PROPORTIONS OF r-REGULAR ELEMENTS IN FINITE CLASSICAL GROUPS

LÁSZLÓ BABAI, SIMON GUEST, CHERYL E. PRAEGER,
AND ROBERT A. WILSON

Abstract. For a prime r, we obtain lower bounds on the proportion of r-regular elements in classical groups and show that these lower bounds are the best possible lower bounds that do not depend on the order of the defining field. Along the way, we also provide new upper bounds and answer some open questions of the first author, Pálfy and Saxl.

1. Introduction

A number of results have appeared recently giving lower bounds for the proportion of r-regular elements, for a prime r, in a group of Lie type G in characteristic $p \neq r$. We denote this proportion by $p_r(G)$. Of particular interest is the dependence on the dimension, in the case of classical groups in dimension d. In [BPS], the first author, Pálfy and Saxl show that the proportion of r-regular elements in any classical group is at least $1/2d$. To complement their lower bounds, they also showed that for all prime powers $q \equiv -1 \pmod 4$ and all $d \geq 2$ such that $(d, q-1) \leq 2$, the proportion of 2-regular elements in $\mathrm{PSL}_d(q)$ is $p_2(\mathrm{PSL}_d(q)) \leq 4q^{-1} + 4(\pi d)^{-1/2}$ (see [BPS, Theorem 6.1]). In a remark after Corollary 22 in [PW], Parker and the fourth author show that, in general, a lower bound of $1/d$ for $p_2(\mathrm{PSL}_d(q))$ would be best possible. On the other hand, in [GP1], the second and third authors improved these $O(d^{-1})$ lower bounds to $O(d^{-3/4})$ in the case $r = 2$ for symplectic and orthogonal groups. We generalize and improve these results here, obtaining lower bounds and upper bounds in terms of the dimension.

Theorem 1.1. (a) *If $G = \mathrm{PSL}_d(q)$ or $\mathrm{PSU}_d(q)$ then*

$$p_r(G) \geq \frac{1}{d}.$$

(b) *Let $G = \mathrm{Sp}_d(q)$, $\mathrm{SO}^{\pm}_d(q)$, $\mathrm{SO}^{\circ}_d(q)$, $\Omega^{\pm}_d(q)$, $\Omega^{\circ}_d(q)$ or one of the corresponding projective groups. There exist explicit constants C_1, C_2 such that*

$$p_r(G) \geq \begin{cases} C_1 d^{-3/4} & \text{if } r = 2; \\ C_2 d^{-1/2} & \text{if } r \text{ is odd.} \end{cases}$$

We show that these lower bounds are essentially best possible for infinitely many orders of the underlying field (and indeed infinitely many fields in infinitely many characteristics).

Theorem 1.2. (a) *Let $G = \mathrm{PSL}_d(q)$ or $\mathrm{PSU}_d(q)$ and let $\epsilon > 0$. Then there exist infinitely many primes, and infinitely many powers q of each, for which $p_r(G) \leq \frac{1}{d} + \epsilon$.*

(b) *Let* $G = \mathrm{Sp}_d(q)$, $\mathrm{SO}^{\pm}_d(q)$, $\mathrm{SO}^{\circ}_d(q)$, $\Omega^{\pm}_d(q)$, $\Omega^{\circ}_d(q)$ *or one of the corresponding projective groups. There exist explicit constants* C_3, C_4, *infinitely many primes, and infinitely many powers* q *of each, for which*

$$p_r(G) \leq \begin{cases} C_3 d^{-3/4} & \text{if } r = 2; \\ C_4 d^{-1/2} & \text{if } r \text{ is odd.} \end{cases}$$

In addition, we prove some general upper bounds in the case $r = 2$. At this stage it is not clear to us whether these are essentially best possible.

Theorem 1.3. (a) *Let* $G = \mathrm{PSL}_d(q)$ *or* $\mathrm{PSU}_d(q)$. *Then, for each odd* q, *there exists an explicit constant* C_q *such that for all* $d \geq 2$ *we have*

$$p_2(G) \leq \frac{C_q}{\sqrt{d}}.$$

(b) *Let* $G = \mathrm{Sp}_d(q)$, $\mathrm{SO}^{\pm}_d(q)$, $\mathrm{SO}^{\circ}_d(q)$, $\Omega^{\pm}_d(q)$, $\Omega^{\circ}_d(q)$ *or one of the corresponding projective groups. Then there exist constants* C_5, C_6 *such that for all odd* q *and all* $d \geq 4$

$$p_2(G) \leq \begin{cases} C_5 d^{-5/8} & \text{if } q \equiv 1 \pmod 4; \\ C_6 d^{-1/2} & \text{if } q \equiv -1 \pmod 4. \end{cases}$$

We note that we can take $C_q = 2(q-1)_2/\sqrt{\pi}$ when $G = \mathrm{PSL}_d(q)$ and $C_q = 2(q+1)_2/\sqrt{\pi}$ when $G = \mathrm{PSU}_d(q)$, where $(q \pm 1)_2$ denotes the largest power of 2 dividing $(q \pm 1)$. In fact, we sharpen our bounds considerably in Theorem 1.4 below. As a notational convenience, we will frequently write the dimension d of the natural module for G in terms of the (untwisted) Lie rank n. In order to express our sharper lower bounds more succinctly, we define the following functions of n:

$$f_r(n) = \begin{cases} \frac{(2n)!}{2^{2n} n!^2} & \text{if } r \text{ is odd;} \\ \frac{\Gamma(n+1/4)}{\Gamma(1/4)\Gamma(n+1)} & \text{if } r = 2; \end{cases} \tag{1}$$

$$g_{odd,r}(n) = \begin{cases} \frac{(2n)!}{2^{2n} n!^2} \cdot \left(\frac{2n}{2n-1}\right) & \text{if } r \text{ is odd;} \\ \frac{\Gamma(n+1/4)}{\Gamma(1/4)\Gamma(n+1)} + \frac{\Gamma(n-1/4)}{4\Gamma(3/4)\Gamma(n+1)} & \text{if } r = 2; \end{cases} \tag{2}$$

and

$$g_{even,r}(n) = \begin{cases} \frac{(2n)!}{2^{2n} n!^2} \cdot \left(\frac{2n-2}{2n-1}\right) & \text{if } r \text{ is odd;} \\ \frac{\Gamma(n+1/4)}{\Gamma(1/4)\Gamma(n+1)} - \frac{\Gamma(n-1/4)}{4\Gamma(3/4)\Gamma(n+1)} & \text{if } r = 2. \end{cases} \tag{3}$$

See section 2 for details. In particular, equations (17) and (18) provide some insight into our choice of notation for the functions $g_{odd,r}$ and $g_{even,r}$.

Theorem 1.4. *Let* q *be a prime power,* r *a prime not dividing* q *and* $\epsilon \in \mathbb{R}$ *with* $0 < \epsilon < 1$. *Let* $G = X_d(q)$ *be a finite classical group with natural module of dimension* d *and (untwisted) Lie rank* n. *Let* $p_r(G)$ *denote the proportion of* r*-regular elements in* G. *Suppose that* X, d *and* $h_{X,r}$ *are defined in one of the rows of Table 1.*
(i) *Then*

$$p_r(G) \geq h_{X,r}(n).$$

(ii) *For infinitely many primes p, there exist infinitely many powers q of p for which*

$$p_r(G) < h_{X,r}(n) + \epsilon.$$

(iii) *Moreover, $p_r(G/Z(G)) = |Z(G)|_r p_r(G)$.*

X_d	Conditions	$h_{X,r}(n)$
PSL_{n+1}		$1/(n+1)$
PSU_{n+1}		$1/(n+1)$
Sp_{2n}		$f_r(n)$
SO°_{2n+1}		$f_r(n)$
Ω°_{2n+1}		$(2,r)f_r(n)$
SO^+_{2n}		$g_{even,r}(n)$
Ω^+_{2n}		$(2,r)g_{even,r}(n)$
SO^-_{2n}	n even	$g_{odd,r}(n)$
SO^-_{2n}	n odd	$g_{even,r}(n)$
Ω^-_{2n}	n even	$(2,r)g_{odd,r}(n)$
Ω^-_{2n}	n odd	$(2,r)g_{even,r}(n)$

TABLE 1. Lower bounds $h_{X,r}(n)$ on $p_r(X_d(q))$

Remark 1.5. (a) For fixed X, d and ϵ, we show, in many cases, that for all primes p (distinct from r), there exist infinitely many powers q of p for which $p_r(X_d(q)) < h_{X,r}(n) + \epsilon$. See Lemmas 3.2, 4.2, 5.3 and 6.3 for details.
(b) For each row of Table 1, the function $h_{X,r}$ is the same for every *odd* prime r.
(c) Much better lower bounds are possible for certain values of q and r. For example, if $G = \mathrm{PSL}_d(q)$ and the multiplicative order m of q modulo r is at least 2, then $p_r(G) \geq Cd^{-1/m}$, where C is an explicit constant. See Lemmas 3.1, 4.1, 5.1 and 6.1 for details.
(d) The lower bound $p_r(G) \geq 1/2d$ in [BPS] also holds when $r|q$, whereas our results do not deal with this case. However, Guralnick and Lubeck [GL] have shown that, when $r|q$, we have

$$p_r(X_n(q)) \geq 1 - 3/(q-1) - 2/(q-1)^2$$

for all simple groups of Lie type $X_n(q)$. It follows that the bounds in Table 1 hold when $r|q$ unless q is small. In any case, the proof of the lower bound $p_r(G) \geq 1/2d$ in [BPS] actually shows that $p_r(G) \geq 1/d$ when $r|q$; thus the lower bounds in Table 1 for $\mathrm{PSL}_d(q)$ and $\mathrm{PSU}_d(q)$ hold for all values of r and q.

Remark 1.5(c) above answers an open question in [BPS, Section 8]; moreover the following theorem answers the main open question in [BPS, Section 8]. See Section 7 for details, the answer to a further open question and for the proof of Theorem 1.6.

