stable on* Ω (or *gf-stable on* Ω) if, and only if, $Q(\Omega) \subset \mathcal{V}$ and the nonlinear *gain-function*
$$\begin{aligned} g\left[Q|_{\Omega}\right] : (r_0, \infty) &\to [0, \infty)\,, \\ r &\mapsto g\left[Q|_{\Omega}\right](r) := \sup\left\{ \|R_\tau Qx\|_\tau \;\middle|\; x \in \Omega\,,\ \|R_\tau x\|_\tau \in (r_0, r]\,,\ \tau > 0 \right\}\,, \end{aligned}$$
is defined, where $r_0 := \inf_{x \in \Omega} \|x\|_{\mathcal{X}} < \infty$.

It is important to observe that in contrast to some treatments of gain functions, we do not require $g[Q](0) = 0$ (if $r_0 = 0$).

The above notions of stability will be applied to the following operators:
$$\Pi_{P//C}\colon \mathcal{W} \to \mathcal{W}_a, \quad w_0 \mapsto w_1, \qquad \text{and} \qquad \Pi_{C//P}\colon \mathcal{W} \to \mathcal{W}_a, \quad w_0 \mapsto w_2$$
in order to define the appropriate notions of stability of the closed loop system (note that $H_{P,C} = \left(\Pi_{P//C}, \Pi_{C//P}\right)$ and $\Pi_{P//C} + \Pi_{C//P} = I$). For normed signal spaces $\mathcal{U}$, $\mathcal{Y}$ and $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$ and the causal operator $P\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C\colon \mathcal{Y}_a \to \mathcal{U}_a$ define the following:

(i) The closed-loop system $[P, C]$ given by (1.1) with the associated operator $\Pi_{P//C}\colon \mathcal{W} \to \mathcal{W}_a$ is said to be BIBO $\mathcal{W}$*-stable* if, and only if, it is globally well posed and $H_{P,C}(\mathcal{W}) \subset \mathcal{W} \times \mathcal{W}$.

(ii) Let $\Omega \subset \mathcal{W}$. The closed-loop system $[P, C]$ given by (1.1) with the associated operator $\Pi_{P//C}\colon \mathcal{W} \to \mathcal{W}_a$ is said to be *regionally gain stable (with (uniform) bias) on* Ω if, and only if, it is globally well posed on Ω and $\Pi_{P//C}$ is gain stable (with (uniform) bias) on Ω.

(iii) Let $\Omega \subset \mathcal{W}$. The closed-loop system $[P, C]$ given by (1.1) with the associated operator $\Pi_{P//C}\colon \mathcal{W} \to \mathcal{W}_a$ is said to be *gain-function stable on* Ω if, and only if, it is globally well posed on Ω and $\Pi_{P//C}$ is gain-function stable on Ω.

For all the above stability definitions, if an object is (gain) stable (with bias) on $\Omega = \mathcal{W}$, then it is said to be globally (gain) stable (with bias).

The notion of gain function stability and regional gain stability with uniform bias are closely related as the following lemma shows:

Lemma 2.1 *The following statements are equivalent:*

1. $[P,C]$ *is gain function stable on* $\mathcal{W}$.
2. $[P,C]$ *is regionally gain stable with uniform bias on* $\mathcal{W}$.

Proof. If $[P,C]$ is $(\gamma(\cdot),\beta)$ regionally gain stable with uniform bias, then $[P,C]$ is gain function stable with gain function $g(r)=\gamma(r)r+\beta$. Conversely suppose $[P,C]$ is gain function stable with gain function $g[\Pi_{P//C}]$. Let $r_0>0$ and let $g(r)=g[\Pi_{P//C}](r_0)$ for $r\le r_0$ and $g(r)=g[\Pi_{P//C}](r)$ for $r>r_0$. Then, for any $R>0,\tau>0$ and any w with $\|R_\tau w\|\le R$, since $g[\Pi_{P//C}]$ is increasing, we have

$$\|R_\tau\Pi_{P//C}w\|\le g[\Pi_{P//C}](\|R_\tau w\|)\le g(r_0)=g(0)\quad \text{if } \|R_\tau w\|\le r_0$$

and

$$\begin{aligned}\|R_\tau\Pi_{P//C}w\| &= \frac{\|R_\tau\Pi_{P//C}R_\tau w\|}{\|R_\tau w\|-g(0)}\|R_\tau w\|+g(0)\\ &\le \frac{g(R)-g(0)}{r_0}\|R_\tau w\|+g(0)\quad \text{if } \|R_\tau w\|>r_0.\end{aligned}$$

Hence $[P,C]$ is $(\gamma(\cdot),g(0))$ regionally gain stable with uniform bias where $\gamma(R)=\frac{g(R)-g(0)}{r_0}$. □

Finally

For the plant operator P and the controller operator C and signal spaces $\mathcal{U}$, $\mathcal{Y}$, we define the *graph* $\mathcal{G}_P$ of the plant and the *graph* $\mathcal{G}_C$ of the controller, respectively, as follows:

$$\mathcal{G}_P:=\left\{\begin{pmatrix}u\\Pu\end{pmatrix}\;\middle|\; u\in\mathcal{U},\, Pu\in\mathcal{Y}\right\}\subset\mathcal{W},\quad \mathcal{G}_C:=\left\{\begin{pmatrix}Cy\\y\end{pmatrix}\;\middle|\; Cy\in\mathcal{U},\, y\in\mathcal{Y}\right\}\subset\mathcal{W}.$$

The graph of a system operator is the set of all bounded input/output pairs and plays a key role in the definition of the nonlinear gap, see below.

3 Gap distances and robust performance functions

Throughout this paper, our principal measure of worst case nominal performance of a closed loop system $[P,C]$ is given by:

$$A_{P,C}(r):=\sup_{\|w_0\|\le r}\|\Pi_{P//C}w_0\|, \tag{3.1}$$

that is, $A_{P,C}(r)$ is the maximum size of the internal signal $w_1=\Pi_{P//C}w_0$, given external signals w_0 of size less than or equal to r.

Let Γ denote a set of causal operators $\mathcal{U}_a\to\mathcal{Y}_a$ corresponding to the admissible plants. Given $P,P_1\in\Gamma$ and a distance measure $\vec{\delta}\colon\Gamma\times\Gamma\to[0,\infty]$, the robust stability margin is defined as follows:

$$B_{P,C}(r):=\sup\left\{\varepsilon\ge 0\;\middle|\;\begin{array}{c}\vec{\delta}(P,P_1)<\varepsilon\ \text{ and } [P_1,C]\ \text{ is causal}\\ \text{and has the uniqueness property}\end{array}\implies A_{P_1,C}(r)<\infty\right\}, \tag{3.2}$$

i.e. $B_{P,C}(r)$ is the size of the largest worst case plant perturbation (as measured by $\vec{\delta}$) which maintains bounded performance for disturbances of size less than or equal to r. The robust performance function is defined to be:

$$AB_{P,C}(r,\eta):=\sup\left\{A_{P_1,C}(r)\mid P_1\in\Gamma,\vec{\delta}(P,P_1)<\eta B_{P,C}(r)\right\},\ \eta\in[0,1). \tag{3.3}$$

The robust performance function therefore quantifies the worst case performance as the permissible plant perturbations are varied from their maximum size ($\eta = 1$) to the nominal case of no perturbation ($\eta = 0$), again at disturbance level r.

Throughout this paper, the distance measure $\vec{\delta}$ is taken to be the gap metric, or one of its variations/generalisations which we next define, and relate to our definitions of nominal performance, robust stability margin and robust performance.

3.1 The Linear Gap

We let $\mathcal{R}(\mathcal{U}, \mathcal{Y})$ denote the set of linear operators $P\colon \mathcal{U}_e \to \mathcal{Y}_e$ specified by rational transfer functions. $\mathcal{H}^\infty$ denotes the Banach space of complex valued functions that are analytic and bounded on the open half plane $\mathbb{C}_+$ with norm: $\|f\|_{\mathcal{H}^\infty} = \operatorname{ess\,sup}_{\omega\in\mathbb{R}^n} |f(j\omega)|$. $\mathcal{RH}^\infty$ is the subset of $\mathcal{H}^\infty$ consisting of rational functions. The pair $N, D \in \mathcal{RH}^\infty$ are said to be normalized right co-prime factors over $\mathcal{RH}^\infty$ of a transfer function P, if $P = ND^{-1}$ and $N^*N + D^*D = I$.[1] The set of all such (N, D) is denoted by NRCF(P).

For the case of $\Gamma = \mathcal{R}(L^2(\mathbb{R}_+), L^2(\mathbb{R}_+))$, $C \in \mathcal{R}(L^2(\mathbb{R}_+), L^2(\mathbb{R}_+))$ and we let $\vec{\delta}_0$ denote the directed H_2 gap metric:

$$\vec{\delta}_0(P_1, P_2) = \inf\left\{ \left\| \begin{pmatrix} \Delta_N \\ \Delta_D \end{pmatrix} \right\|_{\mathcal{H}^\infty} \;\middle|\; \begin{pmatrix} \Delta_N \\ \Delta_D \end{pmatrix} \in \mathcal{RH}^\infty, \; \begin{array}{l} P_2 = (N_1 + \Delta_N)(D_1 + \Delta_D)^{-1} \\ (N_1, D_1) \in \mathrm{NRCF}(P_1) \end{array} \right\}. \tag{3.4}$$

It is well known that $A_{P,C}(r) = b_{P,C}^{-1} r$, $B_{P,C}(r) = b_{P,C}$, where $b_{P,C} = \|\Pi_{P//C}\|^{-1}$. For any $\eta \in [0, 1)$, we see $\eta b_{P,C} < B_{P,C}$. Suppose, as in (3.3) that $\vec{\delta}(P, P_1) < \eta b_{P,C}$. Then by Theorem 1 of [9], we have

$$\|\Pi_{P_1//C}\| \le \frac{1 + \eta b_{P,C}}{1 - \eta b_{P,C} \|\Pi_{P//C}\|} \|\Pi_{P//C}\| \le \frac{1 + \eta b_{P,C}}{1 - \eta} \|\Pi_{P//C}\|.$$

This shows $A_{P_1,C}(r) \le \frac{1+\eta b_{P,C}}{1-\eta} A_{P,C}(r)$. Hence

$$AB_{P,C}(r, \eta) \le \frac{1 + \eta b_{P,C}}{1 - \eta} A_{P,C}(r) \quad \text{for all } \eta, r \ge 0. \tag{3.5}$$

