Appendix of the Submitted Paper "Maximum Average Service Rate and Optimal Queue Scheduling of Delay-Constrained Hybrid Cognitive Radio in Nakagami Fading Channels"

Jie Hu, Student Member, IEEE, Lie-Liang Yang, Senior Member, IEEE, and Lajos Hanzo, Fellow, IEEE

I. THE PROOF OF THEOREM 1

We assume a continuous exponential distributed random variable $T$ with the mean of $1/\mu$, whose cumulative distribution function is $P(T \leq t) = 1 - \exp(-\mu t)$. $T$ is discretized into a new random variable $N = [T/\Delta t]$, where $\Delta t$ is a time unit far less than one $T$. Consequently the probability mass function (PMF) of $N$ is derived as

$$P(N = n) = P(n - 1 < \frac{T}{\Delta t} \leq n) = P((n - 1)\Delta t < T \leq n\Delta t) = \left[1 - (1 - \exp(-\mu\Delta t))^{n-1}\right] \left[1 - \exp(-\mu\Delta t)\right].$$  (1)

Jie Hu, Lie-Liang Yang and Lajos Hanzo are with the School of Electronics and Computer Science, University of Southampton, Southampton, SO17 1BJ, UK, e-mail: {jhl0g11, lly, lh}@ecs.soton.ac.uk.

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According to the fact that $\Delta t$ TS is small enough and $\mu$ is in the region of $[0, 1]$. With the aid of the equation $1 - \exp(-x) = x$ when $x \to 0$, we may rewrite Equation (1) as

$$P(N = n) = (1 - \mu \Delta t)^{n-1} \cdot \mu \Delta t. \tag{2}$$

Obviously Equation (2) is the PMF of the geometric distributed discrete random variable $N$ with parameter $\mu \Delta t$. $\mu \Delta t$ represents the successful packet reception probability in one duration of $\Delta t$ TS. And $N$ represents that the packet is first successfully received by the target at the $N$th duration of $\Delta t$. As a result, the total time spent to transmit this packet is $N \cdot \Delta t$. We can see clearly that if $\Delta t$ is small enough, which make the discrete random variable more continuous, we can model the time spent in transmitting one packet as an exponential distribution. The $Theorem 1$ is proved.

II. THE DERIVATION OF $\Phi(A, B)$ AND $\Phi'(A, B)$

We assume having two random variables, which are $X \sim \text{Gamma}(m_x, 1/m_x)$ with a PDF of $f_X(x)$ and $Y \sim \text{Gamma}(m_y, 1/m_y)$ with a PDF of $f_Y(y)$, respectively. We set out to find the probability of $P[X > A + BY]$, which may be derived as:

$$P[X > A + BY] = \int_{X>A+BY} f_X(x)f_Y(y)dxdy. \tag{3}$$

According to $[?]$, if $m$ is a positive integer, then the following equations hold:

$$\Gamma(m) = (m - 1)! \text{ and } \Gamma(m, x) = (m - 1)!e^{-x} \sum_{n=0}^{m-1} \frac{x^n}{n!}. \tag{4}$$

Given (3) and (4), we succeed in finding the closed-form probability $P[X > A + BY]$ for the following two special cases.

**Case 1: $m_x$ must be a positive integer, but $m_y$ can be any arbitrary real number.**

Let us now define the new function $\Phi(A, B)$ representing the probability $P[X > A + BY]$
as:

\[ \Phi(A, B) = P[X > A + BY] = \int_0^\infty f_Y(y)dy \int_{A+By}^\infty f_X(x)dx, \tag{5} \]

where the second integral in (5) may be calculated as:

\[ \int_{A+By}^\infty f_X(x)dx = P[X > A + By] = \frac{\Gamma(m_X, m_X(A + By))}{\Gamma(m_X)}. \tag{6} \]

Substituting (6) into (5), a general expression of the probability \( P[X > A + BY] \) may be obtained:

\[ P[X > A + BY] = \int_0^\infty \frac{\Gamma[m_X, m_X(A + By)]}{\Gamma(m_X)} \cdot f_Y(y)dy. \tag{7} \]

Given the assumption that \( m_X \) must be a positive integer but \( m_Y \) can be any real number, according to the relationship shown in (4) we may rewrite (6) as:

\[ \int_{A+By}^\infty f_X(x)dx = e^{-m_X(A+By)} \sum_{n=0}^{m_X-1} \frac{B^n(A/B + y^n)m^n_Y}{n!} \sum_{k=0}^n \binom{n}{k} \left( \frac{A}{B} \right)^{n-k} y^k. \tag{8} \]