Theorem 1.6. *For all $\epsilon > 0$, there exists a prime power q and a constant C such that, for all $d \geq 2$,*

$$p_2(\mathrm{PSL}_d(q)) \leq Cd^{\epsilon-1}.$$

We prove Theorem 1.4 for linear, unitary, symplectic and odd-dimensional orthogonal, and even-dimensional orthogonal groups in sections 3, 4, 5 and 6 respectively. Theorem 1.1 is a direct consequence of Theorem 1.4. Theorems 1.2 and 1.3 are proved in Sections 3, 4, 5, and 6. In particular, the constants C_1, C_2, C_3, C_4 involved in the statements can be obtained from Corollaries 5.4 and 6.4.

2. Preliminaries

Let $\overline{G}$ be a connected reductive algebraic group defined over $\bar{\mathbb{F}}_q$, the algebraic closure of a field $\mathbb{F}_q$ of order q. Let F be a Frobenius morphism of $\overline{G}$, and let $G = \overline{G}^F$ be the subgroup of $\overline{G}$ fixed elementwise by F, so that G is a finite group of Lie type. Let $\overline{T}$ be an F-stable maximal torus in $\overline{G}$ and let $W := N_{\overline{G}}(\overline{T})/\overline{T}$ denote the Weyl group of $\overline{G}$. We will say that two elements w, w' in W are F-conjugate if there exists x in W such that $w' = x^{-1}wF(x)$. This is an equivalence relation, and we will refer to the equivalence classes as *F-classes*. Moreover, there is an explicit one-to-one correspondence between the F-classes of W and the G-conjugacy classes of maximal tori in G. A thorough description of the correspondence can be found in [NP], and we will summarize the necessary results below. Although the correspondence is between F-classes in W and G-conjugacy classes of maximal tori in G, we will frequently refer to an *element* of W corresponding to a maximal torus in G, rather than a G-conjugacy class of tori, where there is little possibility of confusion.

Now recall that every element g in G can be expressed uniquely in the form $g = su$, where $s \in G$ is semisimple, $u \in G$ is unipotent and $su = us$. This is the multiplicative Jordan decomposition of g (see [Ca, p. 11]). We now define a quokka set as a subset of a finite group of Lie type that satisfies certain closure properties.

Definition 2.1. Suppose that G is a finite group of Lie type. A nonempty subset Q of G is called a *quokka set*, or quokka subset of G, if the following two conditions hold.

(i) For each $g \in G$ with Jordan decomposition $g = su = us$, where s is the semisimple part of g and u the unipotent part of g, the element g is contained in Q if and only if s is contained in Q; and
(ii) the set Q is a union of G-conjugacy classes.

For G a classical group, and a prime r not dividing q, define the subset

$$Q(r, G) := \{g \in G \,:\, r \nmid |g|\},$$

consisting of all the r-regular elements g in G. We readily see that $Q(r, G)$ is a quokka set.

Applying [NP, Theorem 1.3] we have

Lemma 2.2. *Let G, W, $Q(r,G)$ be as above, let C be a subset of W consisting of a union of F-classes of W. For each F-class C_0 in W, let T_{C_0} denote a maximal torus corresponding to C_0. Then*

$$\frac{|Q(r,G)|}{|G|} = \sum_{C_0 \subset W} \frac{|C_0|}{|W|}.\frac{|T_{C_0} \cap Q(r,G)|}{|T_{C_0}|}, \tag{4}$$

and if there exists a constant $A \in [0,1]$ such that $\frac{|T_{C_0} \cap Q(r,G)|}{|T_{C_0}|} \geq A$ for all F-classes in C, then

$$\frac{|Q(r,G)|}{|G|} \geq \frac{A|C|}{|W|}. \tag{5}$$

In order to obtain an estimate for the proportions of r-regular elements in $\mathrm{PSL}_n(q)$ and $\mathrm{PSU}_n(q)$ we prove the following lemma, which essentially follows the proof of [NP, Theorem 1.6].

Lemma 2.3. *Let G be a classical group, $Q = Q(r,G)$, $Z = Z(G)$ and $QZ := \{xz : x \in Q, z \in Z\}$. Then the proportion of r-regular elements in G/Z is equal to*

$$p_r(G/Z) = \frac{|QZ|}{|G|} = \frac{|Q||Z|}{|G||Q \cap Z|}.$$

Proof. First observe that the set of r-regular elements in G/Z is equal to $QZ/Z := \{xZ : x \in Q\}$. Clearly if $x \in Q$ then xZ is an r-regular element in G/Z. Conversely, suppose xZ is an r-regular element in G/Z. Let m be the order of xZ and $r^k = |x|_r$ the r-part of the order of x. Since r^k and m are coprime, there exist $a, b \in \mathbb{Z}$ such that $1 = ar^k + bm$. But then $x = x^{ar^k}x^{bm}$ and $x^{ar^k} \in Q$ and $x^{bm} \in Z$. So $xZ = x^{ar^k}Z \in QZ/Z$.

So the proportion of r-regular elements in G/Z is equal to

$$\frac{|QZ/Z|}{|G/Z|} = \frac{|QZ|}{|G|}.$$

To calculate $|QZ|$, we count the set $A := \{(z,g,h) : z \in Z, g \in Q, h \in QZ \text{ and } gz = h\}$ in two ways. On the one hand, $|A| = |Z||Q|$ since each pair $(z,g) \in Z \times Q$ occurs exactly once in A and there are $|Z||Q|$ such pairs. On the other hand, for each $h \in QZ$, there exist at least one $z \in Z$ and $g \in Q$ such that $h = gz$. If $z' \in Z$ and $g' \in Q$ also satisfy $h = g'z'$ then $g^{-1}g' = (z')^{-1}z \in Z$. And if we let $y := g^{-1}g' = (z')^{-1}z$, then $z' = zy^{-1}$ and $g' = gy$. Note that for all $y \in Z$, the element $z' = zy^{-1}$ is in Z, but $g' = gy$ is contained in Q if and only if $y \in Q$. Therefore given $h \in QZ$, the number of pairs $(z,g) \in Z \times Q$ for which $h = gz$ is $|Q \cap Z|$, and thus $|A| = |QZ||Q \cap Z|$. Equating the two expression for $|A|$ gives

$$|QZ| = \frac{|Q||Z|}{|Q \cap Z|}$$

and the result follows. □

We now state some number theoretic results, which we will find useful.

Lemma 2.4. *Let $q \geq 2$ be an integer. For all integers $i, j \geq 1$ we have*

$$(q^i - 1, q^j - 1) = q^{(i,j)} - 1;$$

$$(q^i - 1, q^j + 1) = \begin{cases} q^{(i,j)} + 1 & \text{if } 2j_2 \leq i_2; \\ (2, q-1) & \text{otherwise;} \end{cases}$$

$$(q^i + 1, q^j + 1) = \begin{cases} q^{(i,j)} + 1 & \text{if } j_2 = i_2; \\ (2, q-1) & \text{otherwise.} \end{cases}$$

Lemma 2.5. *Let q and i be positive integers, and let r be a prime not dividing q. If q is odd then*

$$(q^i + 1)_2 = \begin{cases} 2 & \text{if } i \text{ is even;} \\ (q+1)_2 & \text{if } i \text{ is odd;} \end{cases} \tag{6}$$

and

$$(q^i - 1)_2 = \begin{cases} (q-1)_2 & \text{if } i \text{ is odd;} \\ i_2(q-1)_2 & \text{if } q \equiv 1 \pmod 4, \text{ and } i \text{ is even;} \\ i_2(q+1)_2 & \text{if } q \equiv 3 \pmod 4, \text{ and } i \text{ is even.} \end{cases} \tag{7}$$

If r is odd then

$$(q^i - 1)_r = \begin{cases} i_r(q-1)_r & \text{if } r | q-1; \\ i_r(q+1)_r & \text{if } r | q+1 \text{ and } i \text{ is even;} \\ 1 & \text{if } r | q+1 \text{ and } i \text{ is odd.} \end{cases} \tag{8}$$

and

$$(q^i + 1)_r = \begin{cases} i_r(q+1)_r & \text{if } r | q+1 \text{ and } i \text{ is odd;} \\ 1 & \text{if } r | q+1 \text{ and } i \text{ is even;} \\ 1 & \text{if } r | q-1. \end{cases} \tag{9}$$

Proof. Equations (6) and (7) are proved in [GP1, Lemma 2.5]. To prove equation (8), Lemma 2.8 of [NPP] implies that if $r|q-1$, then for all integers i, we have

$$(q^{r^j} - 1)_r = r^j (q-1)_r.$$

In particular, for all positive integers i we have $(q^{i_r} - 1)_r = i_r(q-1)_r$. And writing $i = i_r t$, where $(r, t) = 1$, we have

$$(q^i - 1) = (q^{i_r} - 1)(q^{i_r(t-1)} + \cdots + q^{i_r} + 1).$$

But $q^{i_r(t-1)} + \cdots + q^{i_r} + 1 \equiv t \pmod r$ and so $(q^i - 1)_r = (q^{i_r} - 1)_r = i_r(q-1)_r$, which proves the first line of (8). To prove the second line, suppose $r|q+1$ and i is even. Then $r|q^2 - 1$ and

$$(q^i - 1)_r = ((q^2)^{i/2} - 1)_r = (i/2)_r (q^2 - 1)_r$$

by the first line of (8). Moreover, $(i/2)_r(q^2-1)_r = i_r(q+1)_r$ since r is odd and $r|q+1$. This proves the second line. Now suppose $r|q+1$ and i is odd. In particular, $q \equiv -1 \pmod r$, and since i is odd, we have $q^i \equiv -1 \pmod r$; hence $(q^i - 1)_r = 1$. This proves the third line of (8). Now equation (9) follows from (8) since the r-part of $q^i + 1$ is $(q^{2i} - 1)_r / (q^i - 1)_r$. □

We will also need bounds on the proportion of elements in the symmetric group S_d that have cycles of certain lengths. We define $s_{\neg m}(d)$ to be the proportion of elements in S_d that have no cycles of length divisible by m. Erdős and Turán [ET] proved that when m is a prime power, we have the formula

$$s_{\neg m}(d) = \prod_{i=1}^{\lfloor d/m \rfloor} \left(1 - \frac{1}{im}\right). \tag{10}$$