3.2 The Un-Biased Regional Nonlinear Gap

The nonlinear gap metric was introduced in [9], and is defined as follows. Let Γ denote the set of all causal operators $\mathcal{U}_a \to \mathcal{Y}_a$. Given $P_1, P_2 \in \Gamma$ and a subset $\Omega \subset \mathcal{W}$, define the (possibly empty) set

$$\mathcal{O}^\Omega_{P_1,P_2} := \left\{ \Phi\colon \mathcal{G}_{P_1} \cap \Omega \to \mathcal{G}_{P_2} \;\middle|\; \begin{array}{l} \Phi \text{ is causal, and} \\ R_\tau(\Phi - I) \text{ is compact for all } \tau > 0 \end{array} \right\}, \tag{3.6}$$

and the nonlinear directed gap [9]:

$$\vec{\delta}_\Omega(P_1, P_2) = \begin{cases} \inf_{\Phi\in\mathcal{O}^\Omega_{P_1,P_2}} \sup_{\substack{x\in\mathcal{G}_{P_1}\cap\Omega,\, \tau>0 \\ R_\tau x\neq 0}} \frac{\|R_\tau(\Phi-I)|_{\mathcal{G}_{P_1}} x\|_\tau}{\|R_\tau x\|_\tau} & \text{if } \mathcal{O}^\Omega_{P_1,P_2} \neq \emptyset, \\ \infty & \text{if } \mathcal{O}^\Omega_{P_1,P_2} = \emptyset. \end{cases} \tag{3.7}$$

Various equivalent expressions for the (global) nonlinear gap can be found in [1, 9, 12], including formulae based on nonlinear coprime factorisations closely related in form to (3.4). The $L^2(\mathbb{R}_+)$

[1] Here N^* denotes the conjugate transpose of N.

nonlinear gap is a generalisation of the standard definition of the H_2 gap $\vec{\delta}_0(\cdot,\cdot)$ [9, 16], in the sense that if $P_1, P_2 \in \mathcal{R}(L^2(\mathbb{R}_+), L^2(\mathbb{R}_+))$, and either $\vec{\delta}_0(P_1,P_2) < 1$ or $\vec{\delta}_0(P_2,P_1) < 1$, then it has been shown in [9, Proposition 5] that if $\Omega := \{w \in L^2(\mathbb{R}_+) \mid \|w\|_{L^2(\mathbb{R}_+)} \leq R\}$, $R > 0$, then $\vec{\delta}_0(P_1,P_2) = \vec{\delta}_\Omega(P_1,P_2)$.

For nonlinear systems, we have $A_{P,C}(r) \leq \|\Pi_{P//C}|_{\mathcal{B}(r)}\| r$. However, in contrast to the linear setting, the robust stability margin is also in general dependent on the disturbance level $r > 0$, and the parallel projection gain only provides a lower bound:

$$B_{P,C}(r) \geq \left(\sup_{0<\|(u_0,y_0)\|\leq R} \frac{\|\Pi_{P//C}(u_0,y_0)\|}{\|(u_0,y_0)\|} \right)^{-1} =: \|\Pi_{P//C}|_{\mathcal{B}(R)}\|^{-1}, \tag{3.8}$$

for some appropriate choice of $R > r$, see [9].

It follows that the minimization of the gain of $\Pi_{P//C}$ remains, as in the LTI case, a sensible design objective, since a small gain ensures both a good robust stability margin and good nominal performance. However, as we will see in Section 4, it is not always possible to achieve $\|\Pi_{P//C}|_{\mathcal{B}(R)}\| < \infty$, even when a sensible robust stability margin should exist.

Note that the reverse inequality to inequality (3.8) does not necessarily hold. So the robust performance function $AB_{P,C}(r,\eta)$ for nonlinear systems cannot in general be estimated as simply as in the linear case. However, if $A_{P,C}(r)$ satisfies certain growth assumptions, then we are still able to estimate $AB_{P,C}(r,\eta)$ as shown in the following lemma.

Lemma 3.1 *Let $\Omega \subset \mathcal{W}$ and $\|\Pi_{P//C}|_\Omega\| < \infty$. Suppose that for $r > 0, \eta \in [0,1)$ with $\eta B_{P,C}(r) < \|\Pi_{P//C}|_\Omega\|^{-1}$, there exists $R \geq r$ such that*

$$\mathcal{B}(R) \subset \Omega \quad \text{and} \quad \|\Pi_{P//C}|_{\mathcal{B}(R)}\| \leq \frac{R-r}{\eta B_{P,C}(r) R}.$$

Then

$$AB_{P,C}(r,\eta) \leq (1 + \eta B_{P,C}(r)) A_{P,C}\left(\frac{r}{1 - \eta B_{P,C}(r)\|\Pi_{P//C}|_\Omega\|} \right). \tag{3.9}$$

Proof. Let $P_1 \in \Gamma$, and suppose $\vec{\delta}_\Omega(P,P_1) < \eta B_{P,C}(r)$. By assumption, there exists $R > r$ such that

$$r \leq (1 - \eta B_{P,C}(r)\|\Pi_{P//C}|_{\mathcal{B}(R)}\|)R.$$

For any $\tau > 0$ and $w \in \mathcal{W}$ with $\|w\|_\tau \leq r$, by the proof of Theorem 4 in [9], there exists $x \in \mathcal{B}(R)$ such that

$$\|\Pi_{P_1//C} w\|_\tau \leq (1 + \eta B_{P,C}(r))\|\Pi_{P//C} x\|_\tau,$$

and

$$\|x\|_\tau \leq \frac{\|w\|_\tau}{1 - \eta B_{P,C}(r)\|\Pi_{P//C}|_{\mathcal{B}(R)}\|} \leq \frac{r}{1 - \eta B_{P,C}(r)\|\Pi_{P//C}|_\Omega\|}.$$

This shows

$$A_{P_1,C}(r) \leq (1 + \eta B_{P,C}(r)) A_{P,C}\left(\frac{r}{1 - \eta B_{P,C}(r)\|\Pi_{P//C}|_\Omega\|} \right).$$

□

Note also that for linear systems, inequality (3.9) reduces to inequality (3.5), since we can take $\Omega = \mathcal{W}$ and since $B_{P,C}(r)\|\Pi_{P//C}|_\Omega\|$.

4 A motivating example

The potential lack of tightness of the lower bound (3.8) is not pathological [4, 5]; many adaptive controllers have the property:

$$A_{P,C}(r) < \infty; \qquad B_{P,C}(r) > 0 \quad \text{for all } r \geq 0 \tag{4.1}$$

whilst

$$\sup_{\|u_0,y_0\|\leq r} \left(\frac{\|\Pi_{P//C}(u_0, y_0)\|}{\|(u_0, y_0)\|} \right) = \infty \quad \text{for all } r > 0. \tag{4.2}$$

This arises due to a problem with small signal behaviour, where whilst $\Pi_{P//C}(0) = 0$, the operator $\Pi_{P//C}$ is not continuous at 0 – which precludes the existence of a 'local finite gain'.

An explicit example of this (in an L^2 setting) is given by the plant

$$P(\theta)(u_1) = y_1 \text{ where } \dot{y}_1 = \theta y_1 + u_1 \; y_1(0) = 0, \tag{4.3}$$

with $\theta > 0$ and the controller:

$$\begin{aligned} C(y_2)(t) &= u_2(t) \\ u_2(t) &= -ky_2(t) \\ \dot{k}(t) &= \alpha y_2^2, k(0) = 0, \end{aligned} \tag{4.4}$$

It has been shown that this closed loop is BIBO stable in an $L^2(\mathbb{R}_+)$ setting, see [5, 7], indeed $A_{P,C}(r) < \infty$ for all $r \geq 0$. Clearly, for any disturbance (arbitrarily small) which moves $y_1 \neq 0$ there are trajectories which grow without bound unless there exists a time at which $k(t) \geq \theta$, ie. $\|y_2\|_{L^2[0,t]} \geq \frac{\sqrt{\theta}}{\alpha}$. Hence for all $\varepsilon > 0$, $\exists u_0, y_0$, $\|(u_0, y_0)\| \leq \varepsilon$ such that:

$$\|\Pi_{C//P}\| \geq \frac{\|\Pi_{C//P}(u_0, y_0)^T\|}{\|(u_0, y_0)^T\|} = \frac{\|(u_2, y_2)^T\|}{\|(u_0, y_0)^T\|} \geq \frac{\sqrt{\theta}}{\alpha\varepsilon} \to \infty \quad \text{as } \varepsilon \to 0. \tag{4.5}$$

Hence $\|\Pi_{P//C}\| = \infty$, and this is caused by a lack of continuity at 0.

However, let us first note that this discontinuity is addressed in [5, 7] by appending θ onto the input space, for then an inequality of the form:

$$\|(u_1, y_1, \theta)\|_{\mathcal{U}\times\mathcal{Y}\times\mathbb{R}} \leq g(\|(u_0, y_0)\|_{\mathcal{U}\times\mathcal{Y}}, |\theta|), \tag{4.6}$$

was constructed, and from this it was shown in [5, 7] that $B_{P(\theta),C}(r) > 0$, ie. we have a non zero but disturbance dependent robustness margin. Hence both (4.1) and (4.2) hold. This paper will give a clearer analysis of this robustness, showing that this system is stable under the appropriate notion of stability (stability with bias) and this form of stability is robust to gap perturbations.

5 Regional robust stability

In the context of a notion of stability with bias, it is natural to adopt this notion of stability to assess the 'gap' between the graphs of P and P_1. Hence for $\Omega \subset \mathcal{W}$ define the set:

$$\vec{\Delta}(P_1, P_2; \Omega) = \left\{ (\vec{\delta}, \vec{\sigma}) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid \exists\, \Phi \in \mathcal{O}^{\Omega}_{P_1,P_2} \; \|R_\tau(I - \Phi)x\| \leq \vec{\delta}\|x\|_\tau + \vec{\sigma}, \; \forall x \in \mathcal{G}_{P_1} \cap \Omega \right\}.$$

where $\mathcal{O}^{\Omega}_{P_1,P_2}$ is given by (3.6). Thus the set $\vec{\Delta}(P_1, P_2; \Omega)$ consists of all possible gains and biases which describe the deviation from the identity of maps Φ between the graphs of P and P_1 when restricted to the domain Ω. In contrast to the nonlinear gap (3.7), there are now two parameters $\vec{\delta}$, $\vec{\sigma}$ describing this 'gap', and there is no single natural way to reduce this to a scalar distance measure. Since there are different ways of defining a scalar distance measure from these two parameters, see Section 6 below, we elect to present the main regional robust stability result without making such choices, i.e. the description of the 'gap' between P_1 and P_2 remains described by the two dimensional set $\vec{\Delta}(P_1, P_2; \Omega)$ of all possible gains and biases.

