Appendix of the Submitted Paper "Maximum Average Service Rate and Optimal Queue Scheduling of Delay-Constrained Hybrid Cognitive Radio in Nakagami Fading Channels"

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I. The Proof of Theorem 1

We assume a continuous exponential distributed random variable T TS with the mean of $1/\mu$ TS, whose cumulative distribution function is $P(T \leq t) = 1 - \exp(-\mu t)$. T is discretiZed into a new random variable $N = \lceil T/\Delta t \rceil$, where Δt is a time unit far less than one TS. Consequently the probability mass function (PMF) of N is derived as

$$\begin{aligned} P(N = n) &= P(n - 1 < \frac{T}{\Delta t} \leq n) = P((n-1)\Delta t < T \leq n\Delta t) \\ &= \left[1 - \left(1 - \exp\left(-\mu\Delta t\right)\right)\right]^{n-1}\left[1 - \exp\left(-\mu\Delta t\right)\right]. \end{aligned} \tag{1}$$

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According to the fact that Δt TS is small enough and μ is in the region of $[0, 1]$. With the aid of the equation $1 - \exp(-x) = x$ when $x \to 0$, we may rewrite Equation (1) as

$$P(N = n) = (1 - \mu\Delta t)^{n-1} \cdot \mu\Delta t. \tag{2}$$

Obviously Equation (2) is the PMF of the geometric distributed discrete random variable N with parameter $\mu\Delta t$. $\mu\Delta t$ represents the successful packet reception probability in one duration of Δt TS. And N represents that the packet is first successfully received by the target at the Nth duration of Δt. As a result, the total time spent to transmit this packet is $N \cdot \Delta t$. We can see clearly that if Δt is small enough, which make the discrete random variable more continuous, we can model the time spent in transmitting one packet as an exponential distribution. The *Theorem 1* is proved.

II. The Derivation of $\Phi(A, B)$ and $\Phi'(A, B)$

We assume having two random variables, which are $X \sim \text{Gamma}(m_X, 1/m_X)$ with a PDF of $f_X(x)$ and $Y \sim \text{Gamma}(m_Y, 1/m_Y)$ with a PDF of $f_Y(y)$, respectively. We set out to find the probability of $P[X > A + BY]$, which may be derived as:

$$P[X > A + BY] = \iint_{X>A+BY} f_X(x) f_Y(y) dx dy. \tag{3}$$

According to [**?**], if m is a positive integer, then the following equations hold:

$$\Gamma(m) = (m-1)! \text{ and } \Gamma(m, x) = (m-1)! \mathrm{e}^{-x} \sum_{n=0}^{m-1} \frac{x^n}{n!}. \tag{4}$$

Given (3) and (4), we succeed in finding the closed-form probability $P[X > A + BY]$ for the following two special cases.

Case 1: m_X must be a positive integer, but m_Y can be any arbitrary real number.

Let us now define the new function $\Phi(A, B)$ representing the probability $P[X > A + BY]$

as:

$$\Phi(A,B) = P[X > A + BY] = \int_0^\infty f_Y(y)dy \int_{A+By}^\infty f_X(x)dx, \tag{5}$$

where the second integral in (5) may be calculated as:

$$\int_{A+By}^\infty f_X(x)dx = P[X > A + By] = \frac{\Gamma(m_X, m_X(A+By))}{\Gamma(m_X)}. \tag{6}$$

Substituting (6) into (5), a general expression of the probability $P[X > A + BY]$ may be obtained:

$$P[X > A + BY] = \int_0^\infty \frac{\Gamma[m_X, m_X(A+By)]}{\Gamma(m_X)} \cdot f_Y(y)dy. \tag{7}$$

Given the assumption that m_X must be a positive integer but m_Y can be any real number, according to the relationship shown in (4) we may rewrite (6) as:

$$\int_{A+By}^\infty f_X(x)dx = \mathrm{e}^{-m_X(A+By)} \sum_{n=0}^{m_X-1} \frac{B^n(A/B+y)^n m_X^n}{n!} \tag{8}$$

Upon invoking the binomial theorem to further expand Equation (8), we arrive at

$$\int_{A+By}^\infty f_X(x)dx = \mathrm{e}^{-m_X(A+By)} \sum_{n=0}^{m_X-1} \frac{(m_X B)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{A}{B}\right)^{n-k} y^k. \tag{9}$$