In [BLNPS], it is shown that this formula also holds when m is not a prime power, and moreover that there exist constants c_m (depending on m) such that

$$c_m d^{-1/m} \left(1 - \frac{1}{d}\right) \leq s_{\neg m}(d) \leq c_m d^{-1/m} \left(1 + \frac{2}{d}\right). \tag{11}$$

Our functions $g_{odd,r}$ and $g_{even,r}$ involve the Gamma function Γ, which is defined for all $z \in \mathbb{C}$, with $\mathrm{Re}\, z > 0$, by the equation

$$\Gamma(z) := \int_0^\infty y^{z-1} e^{-y}\, dy.$$

Recall that for all positive integers n we have

$$\Gamma(n+1) = n! \tag{12}$$

We denote the number of permutations in S_n with precisely k cycles by $c(n,k)$. These numbers are known as the unsigned Stirling numbers of the first kind (see section 4 of [GP1] for details). Using generating function methods (see (4.4) and (4.6) of [GP1]), we can prove that

$$\begin{aligned} \sum_{k=1}^{n} \frac{c(n,k)}{n! 2^k} &= \frac{(2n)!}{2^{2n} n!^2} = f_r(n) \text{ for } r = 2; \\ \sum_{k=1}^{n} \frac{c(n,k)}{n! 4^k} &= \frac{\Gamma(n+1/4)}{\Gamma(1/4)\Gamma(n+1)} = f_r(n) \text{ for } r \text{ odd}. \end{aligned} \tag{13}$$

More generally, [GP1, (4.3)] and [AS, 6.1.22] show that for $-1 < x < 1$ we have

$$\sum_{k=1}^{n} \frac{c(n,k) x^k}{n!} = \frac{1}{n!} \prod_{k=0}^{n-1} (x+k) = \frac{\Gamma(n+x)}{\Gamma(x)\Gamma(n+1)}. \tag{14}$$

To analyze the asymptotics of (14), we shall make use of an inequality due to Kečkić and Vasić [KV]. It states that if $y > x \geq 1$ then

$$\frac{x^{x-1/2}}{y^{y-1/2}} e^{y-x} < \frac{\Gamma(x)}{\Gamma(y)} < \frac{x^{x-1}}{y^{y-1}} e^{y-x}. \tag{15}$$

Applying the inequality (15) to (14) implies that for a given x in $(-1,1)$ we have

$$\sum_{k=1}^{n} \frac{c(n,k) x^k}{n!} \leq \frac{(n+x)^{n+x-1} e^{1-x}}{(n+1)^n \Gamma(x)} \leq \left(1 + \frac{x}{n}\right)^n (n+x)^{x-1} e^{1-x} \leq C_x n^{x-1} \tag{16}$$

for some constant C_x. We also note that equations (4.8), (4.9), (4.10) and (4.11) of [GP1] give us

$$2\sum_{\substack{k=1\\ k \text{ odd}}}^{n} \frac{c(n,k)}{n!2^k(2,r)^k} = g_{odd,r}(n) \tag{17}$$

and

$$2\sum_{\substack{k=1\\ k \text{ even}}}^{n} \frac{c(n,k)}{n!2^k(2,r)^k} = g_{even,r}(n). \tag{18}$$

Moreover, it will be useful to know which of these functions provides the best lower bound, which is the purpose of the following two lemmas:

Lemma 2.6. *For all integers $n \geq 2$ we have*

$$\frac{1}{\sqrt{\pi n}} \geq \frac{(2n)!}{2^{2n}n!^2} \geq \frac{25}{29\sqrt{\pi n}}. \tag{19}$$

Proof. The inequalities are easily verified using the Stirling approximation

$$\left|\frac{n!}{\sqrt{2\pi n}n^n e^{-n}} - 1 - \frac{1}{12n}\right| \leq \frac{1}{288n^2} + \frac{1}{9940n^3},$$

which holds for all integers $n \geq 2$ (see [M] for example).

□

Lemma 2.7. *For all integers n, m satisfying $n \geq m \geq 2$, we have*

$$s_{\neg m}(n) \geq g_{odd,r}(n) > f_r(n) > g_{even,r}(n). \tag{20}$$

Proof. The inequalities $g_{odd,r}(n) > f_r(n)$ and $f_r(n) > g_{even,r}(n)$ are clear from the definitions of the functions. To prove that $s_{\neg m}(n) \geq g_{odd,r}(n)$, we note that $s_{\neg m}(n) \geq c_m n^{-1/m}(1-1/n)$ by (11). Moreover, Remark 2.2 of [GP2] implies that for all $m \geq 2$, we have $c_m \geq c_2 = (\pi/2)^{-1/2}$. Therefore for all n, m satisfying $n \geq m \geq 2$, we have

$$s_{\neg m}(n) \geq c_2 n^{-1/2}\left(1 - \frac{1}{n}\right) \geq \left(\frac{2}{n\pi}\right)^{1/2}\left(1 - \frac{1}{n}\right).$$

But Lemma 2.6 implies that $\frac{2n}{(2n-1)(n\pi)^{1/2}} \geq g_{odd,r}(n)$ and it is easy to see that $2(1-n^{-1}) \geq \frac{2n}{(2n-1)}$ for $n \geq 5$. Thus $s_{\neg m}(n) \geq g_{odd,r}(n)$ for $n \geq 5$ and it remains to check the cases where n and m satisfy $4 \geq n \geq m \geq 2$. We can do this easily using (10). □

3. Linear Case

Suppose that $G = \mathrm{SL}_d(q)$. Then $W = S_d$ and an F-class in W consisting of cycles of lengths $b_1, \ldots, b_m$ corresponds to a G-class of maximal tori T, which are subgroups of

$$\prod_{i=1}^{m}(q^{b_i} - 1)$$

of index $q-1$. Set $Q = Q(r, G)$.

Lemma 3.1. *Let m be the multiplicative order of q modulo r. Then $p_r(\mathrm{SL}_2(q)) \geq \frac{1}{2(q+1)_r} + \frac{1}{2(q-1)_r}$ and for $d \geq 3$ we have*

$$p_r(\mathrm{SL}_d(q)) \geq \begin{cases} s_{\neg m}(d) & \text{if } m \geq 2; \\ \frac{1}{dd_r} + \frac{1}{(d-1)(d-1)_r(q-1)_r} & \text{if } m = 1 \text{ and } r \text{ is odd;} \\ \frac{1}{d} + \frac{2}{(d-1)(d-1)_2(q^2-1)_2} & \text{if } r = 2 \text{ and } d \text{ is odd;} \\ \frac{2}{dd_2(q+1)_2} + \frac{1}{(d-1)(q-1)_2} & \text{if } r = 2 \text{ and } d \text{ is even.} \end{cases}$$

Moreover, for $G = \mathrm{PSL}_d(q)$, the lower bound $p_r(G) \geq d^{-1}$ in part (i) *of Theorem 1.4 holds.*

Proof. First suppose that r is odd and $d \geq 3$. Note that if $r|(q^{b_i} - 1)$ then $r|(q^{b_i} - 1, q^m - 1) = q^{(b_i,m)} - 1$ and therefore $m|b_i$. It follows that if $w \in S_d$ has no cycle lengths divisible by m, then $r \nmid |T_w|$ and $|T_w \cap Q|/|T_w| = 1$. If $m \geq 2$ then using Theorem 2.2 we find that

$$p_r(\mathrm{SL}_d(q)) = \frac{|Q|}{|\mathrm{SL}_d(q)|} \geq s_{\neg m}(d) \geq c_m d^{-1/m},$$

where the second inequality follows from (11). Furthermore, the bounds in (11) are sufficient to prove that $s_{\neg m}(d) \geq 1/d$ unless $(d, m) = (3, 2)$. But an easy direct calculation verifies that $s_{\neg 2}(3) \geq 1/3$. The proportion of r-regular elements in $\mathrm{PSL}_d(q)$ when $m \geq 2$ is therefore

$$p_r(\mathrm{PSL}_d(q)) = (d, q-1)_r p_r(\mathrm{SL}_d(q)) \geq d^{-1}$$

by Lemma 2.3.

If $m = 1$ then $r|q - 1$. Consider the conjugacy classes of d-cycles and $(d-1)$-cycles in S_d, which correspond to the G-classes of cyclic maximal tori in $\mathrm{SL}_d(q)$ of orders $(q^d - 1)/(q - 1)$ and $q^{d-1} - 1$. Since $r|q - 1$, the r-part of $(q^d - 1)/(q - 1)$ is d_r by Lemma 2.5, and the r-part of $(q^{d-1} - 1)$ is $(d - 1)_r(q - 1)_r$. For such tori T, we therefore have $|Q \cap T|/|T| = 1/d_r$ and $1/\left((d - 1)_r(q - 1)_r\right)$ respectively. Since the proportion of d-cycles in S_d is $1/d$ and the proportion of $(d - 1)$-cycles is $1/(d - 1)$, Theorem 2.2 implies

$$p_r(\mathrm{SL}_d(q)) \geq \frac{1}{dd_r} + \frac{1}{(d-1)_r(q-1)_r(d-1)}.$$

Taking the same two classes of tori when $d \geq 3$ and $r = 2$ implies that

$$p_2(\mathrm{SL}_d(q)) \geq \begin{cases} \frac{2}{dd_2(q+1)_2} + \frac{1}{(d-1)(q-1)_2} & \text{when } d \text{ is even;} \\ \frac{1}{d} + \frac{2}{(d-1)(d-1)_2(q^2-1)_2}. & \text{when } d \text{ is odd.} \end{cases}$$

Using Lemma 2.3, we find that the proportion of r-regular elements in $\mathrm{PSL}_d(q)$ is therefore

$$p_r(\mathrm{PSL}_d(q)) = (d, q-1)_r p_r(\mathrm{SL}_d(q)).$$

We claim that $p_r(\mathrm{PSL}_d(q)) \geq 1/d$ in all cases. If r is odd and $d_r = (d, q-1)_r$, then this is clear. On the other hand, if r is odd and $d_r > (d, q-1)_r = (q-1)_r$, then $(d-1)_r = 1$ and therefore $p_r(\mathrm{PSL}_d(q)) \geq 1/(d-1)$. If $r = 2$ and d is odd, then the result is also clear. If $r = 2$ and $(d, q-1)_2 = (q-1)_2$ (so d is even), then it follows that $p_2(\mathrm{PSL}_d(q)) \geq 1/(d-1)$. If $(q-1)_2 > (d, q-1)_2 = d_2$ and d is even, then observe that $q \equiv 1 \pmod 4$ and $(q+1)_2 = 2$. Now the result follows quickly in this case as well. It remains to prove the bounds for