In the rest of this paper, unless specified otherwise, we always let $\mathcal{U}$, $\mathcal{Y}$ be truncation complete normed signal spaces, let $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$, and suppose that for all $\tau > 0$, there exists a continuous mapping $E_\tau : \mathcal{W}[0, \tau] \to \mathcal{W}$ such that

$$R_\tau x = R_\tau(E_\tau x), \text{ for all } x \in \mathcal{W}[0, \tau).$$

An operator $Q : \mathcal{W} \to \mathcal{W}$ is said to be *relatively continuous* if for all $\tau > 0$ and for all operators $\Psi : \mathcal{W} \to \mathcal{W}$ with $R_\tau \Psi$ compact, the operator $R_\tau \Psi Q : \mathcal{W} \to \mathcal{W}[0, \tau)$ is continuous.

Theorem 5.1 *Consider $P\colon \mathcal{U}_a \to \mathcal{Y}_a$, $P_1\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C\colon \mathcal{Y}_a \to \mathcal{U}_a$. Let $R > 0$, $0 < \varepsilon < 1$. Suppose $(\vec{\delta}, \vec{\sigma}), (\gamma, \beta) \in \mathbb{R}^2_+$ are such that:*

$$(\vec{\delta}, \vec{\sigma}) \in \Delta(P, P_1; \Omega)$$

where $\Omega = \mathcal{B}\left(\gamma \frac{\rho(\varepsilon R)}{\varepsilon} + \beta\right) \subset \mathcal{W}$, $\rho(R) = R + \vec{\sigma} + \vec{\delta}\beta$, $[P, C]$ is (γ, β) gain stable with bias on $\mathcal{B}\left(\frac{\rho(\varepsilon R)}{\varepsilon}\right) \subset \mathcal{W}$ and $\Pi_{P,C}$ is relatively continuous. If $[P_1, C]$ is causal, has the uniqueness property and

$$\vec{\delta}\gamma < 1 - \varepsilon \tag{5.1}$$

then the closed-loop system $[P_1, C]$ is (γ_1, β_1) gain stable with bias on $\mathcal{B}(\varepsilon R)$ and

$$\gamma_1 = \gamma \frac{1 + \vec{\delta}}{\varepsilon}, \tag{5.2}$$

$$\beta_1 = \beta + \left(\vec{\sigma} + \vec{\delta}\beta\right)\left(1 + \gamma \frac{1 + \vec{\delta}}{\varepsilon}\right). \tag{5.3}$$

Proof. Let $w \in \mathcal{W}$, $\|w\| \le \varepsilon R$ and suppose $0 < \tau < \infty$. Since $(\vec{\delta}, \vec{\sigma}) \in \Delta(P, P_1; \Omega)$ it follows that there exists a causal mapping $\Phi\colon \mathcal{G}_P \cap \Omega \to \mathcal{G}_{P_1}$ such that $R_\tau(\Phi - I)$ is compact and for all $x \in \mathcal{W}$, $\|x\| \le \frac{\rho(\varepsilon R)}{\varepsilon}$:

$$\begin{aligned} \|R_\tau(\Phi - I)\Pi_{P//C}|_{\mathcal{B}(\frac{\rho(\varepsilon R)}{\varepsilon})} x\| &\le \vec{\delta} \cdot (\gamma\|x\| + \beta) + \vec{\sigma} \\ &\le (1 - \varepsilon)\|x\| + \vec{\sigma} + \vec{\delta}\beta \\ &= \rho\left((1 - \varepsilon)\|x\|\right). \end{aligned}$$

Consider the equation

$$R_\tau w = R_\tau(I + (\Phi - I)\Pi_{P//C})x = R_\tau(\Pi_{C//P} + \Phi\Pi_{P//C})x. \tag{5.4}$$

Let

$$V = \left\{ x \in \mathcal{W}[0, \tau) \Big| \|x\|_\tau \le \frac{\rho(\|w\|)}{\varepsilon} \right\}$$

and consider the operator

$$Q_w\colon V \to \mathcal{W}[0,\tau) \quad : \quad x \mapsto R_\tau w + R_\tau (I-\Phi)\Pi_{P//C} E_\tau x.$$

By definition of $\mathcal{W}[0,\tau)$ and our assumptions, the operator is well defined and continuous in $\mathcal{W}[0,\tau)$. Let $x \in V$ and $\bar{x} = E_\tau x$. Then $\|\bar{x}\|_\tau \leq \frac{\rho(\|w\|)}{\varepsilon} \leq \frac{\rho(\varepsilon R)}{\varepsilon}$, and $\|R_\tau(I-\Phi)\Pi_{P//C}\bar{x}\|_\tau \leq \rho\left((1-\varepsilon)\|\bar{x}\|_\tau\right)$, so:

$$\begin{aligned}
\|Q_w x\|_\tau &\leq \|R_\tau w\|_\tau + \|R_\tau(I-\Phi)\Pi_{P//C}\bar{x}\|_\tau \\
&\leq \|w\| + \rho\left((1-\varepsilon)\|\bar{x}\|_\tau\right) \\
&\leq \|w\| + (1-\varepsilon)\frac{\rho(\|w\|)}{\varepsilon} + \vec{\sigma} + \vec{\delta}\beta, \\
&\leq \rho(\|w\|) + (1-\varepsilon)\frac{\rho(\|w\|)}{\varepsilon}, \\
&= \frac{\rho(\|w\|)}{\varepsilon}.
\end{aligned} \tag{5.5}$$

Therefore $Q_w(V) \subset V$. Since $R_\tau(I-\Phi)$ is compact and $\Pi_{P//C}$ is bounded, it follows that Q_w is compact. Since $\mathcal{W}$ is truncation complete, $\mathcal{W}[0,\tau)$ is a Banach space, it follows by Schauder's fixed point theorem that Q_w has a fixed point in V. Hence equation (5.4) has a solution $x \in V \subset \mathcal{W}[0,\tau)$ as claimed.

By the uniqueness property for $[P_1, C]$, $\Pi_{P_1//C}\colon \mathcal{W} \to \mathcal{W}_a$ is defined. Since $w_1 = \Phi\Pi_{P//C}\bar{x} \in \mathcal{G}_{P_1}$, $w_2 = \Pi_{C//P}\bar{x} \in \mathcal{G}_C$ and Φ, $\Pi_{P_1//C}$, $\Pi_{P//C}$, $\Pi_{C//P}$ are causal, it follows from equation (5.4) that $(w, R_\tau w_1, R_\tau w_2) = (w, R_\tau\Phi\Pi_{P//C}\bar{x}, R_\tau\Pi_{C//P}\bar{x})$ is a solution for $[P_1, C]$. Since this holds for all $\tau > 0$, it follows that $\omega_w = \infty$ for $[P_1, C]$. Consequently $\mathrm{dom}(\Pi_{P_1//C}) = [0,\infty)$ and thus $[P_1, C]$ is globally well posed. Since $x \in V$ and $R_\tau\Pi_{P_1//C}w = R_\tau\Phi\Pi_{P//C}\bar{x}$,

$$\begin{aligned}
\|\Pi_{P_1//C}w\|_\tau &= \|\Phi\Pi_{P//C}\bar{x}\|_\tau \\
&\leq \|\Pi_{P//C}\bar{x}\|_\tau + \|(\Phi - I)\Pi_{P//C}\bar{x}\|_\tau \\
&\leq (1+\vec{\delta})\|\Pi_{P//C}\bar{x}\|_\tau + \vec{\sigma} \\
&\leq (1+\vec{\delta})\left(\gamma\|x\|_\tau + \beta\right) + \vec{\sigma} \\
&\leq (1+\vec{\delta})\left(\gamma\frac{\rho(\|w\|_\tau)}{\varepsilon} + \beta\right) + \vec{\sigma} \\
&\leq (1+\vec{\delta})\left(\gamma\frac{\|w\|_\tau + \vec{\sigma} + \vec{\delta}\beta}{\varepsilon} + \beta\right) + \vec{\sigma}
\end{aligned} \tag{5.6}$$

hence $\Pi_{P_1//C}$ satisfies

$$\|\Pi_{P_1//C}x\|_\tau \leq \gamma_1\|x\|_\tau + \beta_1. \tag{5.7}$$

where γ_1 and β_1 are given by (5.2), (5.3).

As we have shown $[P_1, C]$ is globally well posed, inequality (5.7) holds for all $\tau > 0$, and the proof is complete. □

We remark that if the operator $R_\tau\Phi$ used to define Q_w is locally incrementally stable, i.e.

$$\sup_{\substack{R_\tau w_1 \neq R_\tau w_2 \\ \|R_\tau w_1\|, \|R_\tau w_2\| \leq r}} \frac{\|R_\tau\Phi w_1 - R_\tau\Phi w_2\|_\tau}{\|R_\tau w_1 - R_\tau w_2\|_\tau} < \infty,$$

then the relative continuity requirement for $\Pi_{P,C}$ can be replaced by the weaker requirement that $R_\tau \Pi_{P,C}$ is continuous.

We also remark that in contrast to the approach to derive the (unbiased) global results in [9], here we impose a compactness requirement in the definition of $\mathcal{O}^{\mathcal{W}}_{P_1,P_2}$. In turn this stronger requirement on the maps Φ results in substantially weaker assumptions on $[P_1, C]$. In [9] it was required that $[P_1, C]$ was globally well posed, and e.g. in [6] the alternative requirement of regularly well posed was used. Here, the requirement is that $[P_1, C]$ satisfies uniqueness and causality properties only. This is weaker than either the assumption of global or regular well posedness, and is often straightforward to verify (in contrast to the existence property of well posedness assumptions which is often hard to verify a-priori).

6 Robust stability and performance

We have already observed that in Theorem 5.1 the set $\vec{\Delta}(P, P_1; \Omega) \subset \mathbb{R}^2_+$ plays the role of the gap distance in the unbiased robust stability theorem, and that as a 2-dimensional description of the 'gap' between P and P_1, it does not define a distance between the two plants. Some possible scalar measures are:

1. *Gap defined with respect to the smallest gain:*

$$\vec{\delta}_R(P_1, P_2) = \inf\left\{\vec{\delta} \geq 0 \mid \exists \vec{\beta} \text{ s.t. } (\vec{\delta}, \vec{\beta}) \in \vec{\Delta}(P_1, P_2; \mathcal{B}(R))\right\}.$$

   Correspondingly we define the bias:

$$\vec{\beta}_R(P_1, P_2) = \inf\left\{\vec{\beta} \geq 0 \mid \exists \vec{\beta} \text{ s.t. } (\vec{\delta}_R(P_1, P_2), \vec{\beta}) \in \overline{\vec{\Delta}(P_1, P_2; \mathcal{B}(R))}\right\}.$$

   Note that in general $(\vec{\delta}_R(P_1, P_2), \vec{\beta}_R(P_1, P_2)) \notin \vec{\Delta}(P_1, P_2; \mathcal{B}(R))$, however, $(\vec{\delta}, \vec{\beta}) \in \vec{\Delta}(P_1, P_2; \mathcal{B}(R))$ for all $\vec{\delta} > \vec{\delta}_R(P_1, P_2)$, $\vec{\beta} > \vec{\beta}_R(P_1, P_2))$.