Upon invoking the binomial theorem to further expand Equation (8), we arrive at

\[ \int_{A+By}^\infty f_X(x)dx = e^{-m_X(A+By)} \sum_{n=0}^{m_X-1} \frac{B^n(A/B + y^n)m^n_Y}{n!} \sum_{k=0}^n \binom{n}{k} \left( \frac{A}{B} \right)^{n-k} y^k. \tag{9} \]

Substituting (9) into (5), we have

\[ \Phi(A, B) = \sum_{n=0}^{m_X-1} \sum_{k=0}^n \frac{m^n_X B^k A^{n-k}}{k!(n-k)!} \frac{m^n_Y}{\Gamma(m_Y)} e^{-m_X A} \cdot \int_0^\infty y^{m_Y-1+k} e^{-(m_X B + m_Y)y}dy \tag{10} \]

As our next step, variable substitution is carried out for the sake of simplifying (10). Upon introducing \( \alpha = (m_X B + m_Y) \), we get the closed-form solution to the integral in (10) as follows:

\[ \int_0^\infty y^{m_Y-1+k} e^{-(m_X B + m_Y)y}dy = \left( \frac{1}{m_X B + m_Y} \right)^{m_Y+k} \int_0^{\alpha} \alpha^{m_Y+k-1} e^{-\alpha} d\alpha = \left( \frac{1}{m_X B + m_Y} \right)^{m_Y+k} \Gamma(m_Y + k) \tag{11} \]
Substituting Equation into (10), we finally arrive at the closed-form formula for \(\Phi(A, B)\) in Case 1, which is seen in Equation (4) of the submitted paper.

**Case 2:** \(m_X\) can be any arbitrary real number, but \(m_Y\) must be a positive integer.

In order to satisfy the requirements of Case 2, we should change the order of the integrals in (5), which may be rewritten as

\[
\Phi'(A, B) = P[X > A + BY] = \int_{A}^{\infty} f_X(x)dx \int_{0}^{\frac{x-A}{B}} f_Y(y)dy,
\]

(12)

where the second integral may be formulated as

\[
\int_{0}^{\frac{x-A}{B}} f_Y(y)dy = 1 - \frac{\Gamma(m_Y, \frac{x-A}{B})}{\Gamma(m_Y)}.
\]

(13)

Given the relationship in (4), when \(m_Y\) is a positive integer, (13) is written as:

\[
\int_{0}^{\frac{x-A}{B}} f_Y(y)dy = 1 - e^{-m_Y(x-A)/B} \sum_{n=0}^{m_Y-1} \frac{m_Y^n (x-A)^n}{B^n n!}.
\]

(14)

The term \((x-A)^n\) may be expanded with the aid of the Binomial theorem under the assumption that \(m_Y\) is a positive integer, which leads Equation (14) to

\[
\int_{0}^{\frac{x-A}{B}} f_Y(y)dy = 1 - e^{-m_Y(x-A)/B} \sum_{n=0}^{m_Y-1} \sum_{k=0}^{n} \frac{m_Y^n (x-A)^n}{B^n n! (n-k)!} (-A)^{n-k}.
\]

(15)

Substituting (15) into (12), we have:

\[
\Phi'(A, B) = \frac{\Gamma(m_X, m_XA)}{\Gamma(m_X)} - \sum_{n=0}^{m_Y-1} \sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{e^{m_Y \frac{x}{B}} (-A)^{n-k} m_X^{m_X}}{n! \Gamma(m_X)} \cdot \int_{A}^{\infty} x^{k+m_X-1} e^{-x(m_X+m_Y/B)} dx.
\]

(16)

Again, as our next step, variable substitution is carried out for simplifying (16). Upon introducing \(\alpha = x(m_X + m_Y/B)\), we arrive at the closed-form representation of the integral in
(16):
\[ \int_{A}^{\infty} x^{k+m_X-1} e^{-x(m_X+m_Y/B)} \, dx = \frac{\int_{A(m_X+m_Y/B)}^{\infty} \alpha^{k+m_X-1} e^{-\alpha} \, d\alpha}{(m_X + m_Y/B)^{k+m_X}} = \left( \frac{1}{m_X + m_Y/B} \right)^{k+m_X} \Gamma[k + m_X, A(m_X + m_Y/B)]. \]

Finally, upon substituting (17) into (16), we arrive at the closed-form function of \( \Phi'(A, B) \) in Case 2, which is seen in Equation (4) of the submitted paper.