$\mathrm{SL}_2(q)$. The tori corresponding to $\{1\}$ in S_2 are cyclic of order $q-1$ and the tori corresponding to $\{(12)\}$ are cyclic of order $q+1$. Theorem 2.2 implies $p_r(\mathrm{SL}_2(q)) \geq \frac{1}{2(q-1)_r} + \frac{1}{2(q+1)_r}$ as required. Finally

$$p_r(\mathrm{PSL}_2(q)) = p_r(\mathrm{SL}_2(q))(2, q-1)_r \geq \frac{1}{2}.$$

□

Lemma 3.2. *Let $\epsilon > 0$, and let p and r be distinct primes. Then there exist infinitely many powers q of p for which $p_r(\mathrm{PSL}_d(q)) < 1/d + \epsilon$. In particular, part* (ii) *of Theorem 1.4 holds when $G = \mathrm{PSL}_d(q)$.*

Proof. Let $\epsilon > 0$ and $d \geq 2$, let p be a prime and let r be a prime distinct from p. Choose a positive integer a such that $1/r^a < \epsilon/d_r$ and $r^a \geq d_r$. Now choose a positive integer j such that $r^a | p^j - 1$. Note that for any positive integer b, if we let $q = p^{jr^b}$, then $r^a | q-1$ by Lemma 2.5 and $(d, q-1)_r = d_r$. Now observe that for all classes in W, other than the class of d-cycles, the corresponding tori in $\mathrm{SL}_d(q)$ all have at least one cyclic factor of order $q^i - 1$. By Lemma 2.5, for any such torus T we have

$$|Q \cap T|/|T| \leq \frac{1}{(q^i-1)_r} \leq \frac{1}{r^a} < \frac{\epsilon}{d_r}.$$

For the cyclic tori T of order $(q^d-1)/(q-1)$ we have $|Q \cap T|/|T| = 1/d_r$. Applying Theorem 2.2 and this inequality gives

$$p_r(\mathrm{SL}_d(q)) = \sum_{C_0 \subset W} \frac{|C_0|}{|W|} \cdot \frac{|T_{C_0} \cap Q(r, G)|}{|T_{C_0}|} \leq \frac{(d-1)\epsilon}{dd_r} + \frac{1}{dd_r} < \frac{1}{d_r}\left(\epsilon + d^{-1}\right).$$

Now applying Lemma 2.3 together with the fact that $(d, q-1)_r = d_r$ implies

$$p_r(\mathrm{PSL}_d(q)) < \epsilon + d^{-1}$$

as required. □

Lemma 3.3. *For each odd prime power q and for all $d \geq 2$ we have*

$$p_2(\mathrm{PSL}_d(q)) \leq \frac{2(d, q-1)_2}{\sqrt{\pi d}}.$$

In particular Theorem 1.3 holds for these groups with $C_q = 2(q-1)_2/\sqrt{\pi}$.

Proof. Note that $p_2(\mathrm{PSL}_d(q)) = (d, q-1)_2 p_2(\mathrm{SL}_d(q))$ by Lemma 2.3 and set $Q = Q(2, \mathrm{SL}_d(q))$. Now observe that for all classes in W with $k \geq 2$ cycles, the corresponding tori in $\mathrm{SL}_d(q)$ have $k-1$ cyclic factors of order $(q^i - 1)$ for various i. By Lemma 2.5, for any such torus T, we have

$$\frac{|Q \cap T|}{|T|} \leq \frac{1}{2^{k-1}}.$$

Applying Theorem 2.2 and this inequality gives

$$p_2(\mathrm{PSL}_d(q)) \leq (d, q-1)_2 \left(\frac{1}{d} + \sum_{k=2}^{d} \frac{c(d,k)}{d!2^{k-1}}\right) = 2(d, q-1)_2 \sum_{k=1}^{d} \frac{c(d,k)}{d!2^k}$$

and the required bound follows from (13). □

Remark 3.4. We note that Lemma 3.3 improves [BPS, Theorem 6.1], which states that for $q \equiv 3 \pmod 4$ such that $(d, q-1) \leq 2$, the proportion of elements of odd order is $p_2(\mathrm{PSL}_d(q)) \leq 4/q + 4/\sqrt{\pi d}$.

4. Unitary case

Suppose that $G = \mathrm{SU}_d(q)$, and since $\mathrm{SU}_2(q) \cong \mathrm{SL}_2(q)$, we may assume $d \geq 3$. Now $W = S_d$ and an F-class in W consisting of cycles of length $b_1, \ldots, b_k$ corresponds to a G-class of maximal tori T, which are subgroups of

$$\prod_{i=1}^{m} (q^{b_i} - (-1)^{b_i}) \tag{21}$$

of index $q+1$. Again set $Q = Q(r, G)$.

Lemma 4.1. *Let m' be the multiplicative order of q modulo r. Define*

$$m = \begin{cases} 1 & \text{if } r = 2; \\ 2m' & \text{if } m' \text{ is odd and } r \neq 2; \\ m' & \text{if } m' \equiv 0 \pmod 4 \text{ and } r \neq 2; \\ m'/2 & \text{if } m' \equiv 2 \pmod 4 \text{ and } r \neq 2. \end{cases}$$

Then

$$p_r(\mathrm{SU}_d(q)) \geq \begin{cases} s_{\neg m}(d) \geq c(m) d^{-1/m} & \text{if } m \geq 2; \\ \frac{1}{dd_r} + \frac{1}{(d-1)(d-1)_r(q+1)_r} & \text{for } m = 1 \text{ and } r \text{ odd}; \\ \frac{1}{d} + \frac{2}{(d-1)(d-1)_2(q^2-1)_2} & \text{for } r = 2 \text{ and } d \text{ odd}; \\ \frac{2}{dd_2(q-1)_2} + \frac{1}{(d-1)(q+1)_2} & \text{for } r = 2 \text{ and } d \text{ even}. \end{cases} \tag{22}$$

and for $G = \mathrm{PSU}_d(q)$ the lower bound $p_r(G) \geq 1/d$ in part (i) *of Theorem 1.4 holds.*

Proof. First we show that m is the smallest positive integer for which $r | q^m - (-1)^m$. If $r = 2$ then $r | q+1$ and $m = 1$ so the claim is true. Suppose now that r is odd. Note that $r | q^i - 1$ if and only if $m' | i$. If m' is odd then $r \nmid q^i + 1$ for any i so we seek the least even i such that $r | q^i - 1$; the least such i is $2m' = m$. So suppose that m' is even. Then $r | q^{m'/2} + 1$ and $m'/2$ is the least integer i such that $r | q^i + 1$. If $m' \equiv 2 \pmod 4$ then $m = m'/2$ is odd, and hence $q^{m'/2} + 1 = q^m - (-1)^m$ and the claim is proved. This leaves $m' \equiv 0 \pmod 4$ in which case $m = m'$, and there is no odd i such that $r | q^i + 1$. So the claim holds in this case also.

First suppose that r is odd. If m is even, and $r | (q^{b_i} - 1)$ (with b_i even) then r divides $(q^{b_i} - 1, q^m - 1)$, which equals $q^{(b_i, m)} - 1$ by Lemma 2.4, and therefore $m | b_i$. If m is even, and $r | (q^{b_i} + 1)$ (with b_i odd), then r divides $(q^{b_i} + 1, q^m - 1)$, which equals $q^{(b_i, m)} + 1$ by Lemma 2.4 and therefore $m | b_i$. It follows that if m is even and $w \in S_n$ has no cycle of length divisible by m, then $r \nmid |T_w|$ and $|T_w \cap Q| / |T_w| = 1$.

Similarly, if $m \geq 3$ is odd, and $r | (q^{b_i} - 1)$ (with b_i even) then r divides $(q^{b_i} - 1, q^m + 1)$, which equals $q^{(b_i, m)} + 1$ and therefore $m | b_i$. If m is odd, and $r | (q^{b_i} + 1)$ (with b_i odd), then r divides $(q^{b_i} + 1, q^m + 1)$, which equals $q^{(b_i, m)} + 1$ by Lemma 2.4, and therefore $m | b_i$. It follows that if m is odd and

$m \geq 3$, and if $w \in S_n$ has no cycles of length divisible by m, then $r \nmid |T_w|$ and $|T_w \cap Q|/|T_w| = 1$.

Using Theorem 2.2 we find that

$$p_r(\mathrm{SU}_d(q)) = \frac{|Q|}{|\mathrm{SU}_d(q)|} \geq s_{\neg m}(d).$$

As in the linear case, we can verify that $s_{\neg m}(d) \geq 1/d$ in all cases. The proportion of r-regular elements in $\mathrm{PSU}_d(q)$ when r is odd and $m \geq 2$ is therefore

$$p_r(\mathrm{PSU}_d(q)) \geq (d, q+1)_r s_{\neg m}(d) \geq d^{-1}$$

by Lemma 2.3.

Now suppose $m = 1$ so that $r|q+1$. As in the linear case, we consider the conjugacy classes of d-cycles and $(d-1)$-cycles in S_d, which correspond to the classes of cyclic maximal tori in $\mathrm{SU}_d(q)$ of orders $(q^d-(-1)^d)/(q+1)$ and $(q^{d-1}-(-1)^{d-1})$ respectively. Since $r|q+1$, when r and d are odd, the r-part of $(q^d+1)/(q+1)$ is d_r by Lemma 2.5. Similarly if r is odd and d is even, then the r-part of $(q^d-1)/(q+1)$ is also d_r. And if r is odd, then the r-part of $(q^{d-1}-(-1)^{d-1})$ is $(d-1)_r(q+1)_r$. Applying Theorem 2.2 proves the lemma in the case $m = 1$ and r odd. The cases with $r = 2$ follow from entirely similar applications of Lemma 2.5 and Theorem 2.2 and we omit the details. Thus, the inequality (22) is proved.