2. *Gap defined with respect to the smallest bias:*

$$\vec{\delta}_R(P_1, P_2) = \inf\left\{\vec{\delta} \geq 0 \mid (\vec{\delta}, \vec{\beta}_R) \in \overline{\vec{\Delta}(P_1, P_2; \mathcal{B}(R))}\right\},$$

   where

$$\vec{\beta}_R(P_1, P_2) = \inf\left\{\vec{\beta} \geq 0 \mid \exists \vec{\delta} \text{ s.t. } (\vec{\delta}, \vec{\beta}) \in \vec{\Delta}(P_1, P_2; \mathcal{B}(R))\right\}.$$

   Note that, similarly to 1. above, in general $(\vec{\delta}_R(P_1, P_2), \vec{\beta}_R(P_1, P_2)) \notin \vec{\Delta}(P_1, P_2; \mathcal{B}(R))$, however, $(\vec{\delta}, \vec{\beta}) \in \vec{\Delta}(P_1, P_2; \mathcal{B}(R))$ for all $\vec{\delta} > \vec{\delta}_R(P_1, P_2)$, $\vec{\beta} > \vec{\beta}_R(P_1, P_2))$.

3. *Gap defined at a given bias level $\vec{\beta} > 0$:*

$$\vec{\delta}_R(P_1, P_2) = \inf\left\{\vec{\delta} \geq 0 \mid (\vec{\delta}, \vec{\beta}) \in \vec{\Delta}(P_1, P_2; \mathcal{B}(R))\right\}.$$

   Note that $\vec{\delta}_R(P_1, P_2)$ at bias level $\beta = 0$ recovers the nonlinear gap (3.7).

Note that it is straightforward to restate Theorem 5.1 using any of the distance measures 1,2,3 above.

To illustrate this we now specialize to the case where the gap between the plant can be described without biases as in (3.7), which incorporates the case of linear plants in $\mathcal{R}(L^2(\mathbb{R}_+), L^2(\mathbb{R}_+))$. That is, in the following, $\vec{\delta}$ is given by (3.7) with $\Omega = \mathcal{W}$. This is a special case of both 2. and

3. above. This special case is important for the adaptive control setting considered in sections 4 and 7, where the plant is linear, hence the gap can be measured by $\vec{\delta}_0$, and the bias arises from the nonlinear controller. As the P and P_1 do have a finite nonlinear gap then this coincides with the distance measure in 2. above, whilst in relation to case 3. we are just considering the gap at bias level 0.

Corollary 6.1 *Let $\mathcal{U}$, $\mathcal{Y}$ be truncation complete normed signal spaces, and let $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$. Consider $P\colon \mathcal{U}_a \to \mathcal{Y}_a$, $P_1\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C\colon \mathcal{Y}_a \to \mathcal{U}_a$. Suppose $[P, C]$ is $(\gamma(\cdot), \beta)$ regionally gain stable with uniform bias on $\mathcal{W}$, where γ is continuous and $\Pi_{P,C}$ is relatively continuous. Let $R > 0, 0 < \varepsilon < 1$ and let $\vec{\delta}(P, P_1) = \vec{\delta}_{\mathcal{W}}(P, P_1)$ be as given in (3.7). If $[P_1, C]$ is causal, has the uniqueness property and*

$$\vec{\delta}(P, P_1)\gamma \left(\frac{R + \vec{\delta}(P, P_1)\beta}{\varepsilon} \right) < 1 - \varepsilon, \tag{6.1}$$

then the closed-loop system $[P_1, C]$ is $(\gamma_1(R), \beta_1)$ gain stable with bias on $\mathcal{B}(R)$ where:

$$\gamma_1(R) = \gamma \left(\frac{R + \vec{\delta}(P, P_1)\beta}{\varepsilon} \right) \left(\frac{1 + \vec{\delta}(P, P_1)}{\varepsilon} \right), \tag{6.2}$$

$$\beta_1 = \beta + \vec{\delta}(P, P_1)\beta \left(1 + \gamma \left(\frac{R + \vec{\delta}(P, P_1)\beta}{\varepsilon} \right) \frac{1 + \vec{\delta}(P, P_1)}{\varepsilon} \right). \tag{6.3}$$

Proof. By assumption (6.1), for sufficiently small $b > 0$ we have

$$\left(\vec{\delta}(P, P_1) + b \right) \gamma \left(\frac{R + (\vec{\delta}(P, P_1) + b)\beta}{\varepsilon} \right) < 1 - \varepsilon. \tag{6.4}$$

By the definition of $\vec{\delta}(P, P_1)$, there exists $\Phi \in \mathcal{O}^{\mathcal{W}}_{P,P_1}$ such that $\|R_\tau(I - \Phi)x\| \leq (\vec{\delta}(P, P_1) + b)\|x\|_\tau$ for all $x \in \mathcal{G}_P$, from which it follows that $(\vec{\delta}, \vec{\sigma}) := (\vec{\delta}(P, P_1) + b, 0) \in \Delta(P, P_1; \Omega)$ for any $\Omega \subset \mathcal{W}$. We let

$$\gamma_b = \gamma \left(\frac{R + \vec{\delta}(P, P_1)\beta + b\beta}{\varepsilon} \right)$$

and let $\Omega = \mathcal{B}\left(\gamma_b \frac{\rho(R)}{\varepsilon} + \beta \right) \subset \mathcal{W}$, $\rho(R) = R + \vec{\sigma} + \vec{\delta}\beta + b\beta$. Then inequality (6.1) implies (5.1) and $[P, C]$ is (γ_b, β) gain stable with bias on $\mathcal{B}\left(\frac{\rho(R)}{\varepsilon} \right)$. Hence, the result follows Theorem 5.1 as b is arbitrarily small and γ is continuous. $\square$

Expressed in terms of the robust stability margin and the robust performance function, we have:

Corollary 6.2 *Let $\mathcal{U}$, $\mathcal{Y}$ be truncation complete normed signal spaces, and let $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$. Consider $P\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C\colon \mathcal{Y}_a \to \mathcal{U}_a$. Suppose $[P, C]$ is $(\gamma(\cdot), \beta)$ regionally gain stable with uniform bias on $\mathcal{W}$, where γ is continuous and $\Pi_{P,C}$ is relatively continuous. Let $\vec{\delta}(P, P_1) = \vec{\delta}_{\mathcal{W}}(P, P_1)$ be as given in (3.7). Then for any $R > 0$, we have:*

$$B_{P,C}(R) \geq \sup \left\{ r > 0 \,\middle|\, r\gamma \left(\frac{R + r\beta}{\varepsilon} \right) < 1 - \varepsilon \ \text{ for some } \varepsilon \in (0, 1) \right\}, \tag{6.5}$$

$$AB_{P,C}(R,\eta) \leq \inf \left\{ \gamma_{\eta,\varepsilon}(R)R + \beta_{\eta,\varepsilon} \;\middle|\; \begin{array}{c} \eta B_{P,C}(R)\gamma\left(\frac{R+\eta B_{P,C}(R)\beta}{\varepsilon}\right) < 1-\varepsilon \\ \textit{for some } \varepsilon \in (0,1) \end{array} \right\}, \tag{6.6}$$

where in (6.6)

$$\begin{aligned} \gamma_{\eta,\varepsilon}(R) &= \gamma\left(\frac{R+\eta B_{P,C}(R)\beta}{\varepsilon}\right)\left(\frac{1+\eta B_{P,C}(R)}{\varepsilon}\right), \\ \beta_{\eta,\varepsilon} &= \beta + \eta B_{P,C}(R)\beta\left(1+\gamma\left(\frac{R+\eta B_{P,C}(R)\beta}{\varepsilon}\right)\frac{1+\eta B_{P,C}(R)}{\varepsilon}\right). \end{aligned}$$

Proof. Inequality (6.5) follows directly from Corollary 6.1. To show inequality (6.6), we suppose $P_1 \in \Gamma$, $\vec{\delta}(P,P_1) \leq \eta B_{P,C}(R)$ and

$$\eta B_{P,C}(R)\gamma\left(\frac{R+\eta B_{P,C}(R)\beta}{\varepsilon}\right) < 1-\varepsilon \text{ for some } \varepsilon \in (0,1).$$

Then, by the monotonicity of $\gamma(\cdot)$, we see $\vec{\delta}(P,P_1)\gamma\left(\frac{R+\vec{\delta}(P,P_1)\beta}{\varepsilon}\right) < 1-\varepsilon$, therefore by Corollary 6.1, $A_{P_1,C}(R) \leq \gamma_1(R)R+\beta_1$. Again by the monotonicity of $\gamma(\cdot)$, $\gamma_1 \leq \gamma_{\eta,\varepsilon}$ and $\beta_1 \leq \beta_{\eta,\varepsilon}$. Hence $A_{P_1,C}(R) \leq \gamma_{\eta,\varepsilon}(R)R + \beta_{\eta,\varepsilon}$ and $AB_{P,C}(R,\eta) \leq \gamma_{\eta,\varepsilon}(R)R + \beta_{\eta,\varepsilon}$, which proves the claim. □

Alternatively, the robust performance function can be expressed as a function of $B_{P,C}$ and $A_{P,C}$ in the spirit of inequality (3.5):

Corollary 6.3 *Let $\mathcal{U}$, $\mathcal{Y}$ be truncation complete normed signal spaces, and let $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$. Consider $P\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C\colon \mathcal{Y}_a \to \mathcal{U}_a$. Suppose $[P,C]$ is $(\gamma(\cdot),\beta)$ regionally gain stable with uniform bias on $\mathcal{W}$ and $\Pi_{P//C}$ is relatively continuous. Let $\vec{\delta}(P,P_1) = \vec{\delta}_{\mathcal{W}}(P,P_1)$ be as given in* (3.7)*. Suppose that for $r > 0$ and $\eta \in [0,1)$, there exists $R := R(r)$ such that*

$$r + \eta B_{P,C}(r)(\gamma(R)R + \beta) \leq R. \tag{6.7}$$

Then

$$AB_{P,C}(r,\eta) \leq (1+\eta B_{P,C}(r))A_{P,C}\left(\frac{r+\eta B_{P,C}(r)\beta}{1-\eta B_{P,C}(r)\gamma(R)}\right). \tag{6.8}$$