Substituting (9) into (5), we have

$$\Phi(A,B) = \sum_{n=0}^{m_X-1} \sum_{k=0}^{n} \frac{m_X^n B^k A^{n-k}}{k!(n-k)!} \frac{m_Y^{m_Y}}{\Gamma(m_Y)} \mathrm{e}^{-m_X A} \cdot \int_0^\infty y^{m_Y-1+k} \mathrm{e}^{-(m_X B + m_Y)y} dy \tag{10}$$

As our next step, variable substitution is carried out for the sake of simplifying (10). Upon introducing $\alpha = (m_X B + m_Y)$, we get the closed-form solution to the integral in (10) as follows:

$$\int_0^\infty y^{m_Y-1+k} \mathrm{e}^{-(m_X B+m_Y)y} dy = \left(\frac{1}{m_X B + m_Y}\right)^{m_Y+k} \int_0^\infty \alpha^{m_Y+k-1} \mathrm{e}^{-\alpha} d\alpha = \left(\frac{1}{m_X B + m_Y}\right)^{m_Y+k} \Gamma(m_Y + k) \tag{11}$$

Substituting Equation into (10), we finally arrive at the closed-form formula for $\Phi(A, B)$ in **Case 1**, which is seen in Equation (4) of the submitted paper.

Case 2: m_X **can be any arbitrary real number, but** m_Y **must be a positive integer.**

In order to satisfy the requirements of **Case 2**, we should change the order of the integrals in (5), which may be rewritten as

$$\Phi'(A, B) = P[X > A + BY] = \int_A^{\infty} f_X(x)dx \int_0^{\frac{x-A}{B}} f_Y(y)dy, \tag{12}$$

where the second integral may be formulated as

$$\int_0^{\frac{x-A}{B}} f_Y(y)dy = 1 - \frac{\Gamma(m_Y, m_Y \frac{x-A}{B})}{\Gamma(m_Y)}. \tag{13}$$

Given the relationship in (4), when m_Y is a positive integer, (13) is written as:

$$\int_0^{\frac{x-A}{B}} f_Y(y)dy = 1 - \mathrm{e}^{-m_Y(x-A)/B} \sum_{n=0}^{m_Y-1} \frac{m_Y^n (x-A)^n}{B^n n!}. \tag{14}$$

The term $(x-A)^n$ may be expanded with the aid of the Binomial theorem under the assumption that m_Y is a positive integer, which leads Equation (14) to

$$\int_0^{\frac{x-A}{B}} f_Y(y)dy = 1 - \mathrm{e}^{-m_Y(x-A)/B} \sum_{n=0}^{m_Y-1} \sum_{k=0}^{n} \frac{m_Y^n}{B^n n!} \binom{n}{k} x^k (-A)^{n-k}. \tag{15}$$

Substituting (15) into (12), we have:

$$\Phi'(A, B) = \frac{\Gamma(m_X, m_X A)}{\Gamma(m_X)} - \sum_{n=0}^{m_Y-1} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{m_Y}{B}\right)^n \frac{\mathrm{e}^{m_Y \frac{A}{B}} (-A)^{n-k} m_X^{m_X}}{n! \Gamma(m_X)} \cdot \int_A^{\infty} x^{k+m_X-1} \mathrm{e}^{-x(m_X+m_Y/B)} dx. \tag{16}$$

Again, as our next step, variable substitution is carried out for simplifying (16). Upon introducing $\alpha = x(m_X + m_Y/B)$, we arrive at the closed-form representation of the integral in

(16):

$$\int_A^\infty x^{k+m_X-1}\mathrm{e}^{-x(m_X+m_Y/B)}dx = \frac{\int_{A(m_X+m_Y/B)}^\infty \alpha^{k+m_X-1}\mathrm{e}^{-\alpha}d\alpha}{(m_X+m_Y/B)^{k+m_X}} = \left(\frac{1}{m_X+m_Y/B}\right)^{k+m_X}\Gamma\left[k+m_X, A\left(m_X+\frac{m_Y}{B}\right)\right]. \tag{17}$$

Finally, upon substituting (17) into (16), we arrive at the closed-form function of $\Phi'(A,B)$ in **Case 2**, which is seen in Equation (4) of the submitted paper.