By Lemma 2.3, we have $p_r(\mathrm{PSU}_d(q)) = (d, q+1)_r p_r(\mathrm{SU}_d(q))$ and the inequality $p_r(\mathrm{PSU}_d(q)) \geq d^{-1}$ follows by the same argument as in the linear case. □

Lemma 4.2. *Let $\epsilon > 0$, and let p and r be distinct primes such that the multiplicative order of p modulo r is even. Then there exist infinitely many prime powers q of p for which $p_r(\mathrm{PSU}_d(q)) < 1/d + \epsilon$. In particular, part* (ii) *of Theorem 1.4 holds when $G = \mathrm{PSU}_d(q)$.*

Proof. Let $\epsilon > 0$ and $d \geq 2$, and let r be a prime. As in the linear case, choose a positive integer a such that $1/r^a < \epsilon/d_r$ and $r^a \geq d_r$. Since the multiplicative order of p modulo r is even, there exists a positive integer j' such that $r|p^{j'}+1$. In the light of Lemma 2.5, there exists a positive integer j such that $r^a|p^j+1$. Now for any positive integer b, if we let $q = p^{jr^b}$ for r odd and $q = p^{j3^b}$ for $r = 2$, then $r^a|q+1$ by Lemma 2.5 and $(d, q+1)_r = d_r$. We observe that for all classes in W, other than the class of d-cycles, the corresponding tori in $\mathrm{SU}_d(q)$ all have at least one cyclic factor of order $q^i-(-1)^i$. By Lemma 2.5, for any such torus T we have

$$\frac{|Q \cap T|}{|T|} \leq \frac{1}{(q^i-(-1)^i)_r} \leq \frac{1}{r^a} < \frac{\epsilon}{d_r}.$$

Similarly, for a cyclic torus T of order $(q^d-(-1)^d)/(q+1)$ we have $|Q \cap T|/|T| = 1/d_r$ by Lemma 2.5 (note that if $r = 2$, then $q \equiv 3 \pmod 4$). Applying Theorem 2.2 and this inequality gives

$$p_r(\mathrm{SU}_d(q)) = \sum_{B \subset W} \frac{|B|}{|W|} \cdot \frac{|T_B \cap Q(r,G)|}{|T_B|} \leq \frac{(d-1)\epsilon}{dd_r} + \frac{1}{dd_r}.$$

Now applying Lemma 2.3 together with the fact that $(d, q+1)_r = d_r$ implies

$$p_r(\mathrm{PSU}_d(q)) < \epsilon + d^{-1}$$

as required. □

Lemma 4.3. *For each prime power q and for all $d \geq 2$ we have*

$$p_2(\mathrm{PSU}_d(q)) \leq \frac{2(d, q+1)_2}{\sqrt{\pi d}}.$$

In particular Theorem 1.3 holds for these groups with $C_q = 2(q+1)_2/\sqrt{\pi}$.

Proof. Note that $p_2(\mathrm{PSU}_d(q)) = (d, q+1)_2 p_2(\mathrm{SU}_d(q))$ by Lemma 2.3 and set $Q = Q(2, \mathrm{SU}_d(q))$. Now observe that for all classes in W with $k \geq 2$ cycles, the corresponding tori in $\mathrm{SU}_d(q)$ have $k-1$ cyclic factors of order $(q^i - (-1)^i)$ for various i. By Lemma 2.5, for any such torus T, we have

$$\frac{|Q \cap T|}{|T|} \leq \frac{1}{2^{k-1}}.$$

Applying Theorem 2.2 and the inequality above gives

$$p_2(\mathrm{PSU}_d(q)) \leq (d, q+1)_2 \left(\frac{1}{d} + \sum_{k=2}^{d} \frac{c(d,k)}{d! 2^{k-1}} \right) \leq 2(d, q+1)_2 \sum_{k=1}^{d} \frac{c(d,k)}{d! 2^k}$$

and the required bound follows from (13). □

5. Symplectic and odd dimensional orthogonal cases

Suppose that $G = \mathrm{Sp}_{2n}(q)$. Then $W = C_2 \wr S_n \leq S_{2n}$, and a partition

$$\beta = (\beta^+, \beta^-) = (b_1^+, \ldots, b_{k_1}^+, b_1^-, \ldots, b_{k_2}^-)$$

of n, representing an F-class in W, corresponds to a G-class of maximal tori T of the form

$$T = \prod_{i=1}^{k_1} (q^{b_i^+} - 1) \times \prod_{j=1}^{k_2} (q^{b_j^-} + 1). \tag{23}$$

For $w \in W = C_2 \wr S_n$, write $\sigma(w)$ for the natural image of w in S_n. The partition β, without the signs, represents the cycle structure of $\sigma(w)$ in S_n; the $+$ sign indicates that the b_1^+ cycle in S_n is the image of two cycles of length b_1 in $W \leq S_{2n}$, and these will be referred to as *positive cycles*. The $-$ sign indicates that the b_1^- cycle in S_n is the image of one cycle of length $2b_1$ in $W \leq S_{2n}$, and these will be referred to as *negative cycles*. Recall the definition of f_r in (1) and that, by (13), we have

$$f_r(n) = \sum_{k=1}^{n} \frac{c(n,k)}{n! 2^k (2,r)^k}.$$

If $G = \mathrm{SO}_{2n+1}(q)$, then the Weyl group W and the correspondence between classes in W and G-classes of maximal tori are exactly the same as in the case $G = \mathrm{Sp}_{2n}(q)$.

Lemma 5.1. *Let $G = \mathrm{Sp}_{2n}(q)$, $\mathrm{SO}_{2n+1}(q)$ or $\Omega_{2n+1}(q)$ where $n \geq 2$. If r is a prime not dividing q and m is the multiplicative order of q modulo r, then*

$$p_r(G) \geq \begin{cases} s_{\neg m}(n) & \text{if } m \text{ is odd and } m \geq 3; \\ s_{\neg \frac{m}{2}}(n) & \text{if } m \text{ is even and } m \geq 4; \\ f_r(n) & \text{otherwise.} \end{cases}$$

Also, $p_2(\Omega_{2n+1}(q)) = 2p_2(\mathrm{SO}_{2n+1}(q))$ and $p_r(\mathrm{PSp}_{2n}(q)) = (2,r)p_r(\mathrm{Sp}_{2n}(q))$, and parts (i) *and* (iii) *of Theorem 1.4 hold for these groups.*

Remark 5.2. Since $-I_{2n+1} \in Z(\mathrm{GO}_{2n+1}(q))$ has determinant -1, it follows that $\mathrm{PSO}_{2n+1}(q) = \mathrm{SO}_{2n+1}(q)$ and $\mathrm{P}\Omega_{2n+1}(q) = \Omega_{2n+1}(q)$.

Proof. First let us assume that $G = \mathrm{Sp}_{2n}(q)$. We may assume that r is odd since the case $r = 2$ is part of Proposition 6.2 in [GP1]. Observe that $r|q^i - 1$ if and only if i is a multiple of m. Similarly, $r|q^i + 1$ if and only if $q^i \equiv -1 \pmod{r}$ if and only if m is even and i is an odd multiple of $m/2$. It follows that if $m \geq 3$ is odd, and $\sigma(w) \in S_n$ has no cycle lengths b_i divisible by m, then $r \nmid (q^{b_i} - 1)$ and $r \nmid (q^{b_i} + 1)$. In particular, $r \nmid |T_w|$ for all such $w \in W$. Applying Theorem 2.2 with C the union of classes in W satisfying this condition implies $p_r(\mathrm{Sp}_{2n}(q)) \geq s_{\neg m}(n)$ when $m \geq 3$ is odd. Similarly, if $m \geq 4$ is even and $\sigma(w) \in S_n$ has no cycle of length divisible by $m/2$, then $r \nmid |T_w|$ and $p_r(\mathrm{Sp}_{2n}(q)) \geq s_{\neg \frac{m}{2}}(n)$.

For each positive integer i, we have $\gcd(q^i - 1, q^i + 1) = (2, q-1)$. Since r is odd, it follows that r cannot divide both $q^i - 1$ and $q^i + 1$. Therefore for all $\tau \in S_n$ it is possible to label each of the cycles of an arbitrary element $\tau \in S_n$ as positive or negative to give $w \in W$ satisfying $\sigma(w) = \tau$ and $r \nmid |T_w|$. We apply Theorem 2.2 with C the union of classes B such that $r \nmid |T_B|$. If τ has k cycles, then at least 2^{n-k} of the 2^n elements in $\{w \in W \mid \sigma(w) = \tau\}$ satisfy the required condition. Therefore if $c(n,k)$ denotes the number of permutations of S_n with exactly k cycles, then we have

$$p_r(\mathrm{Sp}_{2n}(q)) \geq \sum_{k=1}^{n} \frac{2^{n-k}c(n,k)}{|W|} = \sum_{k=1}^{n} \frac{c(n,k)}{2^k n!} = f_r(n).$$

Now we apply Lemma 2.3 (noting that $(2,r) = 1$) to obtain

$$p_r(\mathrm{PSp}_{2n}(q)) = |Z|_r p_r(\mathrm{Sp}_{2n}(q)) = (2,r)p_r(\mathrm{Sp}_{2n}(q)).$$

The same calculations prove the results for $\mathrm{SO}_{2n+1}(q)$ and $\mathrm{PSO}_{2n+1}(q)$. If $G = \Omega_{2n+1}(q)$ and r is odd then it is easy to see that

$$Q_\Omega := Q(r, \mathrm{SO}_{2n+1}(q)) \cap \Omega_{2n+1}(q)$$

is also a quokka set. Moreover for each class B chosen above, we have $T_B \subset Q(r, \mathrm{SO}_{2n+1}(q))$ and thus $|T_B \cap Q_\Omega| = |T_B \cap \Omega_{2n+1}(q)| = |T_B|/2$ (see [BG, Theorem 4] for example). Theorem 2.2 therefore implies

$$\frac{|Q_\Omega|}{|\mathrm{SO}_{2n+1}(q)|} \geq \frac{(2n)!}{2^{2n+1}n!^2}.$$

It now follows easily that if r is odd, then

$$p_r(\Omega_{2n+1}(q)) = \frac{|Q_\Omega|}{|\Omega_{2n+1}(q)|} \geq \frac{(2n)!}{2^{2n}n!^2} = f_r(n).$$