Proof. Let $r,\tau > 0, \eta \in [0,1)$ and let $P_1 \in \Gamma$ with $\vec{\delta}_{\mathcal{W}}(P,P_1) < \eta B_{P,C}(r)$. Denote by $\varepsilon = \eta B_{P,C}(r)$. Then there exists $\Phi \in \mathcal{O}^{\mathcal{W}}_{P,P_1}$ such that

$$\|R_\tau(\Phi - I)x\|_\tau \leq \varepsilon\|R_\tau x\|_\tau \text{ for all } x \in \mathcal{G}_P.$$

Let R be the number given by our assumption satisfying (6.7). Let

$$V = \{x \in \mathcal{W}[0,\tau) \mid \|x\|_\tau \leq R\}.$$

For any $x \in V$, let $\bar{x} = E_\tau \bar{x}$. Then for any $w \in \mathcal{B}(r)$, by inequality (6.7), we have

$$\|R_\tau\big(w - (\Phi - I)\Pi_{P//C}\bar{x}\big)\|_\tau \leq r + \varepsilon(\gamma(R)\|R_\tau\bar{x}\| + \beta) \leq r + \varepsilon(\gamma(R)R + \beta) \leq R$$

for all $x \in V$. Using the same argument as in Theorem 5.1 we can see that $[P_1, C]$ is globally well-posed and there exists $x \in V$ such that

$$x = R_\tau w - R_\tau(\Phi - I)\Pi_{P//C}\bar{x}, \tag{6.9}$$

$$R_\tau\Pi_{P_1//C}w = R_\tau\Phi\Pi_{P//C}\bar{x}, \tag{6.10}$$

By (6.9) and our assumptions,

$$\|x\|_\tau \leq \|w\|_\tau + \varepsilon(\gamma(\|x\|_\tau)\|R_\tau \bar{x}\|_\tau + \beta) \leq r + \varepsilon\gamma(R)\|x\|_\tau + \varepsilon\beta.$$

By inequality (6.7), $\varepsilon\gamma(R) \leq (R - r - \varepsilon\beta)/R < 1$. So

$$\|x\|_\tau \leq \frac{r + \varepsilon\beta}{1 - \varepsilon\gamma(R)}.$$

So it follows from (6.10) that

$$\|\Pi_{P_1//C} w\|_\tau \leq (1+\varepsilon)\|\Pi_{P//C}\bar{x}\|_\tau \leq (1+\varepsilon)A_{P,C}(\|R_\tau \bar{x}\|_\tau) \leq (1+\varepsilon)A_{P,C}\left(\frac{r+\varepsilon\beta}{1-\varepsilon\gamma(R)}\right).$$

The claim (6.8) follows. □

Note that in (6.8), the parameter R is dependent on r. In the case when $[P, C]$ is globally gain stable, then $\gamma(R) = \gamma$, we may let $R = (r + \eta B_{P,C}(r)\beta)/(1 - \eta B_{P,C}(r)\gamma)$.

7 Robust stability for adaptive control

Further to the adaptive controller (4.4) from Section 4, let us consider the controller:

$$\begin{aligned} C(y_2)(t) &= u_2(t) \\ u_2(t) &= -k y_2(t) \\ \dot{k}(t) &= \alpha \frac{1}{(n+1)k^n} y_2^2, k(0) = 0, \end{aligned} \qquad (7.1)$$

where $\alpha > 0$ is the adaptive gain and $n \geq 0$ is an integer. Note that this adaptive update can be alternatively expressed in the form

$$u_2(t) = -\kappa^{\frac{1}{n+1}} y_2(t), \quad \dot{\kappa}(t) = \alpha y_2^2, \kappa(0) = 0$$

by the substitution $\kappa = k^{n+1}$. In an L^2 setting, robust stability of this controller when applied to certain class of linear plants was established in [5] for the case of $n = 3$ and in [7] for the case of $n = 0$.

We now show that, in L^2 setting, and for all integer $n \geq 0$, the closed loop is regionally gain stable with uniform bias and this stability is preserved for sufficiently small gap perturbations of $P(\theta)$, where for concreteness we take $P(\theta)$ to be the linear plant given by (4.3). A scalar plant is chosen as an example purely for simplicity, we note that analogous explicit results with similar trade-offs can be obtained for general linear plants which are minimum phase, relative degree 1 and have positive high frequency gain, see [5, 7] for related results.

Proposition 7.1 *Let the plant $P(\theta)$ be given by* (4.3) *and controller C be given by* (7.1). *Then $[P(\theta), C]$ is globally well posed, gain stable with uniform bias on $\mathcal{W} = L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+)$, and*

$$A_{P(\theta),C}(r) \leq \beta + \gamma(r) \quad \textit{for } r \geq 0,$$

where, for $n = 0$,

$$\begin{aligned} \beta =& \frac{(4\theta+4)}{\sqrt{2\alpha}} + \frac{2(4\theta+4)^3}{\sqrt{\alpha}}, \\ \gamma(r) =& (2\theta+2)^{\frac{1}{2}} + 1 + (4\theta+6)^{\frac{1}{2}} + (6\alpha)^{\frac{1}{2}} r + 2^7 \alpha^{\frac{1}{2}} (\theta+1)^{\frac{3}{2}} r^2 \\ &+ 4\alpha\left(1 + (4\theta+6)^{\frac{1}{2}} + (6\alpha)^{\frac{1}{2}} r\right)^3 r^2, \end{aligned}$$

and for $n \geq 1$,

$$\beta = \left(\frac{(4\theta+4)^{n+2}}{4\alpha}\right)^{\frac{1}{2}} + 2^{\frac{2}{n+1}} \left(\frac{(4\theta+4)^{\frac{(n+2)(n+3)}{n+1}}}{4\alpha}\right)^{\frac{1}{2}},$$

$$\begin{aligned}\gamma(r) = {} & 2^{\frac{n+1}{2}}(\theta+1)^{\frac{1}{2}} + 1 + (4\theta+6)^{\frac{1}{2}} \\ & + 3^{\frac{1}{2}}\alpha^{\frac{1}{2(n+1)}} r^{\frac{1}{n+1}} + 2^{\frac{n^2+4n+12}{2(n+1)}} \alpha^{\frac{1}{2(n+1)}} (\theta+1)^{\frac{n+3}{2(n+1)}} r^{\frac{2}{n+1}} \\ & + 2^{\frac{2}{n+1}} \alpha^{\frac{1}{n+1}} \left(1 + (4\theta+6)^{\frac{1}{2}} + 3^{\frac{1}{2}}\alpha^{\frac{1}{2(n+1)}} r^{\frac{1}{n+1}}\right)^{\frac{n+3}{n+1}} r^{\frac{2}{n+1}}.\end{aligned}$$

Proof. We first show well posedness. As in [5], we need to show that for any $u_0, y_0 \in L^2(\mathbb{R}_+)$, the solution y_1 of the closed loop has finite norm over its domain.

Let $u_i, y_i (i = 1, 2)$ be the solution of the closed loop for given u_0, y_0 on the time interval $(0, \omega)$ with $\omega \in (0, \infty]$. Then for all $0 \leq s \leq \omega$, we have

$$\begin{aligned} y_1(s)\dot{y}_1(s) &= \theta y_1^2(s) + u_1(s)y_1(s) = \theta y_1^2(s) + [u_0(s) + k(s)y_0(s)]y_1(s) - k(s)y_1^2(s) \\ &\leq \theta y_1^2(s) + \frac{1}{2}u_0^2(s) + \frac{1}{2}k(s)y_0^2(s) + \frac{1}{2}y_1^2(s) + \frac{1}{2}k(s)y_1^2(t) - k(s)y_1^2(s) \\ &= \frac{1}{2}\big(2\theta + 1 - k(s)\big)y_1^2(s) + \frac{1}{2}\big(u_0^2(s) + k(s)y_0^2(s)\big). \end{aligned} \tag{7.2}$$

We claim that

$$\|y_1\|^2_{L^2(0,\omega)} \leq \frac{4(\theta+1)}{\alpha}c^{n+1} + 2\|u_0\|^2_{L^2(\mathbb{R}_+)} + 2c\|y_0\|^2_{L^2(\mathbb{R}_+)} + j_n\alpha^{\frac{1}{n+1}}\|y_0\|^{2+\frac{2}{n+1}}_{L^2(\mathbb{R}_+)} \tag{7.3}$$

with

$$c = 2\theta + 2 + j_n\alpha^{\frac{1}{n+1}}\|y_0\|^{\frac{2}{n+1}}_{L^2(\mathbb{R}_+)} \quad \text{and} \quad j_n = \begin{cases} 2 & \text{if } n = 0 \\ 1 & \text{if } n \geq 1. \end{cases} \tag{7.4}$$

In fact, if $k(t) \leq c$ for all $t \in [0, \omega)$, then

$$\begin{aligned} \|y_1\|^2_{L^2(0,t)} &= \|y_0 - y_2\|^2_{L^2(0,t)} \leq 2\max\left\{\|y_0\|^2_{L^2(0,t)}, \|y_2\|^2_{L^2(0,t)}\right\} \\ &= 2\max\left\{\|y_0\|^2_{L^2(0,t)}, \frac{1}{\alpha}k^{n+1}(t)\right\} \leq 2\max\left\{\|y_0\|^2_{L^2(\mathbb{R}_+)}, \frac{1}{\alpha}c^{n+1}\right\} = \frac{2}{\alpha}c^{n+1}, \end{aligned} \tag{7.5}$$

for all $t \in [0, \omega)$, which shows $\|y_1\|^2_{L^2(0,\omega)} \leq 2c^{n+1}/\alpha$ and proves the claim. So we may suppose $k(t) \nleq c$ for all $t \in [0, \omega)$. Then, by the monotonicity and continuity of k, there exists $t^* \geq 0$ such that $k(t^*) = c$ and $k(t^*) \leq k(s) \leq k(t)$ for all $t \geq s \geq t^*$. Similar to (7.5), we have

$$\|y_1\|^2_{L^2(0,t^*)} = \|y_0 - y_2\|^2_{L^2(0,t^*)} \leq \frac{2}{\alpha}c^{n+1}. \tag{7.6}$$

By the controller law

$$k^{n+1}(t) = \alpha\int_0^{t^*} y_2^2(\tau)d\tau + \alpha\int_{t^*}^t y_2^2(\tau)d\tau = k^{n+1}(t^*) + \alpha\|y_0 - y_1\|^2_{L^2(t^*,t)},$$

which implies (twice using the formula $(a+b)^{2/(n+1)} \leq a^{2/(n+1)} + b^{2/(n+1)}$ for $n \geq 1$):

$$k(t) \leq k(t^*) + \alpha^{\frac{1}{n+1}}\|y_0 - y_1\|^{\frac{2}{n+1}}_{L^2(t^*,t)} \leq c + j_n\alpha^{\frac{1}{n+1}}\left(\|y_0\|^{\frac{2}{n+1}}_{L^2(\mathbb{R}_+)} + \|y_1\|^{\frac{2}{n+1}}_{L^2(t^*,t)}\right). \tag{7.7}$$