Finally, observe that all 2-regular elements in $\mathrm{SO}_{2n+1}(q)$ are actually contained in $\Omega_{2n+1}(q)$; thus $p_2(\Omega_{2n+1}(q)) = 2p_2(\mathrm{SO}_{2n+1}(q))$. □

Lemma 5.3. *Let $\epsilon > 0$, $n \geq 2$ an integer, and let p and r be distinct primes. Then there exist infinitely many powers q of p for which $p_r(\Omega_{2n+1}(q))$ is less than $(2,r)f_r(n) + \epsilon$, and $p_r(\mathrm{Sp}_{2n}(q))$ and $p_r(\mathrm{SO}_{2n+1}(q))$ are less than $f_r(n) + \epsilon$. In particular, part* (ii) *of Theorem 1.4 holds for these groups.*

Proof. Choose a positive integer a such that $1/r^a < \epsilon/2$. Now choose a positive integer j such that $r^a | p^j - 1$. Note that for any positive integer b, if we let $q = p^{jr^b}$, then $r^a | q - 1$ by Lemma 2.5. We show that $p_r(G) < f_r(n) + \epsilon/2$ for $G = \mathrm{Sp}_{2n}(q)$ and $\mathrm{SO}_{2n+1}(q)$ so that the result for $p_2(\Omega_{2n+1}(q))$ follows from the equation $p_2(\Omega_{2n+1}(q)) = 2p_2(\mathrm{SO}_{2n+1}(q))$ obtained in Lemma 5.1. Now observe that for all classes in W with at least one positive cycle, that is, classes whose corresponding tori have at least one cyclic factor of order $q^i - 1$, we have

$$\frac{|Q \cap T|}{|T|} \leq \frac{1}{(q^i - 1)_r} \leq \frac{1}{r^a} < \frac{\epsilon}{2}.$$

Arguing as in the previous proof, for each of the $c(n,k)$ permutations $\sigma(w)$ of S_n with exactly k cycles, exactly 2^{n-k} of the corresponding 2^n elements w of W have k negative cycles. For each torus T with k negative cycles and no positive cycles, we have $|Q \cap T|/|T| = 1/(2,r)^k$. Applying this inequality and Theorem 2.2 for $G = \mathrm{Sp}_{2n}(q)$ or $\mathrm{SO}_{2n+1}(q)$ gives

$$p_r(G) = \sum_{B \subset W} \frac{|B|}{|W|} . \frac{|T_B \cap Q(r,G)|}{|T_B|} < \frac{\epsilon}{2} + \sum_{k=1}^{n} \frac{c(n,k)}{2^k (2,r)^k n!} = \frac{\epsilon}{2} + f_r(n).$$

The bound for $G = \Omega_{2n+1}(q)$ follows by the same argument as in Lemma 5.1 for r odd and from the equation $p_2(\Omega_{2n+1}(q)) = 2p_2(\mathrm{SO}_{2n+1}(q))$ for $r = 2$. □

Corollary 5.4. *Let $G = \Omega_{2n+1}(q)$, $\mathrm{SO}_{2n+1}(q)$ or $\mathrm{Sp}_{2n}(q)$, and let r be a prime not dividing q. Then*

$$p_r(G) \geq \begin{cases} \frac{25}{29\sqrt{\pi n}} & \text{for } r \text{ odd;} \\ \frac{1}{4(n+1)^{3/4}} & \text{for } r = 2 \text{ and } G \neq \Omega_{2n+1}(q); \\ \frac{1}{2(n+1)^{3/4}} & \text{for } r = 2 \text{ and } G = \Omega_{2n+1}(q). \end{cases}$$

Moreover, for any prime p, there exist infinitely many powers q of p for which

$$p_r(G) \leq \begin{cases} \frac{1}{\sqrt{\pi n}} & \text{for } r \text{ odd;} \\ \frac{3}{5(n+1)^{3/4}} & \text{for } r = 2 \text{ and } G \neq \Omega_{2n+1}(q); \\ \frac{6}{5(n+1)^{3/4}} & \text{for } r = 2 \text{ and } G = \Omega_{2n+1}(q). \end{cases}$$

Furthermore there exist constants C_5' and C_6' such that

$$p_2(G) \leq \begin{cases} \frac{C_5'}{n^{5/8}} & \text{if } q \equiv 1 \pmod 4; \\ \frac{C_6'}{n^{1/2}} & \text{if } q \equiv -1 \pmod 4; \end{cases}$$

Therefore Theorems 1.1, 1.2 and 1.3 hold for these groups.

Proof. The first two sets of inequalities follow easily from (4.14) in [GP1] and our bounds in Lemma 2.6. For the final set of inequalities first suppose $q \equiv 1 \pmod 4$ and set $Q = Q(2, G)$. Observe that for each positive cycle, we have a cyclic factor $(q^i - 1)$ in the corresponding torus and $(q^i - 1)_2 \geq 4$. Similarly, for each negative cycle, we have a cyclic factor $(q^i + 1)$ in the corresponding torus and $(q^i + 1)_2 = 2$. So for $w \in W$ with i positive cycles and $k - i$ negative cycles, the proportion of 2-regular elements in T is

$$\frac{|Q \cap T|}{|T|} \leq \frac{1}{4^i 2^{k-i}} = \frac{1}{2^{k+i}}.$$

We note that precisely $c(n,k)2^{n-k}\binom{k}{i}$ of the elements in W have i positive cycles and $k - i$ negative cycles and therefore Theorem 2.2 gives

$$p_2(\mathrm{Sp}_d(q)) \leq \sum_{k=1}^{n} \frac{c(n,k)}{n!2^k} \sum_{i=0}^{k} \binom{k}{i} \frac{1}{2^{k+i}} = \sum_{k=1}^{n} \frac{c(n,k)}{n!} \left(\frac{3}{8}\right)^k.$$

Using (16), it follows that $p_2(G) \leq C_5' d^{-5/8}$ for some constant C_5'. A similar calculation for $q \equiv 3 \pmod 4$ with

$$\frac{|Q \cap T|}{|T|} \leq \frac{1}{2^k}$$

yields the upper bound $f_2(n)$ and, by Lemma 2.6, this is at most $C_6' d^{-1/2}$ for some constant C_6'. □

6. Even dimensional orthogonal cases

The structure of the maximal tori in $\mathrm{SO}^{\pm}_{2n}(q)$ is essentially the same as in the symplectic case, but the Weyl group has index 2 in the Weyl group $W_{B_n} := C_2 \wr S_n$ for B_n. Moreover, in the untwisted case, an F-conjugacy class in W is a conjugacy class of permutations in W_{B_n} with an even number of negative cycles (that is, k_2 in (23) is even). In the twisted case, an F-conjugacy class C in W corresponds to a conjugacy class C' of permutations in W_{B_n} with an odd number of negative cycles (that is, k_2 is odd); if we define the 2-cycle $w = (n\, n') \in W_{B_n}$, then $C = wC'$. Recall that $g_{odd,r}$ and $g_{even,r}$ are defined in (2) and (3) in terms of the Gamma function and that we also have from (17) and (18) that

$$g_{odd,r}(n) = 2 \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{c(n,k)}{n!2^k(2,r)^k} \quad \text{and} \quad g_{even,r}(n) = 2 \sum_{\substack{k=1 \\ k \text{ even}}}^{n} \frac{c(n,k)}{n!2^k(2,r)^k}.$$

The *Witt defect* of $\mathrm{SO}^{\pm}_{2n}(q)$ or $\Omega^{\pm}_{2n}(q)$ is 0 if the group is of + type and 1 if the group is of − type.

Lemma 6.1. *Let $G = \mathrm{SO}^{\pm}_{2n}(q)$ or $\Omega^{\pm}_{2n}(q)$ where $n \geq 2$. Let m be the multiplicative order of q modulo r and let l be the Witt defect of G. If r is*

odd, then

(24)

$$p_r(G) \geq \begin{cases} s_{\neg m}(n) & \text{if } m \text{ is odd and } m \geq 3; \\ s_{\neg \frac{m}{2}}(n) & \text{if } m \text{ is even and } m \geq 4; \\ g_{odd,r}(n) & \text{if } m = 2 \text{ and } n+l \text{ is odd, or if } l = m = 1; \\ g_{even,r}(n) & \text{if } m = 2,\ n+l \text{ is even, or if } m = 1 \text{ and } l = 0. \end{cases}$$

And

$$p_2(\mathrm{SO}^{\pm}_{2n}(q)) \geq \begin{cases} g_{even,r}(n) & \text{if } q^n \equiv \pm 1 \pmod 4; \\ g_{odd,r}(n) & \text{if } q^n \equiv \mp 1 \pmod 4. \end{cases} \tag{25}$$

Also $p_r(G/Z(G)) = |Z(G)|_r p_r(G)$ *and* $p_2(\Omega^{\pm}_{2n}(q)) = 2p_2(\mathrm{SO}^{\pm}_{2n}(q))$. *Parts* (i) *and* (iii) *of Theorem 1.4 hold for these groups.*

Remark 6.2. In this case, $Z(\mathrm{SO}^{\pm}_{2n}(q)) = \langle -I_{2n} \rangle$ since $-I_{2n}$ has determinant 1. The centre of $\Omega^{\pm}_{2n}(q)$ is nontrivial if and only if q is odd and $-I_{2n}$ has spinor norm 1. It turns out that $Z(\Omega^{+}_{2n}(q)) = \langle -I_{2n} \rangle$ if and only if $q^n \equiv 1 \pmod 4$, and is trivial otherwise. Similarly, $Z(\Omega^{-}_{2n}(q)) = \langle -I_{2n} \rangle$ if and only if $q^n \equiv -1 \pmod 4$, and is trivial otherwise.

Proof. First of all, we suppose that $G = \mathrm{SO}^{\pm}_{2n}(q)$. Again we may assume r is odd by Proposition 7.1 of [GP1]. Observe that $r | q^i - 1$ if and only if i is a multiple of m. Similarly, $r | q^i + 1$ if and only if $q^i \equiv -1 \pmod r$ if and only if m is even and i is an odd multiple of $m/2$. It follows that if $m \geq 3$ is odd, and $\sigma(w) \in S_n$ has no cycle lengths b_i divisible by m, then $r \nmid (q^{b_i} - 1)$ and $r \nmid (q^{b_i} + 1)$. In particular, $r \nmid |T_w|$ for all such $w \in W$. Applying Theorem 2.2 with C the union of classes in W satisfying this condition implies $p_r(\mathrm{SO}^{\pm}_{2n}(q)) \geq s_{\neg m}(n)$ when $m \geq 3$ is odd. Similarly, if $m \geq 4$ is even and $\sigma(w) \in S_n$ has no cycle of length divisible by $m/2$, then $r \nmid |T_w|$ and $p_r(\mathrm{SO}^{\pm}_{2n}(q)) \geq s_{\neg \frac{m}{2}}(n)$.

Now suppose that $m = 1$ so that $r | q - 1$. Then $r \nmid q^i + 1$ for all positive integers i and therefore if C is the union of classes B in W whose cycles are all negative, then $r \nmid |T_B|$ for all B. If the Witt defect $l = 0$, then there must be an even number of negative cycles and thus $\sigma(w)$ must have an even number of cycles when $w \in C$. By the same argument as in the symplectic case, if $\tau \in S_n$ has k cycles, then 2^{n-k} of the 2^n elements w in W_{B_n} satisfying $\sigma(w) = \tau$ have all cycles negative, and all of these elements are contained in W. Applying Theorem 2.2 we have

$$p_r(\mathrm{SO}^{+}_{2n}(q)) \geq \frac{|C|}{|W|} = \sum_{\substack{k=1 \\ k \text{ even}}}^{n} \frac{c(n,k)2^{n-k}}{n!2^{n-1}} = 2 \sum_{\substack{k=1 \\ k \text{ even}}}^{n} \frac{c(n,k)}{n!2^k} = g_{even,r}(n)$$

when $m = 1$ and $l = 0$. If $m = l = 1$, then in order for r not to divide $|T_w|$, the permutation $\sigma(w)$ must have an odd number of cycles (all of which are negative) and a similar calculation shows

$$p_r(\mathrm{SO}^{-}_{2n}(q)) \geq 2 \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{c(n,k)}{n!2^k} = g_{odd,r}(n).$$

It remains to deal with the case $m = 2$ (so $r|q+1$). Here $r|q^i+1$ if and only if i is odd, and $r|q^i-1$ if and only if i is even. Therefore we let C be the union of classes in W whose elements only have positive cycles of odd length and negative cycles of even length. If n is even and $l = 0$, then we claim that $w \in W$ can be contained in C only if the number k of cycles of $\sigma(w)$ is even. For if n is even, then the number k_{odd} of cycles of $\sigma(w)$ of odd length must be even. But the number of negative cycles of w must also be even since $l = 0$, and this is the same as the number k_{even} of cycles of even length. Thus k_{odd} and k_{even} must both be even and $k = k_{odd} + k_{even}$ must be even. This argument shows that if n is even, then k_{even} is even if and only if k is even. By the same argument as before, if $\sigma(w)$ has k cycles and k is even, then precisely 2^{n-k} of the 2^n elements in W_{B_n} have all of the cycles of the correct signs, and all of these elements are contained in W. Thus when n is even, $m = 1$, and $l = 0$, we have

$$p_r(\mathrm{SO}^+_{2n}(q)) \geq 2 \sum_{\substack{k=1 \\ k \text{ even}}}^{n} \frac{c(n,k)}{n!2^k} = g_{even,r}(n).$$

Similarly, if n is odd, $m = 1$ and $l = 0$, then $w \in W$ can be contained in C only if the number k of cycles of $\sigma(w)$ is odd and the same argument gives

$$p_r(\mathrm{SO}^+_{2n}(q)) \geq 2 \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{c(n,k)}{n!2^k} = g_{odd,r}(n).$$

If n is even, $m = 1$ and $l = 1$, then k must be odd. And if n is odd, $m = 1$ and $l = 1$, then k must be even to satisfy our condition. Theorem 2.2 implies the result in these last two cases.

The theorem for $G = \Omega^{\pm}_{2n}(q)$ and r odd is obtained in exactly the same way as in the odd dimensional orthogonal case. When $r = 2$, we observe as before that all odd order elements in $\mathrm{SO}^{\pm}_{2n}(q)$ are contained in $\Omega^{\pm}_{2n}(q)$ and therefore $p_2(\Omega^{\pm}_{2n}(q)) = 2p_2(\mathrm{SO}^{\pm}_{2n}(q))$.

As usual, Lemma 2.3 implies that $p_r(G/Z(G)) = |Z(G)|_r p_r(G)$ for all of these groups. □

Recall from Lemma 2.7 that for all $m \geq 2$, we have $s_{\neg m}(n) \geq g_{odd,r}(n) \geq g_{even,r}(n)$ and notice that Lemma 6.1 proves that $p_r(\mathrm{SO}^+_{2n}(q)) \geq g_{even,r}(n)$ for all $n \geq 2$, $p_r(\mathrm{SO}^-_{2n}(q)) \geq g_{odd,r}(n)$ if n is even, and $p_r(\mathrm{SO}^-_{2n}(q)) \geq g_{even,r}(n)$ if $n \geq 3$ is odd. Lemma 6.3 below shows that these are the best possible lower bounds that are independent of q. Here we denote the multiplicative order of p modulo r by $o(p \pmod r)$.

Lemma 6.3. *Let $G = X_{2n}(q)$ be an even dimensional orthogonal group defined over a field of characteristic p, and let r be a prime distinct from p. Suppose that X, n, r, and p satisfy the conditions in one of the rows of Table 2 and let $\epsilon > 0$. Then there exist infinitely many powers q of p for which*

$$p_r(G) < h_{X,r}(n) + \epsilon. \tag{26}$$

In particular, part (ii) *of Theorem 1.4 holds for these groups.*

X_d	Conditions	$h_{X,r}(n)$	$o(p \pmod r)$
SO^+_{2n}	-	$g_{even,r}(n)$	any
Ω^+_{2n}	-	$(2,r)g_{even,r}(n)$	any
SO^-_{2n}		$g_{odd,r}(n)$	any
SO^-_{2n}	n odd	$g_{even,r}(n)$	even
Ω^-_{2n}		$(2,r)g_{odd,r}(n)$	any
Ω^-_{2n}	n odd	$(2,r)g_{even,r}(n)$	even

TABLE 2. Multiplicative orders of p modulo r for which there exist infinitely many powers q of p such that (26) holds.

Proof. Set $Q = Q(r,G)$ and choose a positive integer a such that $1/r^a < \epsilon/2$. Now choose a positive integer j such that $r^a | p^j - 1$. Note that for any positive integer b, if we let $q = p^{jr^b}$, then $r^a | q-1$ by Lemma 2.5. As in Lemma 5.3, we show that $p_r(\mathrm{SO}^\pm_{2n}(q)) < h_{X,r}(n) + \epsilon/2$ and hence obtain the bound for $p_2(\Omega^\pm_{2n}(q))$ from the equation $p_2(\Omega^\pm_{2n}(q)) = 2p_2(\mathrm{SO}^\pm_{2n}(q))$ in Lemma 6.1. Now observe that for all classes in W whose corresponding tori have at least one cyclic factor of order $q^i - 1$, we have

$$\frac{|Q \cap T|}{|T|} \leq \frac{1}{(q^i-1)_r} \leq \frac{1}{r^a} < \frac{\epsilon}{2}.$$

For a torus T with precisely k negative cycles, we have $|Q \cap T|/|T| = 1/(2,r)^k$. Therefore if l denotes the Witt defect of G, then Theorem 2.2 implies

$$p_r(\mathrm{SO}^\pm_{2n}(q)) < \begin{cases} \epsilon/2 + g_{even,r}(n) & \text{for } l = 0; \\ \epsilon/2 + g_{odd,r}(n) & \text{for } l = 1. \end{cases}$$

In the light of Lemma 2.3, this proves the lemma under the conditions of lines 1 and 3 of Table 2. To prove it for line 4, suppose that the Witt defect $l = 1$ and $n \geq 3$ is odd. Let $\epsilon > 0$ and choose an integer a such that $r^{-a} < \epsilon/2$. If the multiplicative order of p modulo r is even, then there exists a positive integer j' such that $r | p^{j'} + 1$. In the light of Lemma 2.5, there exists a positive integer j such that $r^a | p^j + 1$. Now for any positive integer b, if we let $q = p^{jr^b}$ for r odd and $q = p^{j3^b}$ for $r = 2$, then $r^a | q+1$ by Lemma 2.5. Moreover, $(r, q^i - 1) = (2,r)$ if and only if i is odd, and $(r, q^i + 1) = (2,r)$ if and only if i is even. Therefore for all $w \in W$ with all odd length cycles positive and all even length cycles negative, we have $|Q \cap T_w|/|T_w| = 1/(2,r)^k$, where k is the number of cycles of $\sigma(w) \in S_n$. Since n is odd, the number k_{odd} of cycles of odd length of $\sigma(w) \in S_n$ is odd. If w satisfies our condition then $k_{odd} = k_1$, the number of positive cycles is odd, and thus $k_{even} = k_2$, the number of negative cycles, is odd if and only if k is even. For all other $w \in W$, T_w has at least one cyclic factor of order $q^i - 1$ with i even, or $q^i + 1$ with i odd; thus $|Q \cap T_w|/|T_w| \leq r^{-a} < \epsilon/2$. Applying Theorem 2.2 gives

$$p_r(\mathrm{SO}^-_{2n}(q)) < \epsilon/2 + g_{even,r}(n),$$

as required. As usual, the same argument as the one in Lemma 5.1 proves (26) under the conditions of lines 2, 5, or 6 of Table 2 when r is odd. Finally,

the equation $p_2(\Omega^{\pm}_{2n}(q)) = 2p_2(\mathrm{SO}^{\pm}_{2n}(q))$ proves (26) under the conditions of lines 2, 5, or 6 when $r = 2$. □