For $s \in [t^*, t)$, from (7.2), it follows

$$\begin{aligned} y_1(s)\dot{y}_1(s) &\leq \frac{1}{2}\big(2\theta + 1 - k(t^*)\big)y_1^2(s) + \frac{1}{2}u_0^2(s) + \frac{1}{2}k(t)y_0^2(s) \\ &\leq \frac{1}{2}\big(2\theta + 1 - c\big)y_1^2(s) + \frac{1}{2}u_0^2(s) + \frac{1}{2}cy_0^2(s) \\ &\quad + \frac{1}{2}j_n\alpha^{\frac{1}{n+1}}\left(\|y_1\|_{L^2(t^*,t)}^{\frac{2}{n+1}}y_0^2(s) + \|y_0\|_{L^2(\mathbb{R}_+)}^{\frac{2}{n+1}}\right)y_0^2(s). \end{aligned}$$

Integrating on both sides over $[t^*, t]$ with respect to s, we have

$$0 \leq y_1^2(t) \leq \; y_1^2(t^*) - \|y_1\|_{L^2(t^*,t)}^2 + \|u_0\|_{L^2(\mathbb{R}_+)}^2 + c\|y_0\|_{L^2(\mathbb{R}_+)}^2 + j_n\alpha^{\frac{1}{n+1}}\|y_0\|_{L^2(\mathbb{R}_+)}^{2+\frac{2}{n+1}} + b(t)$$

where

$$b(t) = \alpha^{\frac{1}{n+1}}\|y_1\|_{L^2(t^*,t)}^{\frac{2}{n+1}}\|y_0\|_{L^2(\mathbb{R}_+)}^2 - \alpha^{\frac{1}{n+1}}\|y_1\|_{L^2(t^*,t)}^2\|y_0\|_{L^2(\mathbb{R}_+)}^{\frac{2}{n+1}}.$$

In the case when $b(t) \leq 0$ we will have

$$\|y_1\|_{L^2(t^*,t)}^2 \leq \; y_1^2(t^*) + \|u_0\|_{L^2(\mathbb{R}_+)}^2 + c\|y_0\|_{L^2(\mathbb{R}_+)}^2 + j_n\alpha^{\frac{1}{n+1}}\|y_0\|_{L^2(\mathbb{R}_+)}^{2+\frac{2}{n+1}},$$

otherwise, $\|y_1\|_{L^2(t^*,t)} \leq \|y_0\|_{L^2(\mathbb{R}_+)}$. Therefore (note $c > 1$)

$$\|y_1\|_{L^2(t^*,\omega)}^2 = \lim_{t\to\omega}\|y_1\|_{L^2(t^*,t)}^2 \leq \; y_1^2(t^*) + \|u_0\|_{L^2(\mathbb{R}_+)}^2 + c\|y_0\|_{L^2(\mathbb{R}_+)}^2 + j_n\alpha^{\frac{1}{n+1}}\|y_0\|_{L^2(\mathbb{R}_+)}^{2+\frac{2}{n+1}}. \quad (7.8)$$

Considering (7.2) over $[0, t^*]$ to obtain

$$\begin{aligned} y_1(s)\dot{y}_1(s) &\leq \frac{1}{2}\big(2\theta + 1 - k(0)\big)y_1^2(s) + \frac{1}{2}\big(u_0^2(s) + k(t^*)y_0^2(s)\big) \\ &\leq \frac{1}{2}\big(2\theta + 1\big)y_1^2(s) + \frac{1}{2}\big(u_0^2(s) + cy_0^2(s)\big) \quad \text{for } s \in [0, t^*] \end{aligned}$$

and therefore by (7.6)

$$\begin{aligned} y_1^2(t^*) &\leq (2\theta + 1)\|y_1\|_{L^2(0,t^*)}^2 + \|u_0\|_{L^2(\mathbb{R}_+)}^2 + c\|y_0\|_{L^2(\mathbb{R}_+)}^2 \\ &\leq \frac{2(2\theta + 1)c^{n+1}}{\alpha} + \|u_0\|_{L^2(\mathbb{R}_+)}^2 + c\|y_0\|_{L^2(\mathbb{R}_+)}^2. \end{aligned} \quad (7.9)$$

From (7.4), (7.6), (7.8), (7.9) and the fact $\|y_1\|_{L^2(0,\omega)}^2 = \|y_1\|_{L^2(0,t^*)}^2 + \|y_1\|_{L^2(t^*,\omega)}^2$, our claim (7.3) follows. This shows that $[P(\theta), C]$ is globally well-posed and $\omega = \infty$.

We now show gain stability.

Let $\|(u_0, y_0)^\top\|_{L^2(\mathbb{R}_+^2)} \leq r$. Then $\|u_0\|_{L^2(\mathbb{R}_+)} \leq r$, $\|y_0\|_{L^2(\mathbb{R}_+)} \leq r$ (the norm in product space is taken the Euclidian norm). By (7.3)

$$\|y_1\|_{L^2(\mathbb{R}_+)}^2 \leq \frac{4(\theta + 1)}{\alpha}\left(2\theta + 2 + j_n\alpha^{\frac{1}{n+1}}r^{\frac{2}{n+1}}\right)^{n+1} + 3j_n\alpha^{\frac{1}{n+1}}r^{2+\frac{2}{n+1}} + (4\theta + 6)r^2 =: d(r),$$

therefore

$$\|y_2\|_{L^2(\mathbb{R}_+)} = \|y_0 - y_1\|_{L^2(\mathbb{R}_+)} \leq \|y_0\|_{L^2(\mathbb{R}_+)} + \|y_1\|_{L^2(\mathbb{R}_+)} \leq r + [d(r)]^{\frac{1}{2}}.$$

Since $k^{n+1}(t) = \alpha\|y_2\|^2_{L^2(0,t)} \leq \alpha\|y_2\|^2_{L^2(\mathbb{R}_+)}$ for all t, we have

$$\begin{aligned}
\|u_2\|_{L^2(\mathbb{R}_+)} = \|ky_2\|_{L^2(\mathbb{R}_+)} = \sup_{t\geq 0} k(t)\|y_2\|_{L^2(\mathbb{R}_+)} = \alpha^{\frac{1}{n+1}}\|y_2\|^{1+\frac{2}{n+1}}_{L^2(\mathbb{R}_+)} \\
\leq \alpha^{\frac{1}{n+1}}\left(r + [d(r)]^{\frac{1}{2}}\right)^{\frac{n+3}{n+1}}, \\
\|u_1\|_{L^2(\mathbb{R}_+)} = \|u_0 - u_2\|_{L^2(\mathbb{R}_+)} \leq \|u_0\|_{L^2(\mathbb{R}_+)} + \|u_2\|_{L^2(\mathbb{R}_+)} \\
\leq r + \alpha^{\frac{1}{n+1}}\left(r + [d(r)]^{\frac{1}{2}}\right)^{\frac{n+3}{n+1}}.
\end{aligned}$$

From these estimates, it follows

$$\begin{aligned}
A_{P(\theta),C}(r) &\leq \sqrt{d(r) + \left(r + \alpha^{\frac{1}{n+1}}\left(r + [d(r)]^{\frac{1}{2}}\right)^{\frac{n+3}{n+1}}\right)^2} \\
&\leq [d(r)]^{\frac{1}{2}} + r + \alpha^{\frac{1}{n+1}}\left(r + [d(r)]^{\frac{1}{2}}\right)^{\frac{n+3}{n+1}}. \qquad (7.10)
\end{aligned}$$

For simple calculation, we denote by

$$b_1 = (2\theta + 2 + j_n\alpha^{\frac{1}{n+1}} r^{\frac{2}{n+1}})^{n+1}, \; b_2 = 3j_n\alpha^{\frac{1}{n+1}} r^{\frac{2}{n+1}} + (4\theta + 6).$$

Write $i_0 = 1$ and $i_n = 2$ for $n \geq 1$. Then, by the inequalities $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ and $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for $a, b \geq 0$, $p \geq 1$,

$$[d(r)]^{\frac{1}{2}} = \left(\frac{4(\theta+1)}{\alpha} b_1 + b_2 r^2\right)^{\frac{1}{2}} \leq \left(\frac{4(\theta+1)}{\alpha}\right)^{\frac{1}{2}} b_1^{\frac{1}{2}} + b_2^{\frac{1}{2}} r, \qquad (7.11)$$

$$\begin{aligned}
\left(r + [d(r)]^{\frac{1}{2}}\right)^{\frac{n+3}{n+1}} &= \left(\left(\frac{4(\theta+1)}{\alpha}\right)^{\frac{1}{2}} b_1^{\frac{1}{2}} + b_2^{\frac{1}{2}} r + r\right)^{1+\frac{2}{n+1}} \\
&\leq 2^{\frac{2}{n+1}}\left(\frac{4(\theta+1)}{\alpha}\right)^{\frac{n+3}{2(n+1)}} b_1^{\frac{n+3}{2(n+1)}} + 2^{\frac{2}{n+1}}\left(b_2^{\frac{1}{2}} + 1\right)^{\frac{n+3}{n+1}} r^{\frac{n+3}{n+1}}, \qquad (7.12)
\end{aligned}$$

$$b_1^{\frac{1}{2}} = \left(2\theta + 2 + j_n\alpha^{\frac{1}{n+1}} r^{\frac{2}{n+1}}\right)^{\frac{n+1}{2}} \leq i_n^{\frac{n-1}{2}}(2\theta+2)^{\frac{n+1}{2}} + i_n^{\frac{n-1}{2}}\left(j_n\alpha^{\frac{1}{n+1}} r^{\frac{2}{n+1}}\right)^{\frac{n+1}{2}}, \qquad (7.13)$$

$$b_1^{\frac{n+3}{2(n+1)}} = \left(2\theta + 2 + j_n\alpha^{\frac{1}{n+1}} r^{\frac{2}{n+1}}\right)^{\frac{n+3}{2}} \leq 2^{\frac{n+1}{2}}\left((2\theta+2)^{\frac{n+3}{2}} + \left(j_n\alpha^{\frac{1}{n+1}} r^{\frac{2}{n+1}}\right)^{\frac{n+3}{2}}\right) \qquad (7.14)$$

$$b_2^{\frac{1}{2}} \leq (3j_n)^{\frac{1}{2}}\alpha^{\frac{1}{2(n+1)}} r^{\frac{1}{n+1}} + (4\theta+6)^{\frac{1}{2}}. \qquad (7.15)$$

Substituting (7.11)-(7.15) into (7.10) and re-arranging terms, we obtain

$$A_{P(\theta),C}(r) \leq \beta + \gamma(r)r.$$

This completes the proof. □

Under the mild assumptions that $[P_1, C]$ is causal and has the uniqueness property (for example if P_1 is linear then this follows as the right hand side of the differential equations governing $[P_1, C]$ are locally Lipschitz), it now follows from Corollary 6.1 that gain stability with uniform bias is preserved for sufficiently small gap perturbations of $P(\theta)$ given by inequality (6.1).

There is an interesting trade-off concerning the adaptive gain parameter $\alpha > 0$. Namely, it can be seen from the explicit construction of β and γ in Proposition 7.1 that increasing α reduces the bias term β (but a non-zero bias is necessary, see Section 4), but increases the gain term γ. This is because the need for the bias term is due to the transient that occurs in the period of time which is required to drive the value of k to a stabilizing value. A large adaptive gain causes k to rapidly reach this stabilizing value, hence reducing the required size of the bias. On the other hand, a large adaptive gain will cause the value of k to 'overshoot' its required asymptotic value, resulting in an overly high gain control, and consequently a large gain $\gamma(r)$, $r > 0$.

Suppose $\theta \in [0, \theta_{\max}]$. If we let the adaptive gain α be such that $\alpha = \frac{1}{4}\alpha^*(4\theta_{\max}+4)^{\frac{(n+2)(n+3)}{n+1}}$ with $\alpha^* > 0$ a constant, then,

$$\beta \leq \frac{1}{\sqrt{\alpha^*}(4\theta_{\max}+4)^{\frac{2(n+2)}{n+1}}} + \frac{2^{\frac{2}{n+1}}}{\sqrt{\alpha^*}} \leq \frac{1}{\sqrt{\alpha^*}}\left(\frac{1}{(4\theta_{\max}+4)^2} + 2^{\frac{2}{n+1}}\right),$$

which can be made arbitrary small by suitably choosing α^*. Then $\gamma(r)$ is a function of θ of approximately order $\theta^{3/2}$. Furthermore, the gain γ depends on a fractional exponent of r, the term with largest power is $r^{\frac{n+3}{(n+1)^2}} r^{\frac{2}{n+1}} = r^{\frac{3n+5}{(n+1)^2}}$ whose power goes to 1 as $n \to \infty$, i.e., $\gamma(r)r$ approach linear growth as n increases. Hence in the case of $\theta \approx \theta_{\max}$ and r large, this is the scaling of α, n, which produces good performance. On the other hand, if $\theta \approx 0$, this produces an overly high gain controller with relatively poor performance and robustness characteristics. This highlights that scaling learning rates can lead to conservative designs in the sense of [5], but that the tuning parameters, α, n, give significant and useful control over the system performance in various uncertainty ranges.

We now consider the robust performance function $AB_{P(\theta),C}(r,\eta)$. Note, due to the integral, $\Pi_{P(\theta)//C}$ is relative continuous. So, for any $R > 0$ satisfying $r + \eta B_{P(\theta),C}(r)(\gamma(R)R + \beta) \leq R$, by Corollary 6.3:

$$AB_{P(\theta),C}(r,\eta) \leq (1+\eta B_{P(\theta),C}(r))\left(\beta + \gamma\left(\frac{r+\eta B_{P(\theta),C}(r)\beta}{1-\eta B_{P(\theta),C}(r)\gamma(R)}\right)\frac{r+\eta B_{P(\theta),C}(r)\beta}{1-\eta B_{P(\theta),C}(r)\gamma(R)}\right).$$

8 Global robust stability

By applying Theorem 5.1 in a global setting we obtain the following global result:

Theorem 8.1 *Let $\mathcal{U}$, $\mathcal{Y}$ be truncation complete normed signal spaces, and let $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$. Consider $P\colon \mathcal{U}_a \to \mathcal{Y}_a$, $P_1\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C\colon \mathcal{Y}_a \to \mathcal{U}_a$. Suppose $(\vec{\delta}, \vec{\sigma}) \in \Delta(P, P_1; \mathcal{W})$ and $[P, C]$ is (γ, β) gain stable with bias on $\mathcal{W}$. If $[P_1, C]$ is causal, has the uniqueness property and*

$$\vec{\delta}\gamma < 1 - \varepsilon \text{ for some } \varepsilon \in (0,1), \tag{8.1}$$

then the closed-loop system $[P_1, C]$ is (γ_1, β) gain stable with bias on $\mathcal{W}$ and

$$\gamma_1 = \gamma\frac{1+\vec{\delta}}{\varepsilon}, \tag{8.2}$$

$$\beta_1 = \beta + \left(\vec{\sigma} + \vec{\delta}\beta\right)\left(1 + \gamma\frac{1+\vec{\delta}}{\varepsilon}\right). \tag{8.3}$$

Proof. Let $w \in \mathcal{W}$ and suppose $0 < \tau < \infty$. Then there exists $R > 0$ such that $\|w\| \leq \varepsilon R$, $(\vec{\delta}, \vec{\sigma}) \in \Delta(P, P_1; \Omega)$ with

$$\Omega = B\left(\gamma\frac{\rho(\varepsilon R)}{\varepsilon} + \beta\right), \quad \rho(R) = R + \vec{\sigma} + \vec{\delta}\beta$$

and $[P,C]$ is (γ,β) gain stable with bias on $B\left(\frac{\rho(\varepsilon R)}{\varepsilon}\right)$. Following the same procedure as in Theorem 5.1, we see that

$$\|\Pi_{P_1//C}w\|_\tau \le \gamma_1\|w\|_\tau + \beta_1.$$

This completes the proof.

□

As a corollary we specialize, as before, to the case where the gap between the plants, $\vec{\delta}$, is given by (3.7).

Corollary 8.2 *Let $\mathcal{U}$, $\mathcal{Y}$ be truncation complete normed signal spaces, and let $\mathcal{W} = \mathcal{U}\times\mathcal{Y}$. Consider $P\colon \mathcal{U}_a \to \mathcal{Y}_a$, $P_1\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C\colon \mathcal{Y}_a \to \mathcal{U}_a$. Suppose $[P,C]$ is globally (γ,β) gain stable with bias and $\vec{\delta}(P,P_1) = \vec{\delta}_{\mathcal{W}}(P,P_1)$ is given by (3.7). If $\vec{\delta}(P,P_1)\gamma < 1$, then for all $r \ge 0$:*

$$\begin{aligned} A_{P_1,C}(r) &\le \gamma_1 r + \beta_1, \\ B_{P,C}(r) &\ge \gamma^{-1}, \end{aligned}$$

where:

$$\begin{aligned} \gamma_1 &\le \gamma\left(\frac{1+\vec{\delta}(P,P_1)}{1-\vec{\delta}(P,P_1)\gamma}\right), \\ \beta_1 &\le \beta + \vec{\delta}(P,P_1)\beta\left(1+\gamma\left(\frac{1+\vec{\delta}(P,P_1)}{1-\vec{\delta}(P,P_1)\gamma}\right)\right). \end{aligned}$$

Furthermore, for $\eta \in [0,(B_{P,C}(r)\gamma)^{-1})$,

$$AB_{P,C}(r,\eta) \le \gamma\frac{1+\eta B_{P,C}(r)}{1-\eta B_{P,C}(r)\gamma}r + \beta + \eta B_{P,C}(r)\beta\left(1+\gamma\frac{1+\eta B_{P,C}(r)}{1-\eta B_{P,C}(r)\gamma}\right).$$

Proof. Let $a > 0$ be small enough such that $(\vec{\delta}(P,P_1)+a)\gamma < 1$. Let $\varepsilon < 1-(\vec{\delta}(P,P_1)+a)\gamma$. Then $(\vec{\delta}(P,P_1)+a,0) \in \Delta(P,P_1;\mathcal{W})$ and $(\vec{\delta}(P,P_1)+a)\gamma < 1-\varepsilon$. By Theorem 8.1, $[P_1,C]$ is globally well posed and

$$\|\Pi_{P_1//C}w\|_\tau \le \gamma\frac{1+\vec{\delta}(P,P_1)+a}{\varepsilon}\|w\|_\tau + \beta + (\vec{\delta}(P,P_1)+a)\beta\left(1+\gamma\frac{1+\vec{\delta}(P,P_1)+a}{\varepsilon}\right)$$

for all $w \in \mathcal{W}, \tau > 0$. Let $\varepsilon \to 1-(\vec{\delta}(P,P_1)+a)\gamma$ and $a \to 0$, we see $\|\Pi_{P_1//C}w\|_\tau \le \gamma_1\|w\|_\tau + \beta_1$ for all $w \in \mathcal{W}, \tau > 0$. This proves $A_{P_1,C}(r) \le \gamma_1 r + \beta_1$. Consequently, $B_{P,C}(r) \ge \gamma^{-1}$.

If $\eta \in [0,(B_{P,C}(r)\gamma)^{-1}$ with $r \ge 0$, then $\eta B_{P,C}(r)\gamma < 1$. Let $w \in \mathcal{W}, \|w\|_\tau \le r$. Repeating the above proof with $\vec{\delta}(P,P_1)$ replaced by $\eta B_{P,C}(r)$, we have

$$\|\Pi_{P_1//C}w\|_\tau \le \gamma\frac{1+\eta B_{P,C}(r)}{1-\eta B_{P,C}(r)\gamma}r + \beta + \eta B_{P,C}(r)\beta\left(1+\gamma\frac{1+\eta B_{P,C}(r)}{1-\eta B_{P,C}(r)\gamma}\right).$$

So, the estimate for $AB_{P,C}(r,\eta)$ follows. □

We now draw attention to a significant difference between the nonlinear biased setting and the linear unbiased case. An important feature of the standard robust stability theory for LTI systems is that the problem of maximizing the robust stability margin $B_{P,C}(r)$ is equivalent to minimizing the nominal performance $A_{P,C}(r)$, since $B_{P,C}(r) = \|\Pi_{P//C}\|^{-1} = rA^{-1}_{P,C}(r)$. Not only is this optimization tractable (in the L^2 setting minimizing $\|\Pi_{P//C}\|$ forms the standard H_∞ problem), but it shows that optimizing for nominal performance and robust stability is equivalent.

This equivalence breaks down in the nonlinear biased setting, as one can see that given a biased gain bound,

$$\|\Pi_{P//C} w_0\| \leq \gamma[\Pi_{P//C}]\|w_0\| + \beta[\Pi_{P//C}] \qquad \text{for all } w_0 \in \mathcal{W},$$

then the robust stability margin $B_{P,C}(r)$ is maximised by minimising the nominal closed loop gain $\gamma[\Pi_{P//C}]$, whereas the nominal performance, $A_{P,C}(r)$, is a function of *both* the gain $\gamma[\Pi_{P//C}]$ and the bias $\beta[\Pi_{P//C}]$. Thus optimizing the robust stability margin, i.e. by minimizing the gain term, will produce sub-optimal nominal performance. We can therefore conclude that robust stability and (nominal) performance constraints must both be specified in any sensible optimization.