Corollary 6.4. *Let r and p be distinct primes with q a power of p. Let $G = \Omega^{\pm}_{2n}(q)$ or $\mathrm{SO}^{\pm}_{2n}(q)$. Then*

$$p_r(G) \geq \begin{cases} \frac{25}{29\sqrt{\pi n}} \cdot \left(\frac{2n-2}{2n-1}\right) & \text{for } r \text{ odd} \\ \frac{1}{8(n+1)^{3/4}} - \frac{9}{25(n+1)^{5/4}} & \text{for } r = 2 \text{ and } G \neq \Omega^{\pm}_{2n}(q); \\ \frac{1}{4(n+1)^{3/4}} - \frac{18}{25(n+1)^{5/4}} & \text{for } r = 2 \text{ and } G = \Omega^{\pm}_{2n}(q). \end{cases}$$

Moreover, given any prime p, there exist infinitely many powers q of p for which

$$p_r(G) \leq \begin{cases} \frac{1}{\sqrt{\pi n}} \cdot \left(\frac{2n}{2n-1}\right) & \text{for } r \text{ odd} \\ \frac{3}{10(n+1)^{3/4}} + \frac{9}{25(n+1)^{5/4}} & \text{for } r = 2 \text{ and } G \neq \Omega^{\pm}_{2n}(q); \\ \frac{3}{5(n+1)^{3/4}} + \frac{18}{25(n+1)^{5/4}} & \text{for } r = 2 \text{ and } G = \Omega^{\pm}_{2n}(q). \end{cases}$$

Furthermore, there exist constants C_5'', C_6'' such that

$$p_2(G) \leq \begin{cases} \frac{C_5''}{n^{5/8}} & \text{if } q \equiv 1 \pmod 4; \\ \frac{C_6''}{n^{1/2}} & \text{if } q \equiv -1 \pmod 4; \end{cases}$$

Therefore Theorems 1.1, 1.2 and 1.3 hold for these groups.

Proof. The first two sets of inequalities follow easily from (4.16) and (4.17) in [GP1], Lemma 2.6, (2) and (3). For the third set of inequalities, we argue in the same way as in Corollary 5.4. □

7. Answers to some open questions

In [BPS, Section 8], the first author, Pálfy and Saxl introduce the notation $\rho(r, X, d, q)$ to denote the proportion of r-regular elements in a classical simple group $X_d(q)$. One of their main results is that $\rho(r, X, d, q) > 1/2d$ for all r, X and q. Moreover if $X_d(q) = \mathrm{PSL}_d(q)$, they show that for infinitely many values of q the bound $\rho(r, X, d, q) \leq 3/\sqrt{d}$ holds. They asked whether it is possible to close the quadratic gap. More specifically, they let

$$\alpha(r, X, q) = \limsup_{d\to\infty} \frac{-\log \rho(r, X, d, q)}{\log d}$$

and $\alpha = \sup_{r,X,q} \alpha(r, X, q)$. They observe that $1/2 \leq \alpha \leq 1$, but ask for the specific value of α. This follows from Theorem 1.6. We combine the proof of Theorem 1.6 with our determination of α in Lemma 7.1.

Lemma 7.1. *Let $\epsilon > 0$. Then there exists q and a constant C such that $p_2(\mathrm{PSL}_d(q)) \leq Cd^{\epsilon-1}$ for all $d \geq 2$. In particular, Theorem 1.6 holds and the supremum $\alpha = \sup_{r,X,q} \alpha(r, X, q)$ is equal to 1.*

Proof. First, choose a positive integer a such that $2^{-a} < \epsilon$ and choose q such that $2^a | q - 1$. Then for every $w \in W$ with $k \geq 2$ cycles, there are, in the corresponding tori of $\mathrm{SL}_d(q)$, at least $k - 1$ cyclic factors of order $q^i - 1$ for

various i. In particular the proportion of 2-regular elements in such a torus T is

$$\frac{|Q \cap T|}{|T|} \leq \frac{1}{2^{a(k-1)}}.$$

Applying Theorem 2.2 implies

$$p_2(\mathrm{SL}_d(q)) \leq \frac{1}{d} + \sum_{k=2}^{d} \frac{c(d,k)}{d!2^{a(k-1)}} = 2^a \sum_{k=1}^{d} \frac{c(d,k)}{d!2^{ak}}$$

and

$$\sum_{k=1}^{d} \frac{c(d,k)}{d!2^{ak}} \leq \frac{C'}{d^{1-2^{-a}}}$$

by (16), where C' depends only on ϵ. Now $p_2(\mathrm{PSL}_d(q)) = (d, q-1)_2 p_2(\mathrm{SL}_d(q))$ and so we have shown that there exists q and a constant C (depending only on ϵ and q) such that

$$p_2(\mathrm{PSL}_d(q)) \leq C/d^{1-2^{-a}} \leq C/d^{1-\epsilon}$$

for all d. This proves Theorem 1.6. It follows that for any $\epsilon > 0$, there exists q such that $\alpha(2, \mathrm{PSL}, q) \geq 1-\epsilon$ and therefore $\alpha = \sup_{r,X,q} \alpha(r, X, q) = 1$. □

The quantity $\alpha(r, X) = \inf_q \alpha(r, X, q)$ is also defined in [BPS] and it is observed that $\alpha(r, \mathrm{PSL}) \geq 1/r$. They ask whether $\inf_r \alpha(r, \mathrm{PSL}) = 0$. We can prove that this is the case; indeed, we claim that for all r we have $1/(r-1) \geq \alpha(r, \mathrm{PSL}) \geq 1/r$. If $r = 2$ then there is nothing to prove so suppose $r \geq 3$. We fix a prime q with multiplicative order $r-1$ modulo r (the existence of q is guaranteed by the Dirichlet prime number theorem, see [Ap, Theorem 7.9] for example). By Remark 1.5(c), we have $p_r(\mathrm{SL}_d(q)) \geq s_{\neg r-1}(d)$ and so (11) implies that there exists a consant C such that $p_r(\mathrm{PSL}_d(q)) \geq Cd^{-\frac{1}{r-1}}$ for all $d \geq 2$. It follows that $\alpha(r, \mathrm{PSL}, q) \leq \frac{1}{r-1}$ and hence $\alpha(r, \mathrm{PSL}) = \inf_q \alpha(r, \mathrm{PSL}, q) \leq \frac{1}{r-1}$ as required. We also note that $\inf_r \alpha(r, X) = 0$ for the other classes of simple group X by Lemmas 4.1, 5.1 and 6.1.

The authors of [BPS] also state that they expect that $\alpha(2, X) \geq 1/2$ for the other types of classical simple group X. This is indeed the case since Theorem 1.3 shows that for every q and each X, there exists a constant C_q (which may depend on q) such that $\rho(2, X, d, q) \leq C_q d^{-1/2}$; thus we have $\alpha(2, X, q) \geq \frac{1}{2}$ for all q as was expected.

References

[AS] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[Ap] T. M. Apostol. *Introduction to analytic number theory*. Springer-Verlag, New York, 1976. Undergraduate Texts in Mathematics.

[BPS] L. Babai, P. P. Pálfy, and J. Saxl. On the number of p-regular elements in finite simple groups. *LMS J. Comput. Math.*, 12:82–119, 2009.

[BLNPS] R. Beals, C. R. Leedham-Green, A. C. Niemeyer, C. E. Praeger, and A. Seress. Permutations with restricted cycle structure and an algorithmic application. *Combin. Probab. Comput.*, 11(5):447–464, 2002.

[BG] A. A. Buturlakin and M. A. Grechkoseeva. The cyclic structure of maximal tori in finite classical groups. *Algebra Logika*, 46(2):129–156, 2007.

[Ca] Roger W. Carter, *Finite groups of Lie type*, Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1985, Conjugacy classes and complex characters, A Wiley-Interscience Publication.

[ET] P. Erdős and P. Turán. On some problems of a statistical group-theory. II. *Acta Math. Acad. Sci. Hungar*, 18:151–163, 1967.

[GP1] S. Guest and C. E. Praeger, Proportions of elements with given 2-part order in finite classical groups of odd characteristic, Submitted.

[GP2] S. Guest and C. E. Praeger, Proportions of elements with given 2-part order in the symmetric group, Submitted.

[GL] R. M. Guralnick and F. Lübeck. On p-singular elements in Chevalley groups in characteristic p. *Groups and computation, III (Columbus, OH, 1999)*, volume 8 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 169–182. de Gruyter, Berlin, 2001.

[KV] J. D. Kečkić and P. M. Vasić. Some inequalities for the gamma function. *Publ. Inst. Math. (Beograd) (N.S.)*, 11(25):107–114, 1971.

[M] R. Michel. On Stirling's formula. *Amer. Math. Monthly*, 109(4):388–390, 2002.

[NP] Alice C. Niemeyer and C. E. Praeger, Estimating proportions of elements in finite simple groups of Lie type, *J. Algebra* **324** (2010), 122–145.

[NPP] Alice C. Niemeyer, Tomasz Popiel and C. E. Praeger, Abundant p-singular elements in finite classical groups, *preprint*.

[PW] C. W. Parker and R. A. Wilson. Recognising simplicity of black-box groups by constructing involutions and their centralisers. *J. Algebra*, 324(5):885–915, 2010.

Department of Computer Science, University of Chicago, 1100 East 58th Street, Chicago, IL 60637, USA
E-mail address: laci@cs.uchicago.edu

Centre for Mathematics of Symmetry and Computation (M019), The University of Western Australia, 35 Stirling Hwy, Crawley, WA 6009, Australia[1]
E-mail address: simon_guest@baylor.edu

Centre for Mathematics of Symmetry and Computation (M019), The University of Western Australia, 35 Stirling Hwy, Crawley, WA 6009, Australia

Also affiliated with:

King Abdulaziz University, Jeddah, Saudi Arabia
E-mail address: cheryl.praeger@uwa.edu.au

School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London, E1 4NS, United Kingdom
E-mail address: r.a.wilson@qmul.ac.uk

[1]Current address for S. Guest: School of Mathematics, University of Southampton, Southampton, SO17 1BJ, UK