Similar reasoning does not directly apply in the regional setting, since the situation is more complex: the bias determines the region over which the gain in computed, and hence the bias does affect the robust stability margin. However it is clear that the resulting optimizations are different, and hence the conclusion that nominal performance and robust stability margin optimization differ still holds. Thus any sensible formulation of an optimal robust design in this setting must independently specify requirements for *both* nominal performance and robust stability margins.

As an example with global gain stability with bias, we consider the following projection controller C_{Proj} for the plant $P(\theta)$ given in Section 4:

$$\begin{aligned} C_{\text{Proj}}(y_2)(t) &= u_2(t) \\ u_2(t) &= -ky_2(t) - y_2(t) \end{aligned} \tag{8.4}$$

and

$$\dot{k}(t) = \begin{cases} \frac{\alpha}{(n+1)k^n} y_2^2, k(0) = 0, & \text{if } k(t) \leq \theta_{\max}, \\ 0, & \text{if } k(t) > \theta_{\max} \end{cases}$$

where $\theta_{\max}$ is an upper bound of the uncertainty θ.

Proposition 8.3 *Let the plant $P(\theta)$ be given by (4.3) and controller C_{Proj} be given by (8.4). Then $[P(\theta), C_{\text{Proj}}]$ is globally well posed, gain stable with bias on $\mathcal{W} = L^2(\mathbb{R}_+) \times L^2(\mathbb{R}_+)$, and*

$$A_{P(\theta),C_{\text{Proj}}}(r) \leq 2(2+\theta_{\max})\left(\frac{\theta^{n+2}}{\alpha}\right)^{\frac{1}{2}} + \left(2 + \theta_{\max} + \left(12 + 4\theta + 2\theta^2 + 2\theta_{\max}^2\right)^{\frac{1}{2}}\right) r.$$

Proof. Let $u_i, y_i (i = 1, 2)$ be the solution of the closed loop $[P(\theta), C_{\text{Proj}}]$ for given u_0, y_0 on the maximal time interval $[0, \omega)$ with $\omega \in (0, \infty]$. Let

$$t^* = \sup\{t \in [0, \omega) : k(t) \leq \theta\}.$$

Then by the monotonicity of k, we see $k(t) \leq \theta \leq \theta_{\max}$ on $[0, t^*]$, from which it follows $\dot{k}(t) = \frac{\alpha}{(n+1)k^n} y_2^2(t)$ for $t \leq t^*$. Therefore

$$\|y_2\|^2_{L^2(0,t^*)} = \frac{k^{n+1}(t^*)}{\alpha} \leq \frac{\theta^{n+1}}{\alpha},$$

and

$$\|y_1\|^2_{L^2(0,t^*)} = \|y_0 - y_1\|^2_{L^2(0,t^*)} \leq 2\|y_0\|^2_{L^2(\mathbb{R}_+)} + 2\frac{\theta^{n+1}}{\alpha}.$$

Since

$$
\begin{aligned}
y_1(s)\dot{y}_1(s) &= \theta y_1^2(s) + u_1(s)y_1(s) \\
&= (\theta - 1 - k(s))y_1^2(s) + [u_0(s) + y_0(s) + k(s)y_0(s)]y_1(s) \quad \text{for all } s \in [0,\omega),
\end{aligned}
\tag{8.5}
$$

using the inequalities $ab \leq a^2 + b^2/4$ and $ab \leq (a^2+b^2)/2$ and the monotonicity of k, we have

$$
\begin{aligned}
y_1(s)\dot{y}_1(s) &\leq (\theta - 1 - k(s))y_1^2(s) + u_0^2(s) + \frac{1}{2}y_1^2(s) + y_0^2 + \frac{1}{4}y_1^2(s) + k^2(s)y_0^2(s) + \frac{1}{4}y_1^2(s) \\
&\leq \theta y_1^2(s) + u_0^2(s) + y_0^2 + \theta^2 y_0^2(s) \quad \text{for all } s \in [0,t^*],
\end{aligned}
$$

from which we obtain

$$
y_1^2(t^*) \leq 2\theta\|y_1\|^2_{L^2(0,t^*)} + 2\|u_0\|^2_{L^2(\mathbb{R}_+)} + 2(1+\theta^2)\|y_0\|^2_{L^2(\mathbb{R}_+)}.
$$

Over the interval $[t^*,\omega)$, $k(s) \geq \theta$ and $k(s) \leq \theta_{\max}$, by (8.5), we have

$$
\begin{aligned}
y_1(s)\dot{y}_1(s) &\leq (\theta - 1 - k(s))y_1^2(s) + (u_0(s) + y_0(s))^2 + k^2(s)y_0^2(s) + \frac{1}{4}y_1^2(s) + \frac{1}{4}y_1^2(t) \\
&\leq -\frac{1}{2}y_1^2(s) + 2u_0^2(s) + (2+\theta_{\max}^2)y_0^2(s).
\end{aligned}
$$

integrating over $[t^*,\omega)$, we have

$$
\|y_1\|^2_{L^2(t^*,\omega)} \leq y_1^2(t^*) + 4\|u_0\|^2_{L^2(\mathbb{R}_+)} + 2(2+\theta_{\max}^2)\|y_0\|^2_{L^2(\mathbb{R}_+)}.
$$

Hence

$$
\begin{aligned}
\|y_1\|^2_{L^2(0,\omega)} &= \|y_1\|^2_{L^2(0,t^*)} + \|y_1\|^2_{L^2(t^*,\omega)} \\
&\leq \frac{4\theta^{n+2}}{\alpha} + 6\|u_0\|^2_{L^2(\mathbb{R}_+)} + 2\left(3 + 2\theta + \theta^2 + \theta_{\max}^2\right)\|y_0\|^2_{L^2(\mathbb{R}_+)} < \infty,
\end{aligned}
$$

$$
\begin{aligned}
\|u_1\|_{L^2(0,\omega)} &= \|u_0 + (1+k)y_0 - (1+k)y_1\|_{L^2(0,\omega)} \\
&\leq \|u_0\|_{L^2(\mathbb{R}_+)} + (1+\theta_{\max})(\|y_1\|_{L^2(t^*,\omega)} + \|y_0\|_{L^2(\mathbb{R}_+)}) < \infty.
\end{aligned}
$$

By the standard argument on global existence of ODEs, this shows $\omega = \infty$. Consequently, in the case when $\|(u_0, y_0)^\top\| \leq r$ with $r \geq 0$, we have

$$
\begin{aligned}
\|y_1\|_{L^2(\mathbb{R}_+)} &\leq \sqrt{\frac{4\theta^{n+2}}{\alpha} + 2\left(6 + 2\theta + \theta^2 + \theta_{\max}^2\right)r^2} =: d_1(r), \\
\|u_1\|_{L^2(\mathbb{R}_+)} &\leq r + (1+\theta_{\max})(d_1(r) + r) =: d_2(r).
\end{aligned}
$$

Since $d_1(r) \leq 2\left(\frac{\theta^{n+2}}{\alpha}\right)^{\frac{1}{2}} + \left(12 + 4\theta + 2\theta^2 + 2\theta_{\max}^2\right)^{\frac{1}{2}} r$ and u_0, y_0 are arbitrary, we see

$$
\begin{aligned}
A_{P(\theta),C_{\mathrm{Proj}}}(r) &\leq \sqrt{[d_1(r)]^2 + [d_2(r)]^2} \leq d_1(r) + d_2(r) = (2+\theta_{\max})(r + d_1(r) \\
&\leq 2(2+\theta_{\max})\left(\frac{\theta^{n+2}}{\alpha}\right)^{\frac{1}{2}} + \left(2 + \theta_{\max} + \left(12 + 4\theta + 2\theta^2 + 2\theta_{\max}^2\right)^{\frac{1}{2}}\right) r,
\end{aligned}
$$

which shows the global gain stability with bias of $[P(\theta), C_{\mathrm{Proj}}]$. □

Under the mild assumption that $[P_1, C_{\mathrm{Proj}}]$ has the uniqueness property, for example if P_1 is linear then this follows as the right hand side of the differential equations governing $[P_1, C_{\mathrm{Proj}}]$ are locally Lipschitz, so by Corollary 8.2:

$$AB_{P(\theta),C_{\mathrm{Proj}}}(r,\eta) \leq (1+\eta B_{P(\theta),C_{\mathrm{Proj}}}(r)) \left(\beta + \gamma \frac{r+\eta B_{P(\theta),C_{\mathrm{Proj}}}(r)}{1-\eta B_{P(\theta),C_{\mathrm{Proj}}}(r)\gamma}\right),$$

where

$$\beta = 2(2+\theta_{\max}) \left(\frac{\theta^{n+2}}{\alpha}\right)^{\frac{1}{2}},$$
$$\gamma = 2+\theta_{\max} + \left(12+4\theta+2\theta^2+2\theta_{\max}^2\right)^{\frac{1}{2}}.$$

9 Other applications and literature

Whilst this paper has been motivated by examples in adaptive control, in the introduction we noted other applications where the presence of a bias is necessary. Within the current literature we observe that in addition to applications to adaptive control [7, 5] (including λ-tracking results [10]), there have also been applications of the results in [5] to nonlinear non-adaptive control, namely the topic of funnel control [11]. Entirely analogously therefore, the gain bounds obtained in [10, 11] are of the form (4.6) and can be used in conjunction with this paper to show that notions of gain stability with bias are preserved by the funnel controller under gap perturbations.

Direct applications of robust stability for biased notions of stability have also been obtained for L^1 adaptive controllers [13, 14]. These papers also include variations on the initial robust stability results of [2], which are developed further in this paper.

10 Concluding remarks

We have presented a generalisation to the nonlinear robust stability theory of Georgiou and Smith [9] which allows a notion of stability with bias terms. This approach contrasts to the alternative biased approach of [8] as it does not measure the gain w.r.t. a single offset 'bias' trajectory, and is applicable in situations where the need for a bias arises for other reasons, e.g. from a lack of continuity of $\Pi_{P//C}$ at 0 as in the adaptive control example. This extension is significant for the provision of robust stability certificates for the many nonlinear systems for which no induced gain exists, and yet robust stability guarantees can still be given. We have illustrated this by an example from adaptive control where an induced gain does not exist.

A significant difference between the nonlinear biased setting and the linear unbiased case is that the task of the optimization of nominal performance and the robust stability margin are no longer in general equivalent. This could be seen explicitly in the adaptive control examples considered.

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